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ROBUST TESTS FOR SPHERICAL SYMMETRY

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Let R^n be Euclidean n-space and let $O(n)$ be the group of $n \times n$ orthogonal matrices. Consider $\mathscr{F}_0 = \{f \mid f$ is a density on R^n, $f(x) = f(gx)$, $x \in R^n$, $g \in O(n)\}$, and let $Q = \{q \mid q\colon [0, \infty) \to [0, \infty)$, q is nonincreasing, $\int_{R^n} q(\|x\|^2)\, dx = 1\}$. If Σ is an $n \times n$ positive definite matrix, set $\mathscr{F}_1(\Sigma) = \{f \mid f(x) = |\Sigma|^{-\frac{1}{2}} q(x'\Sigma^{-1}x), q \in Q\}$. For $\mu \in R^1$ and $a_0 \in R^n$, $\|a_0\| = 1$, let

$$\mathscr{F}_2(\mu) = \{f \mid f(x) = q(\|x - \mu a_0\|^2), q \in Q\}$$

and

$$\mathscr{F}_3(\mu) = \{f \mid f(x) = q(\|x - \mu a_0\|^2), q \in Q, \text{ and } q \text{ convex}\}.$$

Uniformly most powerful tests are derived for testing $\mathscr{F}_0$ versus $\mathscr{F}_1(\Sigma)$ and for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_2(\mu) \mid \mu > 0\}$. A uniformly most powerful unbiased test is derived for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_3(\mu) \mid \mu \neq 0\}$.

1. Introduction and notation. Throughout, R^n denotes Euclidean n-space and $O(n)$ is the group of $n \times n$ orthogonal matrices. If X is an $n \times 1$ random vector, $\mathscr{L}(X)$ denotes the distribution law of X. A random vector X has a *spherically symmetric* distribution if $\mathscr{L}(X) = \mathscr{L}(gX)$ for $g \in O(n)$. If X has a spherically symmetric distribution and if $P(X = 0) = 0$, we write $\mathscr{L}(X) \in S(n)$.

To describe the classes of probability densities (pdf's) treated in this paper, first let

$$Q = \{q \mid q : [0, \infty) \to [0, \infty),\ q \text{ is nonincreasing, and } \int_{R^n} q(\|x\|^2)\, dx = 1\}. \tag{1.1}$$

Also, let

$$\mathscr{F}_0 = \{f \mid f \text{ is a density on } R^n,\ f(x) = f(gx), x \in R^n, g \in O(n)\}. \tag{1.2}$$

If Σ is an $n \times n$ positive definite matrix, let

$$\mathscr{F}_1(\Sigma) = \{f \mid f(x) = |\Sigma|^{-\frac{1}{2}} q(x'\Sigma^{-1}x), q \in Q\}. \tag{1.3}$$

For $\mu \in R^1$ and $a_0 \in R^n$, $\|a_0\| = 1$, set

$$\mathscr{F}_2(\mu) = \{f \mid f(x) = q(\|x - \mu a_0\|^2), q \in Q\} \tag{1.4}$$

and finally, let

$$\mathscr{F}_3(\mu) = \{f \mid f(x) = q(\|x - \mu a_0\|^2, q \in Q, q \text{ convex}\}. \tag{1.5}$$

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The purpose of this paper is to consider the testing problems $\mathscr{F}_0$ versus $\mathscr{F}_1(\Sigma)$, $\mathscr{F}_0$ versus $\{\mathscr{F}_2(\mu) \mid \mu > 0\}$, and $\mathscr{F}_0$ versus $\{\mathscr{F}_3(\mu) \mid \mu \neq 0\}$.

In Section 2, the distributions of $X/||X||$, $a'X/||X||$ and $X'AX/||X||^2$ are described when $\mathscr{L}(X) \in S(n)$. As an application, the distribution of the sample correlation coefficient is derived under assumptions weaker than the usual normality assumptions. The distribution of $X'AX/||X||^2$ has been derived by Kelker (1970) under the assumption that X has a density $f \in \mathscr{F}_0$.

