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# MINIMAX ESTIMATION OF LOCATION VECTORS FOR A WIDE CLASS OF DENSITIES

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Assume $X = (X_1, \cdots, X_p)^t$ has a $p$-variate density, with respect to Lebesgue measure, of the form $f((x-\theta)^t\Sigma^{-1}(x-\theta))$. Here $\Sigma$ is a known positive definite $p \times p$ matrix and $p \geqq 3$. Assume either (i) $f$ is completely monotonic, or (ii) there exist $\alpha > 0$ and $K > 0$ for which $h(s) = f(s)e^{\alpha s}$ is nondecreasing and nonzero if $s > K$. Then for estimating $\theta$ under a known quadratic loss, classes of minimax estimators are found.

**1. Introduction.** Since Stein (1955) first showed that the best invariant estimator of a $p$-dimensional normal mean was inadmissible ($p \geqq 3$), much study has been given to improving upon the best invariant estimator of a location vector. Until recently, classes of good minimax estimators had been found only for the problem of estimating a normal mean under squared error loss and with covariance matrix a multiple of the identity. (See Baranchik (1970), Strawderman (1971), and Alam (1973).) Of course, Brown (1966) answered the theoretical admissibility questions for a very wide class of distributions and loss functions.

Recently, Berger (1974a) found classes of minimax estimators for a normal mean when the covariance matrix and quadratic loss were arbitrary. (Earlier, Bhattacharya (1966) and Bock (1975) had found some particular minimax estimators for this situation.). Strawderman (1974) extended things in a different direction, by finding good classes of minimax estimators for location vectors of certain symmetric densities.

This paper considers two further extensions. Section 3 deals with an extension combining Strawderman (1974) and Berger (1974a). Densities of the form

$$(1.1)\qquad f((x-\theta)^t\Sigma^{-1}(x-\theta)) = \int \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi)^{p/2}\sigma^p} \exp\left[-\frac{1}{2\sigma^2}(x-\theta)^t\Sigma^{-1}(x-\theta)\right] dF(\sigma)$$

are considered, where $F$ is any known cdf on $(0, \infty)$ and $\Sigma$ is positive definite. For estimating $\theta$ under a quadratic loss, classes of minimax estimators are developed. The results parallel those of Strawderman (1974), though of necessity the proof is different.

Also in Section 3, a characterization of the class of densities of the form (1.1) is given. It is shown that $f$ is of the form (1.1) if and only if it is completely monotonic in $(x-\theta)^t\Sigma^{-1}(x-\theta)$. ($f(s)$ is completely monotonic if

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$(-1)^n[(d^n/ds^n)f(s)] \geqq 0$ for every $n$.) This condition is often easy to verify, obviating the often difficult problem of attempting to find $F$ in applications. Section 3 concludes with several applications.

The class of completely monotonic $f$ is rather small, and so in Section 4 a different approach to the problem is considered. It allows us to deal with densities (with respect to Lebesgue measure) of the form $f((x-\theta)^t\Sigma^{-1}(x-\theta))$, where $f$ does not decrease too fast in the tails. For example, suppose there exist $\alpha > 0$ and $K > 0$ for which $h(s) = f(s)e^{\alpha s}$ is nondecreasing and nonzero if $s > K$. Then the theory applies to $f$, and classes of good minimax estimators are found for $\theta$. Section 4 concludes with several applications.

**2. Preliminaries.** Let $A_i > 0$ $(i = 1, \cdots, p)$ be known positive constants. If $x = (x_1, \cdots, x_p)^t$ is a $p$-dimensional vector, define $|x|_1^2 = \sum_{i=1}^p x_i^2$, $|x|_2^2 = \sum_{i=1}^p x_i^2/A_i$, and $|x|_3^2 = \sum_{i=1}^p x_i^2/A_i^2$.

For simplicity only the following "canonical form" of the problem will be considered. Let $X = (X_1, \cdots, X_p)^t$ be an observation from a $p$-dimensional density (with respect to Lebesgue measure) of the form $f(|x-\theta|_2^2)$. Here $\theta = (\theta_1, \cdots, \theta_p)^t$ is an unknown location vector, while the $A_i$ are known positive constants. Assume the loss incurred in estimating $\theta$ by $\delta$ is $|\delta - \theta|^2$. (The general situation, where $f$ is of the form $f((y-\eta)^t\Sigma^{-1}(y-\eta))$ and the loss is the quadratic loss $(\delta-\eta)^tQ(\delta-\eta)$, can be reduced to the case given above. Indeed, there exists a nonsingular matrix $B$ such that $B^tQB = I_p$ (the $p \times p$ identity matrix), while $B^t\Sigma^{-1}B = A^{-1}$ (the diagonal matrix with diagonal elements $1/A_i$). Defining $X = B^{-1}Y$ and $\theta = B^{-1}\eta$, it is easy to check that the $(X, \theta)$ problem is in the "canonical form" given above.)

