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NOTE ON THE PAPER "TRANSFORMATION GROUPS AND SUFFICIENT STATISTICS" BY J. PFANZAGL

BY C. HIPP

University of Cologne

Pfanzagl (1972) has shown that under suitable regularity conditions a family of probability measures which is generated by a transformation group and which for some sample size greater than one admits a sufficient statistic which is continuous, real-valued, and equivariant, is equivalent to the location parameter family of normal distributions or to a scale parameter family of Gamma distributions. This was proved under the assumption that the transformation group is Abelian. In this note commutativity of the group is replaced by local compactness.

0. Summary. In [1] Pfanzagl has shown that under suitable regularity conditions a family of probability measures which is generated by a transformation group and which for some sample size greater than 1 admits a sufficient statistic which is continuous, real-valued, and equivariant, is equivalent to the location parameter family of normal distributions or to a scale parameter family of Gamma distributions.

This was proved under the assumption that the transformation group is Abelian. In this paper we replace commutativity of the group by local compactness. We remark, however, that any family of probability measures for which the assertion is true necessarily fulfills $P_{\vartheta\tau} = P_{\tau\vartheta}$ for all transformations ϑ, τ.

1. The results. In the following we shall use the notations of [1]. In addition, we will use the following assumptions:

(ii′) $(\Theta, \mathscr{W})$ is a connected and locally connected continuous transformation group on X such that the induced transformation group is Abelian;

(ii″) $(\Theta, \mathscr{W})$ is a connected and locally connected, locally compact continuous transformation group on X.

We call the original assumptions of the Theorem in [1] A, and the assumptions where (ii) is replaced by (ii′) or (ii′) A′ or A″, respectively. (ii′) is an immediate consequence of (ii). We remark that in the proof of the Theorem in [1] assumptions A′ only are used, and that the commutativity of the induced transformation group is even necessary. However, assumption (ii′) has the distractive property of being an assumption on both, the transformation group and the sufficient statistic. Hence the following proposition which uses the stronger assumptions A″ is more satisfactory.

THEOREM 1.1. *Under the assumptions* A″ *the assertion of the theorem in* [1] *is true.*

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PROOF. Since in [1] the commutativity of Θ is used only in the proof of Lemma 5 in [1], the assertion follows immediately from our Lemma 2.4.

THEOREM 1.2. *If the assertion of the theorem in* [1] *is true, then the induced group* Θ *is necessarily commutative. In particular*: A″ *implies* A′.

PROOF. If the assertion of the theorem in [1] is true, this implies in particular that for $\vartheta, \tau \in \Theta$, $P_{\vartheta\tau} \mid \mathscr{A} = P_{\tau\vartheta} \mid \mathscr{A}$. Hence for $t \in T(X^n)$ we have $P^n * T(-\infty, \boldsymbol{\vartheta\tau} t) = P^n * T(-\infty, \boldsymbol{\tau\vartheta} t)$. This, together with assumption (iii) of the theorem in [1], implies $\boldsymbol{\vartheta\tau} = \boldsymbol{\tau\vartheta}$ for all $\vartheta, \tau \in \Theta$.

Together with Theorem 1.1 this yields that A″ implies A′.

2. Auxiliary lemmas.

LEMMA 2.1. *Assume that* $(X, \mathscr{U})$ *is a topological space and* $(\Theta, \mathscr{W})$ *a continuous transformation group on* $(X, \mathscr{U})$ *which is connected and locally connected.*

If $T\colon X \to \mathbb{R}$ *is continuous and equivariant and* $t \in T(X)$ *with* $\boldsymbol{\Theta} t \neq \{t\}$, *then the map* $\vartheta \to \boldsymbol{\vartheta} t$ *is open.*

PROOF. (a) At first we show that $\varepsilon \in W \in \mathscr{W}$ implies $t \in (\mathbf{W}t)^0$. Let $W_1 \in \mathscr{W}$ be an open connected neighborhood of ε in W fulfilling $W_1 \subset W \cap W^{-1}$. Then according to Lemma 4 (ii) in [1] $\mathbf{W}_1 t \neq \{t\}$. Hence w.l.g. there exists $\vartheta \in W_1$ such that $t < \boldsymbol{\vartheta} t$. As W_1 is connected and $\vartheta \to \boldsymbol{\vartheta} t$ is continuous by Lemma 1 in [1], $[t, \boldsymbol{\vartheta} t) \subset \mathbf{W} t$. From $\boldsymbol{\vartheta}^{-1} t < t$ and the connectedness of W_1^{-1} we obtain $(\boldsymbol{\vartheta}^{-1} t, t] \subset \mathbf{W}_1^{-1} t \subset \mathscr{W} t$ which implies the assertion (a).

