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INEQUALITIES OF s-ORDERED DISTRIBUTIONS[1]

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Let C be the cone of functions ϕ that are concave-convex about the origin, continuous at the origin, and have $\phi(0) = 0$, and $\phi'(t) \leqq \phi'(-t)$ for $t > 0$. Necessary and sufficient conditions are given for $\phi(\int x(t)\, dH(t)) \leqq \int \phi(x(t))\, dH(t)$ to hold for all $\phi \in C$ and all increasing functions x, with $x(0) = 0$. This inequality is used to develop comparisons (i) between combinations of order statistics, and (ii) between combinations of the expected values of the order statistics, arising from distributions F and G in the case that $G^{-1}F \in C$. If $F(0) = G(0) = \frac{1}{2}$ and the inequality on the gradient of $G^{-1}F$, $(G^{-1}F)'(x) \leqq (G^{-1}F)'(-x)$ for $x > 0$, is satisfied, then $G^{-1}F \in C$ implies $F <_s G$. The inequalities presented preserve the ordering. A weaker ordering of distributions, called r-ordering, is defined: $F <_r G$ if and only if $F(0) = G(0) = \frac{1}{2}$ and $G^{-1}F(x)/x$ is increasing (decreasing) for x positive (negative) on the support of F. For symmetric r-ordered distributions, the ratio of the expected values of the order statistics preserve the ordering.

1. Introduction. In this paper we develop some inequalities of theoretical interest between functions of the order statistics from certain restricted families of random variables. The application of these inequalities to reliability theory and robustness studies will be treated in a separate paper.

If G is a given continuous distribution function with $G(0) = 0$, Barlow and Proschan (1966a) have considered the properties of linear combinations of order statistics arising from the distribution F when $F(0) = 0$ and $G^{-1}F$ is convex or alternatively star-shaped, on the support of F. Van Zwet (1964) has studied some of the theoretical properties of symmetric distributions F and G related by the fact that $G^{-1}F$ is concave-convex about the point of symmetry which he calls "s-ordering" and denotes by $F <_s G$.

We adopt a more general definition of s-ordering than van Zwet: $F <_s G$ if and only if $F(m) = G(m) = \frac{1}{2}$ and $G^{-1}F(x)$ is concave-convex about m on the support of F. For convenience we assume m to be the origin. Examples of s-ordered distributions are: U-shaped density $<_s$ uniform $<_s$ normal $<_s$ logistic $<_s$ Laplace $<_s$ Cauchy. (See van Zwet (1964) pages 70–73.)

We will also be interested in a weaker ordering than s-ordering, which we call r-ordering and define as follows: $F <_r G$ if and only if $x^{-1}G^{-1}F(x)$ is increasing (decreasing) for $x \geqq 0$ ($x < 0$), and $F(0) = G(0) = \frac{1}{2}$.

Although r- and s-ordering are defined for distributions that are not necessarily

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symmetric, the results in this paper indicate that these concepts are too weak to obtain the desired inequalities. For the sake of clarity the definitions are not made stronger, and the appropriate restrictions are stipulated for each result.

Preliminaries. We adopt the following definitions, grouped here for convenience, and assume throughout that F and G are continuous and that G is strictly increasing on its support. We loosely use "increasing" to mean "non-decreasing" and "decreasing" to mean "non-increasing", except when qualified by "strictly".

(i) $F <_c G$ if and only if $G^{-1}F$ is convex on the support of F.

(ii) $F <_s G$ if and only if $F(0) = G(0) = \frac{1}{2}$ and $G^{-1}F$ is concave-convex about the origin on the support of F.

(iii) $F <_r G$ if and only if $F(0) = G(0) = \frac{1}{2}$ and $x^{-1}G^{-1}F(x)$ is increasing (decreasing) for x positive (negative) on the support of F.

(iv) $X \geqq_{st} Y$ if and only if $P(X \leqq a) \leqq P(Y \leqq a)$, $-\infty < a < \infty$.

Throughout, we let $X_{1,n} \leqq X_{2,n} \leqq \cdots \leqq X_{n,n}$ $(Y_{1,n} \leqq Y_{2,n} \leqq \cdots \leqq Y_{n,n})$ be an ordered sample from $F(G)$, and we observe that $Y =_{st} G^{-1}F(X)$.

If F_α is the distribution of $|X|^\alpha$ and F, G are symmetric about the origin, then for $\alpha \geqq 1$,

$$\text{(a)} \quad F <_s G \quad \text{implies} \quad F_\alpha <_c G_\alpha .$$

$$\text{(b)} \quad F <_r G \quad \text{implies} \quad F_\alpha <_r G_\alpha \quad \text{on the positive axis.}$$

If G is symmetric about the origin and $\mathscr{G} = \{G(\theta x) \mid \theta > 0\}$, then a sufficient statistic for θ based on a complete random sample $\mathbf{Y} = (Y_{1,n}, Y_{2,n}, \cdots, Y_{n,n})$ is given by $(|Y_{1,n}|, |Y_{2,n}|, \cdots, |Y_{n,n}|)$. Suppose we are interested in studying the robustness of statistics derived under the assumption that the observations are distributed according to G, when in fact they are distributed according to F (symmetric) where $F <_s G$. Since by (a), $F <_s G$ implies $F_1 <_c G_1$, the results of Barlow and Proschan (1966a) apply to linear combinations of the sufficient statistic for θ. However, their results do *not* apply if the sample is censored or if F and G are not symmetric.

2. Inequalities for concave-convex functions. In this section we develop inequalities for concave-convex functions some of which we will need later on, and some of which are presented for their own interest.

Let C be the cone of functions ϕ that are concave-convex about the origin, continuous at the origin and have $\phi(0) = 0$, and $\phi'(t) \leqq \phi'(-t)$ for $t > 0$. We let H denote a function of bounded variation on $[a, b]$, $-\infty \leqq a \leqq 0 \leqq b \leqq \infty$ (the interval end points are excluded when not finite). We define H and $\bar{H}$ by

$$H(v) = \int_{[a,v)} dH(x) \qquad \text{and} \qquad \bar{H}(v) = \int_{(v,b]} dH(x) .$$

Thus H and $\bar{H}$ are the left- and right-continuous versions respectively. We always assume that $|\int_{[a,b]} x(t)\, dH(t)| < \infty$.

All theorems are stated and proved for the interval $(-\infty, \infty)$ since the restriction to the interval $[a, b]$ is obvious. In the following theorems ϕ need not be concave-convex or even defined over all the interval $(-\infty, \infty)$. But ϕ must be concave-convex on an interval containing the point $\int x(t)\, dH(t)$ and the range of $x(t)$ for t in the support of H.

