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COMPARING SEQUENTIAL AND NON-SEQUENTIAL TESTS[1]

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Sequential tests for one-sided hypotheses are compared, asymptotically, with non-sequential counterparts. An analog of Pitman efficiency is obtained, as is another comparison that has no purely non-sequential analog. With these methods of comparison, the limiting relative efficiency of the sequential test is never less than one and for most parameter values, it is infinite. An asymptotic notion of minimal relative efficiency is also considered.

1. Introduction. Asymptotic comparisons of sequential and non-sequential tests for composite hypotheses have been given for the normal case by Bechhofer (1960) and, somewhat more generally, by Berk (1973). In these studies, it is shown that the Wald SPRT for testing one-sided composite hypotheses can have limiting relative efficiency (l.r.e.) less than one (in fact, zero at some parameter points) against a corresponding non-sequential test. The corresponding non-sequential test is UMP or LMP for testing the hypotheses. It is "matched" with the SPRT by selecting two parameter points (typically, those defining the SPRT) and matching the error rates there of the two tests. The respective (expected) sample sizes required by the two tests are then compared as the error rates are made to approach zero in a prescribed manner.

In this paper we develop alternative ways of comparing sequential and non-sequential tests of one-sided hypotheses. Our comparisons are somewhat more favorable to the SPRT, since its l.r.e. is at least unity and is, in fact, infinite for most parameter points.

We work within the following framework. Let $X_1, X_2, \cdots$ be a sequence of i.i.d. copies of a random variable X, supposed to have a distribution given by the pdf $f(x \mid \theta)$, where θ is a parameter ranging in Θ, a subinterval of R. (The pdfs are with respect to a dominating measure λ.) We consider testing the hypothesis $H_1 : \theta = \theta^*$ against the one-sided alternative $H_2 : \theta > \theta^*$. Among all level α sequential tests of H_1(vs. H_2), we consider those whose expected stopping times under θ^* are bounded by a given finite constant, say ν. There is, in general, no UMP sequential test of H_1 vs. H_2 in this class, although as shown in Berk (1975), under certain regularity conditions [for the model $f(x \mid \theta)$], there is a LMP sequential test. The LMP sequential test has a stopping time of the form

$$N = \inf\{n : S_n \notin (-a_1, a_2)\} ,$$

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where $S_n = \sum_1^n \partial \log f(X_j \mid \theta)/\partial\theta|_{\theta^*}$. H_1 is rejected if $S_N \geqq a_2$ and a_1 and a_2 (> 0) are chosen to satisfy the constraints $P(S_N \geqq a_2) = \alpha$ and $EN = \nu$. (Probabilities and expectations, unless otherwise indicated, are under θ^*.) For convenience, we state the required regularity conditions: Let

$$K(\theta \mid \theta^*) = E \log [f(X \mid \theta^*)/f(X \mid \theta)] .$$

ASSUMPTION 1. $\limsup_{\theta\to\theta^*} K(\theta \mid \theta^*)/(\theta - \theta^*)^2 < \infty$.

ASSUMPTION 2. $f(X \mid \cdot)$ is a.e. λ differentiable at θ^*.

ASSUMPTION 3. The power function of every non-sequential test (of H_1 vs H_2) is differentiable at θ^* under the integral sign.

Let $r(X) = \partial \log f(X \mid \theta)/\partial\theta|_{\theta^*}$ and $I^2 = Er^2(X)$.

ASSUMPTION 4. $0 < I < \infty$.

With these conditions, $\{S_n\}$ is, under θ^*, a zero-mean random walk and we have the following no-overshoot approximations:

$$P(S_N \geqq a_2) \doteq a_1/(a_1 + a_2) , \qquad EN \doteq a_1 a_2/I^2 . \tag{1.1}$$

As shown in Berk (1975), when $f(x \mid \theta)$ is an exponential model, there are parameter points $\theta_1 < \theta^* < \theta_2$ and corresponding error rates α_1 and α_2 so that the LMP sequential test is a Wald SPRT of $H_1' : \theta = \theta_1$ vs $H_2' : \theta = \theta_2$ for the given error rates.