For a measurable estimator $\delta(X) = (\delta_1(X), \cdots, \delta_p(X))^t$, define the risk function $R(\delta, \theta) = E_\theta|\delta(X) - \theta|_1^2$. ($E_\theta$ is, of course, the expectation under $\theta$.)

It is easy to check that the best invariant estimator of $\theta$ is $\delta_0(X) = X$. Since this is a location parameter problem under squared error loss, it is well known that $\delta_0$ is a minimax estimator. Furthermore, $\delta_0$ clearly has constant risk. An estimator $\delta$ is thus minimax if $\Delta_\delta(\theta) = R(\delta_0, \theta) - R(\delta, \theta) \geqq 0$. The search for minimax estimators is, hence, also a search for estimators as good as, or better than $\delta_0$. From Brown (1966) and Brown and Fox (1974) it is clear that we'll need $p \geqq 3$ to hope to find estimators better than $\delta_0$.

The estimators that will be considered in this paper are given componentwise by

$$\delta_i(X) = \left(1 - \frac{r(|X|_3^2)}{|X|_3^2 A_i}\right) X_i\,. \tag{2.1}$$

In terms of the general problem, with arbitrary $\Sigma$ and $Q$, the above estimator corresponds to

$$\delta(X) = \left(I_p - \frac{r(X^t\Sigma^{-1}Q^{-1}\Sigma^{-1}X)Q^{-1}\Sigma^{-1}}{X^t\Sigma^{-1}Q^{-1}\Sigma^{-1}X}\right) X\,. \tag{2.2}$$

For an intuitive justification of estimators of the above form, along with a numerical analysis of the improvement $\Delta_\delta(\theta)$ they give, see Berger (1974b).

**3. Minimax estimators of normal mixtures.** This section deals with densities, with respect to Lebesgue measure, of the form

$$f(|x-\theta|_2^2) = \int_0^\infty (2\pi)^{-p/2}\sigma^{-p}\left(\prod_{i=1}^p A_i^{-\frac{1}{2}}\right)\exp[-|x-\theta|_2^2/(2\sigma^2)]\,dF(\sigma)\,, \tag{3.1}$$

where $F$ is a cdf on $(0, \infty)$.

THEOREM 1. *Let $X$ be an observation from a p-dimensional density of the form (3.1), where $p \geqq 3$. Assume $E_0|X|_1^2$ and $E_0|X|_1^{-2}$ are finite. Let $\delta$ be of the form (2.1) where*

(i) *$r(v)$ is nondecreasing in $v$,*
(ii) *$r(v)/v$ is nonincreasing in $v$, and*
(iii) *$0 \leqq r \leqq 2/(E_0|X|_2^{-2})$.*

*Then $\delta$ is a minimax estimator of $\theta$ under squared error loss.*

COMMENT. Note that $E_0|X|_1^{-2} < \infty$ is a relatively weak assumption. If $f$ is bounded in a neighborhood of zero and $p \geqq 3$, then it is clear that $E_0|X|_1^{-2} < \infty$. This ensures that condition (iii) is not vacuous.

PROOF OF THEOREM 1. Clearly

$$\begin{aligned}\Delta_\delta(\theta) &= E_\theta[|X-\theta|_1^2 - |\delta(X)-\theta|_1^2] \\ &= \sum_{i=1}^p E_\theta\frac{2r(|X|_3^2)X_i(X_i-\theta_i)}{|X|_3^2A_i} - E_\theta\frac{r^2(|X|_3^2)}{|X|_3^2}\,.\end{aligned} \tag{3.2}$$

Defining $c = (2\pi)^{-p/2}(\prod_{i=1}^p A_i^{-\frac{1}{2}})$ and interchanging orders of integration gives

$$\begin{aligned}&E_\theta\frac{2r(|X|_3^2)X_i(X_i-\theta_i)}{|X|_3^2A_i} \\ &\quad= \int_{R^p}\frac{2r(|x|_3^2)x_i(x_i-\theta_i)}{|x|_3^2A_i}\int_0^\infty c\sigma^{-p}\exp\left[-\frac{1}{2\sigma^2}|x-\theta|_2^2\right]dF(\sigma)\,dx \\ &\quad= \int_0^\infty 2c\sigma^{(2-p)}\int_{R^p}\frac{r(|x|_3^2)x_i}{|x|_3^2}\cdot\frac{(x_i-\theta_i)}{A_i\sigma^2}\exp\left[-\frac{1}{2\sigma^2}|x-\theta|_2^2\right]dx\,dF(\sigma)\,.\end{aligned} \tag{3.3}$$