A UMP test of $\mathscr{F}_0$ versus $\mathscr{F}_1(\Sigma)$, for Σ fixed, is given in Section 3. This result is then extended to cover the following situations when Σ is not fixed: (i) $\Sigma = \sigma^2\Sigma_0$, Σ_0 known, (ii) $\Sigma = \lambda_1(I - M) + \lambda_2 M$, $M^2 = M$, M known, where $\lambda_1 > \lambda_2 > 0$ (or $\lambda_2 > \lambda_1 > 0$), and (iii) $\Sigma^{-1} = \lambda_1 I + \lambda_2 A$, A known, $\lambda_1 > 0$. In these cases the UMP test does not depend on the unknown parameters $(\sigma^2, \lambda_1, \lambda_2)$, so the test is UMP over all the unknown parameters as well as the function q. In Section 4, a UMP test is derived for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_1(\mu) \mid \mu > 0\}$. The basic technique used to derive these tests is a modification of a technique due to Lehmann and Stein (1949) (henceforth abbreviated L–S). In Lemma 3.1 of Section 3, a result concerning the hypothesis of invariance under an infinite group is established, whereas the results of L–S (1949) are mainly for the hypothesis of invariance under a finite group. However, L–S (1949) did present a version of their technique for an infinite group, but it seems to be difficult to apply.

The alternatives $\mathscr{F}_1(\Sigma)$ and $\mathscr{F}_2(\mu)$ contain such distributions as the multivariate t-distribution, the multivariate Cauchy distribution in addition to the multivariate normal distribution. The reader is referred to Johnson and Kotz (1972) and Kelker (1970) for further examples of distributions in $\mathscr{F}_0$, $\mathscr{F}_1(\Sigma)$ and $\mathscr{F}_2(\mu)$. In Section 5, the results of previous sections are applied to the general linear hypothesis in a regression model.

The problem of testing $\mathscr{F}_0$ versus $\{\mathscr{F}_3\{\mu) \mid \mu \neq 0\}$ is considered in Section 6. Sufficiency, completeness and the generalized Neyman-Pearson lemma are used here to derive a UMPU (unbiased) test for the above problem. We briefly discuss a conjecture concerning the existence of a UMPU test for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_2(\mu) \mid \mu \neq 0\}$.

2. The distributions of $a'X/||X||$ and $X'AX/||X||^2$. An n-dimensional normal distribution with mean Δ and covariance matrix Σ is denoted by $N_n(\Delta, \Sigma)$. Let $D(a_1, \cdots, a_{n-1}: a_n)$ denote a Dirichlet distribution with pdf

$$(2.1) \qquad p_n(t_1, \cdots, t_{n-1}) = \Gamma(\textstyle\sum_1^n a_i)[\prod_1^n \Gamma(a_i)]^{-1}[\prod_1^{n-1} t_i^{a_i-1}](1 - \sum_1^{n-1} t_i)^{a_n-1}$$

where $0 \leq t_i$, $\sum_1^{n-1} t_i < 1$, and $a_i > 0$. We write $\mathscr{L}(y_1, \cdots, y_n) = D_n(a_1, \cdots, a_{n-1}: a_n)$ to mean $y_n = 1 - \sum_1^{n-1} y_i$ and $(y_1, \cdots, y_{n-1})$ has the pdf (2.1). $\mathscr{B}(a_1, a_2) \equiv D_2(a_1: a_2)$ denotes the beta distribution.

Consider $C_n = \{x \mid x \in R^n, ||x|| = 1\}$ and let U have the uniform distribution on C_n. $\mathscr{L}(U)$ is the unique probability distribution on C_n which is invariant under $O(n)$.

THEOREM 2.1. *If* $\mathscr{L}(Z) = N_n(0, I)$ *and* $\mathscr{L}(X) \in S(n)$ *then* $\mathscr{L}(X/||X||) = \mathscr{L}(Z/||Z||) = \mathscr{L}(U)$.

PROOF. Let $T(X) = X/||X|| \in C_n$. Then $T(gX) = gT(X)$ for $g \in O(n)$ and so $\mathscr{L}(T(gX)) = \mathscr{L}(T(X)) = \mathscr{L}(gT(X))$. The uniqueness of the invariant probability measure on C_n clearly yields the desired conclusion.

THEOREM 2.2. *Suppose* $\mathscr{L}(X) \in S(n)$ *and* A *is an* $n \times n$ *symmetric matrix. Then* $\mathscr{L}(X'AX/||X||^2) = \mathscr{L}(\sum_1^n d_j y_j)$ *where* $\mathscr{L}(y_1, \cdots, y_n) = D_n(\frac{1}{2}, \cdots, \frac{1}{2} : \frac{1}{2})$ *and the* d_j*'s are the latent roots of* A. *In particular, if* $A^2 = A$ *and rank* $(A) = k$, *then* $\mathscr{L}(X'AX/||X||^2) = \mathscr{B}(k/2, (n-k)/2)$.

PROOF. By Theorem 2.1, we can assume $\mathscr{L}(X) = N_n(0, I)$. Hence the result is immediate.

Press (1969) considered $\mathscr{L}(X'AX/||X||^2)$ when $\mathscr{L}(X) = N_n(0, I)$.