We study the l.r.e. of such LMP sequential tests, as compared with the corresponding LMP or UMP non-sequential test which, for a given n, rejects H_1 if $S_n \geqq c_n$. (The constant c_n is chosen to give level α.) Just how to choose a "corresponding" non-sequential test (i.e., n) poses something of a problem. In the Pitman approach to limiting relative efficiencies for non-sequential tests, one can generally adjust the sample sizes of two competing tests to make their power curves uniformly close (i.e., to coincide, asymptotically). The limiting ratio of sample sizes required to do this then gives the Pitman efficiency. Such a matching is generally not possible when comparing a sequential test with a non-sequential test. Unlike the purely non-sequential case, the limiting power curves are not of the same "shape" and cannot be made uniformly close, no matter how one adjusts sample sizes or stopping boundaries. Thus any method of matching a sequential and non-sequential test is bound to have a high degree of arbitrariness associated with it. We consider two possible ways of matching in Section 2. The first we discuss gives (as shown in Section 3) an analog of the Pitman efficiency. The other method seems to have no purely non-sequential analog.

2. Limiting relative efficiencies. We obtain a notion of l.r.e. for the sequential LMP test, compared with its non-sequential analog, by matching the levels of the two tests and also the slopes of their power curves at θ^*. This method of matching is suggested by the fact that both tests are LMP, so that these slopes are natural aspects of the power curve to consider. The matching of slopes is done asymptotically, as $a_1 \wedge a_2 \to \infty$ (and consequently, as EN, n and the slope become large).

Under the assumptions in Section 1, any sequential test of H_1 vs H_2 whose expected stopping time Et is finite has a power curve differentiable at θ^*. Moreover, if φ_t is the critical function of the test, the slope of the power curve at θ^* is given by $E\varphi_t S_t$. This result is due to Abraham (1969). In particular, the LMP sequential test has critical function $1_{(S_N \geqq a_2)}$ and slope

$$m = ES_N^+ \doteq a_2 P(S_N \geqq a_2) = a_1 a_2/(a_1 + a_2) \tag{2.1}$$

while the non-sequential test based on n observations has slope

$$\hat{m} = ES_n 1_{(S_n \geqq c_n)} . \tag{2.2}$$

It is shown in Berk (1973) that the approximations in (1.1) are asymptotic equalities if a_1 and a_2 tend to ∞ so that $a_1/(a_1 + a_2) \to \alpha \in (0, 1)$. (In particular, the LMP sequential test is then asymptotically level α.) We establish a similar result for (2.1).

LEMMA 2.1. *If $a = a_1 \wedge a_2 \to \infty$ and $a_1/(a_1 + a_2) \to \alpha \in (0, 1)$, then $ES_N^+/(a_1 + a_2) \to \alpha(1 - \alpha)$.*

PROOF. It follows from the proof of Theorem 2.4 in Berk (1973) that the overshoot $\Delta^+ = S_N^+ - a_2 1_{(S_N \geqq a_2)}$ satisfies $E\Delta^+ = o(a)$. Thus $ES_N^+/(a_1 + a_2) - a_2 P(S_N \geqq a_2)/(a_1 + a_2) = o(1)$ and the conclusion then follows from the fact that $a_2/(a_1 + a_2) \to 1 - \alpha$ and that $P(S_N \geqq a_2)$ equals (or approaches) α. □