Assume for the moment that $r$ is differentiable. Let $r'$ denote the first derivative of $r$. A simple integration by parts shows that the inner integral in the last expression of (3.3) equals

$$\begin{aligned}&\int_{R^p}\frac{\partial}{\partial x_i}\left[\frac{r(|x|_3^2)x_i}{|x|_3^2}\right]\exp\left[-\frac{1}{2\sigma^2}|x-\theta|_2^2\right]dx \\ &\quad= \int\left[\frac{r(|x|_3^2)}{|x|_3^2} + \frac{2x_i^2r'(|x|_3^2)}{|x|_3^2A_i^2} - \frac{2x_i^2r(|x|_3^2)}{|x|_3^4A_i^2}\right]\exp\left[-\frac{1}{2\sigma^2}|x-\theta_2^2|\right]dx\,.\end{aligned}$$

Using this together with (3.2) and (3.3) gives

$$\begin{aligned}\Delta_\delta(\theta) = \int_0^\infty\int_{R^p}2\sigma^2\left[\frac{(p-2)r(|x|_3^2)}{|x|_3^2} + 2r'(|x|_3^2)\right]c\sigma^{-p}\exp[\quad]\,dx\,dF(\sigma) \\ - E_\theta\frac{r^2(|x|_3^2)}{|x|_3^2}\,.\end{aligned}$$

Combining terms and noting that by assumption (i) $r' \geqq 0$, it is clear that

$$\Delta_\delta(\theta) \geqq \int_0^\infty \int [2(p-2)\sigma^2 - r(|x|_3^2)] \frac{r(|x|_3^2)}{|x|_3^2} c\sigma^{-p} \exp\left[-\frac{1}{2\sigma^2}|x-\theta|_2^2\right] dx\, dF(\sigma) .$$

Defining $b = \sup_v r(v)$ gives

$$(3.4) \qquad \Delta_\delta(\theta) \geqq \int_0^\infty \left[\frac{2(p-2)\sigma^2 - b}{\sigma^2}\right] \int \frac{r(|x|_3^2)}{|x|_3^2/\sigma^2} c\sigma^{-p} \exp[\quad] \, dx\, dF(\sigma) .$$

Consider next the function

$$\begin{aligned} g(\sigma) &= \int \frac{r(|x|_3^2)}{|x|_3^2/\sigma^2} c\sigma^{-p} \exp\left[-\frac{1}{2\sigma^2}|x-\theta|_2^2\right] dx \\ &= \int \frac{r(|y|_3^2\sigma^2)}{|y|_3^2} c \exp\left[-\frac{1}{2}\left|y - \frac{\theta}{\sigma}\right|_2^2\right] dy . \end{aligned}$$

It will be shown that $g(\sigma)$ is nondecreasing in $\sigma$. Differentiating $g(\sigma)$ with respect to $\sigma$ gives

$$\begin{aligned} g'(\sigma) = &\int r'(|y|_3^2\sigma^2) 2\sigma c \exp\left[-\frac{1}{2}\left|y-\frac{\theta}{\sigma}\right|_2^2\right] dy \\ &+ \int \frac{r(|y|_3^2\sigma^2)}{|y|_3^2} c\left[-\sum_{i=1}^p \frac{1}{A_i}\left(y_i - \frac{\theta_i}{\sigma}\right)\left(\frac{\theta_i}{\sigma^2}\right)\right] \exp[\quad] \, dy . \end{aligned}$$

The first integral above is clearly positive since $r' \geqq 0$. Defining $h(v) = r(v)/v$ and integrating by parts gives that the second integral above equals

$$(3.5) \qquad \sum_{i=1}^p \int -h'(|y|_3^2\sigma^2) 2 y_i A_i^{-2} \sigma^2 \theta_i c \exp\left[-\frac{1}{2}\left|y-\frac{\theta}{\sigma}\right|_2^2\right] dy .$$

To show the above expression is positive, note that

$$\begin{aligned} (3.6) \qquad &\int -h'(|y|_3^2\sigma^2) y_i \theta_i \exp\left[-\frac{1}{2}\left|y-\frac{\theta}{\sigma}\right|_2^2\right] dy \\ &\qquad = \int_{\{y: y_i \geqq 0\}} (\quad)\, dy + \int_{\{y: y_i < 0\}} (\quad)\, dy . \end{aligned}$$

