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TESTS FOR THE GENERAL LINEAR HYPOTHESIS UNDER THE MULTIPLE DESIGN MULTIVARIATE LINEAR MODEL

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The generalization of the standard model for MANOVA which allows for the possibility of different "design" matrices for the variates is known as the multiple design multivariate linear model. For example, in multivariate regression analysis we might have polynomial models of different degree in the "dependent" variates.

In this paper, new tests are given for the general linear hypothesis under the multiple design multivariate model and in one case the corresponding critical region is "inverted" to obtain simultaneous confidence intervals on certain functions of the location parameters.

1. Introduction. Consider the multiple design multivariate (MDM) linear model, Roy and Srivastava (1964) and Srivastava (1967),

$$E(\mathbf{y}_i) = \mathbf{X}_i \boldsymbol{\xi}_i\,, \qquad \operatorname{Var}(\mathbf{y}_i) = \sigma_{ii}\mathbf{I}_n \qquad (i = 1, \cdots, p) \tag{1.1}$$

where $\mathbf{y}_i(n \times 1)$ contains the observations on the ith response, $\mathbf{X}_i(n \times m_i)$ is a matrix of known constants, $\boldsymbol{\xi}_i(m_i \times 1)$ is an unknown vector of parameters, and $\boldsymbol{\Sigma} = (\sigma_{ij})$ is the unknown dispersion matrix of a row of the $(n \times p)$ data matrix $\mathbf{Y} = (\mathbf{y}_1, \cdots, \mathbf{y}_p)$. Further, assume that the rows of $\mathbf{Y}$ are independent observations from multivariate normal populations with common dispersion matrix $\boldsymbol{\Sigma}$. If we have $\mathbf{X}_1 = \cdots = \mathbf{X}_p$, then the usual techniques of multivariate regression analysis and MANOVA become available. However, many practical applications arise where the $\mathbf{X}$'s are unequal; see for example, Zellner (1962).

The estimation problem for multivariate regression systems, falling under the general model (1.1), is considered by Mallios (1961) and by Zellner (1962, 1963).

In this paper, Roy's union-intersection principle of test construction is used to justify new tests of the (testable) general linear hypothesis $H_0: \{\mathbf{C}_i \boldsymbol{\xi}_i = \mathbf{0}$ for $i = 1, \cdots, p\}$ under the multiple design multivariate linear model. Advantages of the tests over the step-down procedure proposed by Roy and Srivastava are that the testability conditions are relatively simple and the standard computational techniques of MANOVA are applicable.

The critical region of the largest root test is "inverted" to obtain simultaneous confidence intervals on functions of the location parameters which resemble bilinear forms.

Finally some simple examples are given to illustrate the theory.

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2. Tests for linear hypotheses. Consider the general linear hypothesis,

$$H_0: \{\mathbf{C}_i \boldsymbol{\xi}_i = \mathbf{0};\ i = 1, 2, \cdots, p\} \tag{2.1}$$

where for all i, $\mathbf{C}_i$ is a known $(s_i \times m_i)$ matrix of rank s_i. Here $\mathbf{0}$ denotes a matrix (of appropriate dimension) each of whose elements is zero. For the arbitrary nonzero $(p \times 1)$ vector $\mathbf{a}' = (a_1, a_2, \cdots, a_p)$ we have from (1.1)

$$E(\mathbf{Ya}) = \mathbf{X}_1(a_1 \boldsymbol{\xi}_1) + \cdots + \mathbf{X}_p(a_p \boldsymbol{\xi}_p) \tag{2.2}$$

$$\operatorname{Var}(\mathbf{Ya}) = (\mathbf{a}'\boldsymbol{\Sigma}\mathbf{a})\mathbf{I}_n . \tag{2.3}$$

Rewrite (2.2) as

$$E(\mathbf{Ya}) = \mathbf{X}\boldsymbol{\xi}_a \tag{2.4}$$

where $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_p)$ and $\boldsymbol{\xi}_a' = (a_1\boldsymbol{\xi}_1', a_2\boldsymbol{\xi}_2', \cdots, a_p\boldsymbol{\xi}_p')$. Note that $\mathbf{X}$ is $(n \times \sum_{i=1}^p m_i)$. Let $R(\mathbf{X}) = r \leqq \sum m_i$ and assume that $p \leqq n - r$, where $R(\mathbf{X})$ denotes the rank of $\mathbf{X}$. Let $s = \max(s_1, s_2, \cdots, s_p)$ and