THEOREM 2.3. *For* $\mathscr{L}(X) \in S(n)$ *and* $a \in R^n$, $||a|| = 1$, *let* $W = a'X/||X||$. *Then* $t \equiv (n-1)^{\frac{1}{2}} W/(1-W^2)^{\frac{1}{2}}$ *has a* $t(n-1)$ *distribution—the Student distribution with* $n-1$ *degrees of freedom.*

PROOF. Since $\mathscr{L}(X) \in S(n)$ we can, without loss of generality, take $a' = (1, 0, \cdots, 0)$ and $\mathscr{L}(X) = N_n(0, I)$. If $X' = (X_1, \cdots, X_n)$, a bit of algebra shows that $t = (n-1)^{\frac{1}{2}} X_1/(\sum_2^n X_i^2)^{\frac{1}{2}}$ and the result follows.

Theorem 2.3 is related to a result due to Efron (1969).

EXAMPLE 2.1. Suppose $u' = (u_1, \cdots, u_n)$ and $v' = (v_1, \cdots, v_n)$ are independent with $\mathscr{L}(u) \in S(n)$ and $P(v \in \{e\}) = 0$. Here, $e = (1, \cdots, 1)' \in R^n$ and $\{e\}$ is the span of e. Consider the sample correlation coefficient

$$r = \sum_1^n (u_1 - \bar{u})(v_i - \bar{v})/[\sum_1^n (u_i - \bar{u})^2 \sum_1^n (v_i - \bar{v})^2]^{\frac{1}{2}} .$$

Let $M = ee'/n$ so $r = u'(I-M)v/[u'(I-M)uv'(I-M)v]^{\frac{1}{2}}$. Consider $y = gu$ and $z = gv$ where $g \in O(n)$ and $g(I-M)g = \operatorname{diag}\{1, 1, \cdots, 1, 0\}$ where $\operatorname{diag}\{b_1, \cdots, b_n\}$ denotes a diagonal matrix with diagonal entries $b_1, \cdots, b_n$. Let $\tilde{y}$ and $\tilde{z}$ be the vectors consisting of the first $(n-1)$ coordinates of y and z respectively. Then $r = \tilde{y}'\tilde{z}/||\tilde{y}||\,||\tilde{z}||$ and it is easy to show $\mathscr{L}(\tilde{y}) \in S(n-1)$. Conditioning on $\tilde{z}$ and applying Theorem 2.3, $\mathscr{L}(n-2)^{\frac{1}{2}} r/(1-r^2))^{\frac{1}{2}} = t(n-2)$. Thus, the distribution of r does not depend on either the normality of $\mathscr{L}(u) \in S(n)$ or the distribution of v so long as u and v are independent and $P\{v \in \{e\}\} = 0$.

3. UMP tests for testing $\mathscr{F}_0$ versus $\mathscr{F}_1(\Sigma)$. The first result in this section, designed to cover testing for invariance under infinite groups, is an alternative form of a result due to L–S (1949). Let $(\mathscr{X}, \mathscr{B})$ be a measurable space and let μ be a σ-finite measure on $(\mathscr{X}, \mathscr{B})$. Suppose G is a group acting bimeasurably on the left of $\mathscr{X}$ by $x \to gx$. Let $\mathscr{F}$ be the class of pdf's (with respect to μ) which are invariant under G. Further, suppose $t: \mathscr{X} \to \mathscr{T}$ is a maximal invariant function with range $\mathscr{T}$.

LEMMA 3.1 (Lehmann and Stein). *Suppose that for a given* pdf $h \notin \mathscr{F}$, *there exists a map* s *from* $\mathscr{T}$ *to* $\mathscr{X}$ *such that* $h(s(t(x)))$ *is integrable with respect to* μ. *Then the test* φ *defined by*

$$(3.1)\qquad \begin{aligned} \varphi(x) &= 1 && \text{if} \quad h(x) > kh(s(t(x))) \\ \varphi(x) &= \gamma(x) && \text{if} \quad h(x) = kh(s(t(x))) \\ \varphi(x) &= 0 && \text{if} \quad h(x) < kh(s(t(x))) \end{aligned}$$

is a MP *test of level* α *for testing* $\mathscr{F}$ *versus* h *provided*

$$(3.2)\qquad \mathscr{E}_{f_0}\varphi = \alpha \qquad \text{and} \qquad \mathscr{E}_f\varphi \leqq \alpha \quad \text{for all} \quad f \in \mathscr{F}$$

where $f_0(x) \equiv I^{-1}h(s(t(x)))$, $I \equiv \int h(s(t(x)))\mu(dx)$ *and* k *is a constant.*

PROOF. By construction, φ is a MP test for testing f_0 versus h. If φ_1 is any level α test for testing $\mathscr{F}$ versus h, then $\mathscr{E}_{f_0}\varphi_1 \leqq \alpha$ so $\mathscr{E}_h\varphi \geqq \mathscr{E}_h\varphi_1$ and the conclusion follows from assumption (3.2).