Since $h'(|y|_3^2\sigma^2)$ is a function of $y_i^2$, a simple change of variables shows that the last integral above equals

$$\begin{aligned} &\int_{\{y: y_i > 0\}} h'(|y|_3^2\sigma^2) y_i \theta_i \exp\left[-\left(-y_i - \frac{\theta_i}{\sigma}\right)^2 \Big/ (2A_i)\right] \\ &\qquad \times \exp\left[-\sum_{j \neq i}\left(y_j - \frac{\theta_j}{\sigma}\right)^2 \Big/ (2A_j)\right] dy . \end{aligned}$$

Thus the left hand side of (3.6) equals

$$\begin{aligned} (3.7) \qquad &\int_{\{y: y_i > 0\}} -h'(|y|_3^2\sigma^2) y_i \theta_i \exp\left[-\sum_{j\neq i}\left(y_j - \frac{\theta_j}{\sigma}\right)^2 \Big/ (2A_j)\right] \\ &\qquad \times \left(\exp\left[-\left(y_i - \frac{\theta_i}{\sigma}\right)^2 \Big/ (2A_i)\right] - \exp\left[-\left(y_i + \frac{\theta_i}{\sigma}\right)^2 \Big/ (2A_i)\right]\right) dy . \end{aligned}$$

By condition (ii), $-h' \geqq 0$. Also, for $y_i > 0$, it is clear that $\theta_i$ and $\exp[-(y_i - \theta_i/\sigma)^2/(2A_i)] - \exp[-(y_i + \theta_i/\sigma)^2/(2A_i)]$ have the same sign. Thus the integral in (3.7) is positive. But the expression in (3.5) is then positive. The conclusion is that $g'(\sigma) \geqq 0$.

Note also that $[2(p-2) - b/\sigma^2]$ is nondecreasing in $\sigma$. Using the above facts, together with (3.4), gives

$$\begin{aligned}\Delta_\delta(\theta) &\geqq \textstyle\int_0^\infty [2(p-2) - b/\sigma^2] g(\sigma)\, dF(\sigma) \\ &\geqq \textstyle\int_0^\infty [2(p-2) - b/\sigma^2]\, dF(\sigma) \int_0^\infty g(\sigma)\, dF(\sigma)\,. \end{aligned} \tag{3.8}$$

A straightforward calculation shows that

$$E_0|X|_2^{-2} = \textstyle\int_0^\infty [\sigma^{-2}/(p-2)]\, dF(\sigma)\,.$$

Combining this with (3.8), the definition of b, and Condition (iii), proves that $\Delta_\delta(\theta) \geqq 0$. The theorem has thus been established for differentiable $r$.

If $r$ is not differentiable, the above proof goes through using Riemann integration. Noting that $r_i(x_i) = r(|x|_3^2)$ (considered as a function of $x_i$) is of bounded variation in $x_i$, it is easy to check that integration by parts for the Riemann integrals is valid. Indeed, the terms $r'(|x|_3^2)\, dx_i$ need only be replaced by $dr_i(x_i)$ in the given proof. □

THEOREM 2. *A density* $f(|x-\theta|_2^2)$ *is of the form* (3.1) *if and only if* $f$ *is completely monotonic in* $(0, \infty)$.

PROOF. Define $t = 1/(2\sigma^2)$, $s = |x-\theta|_2^2$, and $G(t) = -\int_0^t c(2v)^{p/2}\, dF(1/(2v)^{\frac{1}{2}})$. It is easy to see that $G$ is positive and nondecreasing in $t$ and that

$$f(s) = \textstyle\int_0^\infty e^{-st}\, dG(t)\,.$$

By a well-known result about Laplace transforms (see Feller (1966), page 439) $f(s)$ is of this form if and only if it is completely monotonic. □

We now give several examples of the application of the above theorems.

EXAMPLE 1. Let $f(|x-\theta|_2^2)$ be a normal density with mean $\theta$ and a diagonal covariance matrix with diagonal elements $A_i$. Clearly $f(s) = c\exp[-s/2]$, which is completely monotonic. Hence Theorem 1 applies. A simple calculation shows that $E_0|X|_2^{-2} = 1/(p-2)$. Note that the class of minimax estimators thus defined by Theorem 1 is essentially the class found in Berger (1974a).