THEOREM 2.1. *The following conditions are necessary and sufficient for*

$$\phi(\textstyle\int x(t)\, dH(t)) \leqq \int \phi(x(t))\, dH(t) \tag{2.1}$$

to hold for all $\phi \in C$, and for all increasing functions x, antisymmetric about the origin with $x(0) = 0$:

There exists $v_0 \geqq 0$ such that

$$\bar{H}(v) - H(-v) \leqq -1 \quad \text{and} \quad \bar{H}(v) \leqq 0\,, \quad \text{for} \quad 0 \leqq v < v_0\,, \tag{2.2}$$

$$0 \leqq \bar{H}(v) - H(-v) \leqq 1 \quad \text{and} \quad H(-v) \leqq 0\,, \quad \text{for} \quad v \geqq v_0\,, \tag{2.3}$$

$$\bar{H}(v_2)H(-v_1) \leqq \bar{H}(v_1)H(-v_2)\,, \quad \text{for} \quad 0 \leqq v_1 < v_0 \leqq v_2\,. \tag{2.4}$$

PROOF. Functions ϕ of the type

$$\begin{aligned} \phi(x) &= x + z'\,, && x < -z' \\ &= 0\,, && -z' \leqq x \leqq z \\ &= x - z\,, && x > z\,, \qquad 0 \leqq z' \leqq z\,, \end{aligned} \tag{2.5}$$

together with the linear decreasing functions ϕ with $\phi(0) = 0$, positively span the convex cone C. We may therefore restrict attention to this type, noting that (2.1) reduces to an identity for ϕ linear.

We first prove the necessity of the conditions. Let

$$\begin{aligned} x(t) &= -a - b\,, && t < -v_2 \\ &= -a\,, && -v_2 \leqq t < -v_1 \\ &= 0\,, && -v_1 \leqq t \leqq v_1 \\ &= a\,, && v_1 < t \leqq v_2 \\ &= a + b\,, && t > v_2\,, \qquad 0 \leqq v_1 \leqq v_2;\ a, b \geqq 0\,. \end{aligned} \tag{2.6}$$

For $0 \leqq z' \leqq z \leqq a$ we have

$$\phi(\textstyle\int x(t)\, dH(t)) = \phi\{b(\bar{H}(v_2) - H(-v_2)) + a(\bar{H}(v_1) - H(-v_1))\}\,, \qquad \text{and}$$

$$\textstyle\int \phi(x(t))\, dH(t) = b(\bar{H}(v_2) - H(-v_2)) - (a - z')H(-v_1) + (a - z)\bar{H}(v_1)\,.$$

For this case, inequality (2.1) implies:

(i) If $b(\bar{H}(v_2) - H(-v_2)) + a(\bar{H}(v_1) - H(-v_1)) \leqq -z'$ then $z\bar{H}(v_1) - z'H(-v_1) \leqq -z'$. For $\bar{H}(v_2) - H(-v_2) < 0$ the assumption may be satisfied by choosing b sufficiently large. Taking $z' = 0$ and then $z' = z$ we see that $\bar{H}(v_2) - H(-v_2) < 0$ implies that $\bar{H}(v_1) \leqq 0$ and $\bar{H}(v_1) - H(-v_1) \leqq -1$ for all $0 \leqq v_1 \leqq v_2$.

(ii) If $b(\bar{H}(v_2) - H(-v_2)) + a(\bar{H}(v_1) - H(-v_1)) \geqq z$ then $z\bar{H}(v_1) - z'H(-v_1) \leqq z$. For $\bar{H}(v_1) - H(-v_1) > 0$ the assumption is satisfied by choosing a sufficiently large. Taking $z' = z$ we find that $\bar{H}(v_1) - H(-v_1) > 0$ implies that $\bar{H}(v_1) - H(-v_1) \leqq 1$.

(iii) If $-z' \leqq b(\bar{H}(v_2) - H(-v_2)) + a(\bar{H}(v_1) - H(-v_1)) \leqq z$ then $b(\bar{H}(v_2) - H(-v_2)) - (a - z')H(-v_1) + (a - z)\bar{H}(v_1) \geqq 0$. For $0 \leqq \bar{H}(v_1) - H(-v_1) \leqq 1$ the assumption is satisfied for $a = z$, $b = 0$. Thus $0 \leqq \bar{H}(v_1) - H(-v_1) \leqq 1$ implies that $H(-v_1) \leqq 0$.

The conclusions of (i)—(iii) together with the right-(left-)continuity of $\bar{H}(v)(H(-v))$ are equivalent to the existence of $v_0 \geqq 0$ such that conditions (2.2) and (2.3) hold. This proves the necessity of (2.2) and (2.3). To prove the necessity of (2.4) we consider the case when $z' = 0$ and $z = a + b$.

Now we observe that

$$\int \phi(x(t))\, dH(t) = -bH(-v_2) - aH(-v_1)\,,$$

and hence inequality (2.1) implies:

(iv) If $b(\bar{H}(v_2) - H(-v_2)) + a(\bar{H}(v_1) - H(-v_1)) \leqq 0$ then $b\bar{H}(v_2) + a\bar{H}(v_1) \leqq 0$. For $0 \leqq v_1 < v_0 \leqq v_2$ we have $\bar{H}(v_1) - H(-v_1) \leqq -1$ and $0 \leqq \bar{H}(v_2) - H(-v_2) \leqq 1$, and hence

$$a = -\frac{b(\bar{H}(v_2) - H(-v_2))}{\bar{H}(v_1) - H(-v_1)} \geqq 0\,,$$

and any $b \geqq 0$ satisfy the assumptions.

It follows that

$$\bar{H}(v_2) - \frac{\bar{H}(v_1)(\bar{H}(v_2) - H(-v_2))}{\bar{H}(v_1) - H(-v_1)} \leqq 0 \qquad \text{for} \quad 0 \leqq v_1 < v_0 \leqq v_2\,,$$

and hence $\bar{H}(v_2)H(-v_1) \leqq \bar{H}(v_1)H(-v_2)$ since $\bar{H}(v_1) - H(-v_1) < 0$. This proves the necessity of (2.4).

We now prove the sufficiency of the conditions.

If H and $\bar{H}$ satisfy the conditions (2.2)—(2.4) then the left- and right-continuous versions of $H(x^{-1})$ and $\bar{H}(x^{-1})$ will also satisfy these conditions for every increasing and antisymmetric x with $x(0) = 0$. It is therefore sufficient to show that conditions (2.2)—(2.4) imply inequality (2.1) for $x(t) \equiv t$ and all ϕ of the form (2.5). Writing $\lambda = \int t\, dH(t)$ we have

$$\lambda = \int_0^\infty (\bar{H}(t) - H(-t))\, dt\,, \qquad \text{and}$$

$$\int \phi(t)\, dH(t) = \int_z^\infty \bar{H}(t)\, dt - \int_{z'}^\infty H(-t)\, dt\,.$$

We have to prove that conditions (2.2)—(2.4) imply $\phi(\lambda) \leqq \int \phi(t)\, dH(t)$ for all $0 \leqq z' \leqq z$.