(b) Now we show that for $t' \in \mathbf{W} t$, we have $t' \in (\mathbf{W}t)^0$. Let $\tau \in W$ with $t' = \boldsymbol{\tau} t$. Then $\varepsilon \in W\tau^{-1} \in \mathscr{W}$ and $\Theta t' = \Theta \boldsymbol{\tau} t = \Theta t$. Hence we can apply (a) for t' and $W\tau^{-1}$ instead of t and W yielding $t' \in (\mathbf{W}\boldsymbol{\tau}^{-1} t')^0 = (\mathbf{W}t)^0$.

LEMMA 2.2. *Assume that* $(X, \mathscr{U})$ *is a topological space,* $(\Theta, \mathscr{W})$ *a continuous transformation group which is connected and locally connected, and* $T\colon X \to \mathbb{R}$ *is continuous and equivariant. Let* $S := \{t \in T(X) : \Theta t \neq \{t\}\}$. *Then for all* $U \in \mathscr{U}$, $TU \cap \bar{S}$ *is open.*

PROOF. Let $t \in TU \cap \bar{S}$, $x \in U$ with $Tx = t$, and $W^t := \{\vartheta \in \Theta : \vartheta x \in U$ and $\boldsymbol{\vartheta} t \in \bar{S}\}$. Lemma 4 (i) and Lemma 1 in [1] imply $W^t \in \mathscr{W}$. As $\mathbf{W}^t t \subset TU \cap \bar{S}$ and $\mathbf{W}^t t$ is open by Lemma 2.1 this implies the assertion.

We remark that this lemma does not imply Lemma 2 of [1] in general. An application of Lemma 2.2 for $U = T^{-1}A$ yields that $T^{-1}A \in \mathscr{U}$ implies $A \cap \bar{S} \in \mathbb{O} \cap T(X)$. Lemmas 1 and 3 of [1] are subsumed in the following

LEMMA 2.3. *Let* $(X, \mathscr{U})$ *be connected and locally connected,* $(\Theta, \mathscr{W})$ *a connected continuous transformation group on* $(X, \mathscr{U})$, *and* $T\colon X \to \mathbb{R}$ *continuous and equivariant. Then the map* $(\vartheta, t) \to \boldsymbol{\vartheta} t$ *is* $\mathscr{W} \times (\mathbb{O} \cap T(X))$ *continuous.*

PROOF. In Lemma 3 of [1] it was shown that the transformations $\boldsymbol{\vartheta}$ on $T(X)$ are continuous. As they are, furthermore, 1–1 they must be strictly monotone.

We show that they are all strictly increasing. W.l.g. we may assume that we can find $t_1, t_2 \in T(X)$ with $t_1 < t_2$. Let $W := \{\vartheta \in \Theta : \boldsymbol{\vartheta} t_1 < \boldsymbol{\vartheta} t_2\}$. The sets W and $\bar{W}$ form a partition of Θ into disjoint open sets. As Θ is connected and $\varepsilon \in W$ we have $\Theta = W$ which proves that all $\boldsymbol{\vartheta}$ are strictly increasing. From Lemma 1 in [1] we know that for all $t \in T(X)$ $\vartheta \to \boldsymbol{\vartheta} t$ is continuous. This together with 2.3.1 implies the assertion:

2.3.1. If Q is a countable set dense in $T(X)$, then for all $r \in \mathbb{R}$

$$\{(\vartheta, t) \in \Theta \times T(X) : \boldsymbol{\vartheta} t > r\} = \bigcup_{q \in Q} (\{\vartheta \in \Theta : \boldsymbol{\vartheta} q < r\} \times [(-\infty, q) \cap T(X)]) .$$