EXAMPLE 2. Consider the "double exponential" density

$$f(|x-\theta|_2^2) = \frac{\exp[-|x-\theta|_2]}{a_p\Gamma(p)} \textstyle\prod_{i=1}^p A_i^{-\frac{1}{2}}\,, \tag{3.9}$$

where $a_p$ is the surface area of the unit $p$-sphere. Here $f(s) = K\exp[-s^{\frac{1}{2}}]$. In determining whether or not this function is completely monotonic, the following well-known lemma is useful.

LEMMA 1. *If $\psi(s)$ is completely monotonic on* $(0, \infty)$, *and* $\varphi(s) \geqq 0$ *has a completely monotonic first derivative on* $(0, \infty)$, *then* $\psi(\varphi(s))$ *is completely monotonic on* $(0, \infty)$.

PROOF. See Feller (1966), page 441. □

Clearly $\psi(v) = K\exp[-v]$ is completely monotonic, and $\varphi(s) = s^{\frac{1}{2}}$ has a completely monotonic first derivative. Hence, $f(s) = K\exp[-s^{\frac{1}{2}}] = \psi(\varphi(s))$ is completely monotonic on $(0, \infty)$. Theorem 1 thus applies to the density in (3.9). A simple calculation shows that $E_0|X|_2^{-2} = [(p-1)(p-2)]^{-1}$.

EXAMPLE 3. Consider the Cauchy like density

$$f(|x-\theta|_2^2) = 2\Gamma(a)[(1+|x-\theta|_2^2)^a(\prod_{i=1}^p A_i^{\frac{1}{2}})a_p\Gamma(p/2)\Gamma(a-p/2)]^{-1},$$

where $a > 1 + p/2$ so that $E_0|X|_1^2 < \infty$. Clearly $f(s) = K(1+s)^{-a}$ is completely monotonic. A calculation shows that $E_0|X|_2^{-2} = (2a-p)/(p-2)$.

**4. Minimax estimators for densities flatter than a normal density.** While the class of densities discussed in the previous section includes many interest ing, ones, it is obviously quite sparse. In this section, a different approach is considered, leading to a theory which handles a much more complete class of densities.

THEOREM 3. *Let $f(|x-\theta|_2^2)$ be a density, with respect to Lebesgue measure, satisfying the following* 3 *conditions*:

(i) $E_0|X|_1^2 < \infty$ *and* $E_0|X|_1^{-2} < \infty$.

(ii) *The set of points, $W$, (in $(0, \infty)$) at which $f(\cdot)$ is discontinuous has Lebesgue measure zero.*

(iii) $c = \inf_{s\in U} \dfrac{\int_s^\infty f(v)\,dv}{f(s)} > 0$, *where* $U = \{s \notin W : f(s) > 0\}$.

*Let $\delta$ be an estimator of the form* (2.1) *where $r(\cdot)$ is nondecreasing and $0 \leqq r \leqq c(p-2)$. Assume also that $p \geqq 3$. Then $\delta$ is a minimax estimator of $\theta$ under squared error loss.*

PROOF. Assume that $r$ is differentiable. The generalization to nondifferentiable $r$ can be carried out as indicated in Theorem 1.

As in (3.2),

$$\Delta_\delta(\theta) = 2\sum_{i=1}^p E_\theta\left[\frac{X_i r(|X|_3^2)}{|X|_3^2}\cdot\frac{(X_i-\theta_i)}{A_i}\right] - E_\theta\frac{r^2(|X|_3^2)}{|X|_3^2}. \tag{4.1}$$

It can easily be checked that if $\varepsilon > 0$, then $\int_\varepsilon^\infty f(v)\,dv < \infty$. Hence if $|x-\theta|_2^2$ is a positive point of continuity of $f$,

$$\frac{\partial}{\partial x_i}\left(-\tfrac{1}{2}\int_{|x-\theta|_2^2}^\infty f(v)\,dv\right) = f(|x-\theta|_2^2)(x_i-\theta_i)/A_i.$$

An integration by parts thus gives

$$E_\theta\left[\frac{X_i r(|X|_3^2)}{|X|_3^2}\cdot\frac{(X_i-\theta_i)}{A_i}\right]$$

$$(4.2)\qquad = \int \frac{x_i r(|x|_3^2)}{|x|_3^2}\cdot\frac{(x_i-\theta_i)}{A_i} f(|x-\theta|_2^2)\,dx$$

$$= \int\left[\frac{r(|x|_3^2)}{|x|_3^2} - \frac{2x_i^2 r(|x|_3^2)}{|x|_3^4 A_i^2} + \frac{2x_i^2 r'(|x|_3^2)}{|x|_3^2 A_i^2}\right]\left(\tfrac{1}{2}\int_{|x-\theta|_2^2}^\infty f(v)\,dv\right)dx\,.$$

(Clearly $\{x: |x|_3^2 = 0 \text{ or } |x-\theta|_2^2 = 0 \text{ or } |x-\theta|_2^2 \in W\}$ has measure 0, and can hence be ignored in the integration by parts.)