We consider three cases:

(i) $0 \leqq z' \leqq z \leqq v_0$. Since $\bar{H}(t) \leqq 0$ and $\bar{H}(t) - H(-t) \leqq -1$ for $0 \leqq t < v_0$, and as $\phi(x) \leqq x + z'$ for all x, we have

$$\begin{aligned}\int \phi(t)\, dH(t) &\geqq \int_{z'}^\infty (\bar{H}(t) - H(-t))\, dt = \lambda - \int_0^{z'} (\bar{H}(t) - H(-t))\, dt \\ &\geqq \lambda + z' \geqq \phi(\lambda)\,.\end{aligned}$$

(ii) $0 \leqq v_0 \leqq z' \leqq z$. Since $H(-t) \leqq 0$ and $\bar{H}(t) - H(-t) \geqq 0$ for $t \geqq v_0$,

$\int \phi(t)\, dH(t) \geqq \int_z^\infty (\bar{H}(t) - H(-t))\, dt \geqq 0$. Also because $\bar{H}(t) - H(-t) \leqq 1$ for all $t \geqq 0$,

$$\int \phi(t)\, dH(t) \geqq \int_z^\infty (\bar{H}(t) - H(-t))\, dt = \lambda - \int_0^z (\bar{H}(t) - H(-t))\, dt \geqq \lambda - z\,.$$

Hence $\int \phi(t)\, dH(t) \geqq \max(0, \lambda - z) \geqq \phi(\lambda)$.

(iii) $0 \leqq z' < v_0 < z$. If $\int_{z'}^z \bar{H}(t)\, dt \leqq 0$, the proof proceeds in the same manner as in (i). If $\int_{z'}^z H(-t)\, dt \leqq 0$, the proof follows the pattern of (ii). It is therefore sufficient to consider the case when

$$\int_{z'}^z \bar{H}(t)\, dt > 0 \qquad \text{and} \qquad \int_{z'}^z H(-t)\, dt > 0\,.$$

We shall show that this case is impossible and thus can never arise. Suppose to the contrary that it is true, then together with conditions (2.2) and (2.3) we observe

$$\int_{v_0}^z \bar{H}(t)\, dt > -\int_{z'}^{v_0} \bar{H}(t)\, dt \geqq 0 \qquad \text{and}$$
$$\int_{z'}^{v_0} H(-t)\, dt > -\int_{v_0}^z H(-t)\, dt \geqq 0\,,$$

which implies

$$\int_{v_0}^z \bar{H}(t)\, dt \cdot \int_{z'}^{v_0} H(-t)\, dt > \int_{z'}^{v_0} \bar{H}(t)\, dt \cdot \int_{v_0}^z H(-t)\, dt\,.$$

However, this contradicts condition (2.4) which ensures that

$$\int_{v_0}^z \bar{H}(t)\, dt \cdot \int_{z'}^{v_0} H(-t)\, dt \leqq \int_{z'}^{v_0} \bar{H}(t)\, dt \cdot \int_{v_0}^z H(-t)\, dt\,.$$

This completes the proof. □

Corollary 2.2. *If in Theorem* 2.1 *the restriction is added that H be continuous, then the necessary and sufficient conditions are*

$$(2.7) \qquad 0 \leqq \bar{H}(v) - H(-v) \leqq 1\,, \qquad H(-v) \leqq 0 \qquad \textit{for} \quad v \geqq 0\,.$$

Proof. Immediate since both $H(-v)$ and $\bar{H}(v)$ tend to zero as $v \to \infty$. □

We let C' denote the cone of functions ϕ that are concave-convex about the origin, continuous at the origin and have $\phi(0) = 0$ and $\phi'(t) \geqq \phi'(-t)$ for $t > 0$. We note that if $\phi(x) \in C$ then $-\phi(-x) \in C'$, and from this we obtain the following corollary.

Corollary 2.3. *The following conditions are necessary and sufficient for*

$$\phi\{\int x(t)\, dH(t)\} \geqq \int \phi(x(t))\, dH(t)$$

to hold for all $\phi \in C'$, *and for all increasing functions* x, *antisymmetric about the origin with* $x(0) = 0$:

There exists $v_0 \geqq 0$ *such that*

$$H(-v) - \bar{H}(v) \leqq -1 \quad \textit{and} \quad H(-v) \leqq 0 \quad \textit{for} \quad 0 \leqq v < v_0\,,$$
$$0 \leqq H(-v) - \bar{H}(v) \leqq 1 \quad \textit{and} \quad \bar{H}(v) \leqq 0 \quad \textit{for} \quad v \geqq v_0\,,$$
$$H(-v_2)\bar{H}(v_1) \leqq H(-v_1)\bar{H}(v_2) \quad \textit{for} \quad 0 \leqq v_1 < v_0 \leqq v_2\,.$$

If in Corollary 2.3 the extra restriction that H be continuous is added, then necessary and sufficient conditions may be found using the same approch as in Corollary 2.2. It will not be presented here.

By noting that if ϕ is concave-convex then $-\phi$ is convex-concave, we can obtain similar inequalities for convex-concave functions. These results follow trivially.

THEOREM 2.4. *Inequality* (2.1) *is true for all* $\phi \in C$ *with* ϕ *antisymmetric and all increasing antisymmetric* $x(t)$ *with* $x(0) = 0$ *if and only if there exists* $v_0 \geqq 0$ *such that*

$$\bar{H}(v) - H(-v) \leqq -1\,, \qquad 0 \leqq v < v_0$$
$$0 \leqq \bar{H}(v) - H(-v) \leqq 1\,, \qquad v \geqq v_0\,.$$

PROOF. Necessity of the conditions. Let $\phi(y) = y^3$ and

$$\begin{aligned} x(t) &= -1\,, && t < -v \\ &= 0\,, && -v \leqq t \leqq v\,, \quad v \geqq 0 \\ &= 1\,, && t > v\,, \end{aligned}$$

then (2.1) implies that $\{-H(-v) + \bar{H}(v)\}^3 \leqq -H(-v) + \bar{H}(v)$. Hence either $0 \leqq \bar{H}(v) - H(-v) \leqq 1$, or $\bar{H}(v) - H(-v) \leqq -1$.

Note that in Theorem 2.1 we assumed ϕ antisymmetric in showing that if $\bar{H}(v_2) - H(-v_2) < 0$ then $\bar{H}(v_1) - H(-v_1) \leqq -1$ for all $0 \leqq v_1 \leqq v_2$. This proves the necessity of the conditions.

We will now prove sufficiency, As in Theorem 2.1 we will assume without loss of generality that x is the identity. If $\lambda = \int t\, dH(t) \leqq 0$, then by noting that (2.1) can be written in terms of a concave function only,

$$\phi\{\textstyle\int_{-\infty}^{0} t\, d(H(t) + H(-t))\} \leqq \int_{-\infty}^{0} \phi(t)\, d(H(t) + H(-t))$$

we see from Theorem 3.2 in Barlow, Marshall and Proschan (1969) that the conditions are sufficient.