We need a corresponding asymptotic expression for $\hat{m}$ in (2.2), for large n. We obtain this by noting that under θ^*, $U_n = S_n/In^{\frac{1}{2}}$ converges in law to $U \sim N(0, 1)$. Thus for a level α test, we have $c_n \sim z_\alpha In^{\frac{1}{2}}$, where z_α is the upper α-point of the $N(0, 1)$ distribution. Then $\hat{m} = In^{\frac{1}{2}} EU_n 1_{(U_n \geqq k_n)}$ where $k_n = c_n/In^{\frac{1}{2}} \sim z_\alpha$. Since $EU_n^2 = 1$ for all n, $\{U_n\}$ is uniformly integrable and consequently $\lim_n \hat{m}/n^{\frac{1}{2}} = IEU1_{(U \geqq z_\alpha)} = I\varphi(z_\alpha)$, where $\varphi(\cdot)$ is the $N(0, 1)$ pdf. Thus

$$\hat{m} \sim In^{\frac{1}{2}}\varphi(z_\alpha) . \tag{2.3}$$

We now proceed to compare the sequential and non-sequential LMP tests, matching levels and slopes asymptotically. As $a_1 \wedge a_2 \to \infty$ with $a_1/(a_1 + a_2) \to \alpha$, it follows from Lemma 2.1 that $m \sim \alpha(1 - \alpha)(a_1 + a_2)$. Combining this with (2.3), to obtain equal slopes, asymptotically, for the sequential and non-sequential tests, we must, for given a_1 and a_2, choose a sample size $\hat{n}$ for the non-sequential test so that

$$I\hat{n}^{\frac{1}{2}}\varphi(z_\alpha) \sim \alpha(1 - \alpha)(a_1 + a_2) . \tag{2.4}$$

Since $EN \sim a_1 a_2/I^2 \sim \alpha(1 - \alpha)(a_1 + a_2)^2/I^2$, we have, in view of (2.4),

$$\hat{n}/EN \sim \alpha(1 - \alpha)/\varphi^2(z_\alpha) . \tag{2.5}$$

Thus the RHS of (2.5), which we denote by $e(\alpha)$, is the l.r.e. under θ^* of the level α sequential LMP test, compared with the LMP non-sequential test (having, asymptotically the same slope). Some values of $e(\alpha)$ are tabled below in Section 4. It is not difficult to verify that $e(\alpha) > 1$ for all $\alpha \in (0, 1)$. (One can equivalently verify, by differentiating both sides, the inequality $\Phi(x)[1 - \Phi(x)] > \varphi^2(x)$, where Φ is the $N(0, 1)$ df.) Hence the l.r.e. at θ^* for the sequential test is greater than one.

For $\theta > \theta^*$, the comparison is even more dramatic. Typically, $\mu(\theta) = E_\theta r(x) > 0$ for $\theta > \theta^*$. In this case, it follows from Theorem 2.1 of Berk (1973) that $E_\theta N \sim a_2/\mu(\theta)$ as $a = a_1 \wedge a_2 \to \infty$. Thus for $\theta > \theta^*$, $E_\theta N = O(a)$, while it can be seen from (2.4) that $\hat{n} = O(a^2)$. Thus for $\theta > \theta^*$, the sequential test has l.r.e. ∞. Hence the l.r.e. for the sequential test exceeds one at all parameter points and is, in fact, infinite except at θ^*.

These considerations may also be extended to distributions outside the model. Suppose $X \sim f$. If $E_f r(X) \neq 0$, it follows again from Theorem 2.1 of Berk (1973) that $E_f N = O(a)$ and hence $\hat{n}/E_f N \to \infty$. If $E_f r(X) = 0$ and $0 < E_f r^2(X) < \infty$, $E_f N \sim a_1 a_2/E_f r^2(X) \sim \alpha(1 - \alpha)(a_1 + a_2)^2/E_f r^2(X)$ and $\hat{n}/E_f N \sim \alpha(1 - \alpha)E_f r^2(X)/I^2\varphi^2(z_\alpha) > 0$.

This comparison, while very favorable to the sequential test, is not entirely fair. For in matching slopes, it turns out in many examples that the non-sequential test has greater power at all $\theta > \theta^*$. Thus the non-sequential test, in essence, performs better, so it is not surprising that it requires a larger sample size. (However, this does not necessarily lead one to anticipate an l.r.e. of ∞ for the sequential test.) For this reason, we examine another way of comparing the sequential and non-sequential tests that, in a sense, better matches the power curves.