Using (4.1), (4.2), and the assumption that $r' \geqq 0$, a little algebra gives

$$\Delta_\delta(\theta) \geqq \int \frac{r(|x|_3^2)}{|x|_3^2}\left[(p-2)\int_{|x-\theta|_2^2}^\infty f(v)\,dv - r(|x|_3^2) f(|x-\theta|_2^2)\right]dx\,.$$

Since $0 \leqq r \leqq c(p-2)$, it is clear from the definition of $c$ that

$$\left[(p-2)\int_{|x-\theta|_2^2}^\infty f(v)\,dv - r(|x|_3^2) f(|x-\theta|_2^2)\right] \geqq 0\,,$$

except possibly on a set of measure 0. Thus $\Delta_\delta(\theta) \geqq 0$ and $\delta$ is minimax. □

The following theorem is useful in verifying condition (iii) of Theorem 3.

THEOREM 4. *Assume that the first two conditions of Theorem* 3 *hold. Then condition* (iii) *holds if and only if the following are true*:

(a) $f(s) < B < \infty$ *for every* $s \in U$.

(b) $U$ *is not contained in any compact subset of* $[0, \infty)$, *and*

$$(4.3)\qquad \liminf_{s\to\infty, s\in U} \frac{\int_s^\infty f(v)\,dv}{f(s)} = b > 0\,.$$

PROOF. Assume first that conditions (a) and (b) hold. From (b), it is clear that there exists a $K$ such that if $s > K$ and $s \in U$, then $\int_s^\infty f(v)\,dv/f(s) \geqq b/2 > 0$. If $s \leqq K$, then $\int_s^\infty f(v)\,dv \geqq \int_K^\infty f(v)\,dv = K' > 0$ (since $U$ is not contained in a compact subset of $[0, \infty)$). But combined with condition (a), this tells us that if $s \leqq K$ and $s \in U$, then $\int_s^\infty f(v)\,dv/f(s) \geqq K'/B > 0$. Hence $\inf_{s\in U}[\int_s^\infty f(v)\,dv/f(s)] > 0$, and condition (iii) of Theorem 3 holds.

Now assume that condition (iii) of Theorem 3 holds. We first show that this implies (a), that $f$ is essentially bounded. Assume (a) does not hold. Then there exists a sequence $\{z_i\}$ of positive numbers in $U$, such that $f(z_i) \to \infty$. It will first be shown that $z_i \to 0$. Assume not. Then $\{z_i\}$ could be chosen so that $z_i > \varepsilon > 0$ for every $i$, and so that $f(z_i) \to \infty$. But then

$$\int_{z_i}^\infty f(v)\,dv/f(z_i) \leqq \int_\varepsilon^\infty f(v)\,dv/f(z_i) \leqq K'/f(z_i) \to 0 \qquad \text{as} \quad i \to \infty\,.$$

This contradicts the assumption that condition (iii) holds. Hence $\{z_i\}$ must converge to 0.

Now let $\varepsilon$ be a positive number, and define $T_n = (1/n, \varepsilon) \cap U$ for $n > 1/\varepsilon$.

By the above result, it is clear that $f$ is bounded on $T_n$, and that there exists a sequence of numbers $d_n$ such that $d_n \in T_n$ and $f(d_n) \to \infty$. Because $f$ is bounded on $T_n$ and $T_n$ includes only continuity points of $f$, there exist $\tau_n \in T_n$ such that $\sup_{v \in T_n} f(v) \leqq f(\tau_n) + 1/n$. Note that $f(d_n) - 1/n \leqq f(\tau_n)$, which implies that $f(\tau_n) \to \infty$. Finally,

$$\frac{\int_{\tau_n}^\infty f(v)\,dv}{f(\tau_n)} \leqq \frac{\int_{T_n} f(v)\,dv + \int_\varepsilon^\infty f(v)\,dv}{f(\tau_n)}$$
$$\leqq \frac{\varepsilon(f(\tau_n) + 1/n) + K_\varepsilon}{f(\tau_n)} \to \varepsilon \qquad \text{as} \quad n \to \infty\,.$$

Since this holds for every $\varepsilon > 0$, condition (iii) of Theorem 3 is again contradicted. Thus condition (a) must hold.