$$\mathbf{C}_i^* = \begin{bmatrix} \mathbf{C}_i \\ \hline \mathbf{0}_i \end{bmatrix}; \qquad i = 1, \cdots, p , \tag{2.5}$$

where $\mathbf{0}_i$ is $((s - s_i) \times m_i)$. Finally, let

$$\mathbf{C} = (\mathbf{E}_1\mathbf{C}_1^*, \cdots, \mathbf{E}_p\mathbf{C}_p^*) \tag{2.6}$$

where $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_p$ are nonsingular $(s \times s)$ matrices, and assume the rank of the $(s \times \sum m_i)$ matrix $\mathbf{C}$ is $R(\mathbf{C}) = s \leqq r$.

Clearly the null hypothesis H_0 holds if and only if $\mathbf{C}\boldsymbol{\xi}_a = \mathbf{0}$ for all $\mathbf{a} \neq \mathbf{0}$, i.e.,

$$H_0 \equiv \textstyle\bigcap_{\mathbf{a}} \{H_{0a}: \mathbf{C}\boldsymbol{\xi}_a = \mathbf{0}\} . \tag{2.7}$$

DEFINITION 2.1. The hypothesis H_0 is testable if there exist nonsingular matrices $\mathbf{E}_1, \cdots, \mathbf{E}_p$ such that

$$R\begin{bmatrix} \mathbf{X} \\ \hline \mathbf{C} \end{bmatrix} = R[\mathbf{X}] . \tag{2.8}$$

It is well known that condition (2.8) is a necessary and sufficient condition for testability of $H_{0a}: \mathbf{C}\boldsymbol{\xi}_a = \mathbf{0}$ against $H: \mathbf{C}\boldsymbol{\xi}_a \neq \mathbf{0}$ under the "univariate" linear model in (2.3) and (2.4).

Following Roy, Gnandesikan and Srivastava (1971) let

$$\mathbf{Q}_e = \mathbf{I}_n - \mathbf{X}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{X}_0' \tag{2.9}$$

$$\mathbf{Q}_h = \mathbf{X}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{C}_0'[\mathbf{C}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{C}_0']^{-1}\mathbf{C}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{X}_0' \tag{2.10}$$

where $\mathbf{X}_0$ is any basis of $\mathbf{X}$ given by $\mathbf{X}_0 = \mathbf{XH}$, $\mathbf{C}_0 = \mathbf{CH}$, and $\mathbf{H}$ is any suitable $(\sum m_i \times r)$ matrix of rank r.

THEOREM 2.1. *Under H_0, the matrices*

$$\mathbf{S}_h = \mathbf{Y}'\mathbf{Q}_h\mathbf{Y} \quad (\textit{the matrix due to the hypothesis}) \tag{2.11}$$

$$\mathbf{S}_e = \mathbf{Y}'\mathbf{Q}_e\mathbf{Y} \quad (\textit{the matrix due to error}) \tag{2.12}$$

are independent central Wishart matrices with degrees of freedom $t = \min(p, s)$ and $n - r$ respectively.

PROOF. The theorem follows immediately from the fact that for every $\mathbf{a} \neq \mathbf{0}$, $\mathbf{a}'\mathbf{S}_h\mathbf{a}$ and $\mathbf{a}'\mathbf{S}_e\mathbf{a}$ are independent central chi-square random variables with degrees of freedom t and $n - r$ respectively. Note that $\mathbf{S}_h$ is positive definite and, under $p \leqq (n - r)$, $\mathbf{S}_e$ is positive definite with probability 1.

Let $c_i(\mathbf{A})$, $i = 1, \cdots, p$, denote the characteristic roots of the symmetric $(p \times p)$ matrix $\mathbf{A}$ ordered $c_1(\mathbf{A}) \geqq \cdots \geqq c_p(\mathbf{A})$. Application of Roy's union-intersection principle of test construction leads immediately to the largest root $c_1 = c_1(\mathbf{S}_h\mathbf{S}_e^{-1})$, as a test statistic for H_0. The test statistic c_1 together with a size α critical region, $R = \{c_1 : c_1 \geqq \lambda_\alpha\}$, is selected because of its connection with the simultaneous confidence intervals to appear at the end of this section. However, as for the standard multivariate model, other criteria may be proposed for the hypotheses H_0; for instance Wilk's generalized likelihood ratio statistic, $|\mathbf{S}_e|/|\mathbf{S}_h + \mathbf{S}_e|$, Hotelling's trace, $\operatorname{tr}(\mathbf{S}_h\mathbf{S}_e^{-1})$ and Pillai's trace, $\operatorname{tr}(\mathbf{S}_h(\mathbf{S}_h + \mathbf{S}_e)^{-1})$.