Our second comparison matches levels and (expected) sample sizes under θ^*. That is, we compare the level α sequential and non-sequential LMP tests, both of which (under θ^*) use EN observations. Our comparison is again asymptotic; the l.r.e. of the sequential test at θ is now $\lim_a EN/E_\theta N$. (The limit is taken as $a \to \infty$.) This comparison is somewhat fairer to the non-sequential test. Since the sequential test is LMP among all tests for the given EN, it follows that in some neigborhood of θ^*, the sequential test is more powerful than the non-sequential test. By contrast, the non-sequential test is typically more powerful for larger parameter values.

As above, we have $EN = O(a^2)$, while if $\theta \neq 0$, $E_\theta N = O(a)$. Thus the l.r.e. of the sequential test is

$$\lim_a EN/E_\theta N = 1 \quad \text{if} \quad \theta = \theta^* \qquad (2.6)$$
$$= \infty \quad \text{if} \quad \theta > \theta^* .$$

Thus the only apparent change from the previous notion of efficiency is that the l.r.e. at θ^* decreases from $e(\alpha)$ to 1. For distributions outside the model, the only change is when $E_f r(X) = 0$. Then $\lim_a EN/E_f N = E_f r^2(X)/I^2 > 0$.

3. Slopes and Pitman efficiency. We show in this section that for non-sequential tests, matching levels and slopes typically gives the Pitman efficiency as the l.r.e. Thus the first l.r.e. obtained in Section 2 can be considered a form of Pitman efficiency for the sequential test, compared with the non-sequential test. However, unlike the purely non-sequential case, this latter l.r.e. depends on α.

Suppose then that $\{T_n\}$ is a sequence of test statistics for testing H_1 vs H_2, T_n being a function of $X_1, \cdots, X_n$. We assume large values of T_n are significant. Suppose further that $ET_n = 0$, $ET_n^2 = 1$ and that T_n converges in law to $N(0, 1)$. In fact, we suppose more: that $(U_n, T_n) \rightarrow_{\mathscr{L}} (U, T)$, where $U_n = S_n/In^{\frac{1}{2}}$ and $(U, T) \sim N(0, 0, 1, 1, \rho)$. Thus a level α test based on T_n rejects if $T_n \geqq k_n$, where $k_n \sim z_\alpha$. It follows from Abraham's theorem that the slope of this test is

$$m_n = ES_n 1_{(T_n \geqq k_n)} = In^{\frac{1}{2}} EU_n 1_{(T_n \geqq k_n)} . \tag{3.1}$$

Since (U_n, T_n) converges in law and $\{U_n\}$ is uniformly integrable, it follows that $EU_n 1_{(T_n \geqq k_n)} \rightarrow EU1_{(T \geqq z_\alpha)}$. Moreover, we may write $U = \rho T + (1 - \rho^2)^{\frac{1}{2}} V$, where T and V are independent $N(0, 1)$. Thus $EU1_{(T \geqq z_\alpha)} = E\rho T 1_{(T \geqq z_\alpha)} = \rho\varphi(z_\alpha)$. It follows from (3.1) and the preceding that

$$m_n \sim In^{\frac{1}{2}} \rho\varphi(z_\alpha) . \tag{3.2}$$

Suppose now $\{\hat{T}_n\}$ is another sequence of statistics for testing H_1 vs H_2. We assume the same normalizations for $\hat{T}_n$ and in particular, that $(U_n, \hat{T}_n)$ converges in law to $N(0, 0, 1, 1, \hat{\rho})$. Thus the slope for the second test sequence satisfies

$$\hat{m}_n \sim In^{\frac{1}{2}} \hat{\rho}\varphi(z_\alpha) . \tag{3.3}$$

It is then clear that to equalize, asymptotically, the levels and slopes of the two test sequences, one must choose sample sizes n and $\hat{n}$, respectively, satisfying

$$\hat{n}/n \sim \rho^2/\hat{\rho}^2 .$$

Thus $\rho^2/\hat{\rho}^2$ is the l.r.e. of $\{T_n\}$ to $\{\hat{T}_n\}$.