Next, we show that condition (iii) implies condition (b). Clearly, it is only necessary to verify that $U$ is not contained in any compact subset of $[0, \infty)$. Assume it is. Then there exists a constant $K > 0$, such that $f(s) = 0$ for $s \in (K, \infty) \cap W^c$. Choose $K$ as small as possible. Let $D_n = (K - 1/n, K)$ and $V_n = \sup_{s \in D_n \cap U} f(s)$. Note that $V_n > 0$ since $K$ was chosen as small as possible. Also, $V_n < \infty$ by condition (a). By construction, there exists $s_n \in D_n \cap U$ such that $f(s_n) > V_n/2$. Hence,

$$\frac{\int_{s_n}^\infty f(v)\,dv}{f(s_n)} = \frac{\int_{s_n}^K f(v)\,dv}{f(s_n)} \leqq \frac{(K - s_n)V_n}{f(s_n)} \leqq \frac{(K - s_n)V_n}{V_n/2}$$
$$= 2(K - s_n) \to 0 \qquad \text{as} \quad n \to \infty\,.$$

Thus condition (iii) is again contradicted, and condition (b) must hold. □

Theorem 4 clearly gives considerable insight into the class of densities to which Theorem 3 applies. The following lemmas are also useful in this regard.

LEMMA 2. *Assume there exist $K > 0$ and $\alpha > 0$, such that if $s > K$, then $f(s)$ is positive, differentiable, and satisfies* $\limsup_{s \to \infty} [-f'(s)/f(s)] \leqq \alpha < \infty$. *Then* (4.3) *is satisfied.*

PROOF. For convenience, define $g(s) = \int_s^\infty f(v)\,dv$. Note that $\lim_{s \to \infty} g(s) = 0$, that $g(s_1) - g(s_2) > 0$ for $s_2 > s_1 > K$, and that $-g'(s) = f(s) > 0$ for $s > K$.

Let $\varepsilon > 0$. By assumption, there exists $K_\varepsilon > K$, such that if $s > K_\varepsilon$ then $[-f'(s)/f(s)] \leqq \alpha + \varepsilon$. Thus, for $K_\varepsilon < a < b < \infty$, the generalized mean value theorem says that there exists $s \in (a, b)$ at which

$$\frac{f(a) - f(b)}{g(a) - g(b)} = \frac{f'(s)}{g'(s)} \leqq \alpha + \varepsilon\,. \tag{4.4}$$

Let $\{b_n\}$ be a sequence of numbers greater than $K_\varepsilon$, such that $b_n \to \infty$ and $f(b_n) \to 0$. Clearly $g(b_n) \to 0$. Thus (4.4) gives that $[f(a)/g(a)] \leqq \alpha + \varepsilon$. Since this holds for every $\varepsilon > 0$, it is clear that $\limsup_{a \to \infty} [f(a)/g(a)] \leqq \alpha$. Hence, $\liminf_{a \to \infty} [g(a)/f(a)] \geqq 1/\alpha > 0$, which verifies (4.3). □

LEMMA 3. *Assume that $f(\cdot)$ is bounded, positive, and differentiable on* $(0, \infty)$, *that* $E_0|X|_1^2 < \infty$, *that* $p \geqq 3$, *and that* $\limsup_{s\to\infty} [-f'(s)/f(s)] < \infty$. *Then the conditions on $f$ in Theorem* 3 *are satisfied.*

PROOF. Trivial using Theorem 4 and Lemma 2. Note that $f$ bounded and $p \geqq 3$ implies that $E_0|X|_1^{-2} < \infty$. □

LEMMA 4. *Assume that $f(\cdot)$ is bounded and continuous* (*except possibly on a set of measure* 0). *Assume also that* $E_0|X|_1^2 < \infty$, *that* $p \geqq 3$, *and that there exist* $K \geqq 0$ *and* $\alpha > 0$ *such that if* $s > K$, *then* $h(s) = f(s)e^{\alpha s}$ *is nonzero and nondecreasing. Then the conditions on $f$ in Theorem* 3 *are satisfied.*

PROOF. It is only necessary to verify condition (iii) of Theorem 3. Theorem 4 will be used to do this. It is obvious that only (4.3) must be checked. Since $h$ is nondecreasing and nonzero for $s > K$,

$$\liminf_{s\to\infty} \frac{\int_s^\infty f(v)\,dv}{f(s)} = \liminf_{s\to\infty} \frac{\int_s^\infty h(v)e^{-\alpha v}\,dv}{h(s)e^{-\alpha s}} = \liminf_{s\to\infty} \int_s^\infty \frac{h(v)}{h(s)} e^{\alpha(s-v)}\,dv$$
$$\geqq \liminf_{s\to\infty} \int_s^\infty e^{\alpha(s-v)}\,dv = 1/\alpha > 0\,.$$