LEMMA 2.4. *Assume that*

(i) *$(X, \mathscr{U})$ is connected and locally connected;*

(ii) *$(\Theta, \mathscr{W})$ is a connected, locally connected, and locally compact continuous transformation group on $(X, \mathscr{U})$;*

(iii) *P is a probability measure on the Borel field $\mathscr{A}$ of $(X, \mathscr{U})$ such that $P(U) > 0$ for all $\emptyset \neq U \in \mathscr{U}$;*

(iv) *there exists an equivariant, real-valued, and continuous statistic T which is sufficient for the generated family of probability measures $P_\vartheta \mid \mathscr{A}$, $\vartheta \in \Theta$.*

Let S be defined as in Lemma 2.2. Let $t_0 \in \bar{S}$ and $U \in \mathscr{U}$ with $t_0 \in T(U)$ be given. Then for any conditional expectation $p(U, \bullet)$ there exists a $P * T$-null set M and an open interval I containing t_0 such that $p(U, t) > 0$ for every $t \in I \cap \bar{M}$.

This result is slightly better than Lemma 5 of [1], but the new assertion $t_0 \in I^0$ is a simple consequence of the fact that for $t \in \bar{S}$ the map $\vartheta \to \boldsymbol{\vartheta} t$ is open (see Lemma 2.1).

PROOF. 1. As $(\Theta, \mathscr{W})$ is locally compact, there exists a right Haar-measure ν on the σ-field $\mathscr{D}$ generated by $\mathscr{W}$. As there exists a compact neighborhood of unity, connectedness of $(\Theta, \mathscr{W})$ implies that Θ is σ-compact (see [2], page 129, Theorem 14). As the Haar-measure is finite on compact sets, ν is σ-finite. We remark that any Haar-measure is positive on all nonempty open sets.

2. Let $x_0 \in U$ with $Tx_0 = t_0$. As $U \cap T^{-1}\bar{S} \in \mathscr{U}$, continuity of $(\vartheta, x) \to \vartheta x$ implies the existence of connected sets $W_0 \in \mathscr{W}$, $U_0 \in \mathscr{U}$ with $\varepsilon \in W_0$, $x_0 \in U_0$ such that $W_0 U_0 \subset U \cap T^{-1}\bar{S}$ and $W_0^{-1} U_0 \subset U \cap T^{-1}\bar{S}$.

3. Let $E :\equiv \{(\vartheta, t) \in W_0 \times T(X) : p(U, t) < p(U_0, \boldsymbol{\vartheta} t)\}$. As $(\vartheta, t) \to \boldsymbol{\vartheta} t$ is $\mathscr{D} \times (\mathbb{B} \cap T(X))$, $\mathbb{B}$-measurable (see (2.3.1) in Lemma 2.3), we have $E \in \mathscr{D} \times (\mathbb{B} \cap T(X))$. Let E_ϑ and E^t denote sections of E. We shall show that

$$P * T(E_\vartheta) = 0 \qquad \text{for all} \quad \vartheta \in W_0 . \tag{3.1}$$

For all $B \in \mathbb{B} \cap T(X)$ and $\vartheta \in \Theta$ we have

$$\begin{aligned} &\int p(U_0, \boldsymbol{\vartheta} t) 1_B(t) P * T(dt) \\ &\quad = \int p(U_0, \boldsymbol{\vartheta} t) 1_{\vartheta B}(\boldsymbol{\vartheta} t) P * T(dt) = \int p(U_0, s) 1_{\vartheta B}(s) P_\vartheta * T(ds) \\ &\quad = P_\vartheta(U_0 \cap T^{-1}\boldsymbol{\vartheta} B) = P(\vartheta^{-1}U_0 \cap T^{-1}B) = \int p(\vartheta^{-1}U_0, t) 1_B(t) P * T(dt) . \end{aligned}$$

Therefore

$$p(U_0, \boldsymbol{\vartheta} t) = p(\vartheta^{-1}U_0, t) \qquad \text{for} \quad P * T\text{-a.a.}$$

$t \in T(X)$ (with the $P * T$-null set depending on ϑ and U_0).