If $\lambda > 0$, we see, using a similar argument as in Theorem 2.1 that we need only show for all $r \geqq 0$ that

(a) $$\lambda - \textstyle\int_0^r (\bar{H}(t) - H(-t))\, dt \geqq \max(0, \lambda - r)\,.$$

We will consider two cases:

(i) $0 \leqq r \leqq \lambda$. Since $\bar{H}(t) - H(-t) \leqq 1$ for all t, $\int_0^r (\bar{H}(t) - H(-t))\, dt \leqq r$, and (a) is true.

(ii) $0 < \lambda < r$. If $v_0 > r$, $\int_0^r (\bar{H}(t) - H(-t))\, dt \leqq -r$ and (a) is obviously true. If $v_0 \leqq r$, then by noting that

$$\lambda - \textstyle\int_0^r (\bar{H}(t) - H(-t))\, dt = \int_r^\infty (\bar{H}(t) - H(-t))\, dt \geqq 0$$

we see that (a) is satisfied. This proves the theorem. □

THEOREM 2.5. *The following conditions are necessary and sufficient for*

(2.1) $$\phi\{\textstyle\int x(t)\, dH(t)\} \leqq \int \phi(x(t))\, dH(t)$$

to hold for all $\phi \in C$ *and for all increasing functions* x *with* $x(0) = 0$:

There exists $u_0 \leqq 0 \leqq v_0$ *such that*

$$-1 \leqq H(u) \leqq 0 \qquad u \leqq u_0\,, \tag{2.8}$$

$$H(u) \geqq 1 \qquad u_0 < u \leqq 0\,, \tag{2.9}$$

$$\bar{H}(v) \leqq -1 \qquad 0 \leqq v < v_0\,, \tag{2.10}$$

$$0 \leqq \bar{H}(v) \leqq 1 \qquad v \geqq v_0\,. \tag{2.11}$$

(2.12) *If* $H(u) > 0$ *for some* $u \leqq 0$, *then* $\bar{H}(v) \leqq 0$ *for all* $v \geqq 0$.

(2.13) *If* $u \leqq 0 \leqq v$ *and either* $H(u) > 0$ *or* $\bar{H}(v) < 0$ *then* $\bar{H}(v) - H(u) \leqq -1$.

(2.14) *If* $u \leqq 0 \leqq v$ *and either* $H(u) < 0$ *or* $\bar{H}(v) > 0$ *then* $\bar{H}(v) - H(u) \leqq 1$.

PROOF. For $\phi \in C$ and increasing x with $x(0) = 0$ we have

$$\textstyle\int x(t)\,dH(t) = \int s\,dH(x^{-1}(s))\,, \qquad \int \phi x(t)\,dH(t) = \int \phi(s)\,dH(x^{-1}(s))\,,$$

where $x^{-1}(s)$ is increasing with $x^{-1}(0) = 0$. On the other hand, for every increasing y and every increasing antisymmetric z with $y(0) = z(0) = 0$, there exists an increasing x with $x(0) = 0$ such that

$$\textstyle\int x(t)\,dH(t) = \int z(s)\,dH(y(s))\,, \qquad \int \phi x(t)\,dH(t) = \int \phi z(s)\,dH(y(s))\,.$$

Hence necessary and sufficient conditions for (2.1) to hold for $\phi \in C$ and all increasing x with $x(0) = 0$ can be obtained by requiring that the left- and right-continuous versions of $H(y(s))$ and $\bar{H}(y(s))$ respectively, satisfy conditions (2.2)—(2.4) for every increasing y with $y(0) = 0$. Writing $u(s) = -y(-s)$ and $v(s) = y(s)$, $s \geqq 0$, necessary and sufficient conditions are:

For every pair of increasing functions u and v with $u(0) = v(0) = 0$, there exists $s_0 \geqq 0$ such that the left- and right-continuous versions of $H(-u(s))$ and $\bar{H}(v(s))$ satisfy:

$$\bar{H}(v(s)) - H(-u(s)) \leqq -1\,, \qquad \bar{H}(v(s)) \leqq 0 \qquad \text{for} \quad 0 \leqq s < s_0\,, \tag{2.15}$$

$$0 \leqq \bar{H}(v(s)) - H(-u(s)) \leqq 1\,, \qquad H(-u(s)) \leqq 0 \quad \text{for} \quad s \geqq s_0\,, \tag{2.16}$$

$$\bar{H}(v(s_2))H(-u(s_1)) \leqq \bar{H}(v(s_1))H(-u(s_2)) \qquad \text{for} \quad 0 \leqq s_1 < s_0 \leqq s_2\,. \tag{2.17}$$

We have to show that conditions (2.8)—(2.14) are equivalent to conditions (2.15)—(2.17). We first note that conditions (2.15)—(2.17) simply state that for every $u \leqq 0 \leqq v$ either $\bar{H}(v) - H(u) \leqq -1$ or $0 \leqq \bar{H}(v) - H(u) \leqq 1$; that whenever $\bar{H}(v) - H(u) \leqq -1$, then $\bar{H}(v') - H(u') \leqq -1$ for all u', v' with $u \leqq u' \leqq 0 \leqq v' \leqq v$, and that whenever $0 \leqq \bar{H}(v) - H(u) \leqq 1$, then $0 \leqq \bar{H}(v'') - H(u'') \leqq 1$ for all $u'' \leqq u \leqq 0 \leqq v \leqq v''$. Because of the left(right)-continuity of $H(\bar{H})$ we may therefore restate conditions (2.15)—(2.17) as follows:

There exists an increasing function ψ on $(-\infty, 0]$ with $0 \leqq \psi(u) \leqq \infty$ for all

$-\infty < u \leqq 0$ such that

$$\bar{H}(v) - H(u) \leqq -1\,, \qquad \bar{H}(v) \leqq 0 \qquad \text{for} \quad 0 \leqq v < \psi(u)\,, \quad u \leqq 0\,,$$
$$0 \leqq \bar{H}(v) - H(u) \leqq 1\,, \qquad H(u) \leqq 0 \qquad \text{for} \quad v \geqq \psi(u)\,, \quad u \leqq 0\,,$$
$$\bar{H}(v_2)H(u_1) \leqq \bar{H}(v_1)H(u_2)$$
$$\text{for} \quad 0 \leqq v_1 < \psi(u_1)\,, \quad v_2 \geqq \psi(u_2)\,, \quad u_1 \leqq 0\,, \quad u_2 \leqq 0\,.$$

Suppose now that conditions (2.8)—(2.14) are satisfied. Noting that (2.12) implies that either $u_0 = 0$ or $\bar{H}(v) = 0$ for all $v \geqq v_0$, one easily verifies that the conditions above are satisfied for a function ψ defined by $\psi(u) = v_0$ for $u \leqq u_0$ and $\psi(u) = \infty$ for $u_0 < u \leqq 0$.

