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ASYMPTOTICALLY EFFICIENT ESTIMATION OF LOCATION FOR A SYMMETRIC STABLE LAW

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A well-known characteristic function representation of the family of symmetric stable distributions $\mathscr{F}$ indexes them with a location, scale, and type parameter. A sample of size n is taken from an unknown member of $\mathscr{F}$. In this paper, an estimator of the location parameter is constructed which is maximum probability. This means that the estimator conventionally normalized converges in distribution to a normal distribution with zero mean and variance the inverse of the Fisher Information.

1. Introduction. The family of symmetric stable distribution functions $\{F_\alpha((x-\theta)/s)\}$ is identified by the corresponding family of characteristic functions $\varphi(t) = e^{i\theta t - |st|^\alpha}$. Here $-\infty < \theta < \infty$, $0 < s < \infty$, $0 < \alpha \leqq 2$. Gnedenko and Kolmogorov (1968) present a good exposition of the properties and mathematical significance of this family of distribution functions. In recent years, the symmetric stable distribution functions have been proposed as a useful class of models for the behavior of the price of a commodity in a speculative market. As a consequence, the problem of estimating some (or all) of the parameters α, θ and s from a sample has been studied. See Mandelbrot (1963), Fama and Roll (1968), (1971), Press (1972) and DuMouchel (1971), (1973). This problem is made difficult by the fact that the density function of $F_\alpha((x-\theta)/s)$ cannot be written in closed form, except for $\alpha = 1$ and $\alpha = 2$. DuMouchel shows that, subject to considering parameter spaces where α is bounded below by a positive constant, the family of symmetric stable density functions is regular; the likelihood function of a sample of identically distributed symmetric stable random variables has a maximum; and the maximizing vector of parameter values $\tilde{\theta}$ when conventionally normalized converges in distribution to a normal distribution with mean vector zero and covariance matrix the inverse of the Fisher Information matrix. Unfortunately, $\tilde{\theta}$ cannot be solved for explicitly, since the likelihood function of the observations cannot be written down explicitly. No estimators have been explicitly displayed for any of the parameters of the family of symmetric stable distribution functions, which are efficient in a formal sense.

In this paper a sequence of estimators $\bar{\bar{\theta}}(n)$, which is maximum probability (see Weiss and Wolfowitz (1967), (1970)), is constructed for the problem of estimating the location parameter of a symmetric stable distribution function.

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2. The problem and a solution, outlined. Consider the family $\mathscr{F} = \{F_\alpha((x - \theta)/s)\}$ of symmetric stable distributions. Here θ is a location parameter, s is a scale parameter, and α is the so-called type parameter. Let $X_1, X_2, \cdots, X_n$ be independent random variables with distribution function $H(x) \in \mathscr{F}$; estimate the location parameter θ. Hereafter, this problem is referred to as Problem A. The estimator $\bar{\bar{\theta}}(n)$ constructed in this paper for Problem A is shown to be maximum probability (with respect to symmetric intervals about the origin); loosely speaking, this means that $\bar{\bar{\theta}}(n)$ has asymptotically the largest probability of falling in a symmetric interval about the parameter being estimated. An outline of the construction of $\bar{\bar{\theta}}(n)$ follows.

Restate Problem A as follows: let $Y_1, Y_2, \cdots, Y_n$ be the order statistics of a sample of size n from $H(x - \theta)$, where $H(x) \in \{F_\alpha(x/s)\}$; estimate the location parameter θ. Assume temporarily that $H(x)$ is known. Denote $H'(x)$ by $h(x)$. Fenech (1973) shows by standard arguments that the family of densities $h(x - \theta)$ satisfies the conditions of Weiss (1971), and therefore that an estimator of θ is maximum probability if when conventionally normalized it converges in distribution to a normal distribution with mean zero and variance $\{\int [h \circ H^{-1}(t)']^2 \, dt\}^{-1}$. It is easy to verify that $\int [h \circ H^{-1}(t)']^2 \, dt$ equals $\int [h'(x)/h(x)]^2 h(x) \, dx$, the familiar Fisher Information for a location parameter. Suppose now that $p_0(n)$ converges to zero and that $p_0(n) < p_1(n) < \cdots < p_{k(n)}(n)$ divide the interval $[p_0(n), 1 - p_0(n)]$ into $k(n)$ subintervals of equal length $\Delta p(n)$, where $k(n)$ diverges to infinity. Then under certain conditions one may assume that the vector $(Y_{np_0(n)}, Y_{np_1(n)}, \cdots, Y_{np_{k(n)}(n)})'$ has a specified multivariate normal distribution, and the resulting error made in probability calculations converges uniformly to zero as n increases. Acting under such an assumption, an estimator of θ can be easily constructed which conventionally normalized has a normal distribution with mean zero and variance asymptotically equivalent to

$$\left\{\sum_{i=1}^{k(n)} \left[\frac{h \circ H^{-1}(p_i(n)) - h \circ H^{-1}(p_{i-1}(n))}{\Delta p(n)}\right]^2 \Delta p(n)\right\}^{-1} .$$

Observe that the sum contained in brackets resembles a Riemann-sum approximation to the Fisher Information. In fact, a proper choice of the sequences $p(n)$ and $k(n)$ will insure that this sum converges to the Fisher Information. It follows that this estimator conventionally normalized converges in distribution to a normal distribution with mean zero and variance $[\int [h \circ H^{-1}(t)']^2 \, dt]^{-1}$; in other words, this estimator of θ is maximum probability. Now the structure of this estimator is important. It is a weighted average of the order statistics $Y_{np_0(n)}, \cdots, Y_{np_{k(n)}(n)}$, where the weights are functions of i, n and $H(x)$. Denote this estimator by $\bar{\theta}(n)$. At first glance, the construction of $\bar{\theta}(n)$ does not seem to help solve Problem A, where $H(x)$ is not known; instead it is only known that $H(x)$ belongs to $\{F_\alpha(x/s)\}$. However, the lack of knowledge of $H(x)$ and consequently of $\bar{\theta}(n)$ is overcome by borrowing an idea used in a similar setting by Weiss and Wolfowitz (1970). Though $H(x)$ is not known, the estimator $\bar{\theta}(n)$ may still be formally written down. A random set of weights with convenient properties is constructed. Denote by $\bar{\theta}(n)$ the estimator which weights the $Y_{np_i(n)}$ using these random weights. It is shown that $n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$ and $n^{\frac{1}{2}}(\bar{\bar{\theta}}(n) - \theta)$ are asymptotically equivalent, regardless of which $H(x) \in \{F_\alpha(x/s)\}$ is true. Therefore, $\bar{\bar{\theta}}(n)$ is a maximum probability sequence of estimators for Problem A.