Tables for the percentage points of the test statistics are readily available as are illustrations of their use; see for example, Sections 5 and 7 of Chapter 4 in Roy, Gnanadesikan and Srivastava (1971).

We now consider the problem of obtaining simultaneous confidence intervals for all (standardized) "bilinear" forms $\mathbf{b}'\mathbf{C}\boldsymbol{\xi}_a$ where all symbols are as previously defined. The intervals are obtained by the well-known technique of inverting the critical region for the largest root test. Consider

$$\hat{\boldsymbol{\theta}} = \mathbf{C}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{X}_0'\mathbf{Y}\,, \tag{2.13}$$

and let $\boldsymbol{\theta} = E(\hat{\boldsymbol{\theta}})$. Clearly $\mathbf{S}_h^* = (\hat{\boldsymbol{\theta}}' - \boldsymbol{\theta}')[\mathbf{C}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{C}_0']^{-1}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})$ has a central Wishart distribution with $t = \min(p, s)$ degrees of freedom. Thus with probability $(1 - \alpha)$ we have

$$c_1(\mathbf{S}_h^*\mathbf{S}_e^{-1}) \leqq \lambda_\alpha\,, \tag{2.14}$$

where λ_α denotes the upper $100(\alpha)$ percentage point of the (null) distribution of the largest characteristic root of $\mathbf{S}_h^*\mathbf{S}_e^{-1}$ with degrees of freedom p, s and $n - r$. From the extremal formulation for the largest characteristic root, it follows that (2.14) is equivalent to

$$\mathbf{b}'\hat{\boldsymbol{\theta}}\mathbf{a} - (\lambda_\alpha\mathbf{a}'\mathbf{S}_e\mathbf{a})^{\frac{1}{2}} \leqq \mathbf{b}'\mathbf{C}\boldsymbol{\xi}_a \leqq \mathbf{b}'\hat{\boldsymbol{\theta}}\mathbf{a} + (\lambda_\alpha\mathbf{a}'\mathbf{S}_e\mathbf{a})^{\frac{1}{2}} \tag{2.15}$$

for all $(p \times 1)$ vectors $\mathbf{a}$ and all $(s \times 1)$ vectors $\mathbf{b}$ subject to the "standardizing" constraint

$$\mathbf{b}'[\mathbf{C}_0(\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{C}_0']\mathbf{b} = 1\,. \tag{2.16}$$

Thus (2.15) supplies us with a basis for multiple comparisons on all standardized "bilinear" forms $\mathbf{b}'\mathbf{C}\boldsymbol{\xi}_a$.

3. Monotone MDM **models.** In this section we develop the theory for a class of hypotheses which are testable under a special MDM model defined in

DEFINITION 3.1. The MDM model (1.1) is monotone in the design matrices if it is possible by reordering the responses to reparametrize to the form

$$E(\mathbf{y}_i) = \mathbf{X}_i^* \boldsymbol{\xi}_i^* ; \qquad i = 1, 2, \cdots, p \tag{3.1}$$

such that

$$\begin{aligned} \mathbf{X}_2^* &= [\mathbf{X}_1^* \mid \mathbf{X}_{22}^*] \\ \mathbf{X}_3^* &= [\mathbf{X}_1^* \mid \mathbf{X}_{22}^* \mid \mathbf{X}_{32}^*] \\ &\vdots \\ \mathbf{X}_p^* &= [\mathbf{X}_1^* \mid \mathbf{X}_{22}^* \mid \mathbf{X}_{32}^* \mid \cdots \mid \mathbf{X}_{p2}^*] \end{aligned} \tag{3.2}$$

where $\mathbf{X}_i^*$ is $(n \times m_i^*)$ and $m_1^* \leqq m_2^* \leqq \cdots \leqq m_p^*$. Note that $\mathbf{X}_{i2}^*$ is $(n \times (m_i^* - m_{i-1}^*))$ for $i = 2, \cdots, p$. We will drop the superscript, *, from the notation for a monotone MDM model in the following. This should cause no confusion since we could just as easily have started with this particular ordering of the responses and parametrization in the definition (1.1).