The measurability of the maps t and s in Lemma 3.1 is implicitly assumed. In spite of its general form, the existence of the map s and condition (3.2) are rather restrictive. However, if $\mathscr{X} = R^n$ and G is a compact subgroup of $Gl(n)$, Lemma 3.1 can ordinarily be applied. In particular, if G is the permutation group or $G = O(n)$, as in our case, the application of Lemma 3.1 is straightwforard.

Condition (3.2) can be replaced by the condition of similarity,

$$(3.3)\qquad \mathscr{E}_f\varphi = \alpha \qquad \text{for all} \quad f \in \mathscr{F}.$$

If $\mathscr{X} = R^n$ and G is a compact subgroup of $Gl(n)$, then (3.3) is implied by

$$(3.4)\qquad \int \varphi(gx)\nu(dg) = \alpha$$

where ν is the invariant probability measure on G (see L–S (1949) for the case of a finite group G). L–S (1949) proved that the class of tests satisfying (3.4) forms an essentially complete class.

For the testing problems discussed in Section 1, $\mathscr{X} = R^n$, $G = O(n)$, and μ is Lebesgue measure. A maximal invariant is $t(x) = ||x||$.

THEOREM 3.1. *For a fixed* $\Sigma(\Sigma \neq cI, c > 0)$, *the test* φ, *defined by*

$$(3.5)\qquad \begin{aligned} \varphi(x) &= 1 && \text{if} \quad x'\Sigma^{-1}x/x'x < k \\ \varphi(x) &= 0 && \text{if} \quad x'\Sigma^{-1}x/x'x \geqq k\,, \end{aligned}$$

is a UMP *test of its level for testing* $\mathscr{F}_0$ *versus* $\mathscr{F}_1(\Sigma)$. *For a given level* α, k *can be determined by*

$$(3.6)\qquad \int \cdots \int_A p_n(t_1, \cdots, t_{n-1}) \prod_1^{n-1} dt_i = \alpha$$

where p_n *is the* pdf *of* $D_n(\frac{1}{2}, \cdots, \frac{1}{2} : \frac{1}{2})$, $A = \{\sum_1^n d_j t_j < k\}$ *with* $t_n = 1 - \sum_1^{n-1} t_i$ *and the* d_j's *are the latent roots of* Σ^{-1}.

PROOF. To apply Lemma 3.1, first consider $h_0 \in \mathscr{F}_1(\Sigma)$, $h_0(x) = |\Sigma|^{-\frac{1}{2}} q_0(x'\Sigma^{-1}x)$ and assume q_0 is strictly decreasing. Define $s : [0, \infty) \to R^n$ by $s(t) = t\Sigma^{\frac{1}{2}}a$ where $a \in R^n$ is fixed, $||a|| = 1$. Then

$$I = \int h_0(s(t(x)))\, dx = \int |\Sigma|^{-\frac{1}{2}} q_0(||x||^2)\, dx < +\infty .$$

Since q_0 is strictly decreasing, it follows that φ given by (3.1) (with $\gamma(x) \equiv 0$), is exactly φ given by (3.5). Since (3.2) holds (Theorem 2.2) and since the test does not depend on h_0, we see that φ in (3.5) is UMP for testing $\mathscr{F}_0$ versus $\tilde{\mathscr{F}}_1(\Sigma)$; here, $\tilde{\mathscr{F}}_1(\Sigma) \subseteqq \mathscr{F}_1(\Sigma)$ consists of those h's with strictly decreasing q's. For an arbitrary $h \in \mathscr{F}_1(\Sigma)$, consider $h_0 \in \tilde{\mathscr{F}}_1(\Sigma)$ and let $h_m = (1 - 1/m)h + (1/m)h_0$ so $h_m \in \tilde{\mathscr{F}}(\Sigma)$. If ψ is any level α test, then

$$\int \varphi h_m \geqq \int \psi h_m , \quad m = 1, 2, \cdots .$$

Letting $m \to \infty$ and applying Scheffé's lemma, $\int \varphi h \geqq \int \psi h$ for all $h \in \mathscr{F}_1(\Sigma)$. Thus φ is UMP for testing $\mathscr{F}_0$ versus $\mathscr{F}_1(\Sigma)$. The assertion concerning the calculation of k follows immediately from Theorem 2.2.