We show next that this l.r.e. is typically the Pitman efficiency. We have $\rho = EUT = \lim EU_n T_n$. The latter equality follows from weak convergence and uniform integrability: $\{U_n^2\}$ uniformly integrable and $\{T_n\}$ L_2-bounded entails the uniform integrability of $\{U_n T_n\}$. Assuming one can differentiate across the expectation, $EU_n T_n = ES_n T_n/In^{\frac{1}{2}} = (In^{\frac{1}{2}})^{-1} \partial E_\theta T_n/\partial\theta|_{\theta^*}$. Assuming the analogous facts hold also for $\{\hat{T}_n\}$,

$$\rho^2/\hat{\rho}^2 = [\lim_n \partial E_\theta T_n/\partial\theta|_{\theta^*}]^2/[\lim_n \partial E_\theta \hat{T}_n/\partial\theta|_{\theta^*}]^2 ,$$

which is the usual expression for the Pitman efficiency of $\{T_n\}$ to $\{\hat{T}_n\}$. This connection between slopes and Pitman efficiency has been previously noted by Stuart (1954).

4. Limiting minimal efficiency. Under the considerations in Section 2, the l.r.e. of the sequential test is never less than one, even under the more stringent of the two notions of l.r.e. discussed there: $\lim_\alpha EN/E_\theta N \geqq 1$ for all θ. We shall see, however, that the sequential test need not be *uniformly* more efficient, asymptotically, by considering its limiting minimal efficiency (l.m.e.) in this case. This is defined as $\lim_a \inf_\theta EN/E_\theta N = \lim_a EN/\sup_\theta E_\theta N$. We show that $\eta(\alpha) = \lim_a \sup_\theta E_\theta N/EN$ can exceed one, so that the maximal (expected) sample size required by the sequential test can exceed that required by the corresponding non-sequential test. Equivalently, $1/\eta(\alpha)$, the l.m.e. of the sequential test, can be less than one. Note that in view of (2.6), the interval $\{\theta : E_\theta N > EN\}$ must shrink to $\{\theta^*\}$ as $a \to \infty$. The analog of Pitman efficiency discussed in Section 2 gives a somewhat different result. The corresponding l.m.e. is $\lim_a \hat{n}/\sup_\theta EN = e(\alpha)/\eta(\alpha)$. In contrast with the preceding, this l.m.e. is typically greater than one. Some values of $\eta(\alpha)$ and the l.m.e.'s $1/\eta(\alpha)$ and $e(\alpha)/\eta(\alpha)$ are tabulated below.

We give specific considerations for the symmetric binomial problem of testing $H_1 : p = \frac{1}{2}$ vs $H_2 : p > \frac{1}{2}$. Because of invariance-principle considerations, which we do not enter into here, we conjecture that our results apply to the class of tests discussed in this paper. For the symmetric binomial problem, $r(X) = 4X - 2$ and we can, for convenience, write $N = \inf\{n : \sum_1^n (2X_j - 1) \notin (-a_1, a_2)\}$. Note that $2X - 1$ assumes only the values ± 1 and is symmetric when $p = p^* = \frac{1}{2}$. Thus N is the stopping time associated with the classical gambler's ruin problem. We clearly can, without loss of generality, take a_1 and a_2 to be positive integers. Then we have the following exact formulas (see, e.g. Feller (1968)). The power of the LMP sequential test at p is given by

$$\beta(p) = [1 - (q/p)^{a_1}]/[1 - (q/p)^{a_1+a_2}]\,, \tag{4.1}$$

$q = 1 - p$ and

$$E_p N = (a_1 + a_2)[\beta(p) - \alpha]/(2p - 1)\,. \tag{4.2}$$

Here α, the level of the test, is given by

$$\alpha = a_1/(a_1 + a_2) \tag{4.3}$$

and

$$EN = a_1 a_2\,. \tag{4.4}$$

(Equations (4.3) and (4.4) can be formally obtained from (4.1) and (4.2) by continuity at $p = \frac{1}{2}$.) Letting $a_1 + a_2 = k$ and, for $p \neq \frac{1}{2}$, $\nu = \frac{1}{2}\log(p/q)$, we have