As $\vartheta^{-1}U_0 \subset U$ for every $\vartheta \in W_0$, this implies $p(U_0, \boldsymbol{\vartheta} t) \leqq p(U, t)$ for every $\vartheta \in W_0$ and $P * T$-a.a. $t \in T(X)$. This establishes (3.1).

From (3.1) we obtain

$$\nu \times P * T(E) = 0 . \tag{3.2}$$

4. Let $M :\equiv \{t \in T(X) : \nu(E^t) > 0\}$. By Fubini's Theorem, (3.2) implies $P * T(M) = 0$. As $\varepsilon \in W_0 \in \mathscr{W}$ there exists an open neighborhood W_1 of ε, such that $W_1^2 \subset W_0$.

Define for $t \in T(X)$

$$\Psi_1(t) :\equiv \int p(U_0, \boldsymbol{\vartheta} t) 1_{W_0}(\vartheta)\nu(d\vartheta) \qquad \text{and}$$
$$\Psi_2(t) :\equiv \int p(U_0, \boldsymbol{\vartheta} t) 1_{W_1}(\vartheta)\nu(d\vartheta) .$$

If $t \in T(X)$ and $\Psi_1(t) = 0$, then

$$\nu\{\vartheta \in W_0 : p(U_0, \boldsymbol{\vartheta} t) > 0\} = 0 . \tag{4.1}$$

Then we have for all $\tau \in W_1$

$$\begin{aligned}
&\{\vartheta \in W_1 : p(U_0, \boldsymbol{\vartheta\tau} t) > 0\} \\
&\quad = \{\sigma\tau^{-1} \in W_1 : p(U_0, \boldsymbol{\sigma} t) > 0\} = \{\sigma \in W_1\tau : p(U_0, \boldsymbol{\sigma} t) > 0\}\tau^{-1} \\
&\quad \subset \{\sigma \in W_0 : p(U_0, \boldsymbol{\sigma} t) > 0\}\tau^{-1} ,
\end{aligned}$$

and by (4.1) and the right invariance of $\nu \mid \mathscr{D}$ we get

$$\nu\{\vartheta \in W_1 : p(U_0, \boldsymbol{\vartheta\tau} t) > 0\} = 0 .$$

This implies:

$$\text{if} \quad \Psi_1(t) = 0 , \qquad \text{then for all} \quad \tau \in W_1 , \qquad \Psi_2(\boldsymbol{\tau} t) = 0 . \tag{4.2}$$

5. Now we can show that for all $t \in T(U_0) \cap \bar{M}$, $p(U, t) > 0$. We remark that by $U_0 \subset U \cap T^{-1}\bar{S}$ $T(U_0)$ is an open interval (see Lemma 2.2). For $t \notin M$ we have $\Psi_1(t) \leqq p(U, t)\nu W_0$. Let now $t \in T(U_0) \cap \bar{M}$ and assume $p(U, t) = 0$. Then $\Psi_1(t) = 0$ and for all $\tau \in W_1$ $\Psi_2(\boldsymbol{\tau} t) = 0$. This will lead to a contradiction.

With $I = \mathbf{W}_1 t$ we have

$$\begin{aligned}
0 &= P * T(\Psi_2 1_I) \\
&= \int 1_I(t) \int p(U_0, \boldsymbol{\vartheta} t) 1_{W_1}(\vartheta)\nu(d\vartheta) P * T(dt) \\
&= \int \left(\int p(U_0, \boldsymbol{\vartheta} t) 1_I(t) P * T(dt)\right) 1_{W_1}(\vartheta)\nu(d\vartheta) \\
&= \int P(\vartheta^{-1}U_0 \cap T^{-1}I) 1_{W_1}(\vartheta)\nu(d\vartheta) .
\end{aligned}$$

As I is an open interval, there exists an open neighborhood of ε in W_1, say W_2, such that for $\vartheta \in W_2$, $\vartheta^{-1}U_0 \cap T^{-1}I \neq \emptyset$. Then $P(\vartheta^{-1}U_0 \cap T^{-1}I) > 0$ for all $\vartheta \in W_2$, and hence the last integral is positive.

This concludes the proof.

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MATHEMATISCHES INSTITUT DER
UNIVERSITÄT ZU KÖLN
5 KÖLN 41, WEYERTAL 86–90
WEST GERMANY