The partitioning of $\mathbf{X}_2, \mathbf{X}_3, \cdots, \mathbf{X}_p$ in (3.2) induces a partitioning of $\boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_p$ as indicated in

$$\boldsymbol{\xi}_2' = [\boldsymbol{\xi}_{12}', \boldsymbol{\xi}_{22}'], \boldsymbol{\xi}_3' = [\boldsymbol{\xi}_{13}', \boldsymbol{\xi}_{23}', \boldsymbol{\xi}_{33}'], \cdots, \boldsymbol{\xi}_p' = [\boldsymbol{\xi}_{1p}', \boldsymbol{\xi}_{2p}', \cdots, \boldsymbol{\xi}_{pp}'] \tag{3.3}$$

where $\boldsymbol{\xi}_{1j}$ is $(m_1 \times 1)$ for $j = 1, \cdots, p$ and $\boldsymbol{\xi}_{ij}$ is $((m_i - m_{i-1}) \times 1)$ for $j = 2, \cdots, p$; $i = 2, \cdots, j$, and we have dropped the superscript as planned.

DEFINITION 3.2. The hypothesis

$$H_0 \colon \{\mathbf{C}_i \boldsymbol{\xi}_i = \mathbf{0} \ \text{ for } \ i = 1, \cdots, p\} \tag{3.4}$$

is monotone if for $i = 1, \cdots, k - 1$, $\mathbf{C}_i = \mathbf{0}$ is $(s \times m_i)$ and for $i = k, \cdots, p$, $\mathbf{C}_i = [\mathbf{0} \mid \mathbf{A}_k \mid \cdots \mid \mathbf{A}_i]$ where $\mathbf{0}$ is $(s \times m_{k-1})$ and $\mathbf{A}_j$ is $(s \times (m_j - m_{j-1}))$ for $j = k, \cdots, p$.

THEOREM 3.1. *The monotone hypothesis* (3.4) *is testable under the monotone* MDM *model* (3.1) *and* (3.2) *if and only if* $\mathbf{C}_p \boldsymbol{\xi}_p = \mathbf{0}$ *is testable under the univariate linear model*

$$E(\mathbf{y}_p) = \mathbf{X}_p \boldsymbol{\xi}_p \tag{3.5}$$

with $\mathbf{y}_p \sim N_n(\mathbf{X}_p \boldsymbol{\xi}_p, \sigma_{pp} \mathbf{I}_n)$.

PROOF. The assumption that $\mathbf{C}_p \boldsymbol{\xi}_p = \mathbf{0}$ is testable under the univariate linear model $E(\mathbf{y}_p) = \mathbf{X}_p \boldsymbol{\xi}_p$ is equivalent to the condition that there exists a $(s \times n)$ matrix $\mathbf{B}$ such that $\mathbf{B}\mathbf{X}_p = \mathbf{C}_p$, i.e. from (3.2) and (3.4),

$$\mathbf{B}(\mathbf{X}_{k-1} \mid \mathbf{X}_{k2} \mid \cdots \mid \mathbf{X}_{p2}) = (\mathbf{0} \mid \mathbf{A}_k \mid \cdots \mid \mathbf{A}_p) \tag{3.6}$$

where $\mathbf{X}_{k-1} = [\mathbf{X}_1 \mid \mathbf{X}_{22} \mid \cdots \mid \mathbf{X}_{k-1,2}]$. By comparing individual terms in (3.6) we have

$$\begin{aligned} \mathbf{B}\mathbf{X}_i &= \mathbf{C}_i = \mathbf{0} \,, & i &= 1, \cdots, k-1 \,, \\ \mathbf{B}\mathbf{X}_i &= \mathbf{C}_i = [\mathbf{0} \mid \mathbf{A}_k \mid \cdots \mid \mathbf{A}_i] \,, & i &= k, \cdots, p \,. \end{aligned} \tag{3.7}$$

Thus, $\mathbf{B}[\mathbf{X}_1, \cdots, \mathbf{X}_p] = [\mathbf{C}_1, \cdots, \mathbf{C}_p]$, which by the definition in Section 2 is a necessary and sufficient condition that the monotone hypothesis in (3.4) is testable.

