The Annals of Statistics
1975, Vol. 3, No. 6, 1361–1363

ESTIMATING GENERATING FUNCTIONS

BY M. H. HOYLE

Peat Marwick Mitchell & Co.

This note shows that, under appropriate conditions, if a function $A(\theta; t)$ of an unknown parameter θ and a real variable t has an infinite series expansion and if there is a function $B(S; t)$ of the sufficient statistic S which is an unbiased estimator of A for every t and which also has an infinite series expansion, then the coefficients of the power of t in the expansion of B are the proper estimators for the coefficients of the corresponding powers in the expansion of A. This result is applied to estimate two functions of the normal parameters, μ and σ^2, which arise in the derivation of expressions for the removal of transformation bias.

1. Both Neymann and Scott (1960) and Hoyle (1968), in the course of deriving results concerned with removing transformation bias, found it necessary to evaluate unbiased estimates of certain complicated functions of the parameters, μ and σ^2, where $\xi \sim N(\mu, \sigma^2)$. Hoyle (1968) needed an estimator for $\alpha_1(\mu, \sigma^2; m, k) = \mu^m\sigma^{2k}(m > -2k)$ and Neyman and Scott (1960) for $\alpha_2(\mu, \sigma^2; m) = E(\xi^m)$. The main purpose of this note is to point out that their expressions may be obtained by estimating the generating function of α_1, α_2 and then formally expanding these estimators as a power series so leading to the required results. It is hoped this approach may have application in other areas; Stigler (1971) has already considered the use of an empirical generating function to estimate extinction probabilities in branching processes.

2. Let S be a sufficient statistic for the parameter θ and let $B(S; t)$ be an unbiased estimator of $A(\theta; t)$ for every t. Further suppose that $A(\theta; t) = \sum_{m=0}^{\infty} \alpha(\theta; m)t^m/m!$ and $B(S; t) = \sum_{m=0}^{\infty} \beta(S; m)t^m/m!$ are convergent for all t. If the condition $\mathscr{C}$: $\sum_{m=0}^{\infty} E_S|\beta(S; m)t^m/m!| < \infty$ in some neighborhood of 0 is satisfied, then $E_S\{\beta(S; m)\} = \alpha(\theta; m)$ for all m. Because of the familiar results of Lehmann and Scheffé (1950) the function $\beta(S; m)$ is the minimum variance estimator of $\alpha(\theta; m)$.

To prove this result we have by supposition $E_S\{B(S; t)\} = A(\theta; t)$, that is

$$E_S\{\textstyle\sum_{m=0}^{\infty} \beta(S; m)t^m/m!\} = \sum_{m=0}^{\infty} \alpha(\theta; m)t^m/m! \,.$$

Thus so long as the operations of expectation and summation can be interchanged

$$\textstyle\sum_{m=0}^{\infty} \{E_S(S; m) - \alpha(\theta; m)\}t^m/m! = 0 \,.$$

Then by standard arguments, (see for example, Rudin (1953) Theorem 8.5), $E_S\{\beta(S; m)\} = \alpha(\theta; m)$ so long as the power series representation of $A(\theta; t)$ and

Received September 1971; revised October 1974.

AMS 1970 *subject classifications.* Primary 62F10; Secondary 60E05.

Key words and phrases. Transformation bias, unbiased estimation, generating functions, sufficient statistics, infinite series expansion.

$B(S; t)$ are convergent for all t. The conditions under which it is permissible to interchange the expectation and summation operations are exactly the condition $\mathscr{C}$ plus the condition that $E_S|\sum_{m=0}^{\infty} \beta(S; m)t^m/m!| < \infty$, that is $E_S|B(S; t)| < \infty$, and this is true by supposition. This proves the result which can clearly be extended very simply to the multivariate case.

3. As an example, consider estimating the function $\alpha_1(\mu, \sigma^2; m, k) = \mu^m\sigma^{2k}$, then

$$A(\mu, \sigma^2; t, r) = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \mu^m\sigma^{2k}(r/2)^k t^m/(m!\, k!) = \exp(\mu t + r\sigma^2/2)\,.$$

Now using the results of Finney (1941)

$$B(\hat{\mu}, \hat{\sigma}^2; t, r) = e^{\hat{\mu}t}\Phi[(r - t^2\lambda^2)S, \nu]$$

where $\hat{\mu}$ and $\hat{\sigma}^2 = \nu^{-1}S$ are the usual optimal estimators of μ and σ^2 based on n observations so that $\hat{\mu} \sim N(\mu, \lambda^2\sigma^2)$, $\nu\hat{\sigma}^2/\sigma^2 \sim \chi_\nu^{\ 2}$ and where

$$\Phi[xS, \nu] = \sum_{i=0}^{\infty} \frac{\Gamma(\nu/2)(xS/4)^i}{\Gamma(\nu/2 + i)i!}\,.$$