3. The solution in detail. In this section, the following notation conventions are observed: $\int f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$, $\int f(t)\,dt = \int_0^1 f(t)\,dt$, $\max Z(i) = \max_{0\leq i\leq k(n)} Z(i)$, $\sum Z(i) = \sum_{i=0}^{k(n)} Z(i)$, $\lim Z(n) = \lim_{n\to\infty} Z(n)$. Denote $F_\alpha'(x)$ by $f_\alpha(x)$. We proceed now to construct the estimator $\bar{\bar{\theta}}(n)$ of θ for Problem A.

Suppose that $(p(n), k(n), l(n))$ is a triple of sequences such that $0 < p(n) < 1$, $k(n) \to \infty$, $l(n) \to \infty$, $np(n)$, $k(n)$, and $l(n)$ are positive integers, and $np(n) + k(n)l(n) = n(1 - p(n))$. Let $Y_1, Y_2, \cdots, Y_n$ be the order statistics of a sample of size n from $H(x) \in \mathscr{F}$. The estimator $\bar{\bar{\theta}}(n)$ will be a weighted average of the order statistics $Y_{np(n)+jl(n)}$, for $0 \leqq j \leqq k(n)$. To verify that $n^{\frac{1}{2}}(\bar{\bar{\theta}}(n) - \theta)$ has the proper limiting behavior, additional conditions relating the triple $(p(n), k(n), l(n))$ and $\mathscr{F}$ are useful. We now list all of these conditions and allow them to define a class of triples $(p(n), k(n), l(n))$ appropriate for the construction of $\bar{\bar{\theta}}(n)$.

Let $S = \{(\Delta(n), \bar{p}(n))\}$ where the pair of sequences $(\Delta(n(, \bar{p}(n))$ possess the following four properties:

(i) $0 < \Delta(n) < 1$, $\lim \Delta(n) = 1$, $\lim n^{1-\Delta(n)} = \infty$,
$0 < \bar{p}(n) < \frac{1}{2}$, $\lim \bar{p}(n) = 0$, $\lim \bar{p}(n)n^{1-\Delta(n)} = \infty$,

(ii) if $H(x) \in \mathscr{F}$, $\lim n^{1-\Delta(n)}[h \circ H^{-1}(\bar{p}(n))]^2 = \infty$,

(iii) if $H(x) \in \mathscr{F}$, $\lim n^{\Delta(n)-1}\{\sup_{\bar{p}(n)\leqq t\leqq 1-\bar{p}(n)} |h \circ H^{-1}(t)''|\}^2 = 0$,

(iv) if $H(x) \in \mathscr{F}$ and $0 < c < 1$, $\lim n^{(1-\Delta(n))3-\frac{1}{2}}[h \circ H^{-1}(c\bar{p}(n))]^{-2} = 0$.

Now S is not empty. Choose δ and ξ, where $0 < \delta < 1$ and $0 < \xi < \frac{1}{2}$. Define $\Delta(n) = 1 - (\log n)^{-\delta}$ and $\bar{p}(n) = (\log n)^{(\delta-1)\xi}$. Using the identity $h \circ H^{-1}(t) = s^{-1}f_\alpha \circ F_\alpha^{-1}(t)$ and the two inequalities from Theorem A.1 (see the Appendix), one can verify that $(\Delta(n), \bar{p}(n)) \in S$. For each pair $(\Delta(n), \bar{p}(n))$ belonging to S, we associate a triple $(p(n), k(n), l(n))$ through the following recipe. For n even (odd), define $k(n)$ to be the largest even (odd) integer in $n^{1-\Delta(n)}\{1 - 2\bar{p}(n)\}$. Define $l(n)$ to be the largest odd integer in $n^{\Delta(n)}$. Define $p(n)$ through the equation $2np(n) = n - k(n)l(n)$, and let $q(n) = 1 - p(n)$. It is easy to verify that the sequence triple $(p(n), k(n), l(n))$ possesses all of the following properties: $0 < p(n) < 1$, $\lim p(n) = 0$, $\bar{p}(n) < p(n)$, $np(n)$, $k(n)$, and $l(n)$ are positive integers, $\lim p(n)/(l(n)/n) = \infty$, $\lim n^{\frac{1}{2}}p(n) = \infty$, and $n^{-\frac{1}{2}} \leqq (l(n)/n)$ for all n large enough.

In Theorem 1, a result of Weiss (1973) on the asymptotic normality of a gradually increasing number of order statistics is adapted to samples from symmetric stable distributions.

THEOREM 1. *Let $Y_1, Y_2, \cdots, Y_n$ be the order statistics of a sample of size n from $H(x) \in \mathscr{F}$. Choose $(\Delta(n), \bar{p}(n)) \in S$. Let $(p(n), k(n), l(n))$ be the sequence triple associated with $(\Delta(n), \bar{p}(n))$. Define $p_i(n) = p(n) + il(n)/n$ for integers i. Denote*

$l(n)/n$ by $\Delta p(n)$; let $Z(n)' = (Z_0(n), Z_1(n), \cdots, Z_{k(n)}(n))$ where $Z_i(n) = n^{\frac{1}{2}} h \circ H^{-1}(p_i(n))(Y_{np_i(n)} - H^{-1}(p_i(n)))$. Let $V(n)' = (V_0(n), V_1(n), \cdots, V_{k(n)}(n))$ be a multivariate normal random variable with mean vector zero and covariance matrix $A(n)$, where

$$A(n)^{-1} = \frac{(l(n)-1)}{l(n)} \Delta p(n)^{-1}$$

$$\times \begin{bmatrix} \frac{\Delta p(n)}{p(n)} + 1 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & 0 \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & & & & \\ & & & \cdot & \cdot & \cdot & & \\ 0 & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & \frac{\Delta p(n)}{p(n)} + 1 \end{bmatrix}.$$

Then for any sequence of measurable sets $D(n) \subseteqq R^{k(n)+1}$,

$$\lim_{n\to\infty} |P(Z(n) \in D(n)) - P(V(n) \in D(n))| = 0 . \tag{3.1}$$

PROOF. Essentially the conditions of the theorem imply that the random variable obtained by evaluating the ratio of the density of $Z(n)$ to the density of $V(n)$ at the random point $V(n)$ converges stochastically to one. This fact implies the conclusion. See Weiss (1969), (1973) for a detailed verification of these points. For our purpose, a straightforward but uniformative proof of the theorem consists in verifying that its assumptions imply the nineteen assumptions of Weiss (1973). This is easy. □

Applying Theorem 1 to Problem A, we discover a random variable $\bar{\theta}(n)$ which suggests the estimator $\bar{\bar{\theta}}(n)$.