The following examples provide some slight extensions of Theorem 3.1 in that Σ is not fixed but depends on some parameters. However, the test φ is shown not to depend on the parameters so φ is UMP over the class $\mathscr{F}_1(\Sigma)$ as well as over the unknown parameters.

EXAMPLE 3.1. $\Sigma = \lambda\Sigma_0$, $\lambda > 0$, Σ_0 known. By absorbing λ into q, Theorem 3.1 is directly applicable with Σ_0 replacing Σ.

EXAMPLE 3.2. $\Sigma = \lambda_1(I - M) + \lambda_2 M$, $\lambda_1 > \lambda_2 > 0$, $M^2 = M$, M known. Since $\Sigma^{-1} = \lambda_1^{-1}(I - M) + \lambda_2^{-1}M$, $x'\Sigma^{-1}x/x'x = \lambda_1^{-1} + (\lambda_2^{-1} - \lambda_1^{-1})x'Mx/x'x$. Thus the test with critical region (c.r.) $x'Mx/x'x < k$ is UMP for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_1(\Sigma) \mid \lambda_1 > \lambda_2 > 0\}$.

The cutoff point, k, for the test is determined from $\mathscr{L}(x'Mx/x'x) = \mathscr{B}(m/2, (n - m)/2)$ where $m = \text{rank}(M)$. For the case at hand, it is also possible to compute the power function of the test φ. Let $\delta = \lambda_2/\lambda_1$ so $0 < \delta < 1$. The power function of φ, say $\pi(\varphi, \delta)$, has the form

$$\pi(\varphi, \delta) = \int_{\delta^*}^{\infty} F(u; m, n - m)\, du \tag{3.7}$$

where δ^* is a function of k and δ (and not of q). Here, $F(u; m, n - m)$ is the pdf of an $F(m, n - m)$ distribution. The details of this are left to the reader.

Consider $\lambda_1 = \sigma^2(1 - \rho)$, $\lambda_2 = \sigma^2(1 - \rho + n\rho)$ where $\rho > 0$, $\sigma^2 > 0$, and $M = ee'/n^{\frac{1}{2}}$. The test φ with c.r. $x'Mx/x'x > k$ provides a test of sphericity versus positive intraclass correlation.

EXAMPLE 3.3. $\Sigma^{-1} = \lambda_1 I + \lambda_2 A$, A known, $\lambda_1 > 0$. Here, λ_2 takes values for which Σ^{-1} is positive definite. Theorem 3.1 shows that φ with c.r. $x'Ax/x'x < k$ is UMP when $\lambda_2 > 0$ and $\mathscr{L}(x'Ax/x'x)$ is given in Theorem 2.2. As a special case, consider Σ of the form

$$\Sigma^{-1} = \tau(1+\rho^2)I - \tau\rho \begin{bmatrix} 0 & 1 & & & 0 \\ 1 & \cdot & & & \\ & \cdot & \cdot & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ 0 & & & 1 & 0 \end{bmatrix}, \quad |\rho| < 1 .$$

This form for Σ^{-1} arises in serial correlation problems. In this case, rejecting for $\sum_2^n x_i x_{i-1}/x'x > k$ is UMP for $\rho > 0$. This test coincides with the test under normality (see Anderson (1948)).

4. UMP tests for testing $\mathscr{F}_0$ versus $\mathscr{F}_2(\mu)$.

THEOREM 4.1. *For testing $\mathscr{F}_0$ versus $\{\mathscr{F}_2(\mu) \mid \mu > 0\}$, the test defined by*

$$\varphi(x) = 1 \quad \text{if} \quad a_0'x/||x|| > k \tag{4.1}$$
$$\varphi(x) = 0 \quad \text{if} \quad a_0'x/||x|| \geqq k$$

is UMP *of its level. For a given α, the cutoff point can be calculated by*

$$\textstyle\int_{k'}^{\infty} t(u; n-1)\, du = \alpha \quad (k' = (n-1)^{\frac{1}{2}}k/(1-k^2)^{\frac{1}{2}}) \tag{4.2}$$

where $t(u; n-1)$ is the pdf *of a $t(n-1)$ distribution.*

PROOF. Fix $\mu > 0$. Consider $h_0 \in \mathscr{F}_2(\mu)$, $h_0(x) = q_0(||x - \mu a_0||^2)$ where q_0 is strictly decreasing. To apply Lemma 3.1, choose $s: [0, \infty) \to R^n$ to be $s(t) = tb$ where $b \in R^n$, $||b|| = 1$ is fixed. Then the test φ defined by (3.1) (with $\gamma(x) = 0$) is equivalent to the test given in 4.1. That (3.2) holds follows from Theorem 2.3. Noting that the test does not depend on q_0 or $\mu > 0$, and arguing as in the proof of Theorem 3.1, φ defined by (4.1) is UMP of its level for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_2(\mu) \mid \mu > 0\}$. The calculation of the cut-off point follows from Theorem 2.3. This completes the proof.