4. Applications and discussion. Consider a multivariate regression model with (for the sake of simplicity) two "dependent" variables Y_1 and Y_2 and one "independent" variable X. Assume that the joint distribution of (Y_1, Y_2) given X is bivariate normal with means

$$E(Y_1 \mid X) = \alpha_{10} + \alpha_{11}X, \qquad E(Y_2 \mid X) = \alpha_{20} + \alpha_{21}X + \alpha_{22}X^2. \tag{4.1}$$

Clearly for a sample of size n, (Y_{i1}, Y_{i2}, X_i), $i = 1, \cdots, n$, we have a monotone MDM model with

$$\mathbf{X}_1 = \begin{bmatrix} 1 & X_1 \\ \vdots & \\ 1 & X_n \end{bmatrix}, \qquad \mathbf{X}_2 = \begin{bmatrix} 1 & X_1 & X_1^2 \\ \vdots & \vdots & \vdots \\ 1 & X_n & X_n^2 \end{bmatrix}, \tag{4.2}$$

$$\boldsymbol{\xi}_1 = \begin{bmatrix} \alpha_{10} \\ \alpha_{11} \end{bmatrix}, \qquad \boldsymbol{\xi}_2 = \begin{bmatrix} \alpha_{20} \\ \alpha_{21} \\ \alpha_{22} \end{bmatrix}, \qquad \text{etc.}$$

Under the assumption that $\mathbf{X}_2$ is of full column rank, any monotone hypothesis is testable by the procedure outlined in Sections 2 and 3. For example,

$$H_{01}: \left\{\alpha_{11} = 0, \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}, \tag{4.3}$$

i.e. $\mathbf{C}_1 = [0 \;\; 1]$, $\mathbf{C}_2 = [0 \;\; 1 \;\; 1]$ in the notation of (3.4), is monotone and testable, as are

$$H_{02}: \{\alpha_{22} = 0\}, \qquad H_{03}: \{\alpha_{11} = 0, \alpha_{21} = 0\}. \tag{4.4}$$

On the other hand, the hypotheses

$$H_{04}: \{\alpha_{11} = 0\}, \qquad \text{and} \qquad H_{05}: \{\alpha_{11} = 0, \alpha_{22} = 0\} \tag{4.5}$$

are not monotone and hence by Theorem 3.1 are not testable under the above monotone MDM model.

For a second example consider again $p = 2$ responses with 2 block systems (for a description of practical situations calling for 2 block systems, see Roy and Srivastava (1964)). Suppose that there are $n = bt$ experimental units with each of t treatments applied to b units. Further assume that the bt units are divided into b "blocks" of t units each and that each treatment occurs once and only once in each "block". Let Y_{rij} denote the value of the rth response from the jth experimental unit receiving treatment i; $i = 1, \cdots, t$; $j = 1, \cdots, b$. Assume that

$$\begin{aligned} E(Y_{1ij}) &= \mu_1 + \tau_{1i} \\ E(Y_{2ij}) &= \mu_2 + \tau_{2i} + \beta_{2j} \qquad (i = 1, \cdots, t; j = 1, \cdots, b). \end{aligned} \tag{4.6}$$

Thus the motivation for the name 2-block systems, is that for response 1 there is prior knowledge indicating no block effects while there are possibly nonzero block effects for response 2. Again the MDM model is monotone and it can be easily checked that the monotone hypotheses $H_{01}: \{\beta_{21} = \beta_{22} = \cdots = \beta_{2b}\}$ and $H_{02}: \{\tau_{11} = \tau_{12} = \cdots = \tau_{1t}$ and $\tau_{21} = \tau_{22} = \cdots = \tau_{2t}\}$ are testable.

As can be seen in the above examples, the test procedures proposed herein are satisfactory (in the sense that the null distribution of the test statistic is known, the union-intersection principle has a good intuitive appeal, etc.) in some cases, e.g. H_{01} in (4.3), and not in others, e.g. H_{04} and H_{05} in (4.5). It is true that H_{04}: $\{\alpha_{11} = 0\}$ and the components of H_{05}: $\{\alpha_{11} = 0, \alpha_{22} = 0\}$ can be tested in the univariate manner by taking only one response into consideration. However, this procedure disregards the correlation between the responses, and, in as far as the author is aware, no procedures exist which do take the correlation into account for hypotheses of the type in (4.5). The same situation exists in the standard MANOVA model (i.e., $\mathbf{X}_1 = \cdots = \mathbf{X}_p$, where $\mathbf{C}_i \neq \mathbf{C}_j$, $i \neq j$; $i, j = 1, \cdots, p$). The author hopes to be able to consider these problems and related material in a later communication.

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