THEOREM 2. *Let $Y_1, Y_2, \cdots, Y_n$ be the order statistics of a sample of size n from $H(x) \in \mathscr{F}$. Choose $(\Delta(n), \bar{p}(n)) \in S$ and let $(p(n), k(n), l(n))$ be the sequence triple associated with $(\Delta(n), \bar{p}(n))$. Define $\Delta p(n) = l(n)/n$, $p_i(n) = p(n) + i\Delta p(n)$ for integers i, and $\alpha_i(n) = f_\alpha \circ F_\alpha^{-1}(p_i(n))$. Define*

$$a_0(n) = \frac{[(\Delta p(n)/p(n) + 1)\alpha_0(n) - \alpha_1(n)]\alpha_0(n)}{\Delta p(n)},$$

$$a_j(n) = \frac{(-\alpha_{j-1}(n) + 2\alpha_j(n) - \alpha_{j+1}(n))\alpha_j(n)}{\Delta p(n)} \quad \textit{for} \quad 1 \leqq j \leqq k(n) - 1 ,$$

$$a_{k(n)}(n) = \frac{[(\Delta p(n)/p(n) + 1)\alpha_{k(n)}(n) - \alpha_{k(n)-1}(n)]\alpha_{k(n)}(n)}{\Delta p(n)}, \quad \textit{and}$$

$$b(n) = \frac{1}{p(n)} (\alpha_0^2(n) + \alpha_{k(n)}^2(n)) + \sum_{j=1}^{k(n)} \frac{(\alpha_j(n) - \alpha_{j-1}(n))^2}{(\Delta p(n))^2} \Delta p(n) .$$

Let $\bar{\theta}(n) = \sum b(n)^{-1}a_i(n)Y_{np_i(n)}$. *Then* $n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$ *converges in distribution to a normal distribution with mean zero and variance* $s^2/\int [f_\alpha'(x)/f_\alpha(x)]^2 f_\alpha(x)\,dx$.

PROOF. Define $Z'(n) = (Z_0(n), Z_1(n), \cdots, Z_{k(n)}(n))$ where $Z_i(n) = n^{\frac{1}{2}} f_\alpha \circ F_\alpha^{-1}(p_i(n))s^{-1}(Y_{np_i(n)} - \theta - sF_\alpha^{-1}(p_i(n)))$. Since $(\Delta(n), \bar{p}(n)) \in S$, Theorem 1 applies. We proceed to find the distribution function of $n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$, under the artificial assumption that $Z(n)$ has precisely the distribution function of $V(n)$, where $V(n)$ is as in Theorem 1. Define $\alpha'(n) = (\alpha_0(n), \alpha_1(n), \cdots, \alpha_{k(n)}(n))$, $\beta'(n) = (\beta_0(n), \beta_1(n), \cdots, \beta_{k(n)}(n))$, where $\beta_i(n) = f_\alpha \circ F_\alpha^{-1}(p_i(n))F_\alpha^{-1}(p_i(n))$, and $D(n)' = (D_0(n), D_1(n), \cdots, D_{k(n)}(n))$ where $D_i(n) = \alpha_i(n)Y_{np_i(n)}$. Then we have the linear model $E(D(n)) = (\alpha(n), \beta(n))(\theta, s)'$, $\operatorname{Cov}(D(n)) = s^2A(n)n^{-1}$. Using standard methods, one may verify that the best linear unbiased estimator of θ based on $D(n)$ is $\bar{\theta}(n)$, and that $n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$ is normally distributed with mean zero and variance $s^2/\alpha(n)'A(n)^{-1}\alpha(n)$. We use this artificial result, based on the artificial assumption that $Z(n)$ has the same distribution as $V(n)$, to discover the asymptotic distribution of $n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$. Let $F_n(x)$ be a normal distribution function with mean zero and variance $s^2/\alpha(n)'A(n)^{-1}\alpha(n)$; let $F(x)$ be a normal distribution function with mean zero and variance $s^2/\int [f_\alpha'(x)/f_\alpha(x)]^2 f_\alpha(x)\,dx$. Suppose first that

$$\lim \alpha(n)'A(n)^{-1}\alpha(n) = \int [f_\alpha'(x)/f_\alpha(x)]^2 f_\alpha(x)\,dx\,. \tag{3.2}$$

Now (3.1) implies that $\lim \sup_{-\infty<x<\infty} |P(n^{\frac{1}{2}}(\bar{\theta}_n - \theta) \leqq x) - F_n(x)| = 0$ and (3.2) implies that $\lim \sup_{-\infty<x<\infty} |F_n(x) - F(x)| = 0$. Therefore $n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$ converges in distribution to a normal distribution with mean zero and variance $s^2/\int [f_\alpha'(x)/f_\alpha(x)]^2 f_\alpha(x)\,dx$. The proof will be complete, when we verify (3.2). We have, by Theorem A.5,

$$\alpha(n)'A(n)^{-1}\alpha(n) = \frac{(l(n)-1)}{l(n)}\left\{\frac{\alpha_0^2(n) + \alpha_{k(n)}^2(n)}{p(n)} + \sum_{j=1}^{k(n)} \frac{(\alpha_j(n) - \alpha_{j-1}(n))^2}{(\Delta p(n))^2}\Delta p(n)\right\}.$$

Note that $\alpha_0(n) = \alpha_{k(n)}(n)$. Also for $\alpha \in (0, 2]$, $\lim_{x\to-\infty} [f_\alpha(x)]^2/F_\alpha(x) = 0$ follows for $\alpha \in (0, 2)$ from (A.1) and for $\alpha = 2$ from the fact that $|x|^{-1}\Phi'(x) \sim \Phi(x)$ (Feller (1968), page 175). Therefore $\lim p(n)^{-1}[\alpha_0^2(n) + \alpha_{k(n)}^2(n)] = 0$. Consider now $\sum_{j=1}^{k(n)} (\alpha_j(n) - \alpha_{j-1}(n))^2(\Delta p(n))^{-2}\Delta p(n)$, which equals

$$\sum_{j=1}^{k(n)} \frac{(f_\alpha \circ F_\alpha^{-1}(p_j(n)) - f_\alpha \circ F_\alpha^{-1}(p_{j-1}(n)))^2}{(\Delta p(n))^2}\Delta p(n)\,. \tag{3.3}$$