To test $\mathscr{F}_0$ versus $\{\mathscr{F}_2(\mu) \mid \mu < 0\}$, one simply changes a_0 to $-a_0$ and uses the test defined by (4.1). In the case that $a_0' = (1, 1, \cdots, 1)/n^{\frac{1}{2}}$, the test φ is clearly equivalent to the one-sided t-test for testing $\mu = 0$. Thus, Theorem 4.1 establishes a robustness property of the one-sided t-test.

5. An application to linear models. Consider a regression model $y = X\beta + u$ where $y \in R^n$, $X: n \times k$ with rank $(X) = k < n$. Ordinary least squares theory is applicable if $\mathscr{E}(u) = 0$ and $\mathscr{E}(uu') = \sigma^2 I$. Customarily, it is thought that a normality assumption for u must be made to carry out tests of linear hypotheses on $\beta \in R^k$. In this example, we assume $\mathscr{L}(u) \in S(n)$, $\mathscr{E}(||u||^2) < +\infty$ so $\mathscr{E}(u) = 0$ and $\mathscr{E}uu' = \sigma^2 I$, $\sigma^2 > 0$. Consider the problem of testing $A\beta = 0$ where $A: r \times k$ has rank r. (See Scheffè (1959), Lehmann (1959) or Eaton (1972).) After the usual reduction to canonical form, the model is

$$(\underset{k-r}{Z_1'}, \underset{r}{Z_2'}, \underset{n-k}{Z_3'})' = (\gamma_1', \gamma_2', 0)' + (v_1', v_2', v_3')' \tag{5.1}$$

where $(Z_1', Z_2', Z_3')' \equiv Z = Py$ for some $P \in 0(n)$. The γ_i''s and v_i''s have the orders of the corresponding Z_i''s. Clearly, $\mathscr{L}(v) \in S(n)$. The null hypothesis is $H_0: \gamma_2 = 0$. Applying a standard invariance argument to this problem (see Lehmann (1959) or Eaton (1972), Chapter 4) yields the usual F-ratio $Q \equiv (n-k)Z_2'Z_2/rZ_3'Z_3$ as a maximal invariant. From Theorem 2.2, $\mathscr{L}(Q) = F(r, k-r)$. Thus, if attention is restricted to invariant tests, the standard F ratio arises and has (*under* H_0) the F-distribution as long as $\mathscr{L}(u) \in S(n)$. The robustness of the F-test has been studied by Box and Watson (1962) when the error vector $u = (u_1, \cdots, u_n)'$ is a random sample from a symmetric distribution. The assumption of independence of $u_1, \cdots, u_n$ together with $\mathscr{L}(u) \in S(n)$ implies that $\mathscr{L}(u)$ is normal. In the situation treated by Box and Watson, Q no longer has an F-distribution.

If it is assumed that the pdf of u is in $\mathscr{F}_1(\Sigma)$, then one can test $H_0: \mathscr{E}uu' = \sigma^2 I$ versus $H_1: \mathscr{E}uu' = \sigma^2(1-\rho)I + \sigma^2\rho ee'$, $\rho > 0$. In terms of the above canonical form, H_0 remains the same and H_1 becomes $\tilde{H}_1: \mathscr{E}(vv') = \sigma^2(1-\rho)I + \sigma^2\rho aa'$ where $a = Pe$. In this case, the structure of the design matrix X affects the test (see Anderson (1948).) Applying an invariance argument, a UMP invariant test with c.r. $(Z_3'a_3)^2/Z_3'Z_3 < k$ is obtained, provided $a_3 \neq 0$, $a = (a_1', a_2', a_3')'$. The cutoff point and the power function are calculated as in Example 3.2.

6. Testing $\mathscr{F}_0$ versus $\{\mathscr{F}_3(\mu) \mid \mu \neq 0\}$. For a pdf $f \in \mathscr{F}_2(\mu)$, $f(x) = q(||x - \mu a_0||^2) = q(||x||^2 - 2\mu a_0'x + \mu^2)$, where q is nonincreasing. If X has pdf $f \in \mathscr{F}_2(\mu)$, let $T = a_0'X$ and $W = ||X||^2$.

LEMMA 6.1. *The pair* (T, W) *is a complete sufficient statistic for the family* $\{\mathscr{F}_2(\mu) \mid \mu \in R^1\}$. *Further,* W *is a complete sufficient statistic for the family* $\mathscr{F}_0$.