$$\begin{aligned} \beta(p) &= (1 - e^{-2\nu\alpha k})/(1 - e^{-2\nu k})\,, & p &\neq \tfrac{1}{2} \\ &= \alpha\,, & p &= \tfrac{1}{2} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} E_p N &= k[\beta(p) - \alpha]/(2p - 1)\,, & p &\neq \tfrac{1}{2} \\ &= k^2\alpha(1 - \alpha)\,, & p &= \tfrac{1}{2}\,. \end{aligned} \tag{4.6}$$

We show that $\eta(\alpha) = \lim_a \sup_p E_p N/EN$ exists and is related to a similar quantity for a Wiener process.

Since $2p - 1 = (e^{2\nu} - 1)/(e^{2\nu} + 1)$ and $(2/x) - 1 \leqq (e^x + 1)/(e^x - 1) \leqq (2/x) + 1$, $|(2p - 1)^{-1} - \nu^{-1}| \leqq 1$ and it then follows from (4.5) and (4.6) that

$$(4.7) \qquad |E_p N - k[\beta_w(k\nu) - \alpha]/\nu| \leqq k\,,$$

where

$$(4.8) \qquad \begin{aligned} \beta_w(\mu) &= (1 - e^{-2\mu\alpha})/(1 - e^{-2\mu})\,, & \mu &\neq 0 \\ &= \alpha\,, & \mu &= 0 \end{aligned}$$

is the probability that a standard Wiener process with drift μ per unit time hits $1 - \alpha$ before hitting $-\alpha$. We note too that for these boundaries, the expected hitting time for the Wiener process is

$$(4.9) \qquad \begin{aligned} E_\mu \tau &= [\beta_w(\mu) - \alpha]/\mu\,, & \mu &\neq 0 \\ &= \alpha(1 - \alpha)\,, & \mu &= 0\,. \end{aligned}$$

(These formulas for the Wiener process can be obtained from Wald's no-overshoot expressions, which are exact in this case.) It follows from (4.6) and (4.7) that

$$(4.10) \qquad |\sup_p E_p N/EN - \sup_\mu [\beta_w(\mu) - \alpha]/\mu\alpha(1 - \alpha)| \leqq 1/k\alpha(1 - \alpha)\,.$$

Since $\lim_a k = \infty$, it follows that

$$(4.11) \qquad \begin{aligned} \eta(\alpha) &= \lim_a \sup_p E_p N/EN \\ &= \sup_\mu [\beta_w(\mu) - \alpha]/\mu\alpha(1 - \alpha) = \sup_\mu E_\mu \tau / E_0 \tau\,. \end{aligned}$$

Thus in the symmetric binomial case, the ratio $\sup_p E_p N/EN$ behaves, asymptotically, like the corresponding (non-asymptotic) ratio for the Wiener process given by the RHS of (4.11).

Some values of $\eta(\alpha)$ are tabulated below, together with μ_α, the value of μ for which $E_\mu \tau$ attains its maximum. (The tabulated values for μ_α were obtained by an iterative method.) Values of $e(\alpha)$, the l.r.e. at θ^* (see Section 2) are also given, together with the l.m.e.s $1/\eta(\alpha)$ and $e(\alpha)/\eta(\alpha)$. As an aside, we remark that it can be seen from (4.7) that when k is large, for p_α, the value maximizing $E_p N$, we have $\nu_\alpha = \frac{1}{2}\log(p_\alpha/q_\alpha) = \mu_\alpha/k + o(1/k)$, hence that $p_\alpha = \frac{1}{2} + \mu_\alpha/2k + o(1/k)$.