Refer to Theorem A.2. Allowing $f_\alpha \circ F_\alpha^{-1}(t)$ to play the role of f, we may conclude that (3.3) differs from $\int_{p(n)}^{1-p(n)} [f_\alpha \circ F_\alpha^{-1}(t)']^2\,dt$ by less than $9[\max_{p(n)\leqq t\leqq 1-p(n)} |f_\alpha \circ F_\alpha^{-1}(t)''|]^2 l(n)n^{-1}$, which converges to zero by property (iii) of the pairs $(\Delta(n), \bar{p}(n)) \in S$. It is not difficult to verify that $\int_{p(n)}^{1-p(n)} [f_\alpha \circ F_\alpha^{-1}(t)']^2\,dt$ converges to $\int [f_\alpha'(x)/f_\alpha(x)]^2 f_\alpha(x)\,dx$. Therefore $\sum_{j=1}^{k(n)} ((\alpha_j(n) - \alpha_{j-1}(n))^2/(\Delta p(n))^2)\Delta p(n)$ converges to $\int [f_\alpha'(x)/f_\alpha(x)]^2 f_\alpha(x)\,dx$. □

We now construct the estimator $\bar{\bar{\theta}}(n)$ for Problem A. Denote the coefficient of $Y_{np_i(n)}$ in the random variable $\bar{\theta}(n)$ by $l(n, i, \alpha)$. Then we have $\bar{\theta}(n) = \sum l(n, i, \alpha) Y_{np_i(n)}$. Now by Theorem 2, $n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$ converges in distribution to a normal distribution with mean zero and variance the inverse of the Fisher Information. However, the random variable $\bar{\theta}(n)$ is not an estimator of θ for Problem A, because the $l(n, i, \alpha)$ are functions through the $\alpha_i(n)$ of the unknown type parameter α. Theorem 3 and Theorem 4, which follow, show that estimates of the unknown $l(n, i, \alpha)$ can be constructed from $Y_1, Y_2, \cdots, Y_n$ in such a way that the estimator $\bar{\bar{\theta}}(n)$ of θ for Problem A obtained by weighting the $Y_{np_i(n)}$ by the estimates of the $l(n, i, \alpha)$ has the same asymptotic properties as the random variable $\bar{\theta}(n)$.

THEOREM 3. *Suppose that $Y_1, Y_2, \cdots, Y_n$ are the order statistics of a sample of size n from $F_\alpha((x - \theta)/s) \in \mathscr{F}$. Suppose that estimators $\hat{l}(n, i)$ of the $l(n, i, \alpha)$ are such that $\hat{l}(n, i) = \hat{l}(n, k(n) - i)$ for integers i, $\sum \hat{l}(n, i) = 1$, and such that for $c \in (0, 1)$*

$$\max |\hat{l}(n, i) - l(n, i, \alpha)| = O_p\left\{\frac{n^{\frac{3}{4}}k(n)}{l(n)f_\alpha \circ F_\alpha^{-1}(cp(n))}\right\}, \tag{3.4}$$

regardless of the value of α. Define $\bar{\bar{\theta}}(n) = \sum \hat{l}(n, i) Y_{np_i(n)}$. Then $\bar{\bar{\theta}}(n)$ is a maximum probability estimator of θ for Problem A.

PROOF. It is enough to show that $n^{\frac{1}{2}}(\bar{\bar{\theta}}(n) - \theta) - n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$ converges stochastically to zero. The $Y_{np_i(n)}$ are estimators of the $H^{-1}(p_i(n))$; in particular, since $(\Delta(n), \bar{p}(n)) \in S$, the conditions of the Theorem A.3 are satisfied, and so

$$Y_{np_i(n)} = \theta + sF_\alpha^{-1}(p_i(n)) + \frac{n^{-\frac{1}{2}}}{f_\alpha \circ F_\alpha^{-1}(cp(n))} \lambda(n, i), \tag{3.5}$$

where $\max |\lambda(n, i)| = O_p(1)$. By assumption, we may write

$$\hat{l}(n, i) = l(n, i, \alpha) + \frac{n^{\frac{3}{4}}k(n)}{l(n)f_\alpha \circ F_\alpha^{-1}(cp(n))} \omega(n, i, \alpha), \tag{3.6}$$

where $\omega(n, i, \alpha) = \omega(n, k(n) - i, \alpha)$ for i $(0 \leqq i \leqq k(n))$, $\sum \omega(n, i, \alpha) = 0$, and $\max |\omega(n, i, \alpha)| = O_p(1)$. Using (3.5) and (3.6), we write

$$\begin{aligned}\bar{\bar{\theta}}(n) = \sum &\left(l(n, i, \alpha) + \frac{n^{\frac{3}{4}}k(n)}{l(n)f_\alpha \circ F_\alpha^{-1}(cp(n))} \omega(n, i, \alpha)\right) \\ &\times \left(\theta + sF_\alpha^{-1}(p(n)) + \frac{n^{-\frac{1}{2}}}{f_\alpha \circ F_\alpha^{-1}(cp(n))} \lambda(n, i)\right),\end{aligned}$$

which equals $\bar{\theta}(n) + (n^{\frac{1}{4}}k(n)/l(n)[f_\alpha \circ F_\alpha^{-1}(cp(n))]^2) \sum \omega(n, i, \alpha)\lambda(n, i)$. Therefore $n^{\frac{1}{2}}(\bar{\bar{\theta}}(n) - \theta) - n^{\frac{1}{2}}(\bar{\theta}(n) - \theta)$ is absolutely bounded by

$$\frac{n^{\frac{3}{4}}k(n)(k(n) + 1)}{l(n)[f_\alpha \circ F_\alpha^{-1}(cp(n))]^2} \max |\omega(n, i, \alpha)| \max |\lambda(n, i)| .$$

Replacing $l(n)$ and $k(n)$ by their asymptotic equivalents $n^{\Delta(n)}$ and $n^{1-\Delta(n)}$, property

(iv) of pairs $(\Delta(n), \bar{p}(n))$ in S implies that this last random variable converges stochastically to zero. □

We are left now with the task of producing a set of estimators $\hat{l}(n, i)$ of the $l(n, i, \alpha)$ satisfying the conditions listed in Theorem 3.