It remains to be shown that (2.15)—(2.17) imply (2.8)—(2.14). Suppose that (2.15)—(2.17) hold. Let $u(s)$ be defined as follows

$$\begin{aligned} u(s) &= 0\,, \quad s = 0 \\ &= \infty\,, \quad s > 0\,. \end{aligned}$$

Then since $H(u) \to 0$ for $u \to -\infty$, (2.15)—(2.16) reduce to

$$\bar{H}(v(s)) \leqq -1\,, \quad 0 \leqq s < s_0$$
$$0 \leqq \bar{H}(v(s)) \leqq 1\,, \quad s \geqq s_0\,,$$

which can be written as $\bar{H}(v) \leqq -1$, $0 \leqq v < v_0$ and $0 \leqq \bar{H}(v) \leqq 1$, $v \geqq v_0$. In a similar way we may show that (2.15) and (2.16) imply that $H(u) \geqq 1$ for $u_0 < u \leqq 0$, and $-1 \leqq H(u) \leqq 0$ for $u \leqq u_0$. This shows that (2.8)—(2.11) are satisfied.

Assume that $H(-u(s)) > 0$ for some $s \geqq 0$, then from (2.8) and (2.9), it follows that $H(-u(s)) \geqq 1$. We can choose the function v such that $v(s)$ is very large (hence $\bar{H}(v(s))$ is small) and $\bar{H}(v(s)) - H(-u(s)) < 0$. Now since (2.15) and (2.16) are assumed true, we have that $\bar{H}(v(s)) - H(-u(s)) \leqq -1$, and $\bar{H}(v(s)) \leqq 0$ for $0 \leqq s < s_0$. Since v is an increasing function with $v(0) = 0$, we have that $\bar{H}(v) \leqq 0$ and $\bar{H}(v) - H(-u(s)) \leqq -1$ for all $v \geqq 0$. The former inequality proves that (2.12) is satisfied. Similarly, $\bar{H}(v(s)) < 0$, $s \geqq 0$, yields $\bar{H}(v(s)) - H(-u(s)) < 0$ for all sufficiently large $u(s)$, which together with (2.15) and (2.16) ensures that $\bar{H}(v(s)) - H(-u(s)) \leqq -1$ for all $u(s)$; or alternatively, $\bar{H}(v) - H(u) \leqq -1$ for all $u \leqq 0$ if $\bar{H}(v) < 0$. This, together with the previous result proves that (2.13) is satisfied. Since (2.15) and (2.16) together imply that $\bar{H}(v) - H(u) \leqq 1$ for all $u \leqq 0 \leqq v$, condition (2.14) is satisfied. □

REMARK. In proving that (2.15)—(2.17) imply (2.8)—(2.14) we have not used (2.17). Hence (2.15) and (2.16) together imply (2.17). However, it does not seem worthwhile to remove (2.17) from the proof, as you essentially need to prove (2.12) first to do it. Note that in the symmetric case (2.2) and (2.3) do not imply (2.4). A simple counterexample is easy to construct.

COROLLARY 2.6. *If in Theorem* 2.5 *the restriction is added that H be continuous,* *then the necessary and sufficient conditions are*

$$\begin{aligned} H(u) &\leqq 0 \\ \bar{H}(v) &\geqq 0 \\ \bar{H}(v) - H(u) &\leqq 1 \qquad\qquad u \leqq 0 \leqq v\,. \end{aligned}$$

PROOF. $\bar{H}(v)$ and $H(u)$ tend to zero for $v \to \infty$ and $u \to -\infty$. □

COROLLARY 2.7. *The following conditions are necessary and sufficient for*

$$\phi\{\textstyle\int x(t)\,dH(t)\} \geqq \int \phi x(t)\,dH(t)$$

to hold for all $\phi \in C'$ and for all increasing functions x with $x(0) = 0$:
There exists $u_0 \leqq 0 \leqq v_0$ such that:

$$\begin{aligned} 0 \leqq H(u) &\leqq 1\,, & u &\leqq u_0 \\ H(u) &\leqq -1\,, & u_0 &< u \leqq 0 \\ \bar{H}(v) &\geqq 1\,, & 0 &\leqq v < v_0 \\ -1 \leqq \bar{H}(v) &\leqq 0\,, & v &\geqq v_0\,. \end{aligned}$$

If $\bar{H}(v) > 0$ for some $v \geqq 0$, then $H(u) \leqq 0$ for all $u \leqq 0$.

If $u \leqq 0 \leqq v$ and either $\bar{H}(v) > 0$ or $H(u) < 0$ then $H(u) - \bar{H}(v) \leqq -1$.

If $u \leqq 0 \leqq v$ and either $\bar{H}(v) < 0$ or $H(u) > 0$ then $H(u) - \bar{H}(v) \leqq 1$.

The following corollary can be derived from Theorem 2.4 in a similar way to that in which Theorem 2.5 was derived from Theorem 2.1. The proof will be omitted.

COROLLARY 2.8. *Inequality* (2.1) *holds for all antisymmetric $\phi \in C$ and for all increasing $x(t)$ with $x(0) = 0$ if and only if there exists $u_0 \leqq 0 \leqq v_0$ such that*

$$\begin{aligned} -1 \leqq H(u) &\leqq 0\,, & u &\leqq u_0 \\ H(u) &\geqq 1\,, & u_0 &< u \leqq 0 \\ \bar{H}(v) &\leqq -1\,, & 0 &\leqq v < v_0 \\ 0 \leqq \bar{H}(v) &\leqq 1\,, & v &\geqq v_0\,. \end{aligned}$$

If $u \leqq 0 \leqq v$ and either $H(u) > 0$ or $\bar{H}(v) < 0$ then $\bar{H}(v) - H(u) \leqq -1$.

If $u \leqq 0 \leqq v$ and either $H(u) < 0$ or $\bar{H}(v) > 0$ then $\bar{H}(v) - H(u) \leqq 1$.

We will be interested in a discrete version of Theorem 2.5. We use the notation $A_i = \sum_1^i a_j$ and $\bar{A}_i = \sum_i^n a_j$.

COROLLARY 2.9. *The following inequality*

$$\phi(\textstyle\sum_1^n a_i x_i) \leqq \sum_1^n a_i \phi(x_i) \tag{2.18}$$

is true for all $\phi \in C$ *and for all* $x_1 \leqq x_2 \leqq \cdots \leqq x_k < 0 < x_{k+1} \leqq \cdots \leqq x_n$ *if and only if there exists* $0 \leqq i_0 \leqq k < j_0 \leqq n+1$, *such that*

$$\begin{aligned} -1 \leqq A_i &\leqq 0\,, & 1 \leqq i &\leqq i_0 \\ A_i &\geqq 1\,, & i_0 < i &\leqq k \\ \bar{A}_j &\leqq -1\,, & k < j &< j_0 \\ 0 \leqq \bar{A}_j &\leqq 1\,, & j_0 \leqq j &\leqq n\,. \end{aligned}$$

(2.19) *If* $A_i > 0$ *for some* $i \leqq k$ *then* $\bar{A}_j \leqq 0$ *for all* $k < j \leqq n$.
If $1 \leqq i \leqq k < j \leqq n$ *and either* $A_i > 0$ *or* $\bar{A}_j < 0$ *then* $\bar{A}_j - A_i \leqq -1$.
If $1 \leqq i \leqq k < j \leqq n$ *and either* $A_i < 0$ *or* $\bar{A}_j > 0$ *then* $\bar{A}_j - A_i \leqq 1$.