Hence (4.3) is satisfied. □

LEMMA 5. *If* $f(s) = e^{-\alpha s g(s)}$, *where $g(s)$ is nondecreasing,* $g(s) \to \infty$, *and* $\alpha > 0$, *then condition* (iii) *of Theorem* 3 *is violated.*

PROOF. Using the assumptions on $g(s)$ gives

$$\begin{aligned}\liminf_{s\to\infty} \frac{\int_s^\infty f(v)\,dv}{f(s)} &= \liminf_{s\to\infty} \int_s^\infty \exp[-\alpha(vg(v) - sg(s))]\,dv \\ &= \liminf_{s\to\infty} \int_0^\infty \exp[-\alpha((t+s)g(t+s) - sg(s))]\,dt \\ &\leqq \liminf_{s\to\infty} \int_0^\infty \exp[-\alpha t g(s)]\,dt \\ &= \liminf_{s\to\infty} [\alpha g(s)]^{-1} = 0\,.\end{aligned}$$

Condition (iii) of Theorem 3 is thus violated. □

Lemmas 4 and 5 indicate that the density must not go to zero much faster than a normal density (recall $s = |x - \theta|_2^2$), in order for the theory to apply. Most important densities are in this class, however.

We now give some applications. The examples given in Section 3 are not repeated, though they could be handled by the theory of this section also. Clearly, it is only necessary to indicate the function $f(s)$ in the examples. The normalizing constant will be unimportant so it will be denoted by $K$.

EXAMPLE 1. $f(s) = Ks^n \exp[-s/2]$, $(n \geqq 0)$. (Note that if $n = 0$, then $f(|x - \theta|_2^2)$ is a normal density.) Clearly $f$ is not completely monotonic unless $n = 0$. Thus Section 3 could not, in general, be applied. A simple calculation shows that $c = \inf_s [\int_s^\infty f(v)\,dv/f(s)] = 2 > 0$. Hence Theorem 3 applies and gives a wide class of minimax estimators.

EXAMPLE 2. $f(s) = K/\cosh s$. This is a density similar to a hyperbolic cosine density. To show that Theorem 3 applies to this density, Lemma 4 can be used. Choosing $\alpha = 1$ gives $h(s) = [2K/(e^s + e^{-s})]e^s = 2K/(1 + e^{-2s})$, which is clearly nondecreasing. The remaining conditions of Theorem 3 are trivial to check for this density. Numerical calculation showed that $c = 1$.

EXAMPLE 3. $f(s) = Ks(1 + s^2)^{-(m+1)}$, where $m > p/4$. The condition on $m$ ensures that $E_0|X|_1^2 < \infty$. An easy calculation shows that $c = 1/m$.

EXAMPLE 4. $f(s) = Ke^{(-\alpha s-\beta)}(1 + e^{(-\alpha s-\beta)})^{-2}$. This is a density similar to the logistic. It can easily be calculated that $c = 1/\alpha$.

Many other examples could be given. Indeed most densities work fairly easily.

A word is in order as to how the results from Section 4 relate to those from Section 3. There do exist some densities covered by results in Section 3, which cannot be handled by the methods of Section 4. Such densities are not very interesting, however, and so as a whole Section 4 has a much wider range of applicability than Section 3.

On the other hand, when Section 3 does apply, it tends to give stronger results than Section 4 (in the sense that the upper bound on $r$ is often larger). Hence one should use the results from Section 3, if they happen to apply, rather than those from Section 4.

Certain of the above results can be generalized to the multiobservational situation. Assume $n$ observations are taken, and that an invariant estimator, $\delta_1$, is to be used to estimate $\theta$. To attempt to improve upon $\delta_1$, merely set $X = \delta_1$ and proceed. Note that if the original density is of the form (1.1), there is no reason to expect that the density of $\delta_1$ will also be of the form (1.1). Hence the results of Section 3 are of limited usefulness for the multiobservational situation. If the original density satisfies the conditions of Theorem 3, however, then the density of $\delta_1$ will, often, satisfy these conditions itself. Section 4 will thus be very useful for the multiobservational problem. Note, finally, that if $\delta_1$ is Pitman's estimator, than the improved estimators will be minimax.

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