THEOREM 4. *Assume that $Y_1, Y_2, \cdots, Y_n$ are the order statistics of a sample of size n from $H(x) \in \mathscr{F}$. Define $\delta(n)$ through $n\delta(n) = [n^{\frac{3}{4}}]$. Define $\hat{g}_i(n)$ to be*

$$\frac{1}{2}\left\{\frac{\delta(n)}{Y_{np_i(n)+n\delta(n)} - Y_{np_i(n)}} + \frac{\delta(n)}{Y_{np_{k(n)-i}(n)+n\delta(n)} - Y_{np_{k(n)-i}(n)}}\right\}.$$

Define the collection of functions $f(i, n)$ by $l(n, i, \alpha) = f(i, n)(\alpha_0(n), \alpha_1(n), \cdots, \alpha_{k(n)}(n))$, and let $\hat{l}(n, i) = f(i, n)(\hat{g}_0(n), \hat{g}_1(n), \cdots, \hat{g}_{k(n)}(n))$. Then $\hat{l}(n, i) = \hat{l}(n, k(n) - i)$ for i $(0 \leqq i \leqq k(n))$, $\sum \hat{l}(n, i) = 1$, and

$$\max |l(n, i, \alpha) - \hat{l}(n, i)| = O_p\left\{\frac{k(n)n^{\frac{3}{4}}}{l(n)f_\alpha \circ F_\alpha^{-1}(cp(n))}\right\}.$$

PROOF. $H(x) = F_\alpha((x-\theta)/s)$; denote $H'(x)$ by $h(x)$; define $g_i(n) = h \circ H^{-1}(p_i(n))$ and $g(n)' = (g_0(n), g_1(n), \cdots, g_{k(n)}(n))$. To begin with, note the form of the functions $f(i, n)$. For example,

$$l(n, 0, \alpha) = f(0, n)(\alpha(n)') = \frac{\dfrac{[(l(n)/np(n) + 1)\alpha_0(n) - \alpha_1(n)]\alpha_0(n)}{l(n)/n}}{l(n)/(l(n) - 1)[\alpha(n)'A(n)^{-1}\alpha(n)]}.$$

It is easy to verify that $f_\alpha \circ F_\alpha^{-1}(p_i(n)) = sh \circ H^{-1}(p_i(n))$, that is $\alpha_i(n) = sg_i(n)$, and so $l(n, 0, \alpha) = f(0, n)(\alpha(n)') = f(0, n)(g(n)')$. In general, $l(n, i, \alpha) = f(i, n)(\alpha(n)') = f(i, n)(g(n)')$. In particular, we may write

$$l(n, 0, \alpha) = \frac{\dfrac{[(l(n)/np(n) + 1)g_0(n) - g_1(n)]}{l(n)/n} g_0(n)}{l(n)/(l(n) - 1)g(n)'A(n)^{-1}g(n)}$$

$$l(n, i, \alpha) = \frac{\dfrac{(-g_{i-1}(n) + 2g_i(n) - g_{i+1}(n))}{l(n)/n} g_i(n)}{l(n)/(l(n) - 1)g(n)'A(n)^{-1}g(n)} \tag{3.7}$$

$$l(n, k(n), \alpha) = \frac{\dfrac{[(l(n)/np(n) + 1)g_{k(n)}(n) - g_{k(n)-1}(n)]}{l(n)/n} g_{k(n)}(n)}{l(n)/(l(n) - 1)g(n)'A(n)^{-1}g(n)}.$$

Since the $g_i(n)$ are simple quantities defined in terms of $H(x)$ and the sample comes from $H(x)$, we can estimate the $g_i(n)$ from the sample. Specifically, since $(\Delta(n), \bar{p}(n)) \in S$, the conditions of Theorem A.4 are satisfied, so

$$\frac{\delta(n)}{Y_{np_i(n)+n\delta(n)} - Y_{np_i(n)}} = g_i(n) + \frac{n^{-\frac{1}{4}}}{f_\alpha \circ F_\alpha^{-1}(cp(n))}\omega(n, i) \tag{3.8}$$

where $c \in (0, 1)$ and $\max |\omega(n, i)| = O_p(1)$. Notice that $g_i(n) = g_{k(n)-i}(n)$ for i $(0 \leqq i \leqq k(n))$. This leads naturally to the estimators $\hat{g}_i(n)$, which are an average of the estimators given by (3.8) for $g_i(n)$ and $g_{k(n)-1}(n)$. Clearly $\hat{g}_i(n) = \hat{g}_{k(n)-i}(n)$ for i $(0 \leqq i \leqq k(n))$ and

$$\max |g_i(n) - \hat{g}_i(n)| = O_p \left\{ \frac{n^{-\frac{1}{4}}}{f_\alpha \circ F_\alpha^{-1}(cp(n))} \right\} . \tag{3.9}$$

The collection of $\hat{l}(n, i)$ are obtained by substituting the $\hat{g}_i(n)$ for the $g_i(n)$ in (3.7). Since $\hat{g}_i(n) = \hat{g}_{k(n)-i}(n)$, $\hat{l}(n, i) = l(n, k(n) - i)$. Using Theorem A.5, $\sum \hat{l}(n, i) = 1$. Denote the numerator in (3.7) by $e(n, i)$; it follows that the denominator of $l(n, i)$ equals $\sum e(n, i)$. Define the $\hat{e}(n, i)$ as the $e(n, i)$ with the $g_j(n)$ replaced by the $\hat{g}_j(n)$. Of course, $\hat{l}(n, i) = \hat{e}(n, i)/\sum \hat{e}(n, j))$. Using (3.9), it is not difficut to verify that

$$\max |\hat{e}(n, i) - e(n, i)| = O_p \left\{ \frac{n^{\frac{3}{4}}}{l(n) f_\alpha \circ F_\alpha^{-1}(cp(n))} \right\}$$

and

$$\sum \hat{e}(n, i) = \sum e(n, i) + O_p \left\{ \frac{k(n) n^{\frac{3}{4}}}{l(n) f_\alpha \circ F_\alpha^{-1}(cp(n))} \right\} .$$

Using these two facts and the observation that $\sum e(n, j)$ converges to a positive constant, it is not difficult to prove that

$$\max |\hat{l}(n, i) - l(n, i, \alpha)| = O_p \left\{ \frac{k(n) n^{\frac{3}{4}}}{l(n) f_\alpha \circ F_\alpha^{-1}(cp(n))} \right\} . \qquad \square$$

This completes the construction of an estimator $\bar{\bar{\theta}}(n)$ for θ for Problem A which is maximum probability.

APPENDIX

Some properties of the symmetric stable densities, used in this paper and in the appendix, are as follows. For $\alpha \in (0, 2]$, $f_\alpha(x)$ is symmetric about zero, strictly positive, nonincreasing for positive x, and possesses derivatives of all orders. For $\alpha \in (0, 2)$, as $x \to \infty$,

$$f_\alpha^{(k)}(x) \sim C(k, \alpha)/x^{\alpha+1+k} \tag{A.1}$$

where $C(k, \alpha)$ is a nonzero constant; $f_2(x)$ is the normal density function with mean zero and variance 2. These properties are referenced or verified in Fenech (1973), pages 81–88. One consequence to note: $F_\alpha^{-1}(t)$ is defined on (0, 1), maps onto $(-\infty, \infty)$, and is differentiable.