The following corollary follows from the previous one by noting as before that if $\phi \in C$ and $\phi^*(x) = -\phi(-x)$, then $\phi^* \in C'$.

COROLLARY 2.10. *The following inequality*

$$\phi(\textstyle\sum_1^n a_i x_i) \geqq \sum_1^n a_i \phi(x_i) \tag{2.20}$$

holds for all $\phi \in C'$ *and for all* $x_1 \leqq x_2 \leqq \cdots \leqq x_k < 0 < x_{k+1} \leqq \cdots \leqq x_n$ *if and only if there exists* $0 \leqq i_0 \leqq k < j_0 \leqq n+1$, *such that*

$$\begin{aligned} 0 \leqq A_i &\leqq 1\,, & 1 \leqq i &\leqq i_0 \\ A_i &\leqq -1\,, & i_0 < i &\leqq k \\ \bar{A}_j &\geqq 1\,, & k < j &< j_0 \\ -1 \leqq A_j &\leqq 0\,, & j_0 \leqq j &\leqq n\,. \end{aligned}$$

(2.21) *If* $\bar{A}_j > 0$ *for some* $j > k$ *then* $A_i \leqq 0$ *for all* $1 \leqq i \leqq k$.
If $i \leqq k < j$ *and either* $\bar{A}_j > 0$ *or* $A_i < 0$ *then* $A_i - \bar{A}_j \leqq -1$.
If $i \leqq k < j$ *and either* $\bar{A}_j < 0$ *or* $A_i > 0$ *then* $A_i - \bar{A}_j \leqq 1$.

3. Inequalities for combinations of order statistics. In this section we obtain stochastic comparisons between combinations of order statistics arising from distributions F and G in the case that $G^{-1}F \in C$ or $G^{-1}F \in C'$. We see immediately that if $G^{-1}F \in C(C')$ and $F(0) = G(0) = \frac{1}{2}$, then $F <_s G$ with the added stipulation on the gradient of $G^{-1}F$ that $(G^{-1}F)'(x) \leqq (G^{-1}F)'(-x)$ for $x > 0$ $((G^{-1}F)'(x) \geqq (G^{-1}F)'(-x)$ for $x > 0)$. One notes in the special case where F and G are both symmetric that $(G^{-1}F)'(x) = (G^{-1}F)'(-x)$ for all $x > 0$.

Let $X_{1,n} \leqq X_{2,n} \leqq \cdots \leqq X_{n,n}$ be a random vector of ordered observations, and for any outcome $x_{1,n} \leqq x_{2,n} \leqq \cdots \leqq x_{n,n}$ let $k(x) = k(x_{1,n}, x_{2,n}, \cdots, x_{n,n})$ denote the largest index i with $x_{i,n} < 0$. If $X_{1,n}, X_{2,n}, \cdots, X_{n,n}$ is substituted for $x_1, x_2, \cdots, x_n$ in say (2.18), then the weights $a_1, a_2, \cdots, a_n$ need to be chosen as a function of the random variable $k(X)$ in satisfying inequality (2.19). That is we require $a_1(k), a_2(k), \cdots, a_n(k)$ to satisfy (2.19) for every $k = 0, 1, 2, \cdots, n$.

In this case we say that $[a]$ satisfies (2.19). In a later paper some useful inequalities will be developed by using simple weights which satisfy these conditions.

We now use the previous inequalities to construct stochastic comparisons between the order statistics from F and G when $F <_s G$ and $G^{-1}F \in C$ or $G^{-1}F \in C'$. These find use in constructing tolerance limits.

THEOREM 3.1. *If $G^{-1}F \in C$, and $[a]$ satisfies* (2.19) *then*

$$G(\textstyle\sum_1^n a_i(k(Y))Y_{i,n}) \geqq_{st} F(\sum_1^n a_i(k(X))X_{i,n}) .$$

PROOF. From Corollary 2.9 we have that if $[a]$ satisfies (2.19) then

$$G^{-1}F(\textstyle\sum_1^n a_i(k(X))X_{i,n}) \leqq \sum_1^n a_i(k(X))G^{-1}F(X_{i,n}) =_{st} \sum_1^n a_i(k(Y))Y_{i,n} .$$

The stochastic equality follows from the fact that $G^{-1}F$ preserves order with respect to the origin, and $G^{-1}F(X_{1,n}), G^{-1}F(X_{2,n}), \cdots, G^{-1}F(X_{n,n})$ are jointly distributed as the order statistics from G. □

COROLLARY 3.2. *If $G^{-1}F \in C'$, and $[a]$ satisfies* (2.21), *then*

$$G(\textstyle\sum_1^n a_i(k(Y))Y_{i,n}) \leqq_{st} F(\sum_1^n a_i(k(X))X_{i,n}) .$$

4. Inequalities on the expected values of the order statistics from r- and s-ordered distributions. Marshall, Olkin and Proschan (1965) and Barlow and Proschan (1966a) have developed inequalities for the expectations of order statistics arising from distributions F and G in the case that $G^{-1}F$ is starshaped on the support of F and $G(0^-) = F(0^-) = 0$. Van Zwet (1964) has extensively treated inequalities for the expectations of order statistics arising from c-ordered and symmetric s-ordered distributions. We shall develop some new inequalities for the expectations of order statistics and for power combinations of the random variables in the case of two symmetric r-ordered distributions, except for the following inequality where we require the stronger s-ordering.

Van Zwet has obtained necessary and sufficient conditions on $a_1, a_2, \cdots, a_n$ such that

$$F(\textstyle\sum_{i=1}^n a_i EX_{i,n}) \leqq G(\sum_{i=1}^n a_i EY_{i,n})$$

for $F <_c G$. (Personal communication.) We derive necessary and sufficient conditions on $a_1, a_2, \cdots, a_n$ such that the above inequality is true for some special cases of s-ordering.