PROOF. Both of the sufficiency assertions follow from the factorization theorem. The completeness of (T, W) follows by noting that: (i) if $\mathscr{L}(X) = N_n(\mu a_0, \sigma^2 I)$, then the density of X is in $\mathscr{F}_1(\mu)$; (ii) (T, W) is complete for the set of distributions in (i); and (iii) the joint distribution of (T, W), under any distribution in $\{\mathscr{F}_2(\mu) \mid \mu \in R^1\}$ is absolutely continuous with respect to the distribution of (T, W) under (i). The completeness of W under $\mathscr{F}_0$ follows similarly.

If $f \in \mathscr{F}_0$, it is clear that $f(x) = q(||x||^2)$ for some function q on $[0, \infty)$.

LEMMA 6.2. *Under* $\mathscr{F}_0$, T *has a density on* $[-1, 1]$ *given by*

$$r_0(t) = 2\left[\mathscr{B}\left(\frac{1}{2}, \frac{n-1}{2}\right)\right]^{-1}(1-t^2)^{(n-3)/2} \tag{6.1}$$

and W *has a density on* $[0, \infty)$ *given by*

$$r_1(\omega) = q(\omega)\frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(n/2)}\,\omega^{n/2-1}. \tag{6.2}$$

Further, under $\mathscr{F}_0$, T *and* W *are independent.*

PROOF. Under $\mathscr{F}_0$, $X/||X||$ has a uniform distribution on C_n so $X_1^2/||X||^2$ has a $\mathscr{B}(\frac{1}{2}, (n-1)/2))$ distribution. That T has the density (6.1) is now clear. The density of W is derived by changing to polar coordinates. Since the density of T does not depend on $f \in \mathscr{F}_0$ and since W is a complete sufficient statistic for $\mathscr{F}_0$, the independence of T and W follows from a result due to Basu (1955).

Define the probability measure λ_0 on $[-1, 1]$ by

$$\lambda_0(dt) = r_0(t)\, dt \tag{6.3}$$

and define the measure ν_0 on $[0, \infty)$ by

$$\nu_0(d\omega) = \frac{(\Gamma(\frac{1}{2}))^n}{\Gamma(n/2)} \omega^{(n/2)-1}\, d\omega\,. \tag{6.4}$$

LEMMA 6.3. *If X has a density $f \in \mathscr{F}_2(\mu)$, then the joint density of T and W with respect to $\lambda_0 \times \nu_0$ when $f(x) = q(||x - \mu a_0||^2)$ is*

$$g(t, \omega; \mu) = q(\omega - 2(\omega)^{\frac{1}{2}} t\mu + \mu^2)\,. \tag{6.5}$$

PROOF. This follows from Lemma 6.2 and an application of Proposition 7.39 in Eaton (1972).

Now, set

$$k(t; \mu, \omega) = \frac{g(t, \omega; \mu)}{\int g(t, \omega; \mu)\lambda_0(dt)}\,. \tag{6.6}$$

Thus, $k(t; \mu, \omega)$ is the conditional density of T given W with respect to λ_0 and $k(t; 0, \omega) = 1$. We want to find a UMPU test for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_3(\mu) \mid \mu \neq 0\}$. For $0 < \alpha < 1$, let $\mathscr{D}_\alpha$ be the class of test functions which are unbiased. Let $\mathscr{E}_0^T$ denote expectation under $\mathscr{F}_0$ with respect to λ_0.

LEMMA 6.4. *If $\varphi \in \mathscr{D}_\alpha$, then*

$$\mathscr{E}_0^T \varphi(T, W) = \alpha \qquad \text{a.e.} \quad (W)(\mathscr{L}(W) \in \mathscr{F}_3(0)) \tag{6.7}$$

and

$$\mathscr{E}_0^T T\varphi(T, W) = 0 \qquad \text{a.e.} \quad (W)(\mathscr{L}(W) \in \mathscr{F}_3(0))\,. \tag{6.8}$$

PROOF. Since $\varphi \in \mathscr{D}\alpha$, $\mathscr{E}_h \varphi \geqq \alpha$ for all $h \in \{\mathscr{F}_3(\mu) \mid \mu \neq 0\}$ and $\mathscr{E}_h \varphi \leqq \alpha$ for all $h \in \mathscr{F}_0$. Hence $\mathscr{E}_h \varphi = \alpha$ for all $h \in \mathscr{F}_3(0)$ by a continuity argument. Thus

$$\mathscr{E}_W[\mathscr{E}_0^T(\varphi(T, W)' - \alpha) \mid W] = 0\,.$$

The completeness of W under $\mathscr{F}_3(0)$ implies (6.7). Assuming $\mathscr{L}(X) = N_n(\mu a_0, \sigma^2 I)$ and arguing as in Lehmann (1959), Chapter 4, shows that (6.8) holds.