THEOREM A.1. *Consider a symmetric stable density* $f_\alpha(x)$. *Then for all positive* p *small enough*, $\min_{t \in (p, 1-p)} f_\alpha \circ F_\alpha^{-1}(t) > \exp(-p^{-2})$ *and* $\max_{t \in (p, 1-p)} |f_\alpha \circ F_\alpha^{-1}(t)''| < \exp(p^{-1})$.

PROOF. First, for $\alpha \in (0, 2)$, we derive bounds on the approach of $F_\alpha^{-1}(\cdot)$ to $-\infty$ and $f_\alpha(\cdot)$ to 0. By (A.1), as $x \to -\infty$, $F_\alpha(x) \sim C(0, \alpha)\alpha^{-1}|x|^{-\alpha}$. For $\beta \in (0, 2)$, define the function $H_\beta(x) = |x|^{-\beta}$ for $x \in (-\infty, -1)$. $H_\beta(x)$ is monotonic and maps onto (0, 1); therefore $H_\beta^{-1}(p)$ exists, and $H_\beta^{-1}(p) = -p^{-\beta^{-1}}$. Suppose

$\beta < \alpha$. Then as $x \to -\infty$, $x \in (-\infty, -1)$, $H_\beta(x)/F_\alpha(x) \to +\infty$; and so for all p small enough $H_\beta^{-1}(p) < F_\alpha^{-1}(p)$. Or, for a choice of $\beta < \alpha$, for all p small enough,

$$(A.2) \qquad -p^{-\beta^{-1}} < F_\alpha^{-1}(p) \, .$$

Now for $\gamma \in (0, 2)$, define the function $g_\gamma(x) = |x|^{-(\gamma+1)}$ for $x \in (-\infty, 0)$. Suppose $\gamma > \alpha$. Then by (A.1), $g_\gamma(x)/f_\alpha(x) \to 0$ as $x \to -\infty$. Or, for a choice of $\gamma > \alpha$, for all x small enough,

$$(A.3) \qquad |x|^{-(\gamma+1)} < f_\alpha(x) \, .$$

We obtain the first inequality. By the monotonicity and symmetry of $f_\alpha(x)$, $\min_{t \in (p,1-p)} f_\alpha \circ F_\alpha^{-1}(t) = f_\alpha \circ F_\alpha^{-1}(p)$. Consider $\alpha \in (0, 2)$; choose α_1 and α_2 satisfying $0 < \alpha_1 < \alpha < \alpha_2 < 2$. By the monotonicity of $f_\alpha(x)$ and (A.2), for all p small enough, $f_\alpha \circ F_\alpha^{-1}(p) \geqq f_\alpha(-p^{-\alpha_1^{-1}})$; by (A.3), for all p small enough, $f_\alpha(-p^{-\alpha_1^{-1}}) > (p^{-\alpha_1^{-1}})^{-(\alpha_2+1)}$. Or, for all p small enough, $f_\alpha \circ F_\alpha^{-1}(p) > p^{(\alpha_2+1)/\alpha_1}$. Recalling that $\exp(x)$ dominates any polynomial in x as $x \to \infty$, for all p small enough, $f_\alpha \circ F_\alpha^{-1}(p) > \exp(-p^{-2})$. Consider $\alpha = 2$. In this case, $f_2(x) = \frac{1}{2}\pi^{\frac{1}{2}} \exp(-x^2/4)$. By (A.1), $f_{\frac{3}{2}}(x) \sim C(0, \frac{3}{2})|x|^{-(\frac{3}{2}+1)}$ as $x \to -\infty$, so for all p small enough $F_{\frac{3}{2}}^{-1}(p) < F_2^{-1}(p)$. By (A.2), $-p^{-1} < F_{\frac{3}{2}}^{-1}(p)$ for all p small enough, and so $f_2(-p^{-1}) < f_2 \circ F_2^{-1}(p)$ for all p small enough.

Now we obtain the second inequality. To begin with, $f_\alpha \circ F_\alpha^{-1}(t)''$ equals

$$(A.4) \qquad f_\alpha'' \circ F_\alpha^{-1}(t)/[f_\alpha \circ F_\alpha^{-1}(t)]^2 - [f_\alpha' \circ F_\alpha^{-1}(t)]^2/[f_\alpha \circ F_\alpha^{-1}(t)]^3 \, .$$

Note that (A.4) is continuous for $t \in (0, 1)$, and is symmetric about $t = \frac{1}{2}$. Using (A.1), as t goes to zero, we obtain the equivalences $f_\alpha'' \circ F_\alpha^{-1}(t)/[f_\alpha \circ F_\alpha^{-1}(t)]^2 \sim C(2, \alpha)[C(0, \alpha)]^{-2}|F_\alpha^{-1}(t)|^{\alpha-1}$ and $[f_\alpha' \circ F_\alpha^{-1}(t)]^2/[f_\alpha \circ F_\alpha^{-1}(t)]^3 \sim [C(1, \alpha)]^2[C(0, \alpha)]^{-3}|F_\alpha^{-1}(t)|^{\alpha-1}$. Consider $\alpha \in (0, 1]$. The above equivalence relations together with the continuity and symmetry of (A.4) imply that $|f_\alpha \circ F_\alpha^{-1}(t)''|$ is bounded for $t \in (0, 1)$; the inequality follows. Consider $\alpha \in (1, 2)$. In this case, the equivalence relations imply that for all p small enough, $\max_{t \in (p,1-p)} |f_\alpha \circ F_\alpha^{-1}(t)''| < |F_\alpha^{-1}(p)|$. Using (A.2) with $\beta = 1$, we have for all p small enough, $\max_{t \in (p,1-p)} |f_\alpha \circ F_\alpha^{-1}(t)''| < 1/p$; the inequality follows. Finally, consider $\alpha = 2$. Substituting $(\frac{1}{2}\pi^{\frac{1}{2}}) \exp(-x^2/4)$ into (A.4), we find $f_2 \circ F_2^{-1}(t)'' = [a + bF_2^{-1}(t)^2] \exp[F_2^{-1}(t)^2/4]$ where a and b are nonzero constants. Therefore, for all p small enough, $\max_{t \in (p,1-p)} |f_2 \circ F_2^{-1}(t)''| \leqq \exp[F_2^{-1}(p)^2]$. Using $\Phi(x) < |x|^{-1}\Phi'(x)$ for $x < 0$ (see Feller (1968), page 85), for all x small enough $F_2(x) < \exp(-x^2)$, and so for all p small enough, $F_2^{-1}(p)^2 < -\log p$. Therefore, for all p small enough $\max_{t \in (p,1-p)} |f_2 \circ F_2^{-1}(t)''| \leqq \exp[-\log p]$; the inequality follows. □