Thus the unbiased estimator of $\mu^m\sigma^{2k}$ is the coefficient of $t^m r^k/(m!\, k!\, 2^k)$ in the power series expansion of $B(\hat{\mu}, \hat{\sigma}^2; t, r)$ provided that the condition $\mathscr{C}$ holds. It is not too difficult, though not trivial, to show that the required coefficient is

$$\beta_1(\hat{\mu}, \hat{\sigma}^2; n, k) = \Gamma\left(\frac{\nu}{2}\right) m! \left(\frac{S}{2}\right)^k \sum_{i=0}^{[m/2]} \frac{(-S\lambda^2/2)^i \hat{\mu}^{m-2i}}{i!\,(m - 2i)!\,\Gamma(\nu/2 + k + i)}$$

where $[p]$ is the integral part of p. That the condition $\mathscr{C}$ holds so long as $m > -2k$ follows from the application of the results of Hoyle (1968) in particular those of Sections 4 (iv) and 4 (v), page 1129.

4. As a further example, consider $\alpha_2(\theta; m) = E(\xi^m)$. Then

$$A(\theta; t) = \sum_{m=0}^{\infty} E(\xi^m)t^m/m! = E(\sum_{m=0}^{\infty} \xi^m t^m/m!) = \exp(\mu t + \sigma^2 t^2/2)$$

so long as the summation and expectation operations can be interchanged. In general terms, let $\alpha(\theta; m) = E\{f(\xi; m)\}$, then the required conditions are

$$\sum_{m=0}^{\infty} E|f(\xi; m)|t^m/m! < \infty \qquad \text{and} \qquad \sum_{m=0}^{\infty} |E\{f(\xi; m)\}t^m/m!| < \infty\,.$$

In the particular example considered, these conditions can be readily shown to hold by using the results of Neyman and Scott (1960), particularly equations (19), (21) and (24), page 647.

Using results of Finney (1941), $E\{B(\hat{\mu}, \hat{\sigma}^2; t)\} = A(\theta; t)$ where

$$B(\hat{\mu}, \hat{\sigma}^2; t) = e^{\hat{\mu}t}\Phi[(1 - \lambda^2)t^2 S, \nu]\,.$$

The coefficient of t^{2m} and t^{2m+1} in the series expansion of $B(\hat{\mu}, \hat{\sigma}^2; t)$ are readily found to be

$$\beta_2(\hat{\mu}, \hat{\sigma}^2; 2m) = \sum_{k=0}^{m} \frac{\Gamma(\nu/2)(2m)!}{\Gamma(\nu/2 + k)(2m - 2k)!\, k!} \left\{\frac{(1 - \lambda^2)S}{4}\right\}^k \hat{\mu}^{2m-2k}$$

$$\beta_2(\hat{\mu}, \hat{\sigma}^2; 2m + 1) = \sum_{k=0}^{m} \frac{\Gamma(\nu/2)(2m + 1)!}{\Gamma(\nu/2 + k)(2m + 1 - 2k)!\, k!} \left\{\frac{(1 - \lambda^2)S}{4}\right\}^k \hat{\mu}^{2m+1-2k}\,.$$

Reference to Theorem 3, page 650, of Neyman and Scott (1960) shows that the conditions above apply so that $\beta_2(\hat{\mu}, \hat{\sigma}^2; m)$ is the required estimator of $E(\xi^m)$.

REFERENCES

[1] FINNEY, D. J. (1941). On the distribution of a variate whose logarithm is normally distributed. *J. Roy. Stat. Soc. Suppl.* **7** 155–161.

[2] HOYLE, M. H. (1968). The estimation of variances after using a Gaussianating transformation. *Ann. Math. Statist.* **39** 1125–1143.

[3] LEHMANN, E. L. and SCHEFFÉ, H. (1950). Completeness, similar regions and unbiased estimation, Part I. *Sankhyā* **10** 305–340.

[4] NEYMAN, J. and SCOTT, E. L. (1960). Correction for bias introduced by a transformation of variables. *Ann. Math. Statist.* **31** 643–655.

[5] RUDIN, W. (1953). *Principles of Mathematical Analysis.* McGraw-Hill, New York.

[6] STIGLER, S. M. (1971). The estimation of the probability of extinction and other parameters associated with branching processes. *Biometrika* **53** 499–508.

AUSTRAL HOUSE
BASINGHALL AVENUE
LONDON EC 2V 5EU
ENGLAND