THEOREM 4.1. *If $EY_{i,n}$ exists for all $i = 1, 2, \cdots, n$, then for all F and G such that $F <_s G$ with $G^{-1}F \in C$,*

$$F(\textstyle\sum_1^n a_i EX_{i,n}) \leqq G(\sum_1^n a_i EY_{i,n}) \tag{4.1}$$

if and only if for all $0 \leqq u \leqq \frac{1}{2} \leqq v \leqq 1$

$$A_n - \textstyle\sum_1^n A_i\binom{n}{i}\{u^i(1-u)^{n-i} + v^i(1-v)^{n-i}\} \leqq 1 , \tag{4.2}$$

$$\textstyle\sum_1^n A_i\binom{n}{i}u^i(1-u)^{n-i} \leqq 0 , \tag{4.3}$$

$$A_n - \textstyle\sum_1^n A_i\binom{n}{i}v^i(1-v)^{n-i} \geqq 0 . \tag{4.4}$$

PROOF. Let

$$\rho(y) = \sum_1^n a_i \frac{n!}{(n-i)!\,(i-1)!} y^{i-1}(1-y)^{n-i}$$

and $\phi = G^{-1}F$. Hence (4.1) can be written as

$$\phi\{\textstyle\int_0^1 \rho(y)F^{-1}(y)\,dy\} \leqq \int_0^1 \phi(F^{-1}(y))\rho(y)\,dy \cdot \tag{4.5}$$

Note that we have already assumed that $F^{-1}(\frac{1}{2}) = 0$. From Corollary 2.6 we see that the necessary and sufficient conditions for (4.5) to be true are given by

$$\int_v^1 \rho(y)\,dy - \int_0^u \rho(y)\,dy \leqq 1\,,$$
$$\int_v^1 \rho(y)\,dy \geqq 0 \qquad \text{and} \qquad \int_0^u \rho(y)\,dy \leqq 0$$

for $0 \leqq u \leqq \frac{1}{2} \leqq v \leqq 1$. The theorem follows from the fact that

$$\int_0^p \frac{n!}{(k-1)!\,(n-k)!} x^{k-1}(1-x)^{n-k}\,dx = \sum_{w=k}^n \binom{n}{w} p^w(1-p)^{n-w}\,. \qquad \square$$

COROLLARY 4.2. *If $EY_{i,n}$ exists for all $i = 1, 2, \cdots, n$, then for all F and G such that $F <_s G$ with $G^{-1}F \in C'$,*

$$F(\textstyle\sum_1^n a_i EX_{i,n}) \geqq G(\sum_1^n a_i EY_{i,n}) \tag{4.6}$$

if and only if for all $0 \leqq u \leqq \frac{1}{2} \leqq v \leqq 1$

$$\begin{aligned} &A_n - \textstyle\sum_1^n A_i\binom{n}{i}\{u^i(1-u)^{n-i} + v^i(1-v)^{n-i}\} \geqq -1\,,\\ &\textstyle\sum_1^n A_i\binom{n}{i}u^i(1-u)^{n-i} \geqq 0\,,\\ &A_n - \textstyle\sum_1^n A_i\binom{n}{i}v^i(1-v)^{n-i} \leqq 0\,. \end{aligned} \tag{4.7}$$

Using the inequalities developed in Section 2 many alternative cases of Theorem 4.1 may be proved, but we shall here only treat two relatively common cases, F symmetric and both F and G symmetric.

COROLLARY 4.3. *If $EY_{i,n}$ exists for all $i = 1, 2, \cdots, n$, then for all F and G such that $G^{-1}F \in C$ and F symmetric about the origin,*

$$F(\textstyle\sum_1^n a_i EX_{i,n}) \leqq G(\sum_1^n a_i EY_{i,n})$$

if and only if for all $\frac{1}{2} \leqq y \leqq 1$,

$$0 \leqq A_n - \textstyle\sum_1^n A_i\binom{n}{i}\{y^i(1-y)^{n-i} + y^{n-i}(1-y)^i\} \leqq 1\,,$$
$$\textstyle\sum_1^n A_i\binom{n}{i}y^{n-i}(1-y)^i \leqq 0\,.$$

COROLLARY 4.4. *If $EY_{i,n}$ exists for all $i = 1, 2, \cdots, n$, then for all F and G such that $G^{-1}F \in C'$ and F symmetric about the origin,*

$$F(\textstyle\sum_1^n a_i EX_{i,n}) \geqq G(\sum_1^n a_i EY_{i,n}) \tag{4.1}$$

if and only if for all $\frac{1}{2} \leqq y \leqq 1$,

$$-1 \leqq A_n - \textstyle\sum_1^n A_i\binom{n}{i}\{y^i(1-y)^{n-i} + y^{n-i}(1-y)^i\} \leqq 0\,,$$
$$A_n - \textstyle\sum_1^n A_i\binom{n}{i}y^i(1-y)^{n-i} \leqq 0\,.$$

From Theorem 2.4 we may prove the following corollary.

COROLLARY 4.5. *If $EY_{i,n}$ exists for all $i = 1, 2, \cdots, n$, then for all F and G both symmetric about the origin such that $G^{-1}F \in C$*

$$F(\sum_1^n a_i EX_{i,n}) \leqq (\geqq) G(\sum_1^n a_i EY_{i,n})$$

if and only if for all $\frac{1}{2} \leqq y \leqq 1$

$$0 \leqq (\geqq) A_n - \sum_1^n A_i \binom{n}{i} \{y^{n-i}(1-y)^i + y^i(1-y)^{n-i}\} \leqq 1 (\geqq -1) .$$

Barlow and Proschan (1966a) showed that if $F(0) = G(0) = 0$ and $x^{-1}G^{-1}F(x)$ is non-decreasing in $x \geqq 0$, then the ratio of order statistics $EY_{i,n}/EX_{i,n}$ is also increasing in $i = 1, 2, \cdots, n$ for all n; i.e., r-ordering on the positive axis is preserved by the expected values of the order statistics. Van Zwet (1967) showed that c-ordering and symmetric s-ordering are both preserved by the expected values of the order statistics. Independently of van Zwet we have proved a related result, namely that for r-ordered symmetric distributions the expected values of the order statistics preserve the ordering.

Define

$$K(i, n, x) = \frac{\Gamma(n+1)}{\Gamma(i)\Gamma(n-i+1)} F^{i-1}(x)(1-F(x))^{n-i} .$$

In a similar way to Barlow and Proschan (1966a) we use the fact that $K(i, n, x)$ is TP_∞ in $1 \leqq i \leqq n$, and $-\infty < x < \infty$. (For a treatment of total positivity see Karlin (1964).)

We observe that

$$\frac{EX_{i,n}}{EY_{i,n}} = \frac{\int xK(i, n, x)\, dF(x)}{\int G^{-1}F(x)K(i, n, x)\, dF(x)} ,$$

and we define the value of this expression for $i = \frac{1}{2}(n+1)$ by continuity.