THEOREM A.2. *Assume that f is real-valued on* (0, 1), *that f is twice differentiable on* (0, 1), *and* $f(\frac{1}{2}) = 0$. *Choose p where* $0 < p < \frac{1}{2}$; *choose k, a positive integer. Define* $\Delta p = (1 - 2p)/k$, $p_j = p + j\Delta p$, *and* $E = \sup_{p \leq t \leq 1-p} |f''(t)|$. *Then*

$$\left| \sum_{j=1}^{k} \frac{[f(p_j) - f(p_{j-1})]^2}{\Delta_p} - \int_p^{1-p} [f'(t)]^2 \, dt \right| \leqq 9E^2 \Delta p \, .$$

PROOF. By the continuity of $f'(t)$ and the mean value theorem,

$$\int_p^{1-p} [f'(t)]^2\, dt = \sum_{j=1}^k [f'(\theta(j))]^2 \Delta p\,, \tag{A.5}$$

where $p_{j-1} \leqq \theta(j) \leqq p_j$. Now, using Taylor's theorem,

$$\frac{f(p_j) - f(p_{j-1})}{\Delta p} = f'(p_{j-1}) + \frac{f''(\omega(j))}{2} \Delta p\,, \tag{A.6}$$

where $p_{j-1} \leqq \omega(j) \leqq p_j$, and

$$f'(p_{j-1}) = f'(\theta(j)) + E(j)\,, \tag{A.7}$$

where $|E(j)| \leqq E\Delta p$. Combining (A.6) and (A.7),

$$\frac{f(p_j) - f(p_{j-1})}{\Delta p} = f'(\theta(j)) + R(j)\,, \tag{A.8}$$

where $|R(j)| \leqq \frac{3}{2}E\Delta p$. Now substitute (A.8) into

$$\sum_{j=1}^k \frac{[f(p_j) - f(p_{j-1})]^2}{\Delta p^2} \Delta p\,.$$

The resulting sum expands naturally into four terms; one of them is identical to the right-hand side of (A.5), and the other three are each easily bounded in absolute value by $3E^2\Delta p$. □

THEOREM A.3. *Assume that $H(x)$ is a distribution function, such that $H'(x) = h(x)$ exists, is strictly positive, and is symmetric and monotone about some point. Assume that $0 < p(n) < 1$, $\lim p(n) = 0$, $np(n)$ and $l(n)$ are positive integers, $\lim np(n)/l(n) = \infty$. Choose $c \in (0, 1)$. Let $Y_1(n), \cdots, Y_n(n)$ be the order statistics of a sample of size n from $H(x)$. Define $C(n) = \{i\colon np(n) - l(n) \leqq i \leqq n - np + l(n)\}$, and $\lambda(n, i)$ through*

$$Y_i(n) = H^{-1}\left(\frac{i}{n}\right) + \frac{n^{-\frac{1}{2}}}{h \circ H^{-1}(cp(n))} \lambda(n, i)\,.$$

Then

$$\max_{j \in C(n)} |\lambda(n, j)| = O_p(1)\,.$$

PROOF. Define $W_i(n) = n^{\frac{1}{2}}(H(Y_i(n)) - i/n)$. Applying Taylor's theorem to the left-hand side of $H^{-1}(n^{-\frac{1}{2}} W_i(n) + i/n) = Y_i(n)$,

$$H^{-1}(i/n) + \frac{n^{-\frac{1}{2}} W_i(n)}{h \circ H^{-1}(\theta(i, n))} = Y_i(n)\,, \tag{A.9}$$

where $\theta(i, n)$ depends on i, n, and $W_i(n)$, and $\theta(i, n)$ is between i/n and $i/n + n^{-\frac{1}{2}} W_i(n)$.

We proceed now to bound with probability approaching one the approach of $\min_{i \in C(n)} h \circ H^{-1}(\theta(i, n))$ to zero. By Kolmogorov's theorem (see Fisz (1963), page 394), $\max |W_i(n)| = O_p(1)$. Choose $c \in (0, 1)$ and define the sequence $\alpha(n)$ through $n^{-\frac{1}{2}}\alpha(n) = (1 - c)p(n) - l(n)/n$; note that $n^{-\frac{1}{2}}\alpha(n)$ goes to zero, while $\alpha(n)$ goes to infinity. Therefore $\lim P\{\max |n^{-\frac{1}{2}} W_i(n)| < n^{-\frac{1}{2}}\alpha(n)\} = 1$. Note that

$\{\max|n^{-\frac{1}{2}}W_i(n)| < n^{-\frac{1}{2}}\alpha(n)\} \subseteqq \{\max|\theta(i,n) - i/n| < n^{-\frac{1}{2}}\alpha(n)\} \subseteqq \{[np(n) - l(n)]n^{-1} - n^{-\frac{1}{2}}\alpha(n) < \theta(i,n) < [n - np(n) + l(n)]n^{-1} + n^{-\frac{1}{2}}\alpha(n)$ for $i \in C(n)\}$. Substituting the definition of $n^{-\frac{1}{2}}\alpha(n)$ into the preceding event, we conclude that $\lim P\{cp(n) < \theta(i,n) < 1 - cp(n)$ for $i \in C(n)\} = 1$. Therefore, $\min_{i\in C(n)} h \circ H^{-1}(\theta(i,n))$ is bounded below by $h \circ H^{-1}(cp(n))$ with probability approaching one.

Referring to (A.9),

$$\max_{i\in C(n)}\left|Y_i(n) - H^{-1}\left(\frac{i}{n}\right)\right|$$

$$= n^{-\frac{1}{2}} \max_{i\in C(n)} \frac{|W_i(n)|}{h \circ H^{-1}(\theta(i,n))}$$

$$\leqq \frac{n^{-\frac{1}{2}}}{h \circ H^{-1}(cp(n))} \max_{i\in C(n)} \frac{h \circ H^{-1}(cp(n))}{h \circ H^{-1}(\theta(i,n))} \max|W_i(n)|\,.$$

The conclusion of the theorem follows. □

The basic idea of Theorem A.3, that of using the Kolmogorov theorem to bound the maximum difference between a collection of order statistics and the quantiles they estimate, is found in Weiss and Wolfowitz (1970).