Let $\tilde{\mathscr{D}}_\alpha$ denote the set of test functions which satisfy (6.7) and (6.8). Define $\varphi_0 \in \mathscr{D}_\alpha$ by

$$\begin{aligned} \varphi_0(T) &= 1 \quad \text{if } |T| > c \\ &= 0 \quad \text{if } |T| \leqq c \end{aligned} \tag{6.9}$$

where c is chosen so that $\mathscr{E}_0^T \varphi_0 = \alpha$.

THEOREM 6.1. *If* $\varphi \in \tilde{\mathscr{D}}_\alpha$, *then*

$$\mathscr{E}_h \varphi_0 \geqq \mathscr{E}_h \varphi \tag{6.10}$$

for all $h \in \{\mathscr{F}_3(\mu) \mid \mu \in R'\}$.

PROOF. If $h \in \mathscr{F}_2(0)$, then equality holds in (6.10). Fix $h(x) = q(||x - \mu a_0||^2)$ where q is convex and nonincreasing so $h \in \mathscr{F}_3(\mu)$. For a fixed value of W, consider the problem of testing $H_0: \mu = 0$ versus $H_1: \mu = \mu_0 \neq 0$. Applying the generalized Neyman-Pearson lemma (Lehmann (1959)), the supremum of $\mathscr{E}_{\mu_0}(\varphi(T, W) \mid W)$ over the set $\tilde{\mathscr{D}}_\alpha$ is achieved by test functions of the form

$$\begin{aligned} \varphi_1(t) &= 1 \quad \text{if} \quad k(t; \mu_0, \omega) > c_1 + c_2 t \\ &= 0 \quad \text{if} \quad k(t; \mu_0, \omega) \leqq c_1 + c_2 t \end{aligned} \tag{6.11}$$

where k is given by (6.6) and c_1 and c_2 are chosen so that $\varphi_1 \in \tilde{\mathscr{D}}_\alpha$. Since q is convex, $k(t; \mu_0, \omega) - c_2 t$ is a convex function of t. Thus φ_1 can be written as

$$\begin{aligned} \varphi_1(t) &= 0 \quad \text{if} \quad a \leqq t \leqq b \\ &= 1 \quad \text{otherwise} \end{aligned} \tag{6.12}$$

where a and b are chosen so that $\varphi_1 \in \tilde{\mathscr{D}}_\alpha$. However, the only values of a and b such that $\varphi_1 \in \tilde{\mathscr{D}}$ are $-a = b = c$ where c is defined by (6.9). Thus, φ_0 maximizes $\mathscr{E}_{\mu_0}(\varphi(T, W) \mid W)$ over $\tilde{\mathscr{D}}_\alpha$. If $\varphi \in \tilde{\mathscr{D}}_\alpha$, $\mathscr{E}_{\mu_0}(\varphi_0(T) \mid W) \geqq \mathscr{E}_{\mu_0}(\varphi(T, W) \mid W)$ a.e. (W). Integrating on W then yields $\mathscr{E}_h \varphi_0 \geqq \mathscr{E}_h \varphi$ for all $\varphi \in \tilde{\mathscr{D}}_\alpha$. Since φ_0 did not depend on the particular $h \in \mathscr{F}_3(\mu_0)$ or on μ_0, (6.10) holds.

THEOREM 6.2. *The test* φ_0 *in* (6.9) *is* UMPU *for testing* $H_0: h \in \mathscr{F}_0$ *versus* $H_1: f \in \{\mathscr{F}_3(\mu) \mid \mu \neq 0\}$.

PROOF. Since $\mathscr{D}_\alpha \subseteqq \tilde{\mathscr{D}}_\alpha$, the result follows from Theorem 2.1.

Theorem 6.1 is substantially stronger than Theorem 6.2; i.e., φ_0 actually maximizes the conditional power (for W fixed) over all tests in $\mathscr{D}_\alpha$. When $a_0' = (1, 1, \cdots, 1)/(n)^{\frac{1}{2}}$, then φ_0 is just the two-sided t-statistic. Thus, we have another robustness property of the t-test.

Originally, we had set out to prove that φ_0 was UMPU for testing $\mathscr{F}_0$ versus $\{\mathscr{F}_2(\mu) \mid \mu \neq 0\}$, and we conjecture that this result is true. A main difficulty in attempting to establish this conjecture is obtaining a reasonable analytic description of what unbiasedness means.

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