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INDUCED PRIORS IN DECISION PROBLEMS

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The well-known principle that a decision maker's subjective probabilities are determined from his preferences among finite compound lotteries is developed in an especially transparent approach. This development can be easily modified to give versions of the result in a variety of infinite contexts.

1. Introduction. A definition of subjective probability in terms of a decision maker's utilities is given by Anscombe and Aumann [1]. Let $\mathscr{P}$ be a set of payoffs, $\leqq$ a preference relation on the set $\mathscr{P}^*$ of finite probability distributions on $\mathscr{P}$, and $\Theta = \{\theta_1, \cdots, \theta_m\}$ a finite sample space. Let $\mathscr{G}$ be the set of all functions from Θ into $\mathscr{P}^*$ and let $\leqq_g$ be the decision maker's preference relation on the set $\mathscr{G}^*$ of all finite probability distributions on $\mathscr{G}$. Anscombe and Aumann show that if the two preference relations satisfy certain assumptions, then there exist utilities u on $\mathscr{P}^*$ and u_g on $\mathscr{G}^*$, and unique probability masses $\pi_1, \cdots, \pi_m$, such that u agrees with $\leqq$, u_g agrees with $\leqq_g$, and, for all $G \in \mathscr{G}$,

$$u_g(G) = \sum u(G(\theta_i))\pi_i .$$

Fishburn [4, 5] and Ferreira [3] obtain similar results and some extensions. With different sets of underlying assumptions they remove the restriction that Θ is finite; they show that the utility functions are bounded; and they obtain a "subjective" measure that is countably additive.

In this paper, we remark that the above result is essentially an application of the representation theorem for linear functionals on finite-dimensional linear spaces. This remark allows easy generalizations of the result to a variety of conditions on Θ and $\mathscr{G}$, using corresponding representation theorems for linear functionals in infinite-dimensional spaces.

2. Development. We follow as closely as possible the notation of Ferguson ([2], pages 11–21). Assume that there is a utility function u from $\mathscr{P}^*$ onto $[0, 1]$ agreeing with $\leqq$, an enjoying the expected utility property. Let $\mathscr{D}$ be the set of random variables $u \circ G$ where G runs over $\mathscr{G}$; $\mathscr{D}$ is essentially the unit cube in m dimensions. Let $\mathscr{D}^*$ be the set of all finite probability distributions on $\mathscr{D}$ and let $\leqq_d$ be a preference relation on $\mathscr{D}^*$. We consider $\mathscr{D}$ to be embedded in $\mathscr{D}^*$ by identifying degenerate distributions in $\mathscr{D}^*$ with points in $\mathscr{D}$. Assume that there is a utility function u_d on $\mathscr{D}^*$ agreeing with $\leqq_d$, and enjoying the expected utility property. Suppose the following further assumptions

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(essentially those introduced by Anscombe and Aumann and labeled A_1—A_3 by Ferguson, with $\leqq_d$ in place of $\leqq_g$) are satisfied:

A_1 (monotonicity): If D' and D'' are points of $\mathscr{D}$ such that $D'(\theta) \leqq D''(\theta)$ for all θ, then $D' \leqq_d D''$.

A_2 (nondegeneracy): If D' and D'' are points of $\mathscr{D}$ such that $D'(\theta) \equiv d' < d'' \equiv D''(\theta)$, then $D' <_d D''$.

A_3 (equivalence): If D^* is a point of $\mathscr{D}^*$ assigning masses α_i to points D_i in $\mathscr{D}$, and if $D \in \mathscr{D}$ is the convex combination $D = \sum \alpha_i D_i$, then $D \sim_d D^*$.

By A_2, u_d is nonconstant over $\mathscr{D}$ and by A_1, $u_d(D_0) \leqq u_d(D) \leqq u_d(D_1)$, where $D_0 \equiv 0$, $D_1 \equiv 1$ and D is any element of $\mathscr{D}$. Let us choose that version of u_d for which $u_d(D_0) = 0$ and $u_d(D_1) = 1$.

THEOREM. *There exists a unique mass function π over Θ such that for any $D \in \mathscr{D}$, $u_d(D)$ is the expected value of D with respect to π.*

PROOF. Let $0 \leqq \alpha \leqq 1$ and $D, D' \in \mathscr{D}$. By A_3 and the expected utility property, $u_d(\alpha D + (1 - \alpha)D') = \alpha u_d(D) + (1 - \alpha)u_d(D')$, since both sides are the utility of the point in $\mathscr{D}^*$ that assigns mass α to D and mass $1 - \alpha$ to D'. Hence u_d has a unique linear extension to the linear space $\mathscr{D}_e$ spanned by $\mathscr{D}$. Let D^i be the ith unit vector in this space, i.e., $D^i(\theta) = 1$ if $\theta = \theta_i$, and $D^i(\theta) = 0$ otherwise. Set $\pi_i = \pi(\theta_i) = u_d(D^i)$; then $\pi_i \geqq 0$ by A_1. From the linearity we get $\sum \pi_i = u_d(\sum D^i) = u_d(D_1) = 1$; and for D in $\mathscr{D}$, $u_d(D) = u_d(\sum D(\theta_i)D^i) = \sum D(\theta_i)\pi_i$, as asserted.

3. Extensions. This approach to defining subjective probabilities can be readily generalized to more general sample spaces, using in each case an appropriate representation theorem. For example, let Θ consist of the unit interval with its Borel subsets; for simplicity let $\mathscr{P}$ be finite, and let $\mathscr{G}$ be the set of all measurable functions from Θ into $\mathscr{P}^*$. In this case $\mathscr{D}$ is the set of measurable functions from Θ to $[0, 1]$, $\mathscr{D}_e$ is the space of bounded measurable functions on Θ, and the subjective probability π turns out to be a finitely additive measure (cf. Fishburn [4]).

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