THEOREM A.4. *Assume that $H(x)$ is a twice-differentiable distribution function such that $H'(x) = h(x)$ is strictly positive, is symmetric and monotone about some point, and* $\sup|h'(x)|$ *is finite. Assume that* $0 < p(n) < 1$; *that* $\lim p(n) = 0$; *that $np(n)$, $k(n)$, and $l(n)$ are positive integers; that* $\lim np(n)/l(n) = \infty$; *that* $np(n) + k(n)l(n) = n - np(n)$; *that* $\lim n^{\frac{1}{4}}p(n) = \infty$; *and that $n^{-\frac{1}{4}} \leqq l(n)/n$ for all n large enough. Define $p_i(n) = p(n) + il(n)/n$ and $\delta(n)$ through $n\delta(n) = [n^{\frac{3}{4}}]$. Let $Y_1(n), \cdots, Y_n(n)$ be the order statistics of a sample of size n from $H(x)$. Choose $c \in (0,1)$. Then*

$$\frac{\delta(n)}{Y_{np_i(n)+n\delta(n)} - Y_{np_i(n)}} = h \circ H^{-1}(p_i(n)) + \frac{n^{-\frac{1}{4}}}{h \circ H^{-1}(cp(n))}\,\omega(n,i)\,,$$

where $\max|\omega(n,i)| = O_p(1)$.

PROOF. Define $\delta(n)$ by $n\,\delta(n) = [n^{1+\beta}]$, where $-1 < \beta < 0$. Note that $\delta(n) \sim n^{\beta}$. Later we will choose $\beta = -\frac{1}{4}$. By Taylor's theorem,

$$\frac{H(y) - H(x)}{y - x} = H'(\theta)\,,$$

where θ is between y and x. It follows that

$$\frac{\delta(n)}{H^{-1}(p_i(n) + \delta(n)) - H^{-1}(p_i(n))} = h(\theta(i,n))\,,$$

where $H^{-1}(p_i(n)) \leqq \theta(i,n) \leqq H^{-1}(p_i(n) + \delta(n))$. Denote $\sup|h'(x)|$ by M. Assume that $\delta(n)/p(n)$ goes to zero; note that this is true for $\delta(n) \sim n^{-\frac{1}{4}}$. Recall that $c \in (0,1)$. Then for all n large enough, $cp(n) \leqq p_i(n) + \delta(n) \leqq 1 - cp(n)$. Now by Taylor's theorem, $h(\theta(i,n)) = h(H^{-1}(p_i(n))) + h'(\psi(i,n))(\theta(i,n) - H^{-1}(p_i(n)))$ where $H^{-1}(p_i(n)) \leqq \psi(i,n) \leqq \theta(i,n)$. So for all n large enough,

$$|h'(\psi(i,n))(\theta(i,n) - H^{-1}(p_i(n)))| \leqq M\,\delta(n)/(h \circ H^{-1}(cp(n)))\,.$$

We conclude that for all n large enough,

$$\left|\frac{\delta(n)}{H^{-1}(p_i(n)+\delta(n))-H^{-1}(p_i(n))} - h\circ H^{-1}(p_i(n))\right| \leqq \frac{M\,\delta(n)}{h\circ H^{-1}(cp(n))} \tag{A.10}$$

for $0 \leqq i \leqq k(n)$. We proceed now to estimate

$$\frac{\delta(n)}{H^{-1}(p_i(n)+\delta(n))-H^{-1}(p_i(n))}\,. \tag{A.11}$$

Notice that the conditions of Theorem A.3 are written into the conditions of Theorem A.4. Note that for all n large enough,

$$np(n) - l(n) \leqq np_i(n) < np_i(n) + n\,\delta(n) \leqq n(1-p(n)) + l(n)$$

for $0 \leqq i \leqq k(n)$. Therefore Theorem A.3 implies that, defining $C(n) = \{np_i(n), np_i(n) + n\,\delta(n), 0 \leqq i \leqq k(n)\}$, for $j \in C(n)$

$$Y_j(n) = H^{-1}\left(\frac{j}{n}\right) + \frac{n^{-\frac{1}{2}}}{h\circ H^{-1}(cp(n))}\lambda(n, j)\,,$$

where $\max_{j\in C(n)} |\lambda(n, j)| = O_p(1)$. Substitute these estimates of $H^{-1}(p_i(n) + \delta(n))$ and $H^{-1}(p_i(n))$ into (A.11). Tedious but straightforward manipulation of the result yields

$$\frac{\delta(n)}{Y_{n(p_i(n)+\delta(n))} - Y_{np_i(n)}} = \frac{\delta(n)}{H^{-1}(p_i(n)+\delta(n))-H^{-1}(p_i(n))} + \frac{n^{-\frac{1}{2}}}{h\circ H^{-1}(cp(n))}\frac{1}{\delta(n)}\eta(n, i)\,, \tag{A.12}$$

where $\max |\eta(n, i)| = O_p(1)$. Now (A.10) and (A.12) together imply that

$$\left|\frac{\delta(n)}{Y_{n(p_i(n)+\delta(n))} - Y_{np_i(n)}} - h\circ H^{-1}(p_i(n))\right| \leqq \frac{M\,\delta(n)}{h\circ H^{-1}(cp(n))} + \frac{n^{-\frac{1}{2}}}{h\circ H^{-1}(cp(n))\,\delta(n)}\eta(n, i)\,,$$

where $\max|\eta(n, i)| = O_p(1)$. Recall that $\delta(n) \sim n^{\beta}$, where $-1 < \beta < 0$. Clearly, in order to maximize the rate of convergence of the above error bound to zero, choose $\beta = -\frac{1}{4}$. The conclusion of the theorem follows. □

Recall the definition of $A(n)^{-1}$ given in Theorem 1.

THEOREM A.5. *Consider the* $(k(n) + 1)$ *vector* $a(n)$, *where* $a(n)' = (a_0(n), a_1(n), \cdots, a_{k(n)}(n))$. *Then*

$$\{l(n)/(l(n)-1)\}a(n)'A(n)^{-1}a(n) = \frac{\{(\Delta p(n)/p_0(n) + 1)a_0(n) - a_1(n)\}a_0(n)}{\Delta p(n)} + \sum_{j=1}^{k(n)} \frac{(-a_{j-1}(n) + 2a_j(n) - a_{j+1}(n))a_j(n)}{\Delta p(n)}$$

$$+\frac{\{(\Delta p(n)/p_0(n)+1)a_{k(n)}(n)-a_{k(n)-1}(n)\}a_{k(n)}(n)}{\Delta p(n)}$$

$$=\frac{1}{p_0(n)}(a_0^2(n)+a_{k(n)}^2(n))+\sum_{j=1}^{k(n)}\frac{(a_j(n)-a_{j-1}(n))^2}{\Delta p(n)}.$$

PROOF. Note that $a'(n)A(n)^{-1}a(n)=(a'(n)A(n)^{-1})a(n)$; this gives the first equality. Multiplying out the middle expression above and rearranging the terms of the resulting sum gives

$$\frac{a_0^2(n)+a_{k(n)}^2(n)}{p_0(n)}+\sum_{j=1}^{k(n)}\frac{a_j^2(n)-2a_{j-1}(n)a_j(n)+a_{j-1}^2(n)}{\Delta p(n)}. \qquad \square$$

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