THEOREM 4.6. *Let $F <_r G$, F and G symmetric about the origin, then*

(i) $\dfrac{EX_{i,n}}{EY_{i,n}}$ *as a function of i is increasing for $i \leqq \frac{1}{2}(n+1)$ and decreasing for $i \geqq \frac{1}{2}(n+1)$,*

(ii) $\dfrac{EX_{i,n}}{EY_{i,n}}$ *as a function of n (integer) is decreasing for $n \geqq 2i$ and increasing for $i \leqq n \leqq 2i$.*

PROOF. Let

$$\begin{aligned} h(i) &= \int_{-\infty}^{\infty} (x - cG^{-1}F(x))K(i, n, x)\, dF(x) \\ &= EY_{i,n}\left(\frac{EX_{i,n}}{EY_{i,n}} - c\right). \end{aligned}$$

Since F and G are symmetric, we have that

$$h(i) = -h(n-i+1) \qquad \text{for} \quad i \geqq \tfrac{1}{2}(n+1) ;$$

i.e., $h(i)$ is antisymmetric about $\frac{1}{2}(n+1)$. Since $K(i, n, x)$ is TP_∞ for $1 \leqq i \leqq n$, $-\infty < x < \infty$, and $(x - cG^{-1}F(x))$ changes sign at most three times for $c \geqq 0$, we have by the variation diminishing property of TP_∞ functions that $h(i)$ must change sign at most three times. If $h(i)$ does change sign three times, then the order of the signs must be the same as for $(x - cG^{-1}F(x))$; viz. $+ - + -$.

Since $h(i)$ is antisymmetric about $\frac{1}{2}(n+1)$ we see that $EX_{i,n}/EY_{i,n}$ is increasing as a function of i for $i < \frac{1}{2}(n+1)$ and hence by continuity for $i \leqq \frac{1}{2}(n+1)$ and decreasing for $i > \frac{1}{2}(n+1)$ and hence by continuity for $i \geqq \frac{1}{2}(n+1)$. This proves (i).

We can verify the recurrence relationship for all i in $[1, n]$

$$r(i, n-1) - r(i, n) = \frac{b}{a+b}[r(i+1, n) - r(i, n)],$$

where $r(i, n) = EX_{i,n}/EY_{i,n}$, $a = (1 - i/n)EY_{i,n}$ and $b = (i/n)EY_{i+1,n}$.

Take $n > 2i$; Observe that $a + b < 0$ and $r(i+1, n) - r(i, n) \geqq 0$. For $n \geqq 2i + 1$, since $b \leqq 0$, the recurrence relationship above is nonnegative and hence $r(i, n)$ is decreasing in n (integer) for $n \geqq 2i$.

Take $n < 2i$; In a similar way and from continuity it follows that $r(i, n)$ increases in n (integer) for $n \leqq 2i$. □

Van Zwet (1964) proves that if $F <_s G$ and F, G symmetric about the origin, then

$$\frac{(E|X|^b)^a}{(E|X|^a)^b} \geqq \frac{(E|Y|^b)^a}{(E|Y|^a)^b} \quad \text{for} \quad 0 \leqq a \leqq b \tag{4.8}$$

for those values of b such that $E|Y|^b$ exists. We will prove a stronger result; namely that given $F <_r G$ and F, G symmetric, the inequality (4.8) holds stochastically for the usual estimates of the expectations, and hence by the strong law of large numbers, for the expectations themselves.

We need to introduce the concept of majorization, and one of the theorems applying this concept. For a fuller treatment see Hardy, Littlewood and Pólya (1959) and Ostrowski (1952).

DEFINITION. A sequence $\mathbf{a} = (a_1, \cdots, a_n)$ is said to majorize a sequence $\mathbf{b} = (b_1, \cdots, b_n)$ (written $\mathbf{a} \succ \mathbf{b}$) if $a_1 \geqq \cdots \geqq a_n$, $b_1 \geqq \cdots \geqq b_n$, and $\sum_1^r a_i \geqq \sum_1^r b_i$ for $r = 1, \cdots, n-1$, while $\sum_1^n a_i = \sum_1^n b_i$.

THEOREM 4.7. (Hardy, Littlewood and Pólya). *If f is convex on the interval I and $\mathbf{x} \succ \mathbf{y}$ where $x_1, \cdots, x_n$; $y_1, \cdots, y_n$ belong to I, then*

$$\sum_1^n f(x_i) \geqq \sum_1^n f(y_i).$$

THEOREM 4.8. *If $F <_r G$, F and G symmetric about the origin, then*

$$\text{(i)} \qquad \frac{(\sum_{i=1}^n |X_{i,n}|^b)^a}{(\sum_{i=1}^n |X_{i,n}|^a)^b} \leqq_{st} \frac{(\sum_{i=1}^n |Y_{i,n}|^b)^a}{(\sum_{i=1}^n |Y_{i,n}|^a)^b} \qquad 0 \leqq a \leqq b,$$

and if $E|Y|^b$ exists then

$$\text{(ii)} \qquad \frac{(E|X|^b)^a}{(E|X|^a)^b} \leqq \frac{(E|Y|^b)^a}{(E|Y|^a)^b}.$$

PROOF. Raise to the ath power the absolute value of the observations from F, and order so that

$$|X|_{1,n}^a \geqq |X|_{2,n}^a \geqq \cdots \geqq |X|_{n,n}^a .$$

Now if $F_a(x) = P_F(|X|^a \leqq x)$, $a \geqq 0$ we see that $G_a^{-1}F_a(x^a) = [G^{-1}F(x)]^a$ and since $x^{-1}G^{-1}F(x) \uparrow x \geqq 0$ we have

$$\frac{G_a^{-1}F_a(x)}{x} \uparrow x \geqq 0 .$$

It follows from Marshall, Olkin and Proschan (1965) (cf. Barlow and Proschan (1966a) Theorem 3.12) that for $Y_{i,n} = G^{-1}F(X_{i,n})$

$$\sum_{i=1}^k \left(\frac{|X|_{i,n}^a}{\sum_{i=1}^n |X|_{i,n}^a}\right) \leqq \sum_{i=1}^k \left(\frac{|Y|_{i.n}^a}{\sum_{i=1}^n |Y|_{i,n}^a}\right)$$

for $k = 1, 2, \cdots, n$.

Now from Theorem 4.7, by considering the convex function $f(x) = x^c$, $x \geqq 0$, $c \geqq 1$ we obtain the stochastic inequality

$$\frac{\sum_1^n |X_{i,n}|^{ac}}{(\sum_1^n |X_{i,n}|^a)^c} \leqq_{st} \frac{\sum_1^n |Y_{i,n}|^{ac}}{(\sum_1^n |Y_{i,n}|^a)^c} ,$$

thus yielding (i).

Now if $E|Y|^b$ exists, then $E|Y|^a$ exists, and by a limiting argument we can see that $E|X|^b$ exists. (ii) is then true by the strong law of large numbers. □

COROLLARY 4.9. *If $F <_r G$, F and G symmetric about the origin, then if $EY_{i,n}^{2rk}$ exists*

$$\frac{EX_{i,n}^{2rk}}{(EX_{i,n}^{2r})^k} \leqq \frac{EY_{i,n}^{2rk}}{(EY_{i,n}^{2r})^k} \qquad \textit{for } k = 1, 2, \cdots .$$

PROOF. The proof follows in the same way as for Theorem 4.8 and by the observation that

$$G_{(i)}^{-1}F_{(i)}(x) = G^{-1}F(x) ,$$

where $F_{(i)}(x) = P(X_{i,n} \leqq x)$. □

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