α	.50	.25	.10	.05	.025	.01	.005	.001
μ_α	0	1.12	2.39	3.55	4.8	7.4	10.3	22.7
$\eta(\alpha)$	1	1.09	1.32	1.48	1.61	1.74	1.81	1.91
$e(\alpha)$	1.57	1.86	2.92	4.54	7.14	13.9	23.8	87.9
$1/\eta(\alpha)$	1	.92	.76	.68	.62	.58	.55	.52
$e(\alpha)/\eta(\alpha)$	1.57	1.70	2.21	3.07	4.43	8.01	13.2	46.0

As suggested by the table, $\eta(\alpha)$ increases from 1 as α decreases from $\frac{1}{2}$. Moreover:

LEMMA 4.1. $\lim_{\alpha \to 0} \eta(\alpha) = 2$.

PROOF. From (4.8) and (4.9) we have

$$E_\mu\tau/E_0\tau = [2/(1-\alpha)](2\alpha\mu)^{-1}[(1-e^{-2\alpha\mu})/(1-e^{-2\mu})-\alpha] = [2/(1-\alpha)]c(\alpha, 2\mu)\,, \tag{4.12}$$

where

$$c(\alpha, x) = (\alpha x)^{-1}[(1-e^{-\alpha x})/(1-e^{-x})-\alpha] = (1-e^{-x})^{-1}[(1-e^{-\alpha x})/\alpha x - (1-e^{x})/x]\,.$$

It is easily seen intuitively and may be checked from (4.12) that when $\alpha < \frac{1}{2}$, $E_\mu\tau < E_0\tau$ for $\mu < 0$. We need only consider $\alpha < \frac{1}{2}$ and thus, in determining $\sup_\mu E_\mu\tau$, confine attention to $\mu > 0$. Since for $y > 0$, $(1-e^{-y})/y \leqq 1$, we have for $x > 0$,

$$c(\alpha, x) \leqq (1-e^{-x})^{-1}[1-(1-e^{-x})/x] = (1-e^{-x})^{-1} - 1/x \leqq 1\,.$$

Thus for $\alpha \leqq \frac{1}{2}$, $\eta(\alpha) \leqq 2/(1-\alpha)$; hence $\limsup_{\alpha\to 0}\eta(\alpha) \leqq 2$. Also, since $c(\alpha, x) > 0$,

$$c(\alpha, (2/\alpha)^{\frac{1}{2}}) \geqq (1-e^{-(2\alpha)^{\frac{1}{2}}})/(2\alpha)^{\frac{1}{2}} - (\alpha/2)^{\frac{1}{2}} \geqq 1-(2\alpha)^{\frac{1}{2}}\,,$$

since for $y > 0$, $(1-e^{-y})/y \geqq 1 - y/2$. Thus $\eta(\alpha) \geqq 2c(\alpha, (2/\alpha)^{\frac{1}{2}})/(1-\alpha) \geqq 2[1-(2\alpha)^{\frac{1}{2}}]/(1-\alpha)$ and $\liminf_{\alpha\to 0}\eta(\alpha) \geqq 2$. □

By a more careful analysis of $c(\alpha, x)$, the following asymptotic expansions (for small α) may be obtained.

$$\mu_\alpha = (2\alpha)^{-\frac{1}{2}} + \tfrac{1}{3} + 11(2\alpha)^{\frac{1}{2}}/72 + 43\alpha/270 + O(\alpha^{\frac{3}{2}}) \tag{4.13}$$

and

$$\eta(\alpha) = 2[1-(2\alpha)^{\frac{1}{2}} + 4\alpha/3 - 35\alpha(2\alpha)^{\frac{1}{2}}/36 + 182\alpha^2/135] + O(\alpha^{\frac{5}{2}})\,. \tag{4.14}$$

The bracketed term of (4.14) gives a reasonable approximation for $\alpha \leqq .1$. E.g., we obtain $\eta(.1) \doteq 1.312$, compared with the more precise value 1.316.

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