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A SEQUENTIAL SIGNED-RANK TEST FOR SYMMETRY[1]

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A sequential procedure for testing the hypothesis that the distribution of a sequence of i.i.d. random variables is symmetric about zero is given, where the test statistic is a function of the signs and the ranks of the absolute values of the observations. Necessary and sufficient conditions that the individual signed ranks be independent are given. The critical region, power, and expected sample size of the test are determined approximately by using the fact that the test statistic behaves asymptotically like a Brownian motion process.

1. Introduction. Consider the problem of testing that a sequence $X_1, X_2, \cdots$ of independent and identically distributed random variables has a distribution which is symmetric about some given value. This problem arises, for example, in testing the effects of two treatments where the subjects are arranged in pairs; each of the treatments is given to one member of the pair chosen at random, and the random variables under consideration are the differences between the responses of the subjects within the pairs. For the problem of testing the hypothesis of symmetry, sequential parametric tests (e.g., the sequential t-test) and nonparametric fixed sample size tests (e.g., the sign test and the Wilcoxon signed-rank test) are available, but there appear to be very few sequential nonparametric tests available. This paper develops a sequential nonparametric test based on a signed-rank statistic proposed by Parent (1965).

Among the existing sequential nonparametric tests for symmetry, a sequential probability ratio test based on the signs and ranks of the observations was developed by Weed, Bradley and Govindarajula (1969). This test assumes that the alternatives are Lehmann alternatives. Miller (1970) proposed a truncated sequential test based on the Wilcoxon signed-rank statistic. If $X_1, X_2, \cdots$ is a sequence of independent and identically distributed random variables with continuous cdf F, R_{ij}^+ is defined to be the rank of $|X_i|$ in the set $\{|X_1|, |X_2|, \cdots, |X_j|\}$, $i \leqq j$, and $\operatorname{sgn}(X_i)$ is defined to be 1 if $X_i \geqq 0$ and -1 otherwise, then one form of the Wilcoxon signed-rank statistic is

$$\mathrm{SR}_n = \sum_{i=1}^n \operatorname{sgn}(X_i) R_{in}^+ .$$

Parent (1965) developed a statistic which, like the Wilcoxon statistic, is based on the signs and ranks of the absolute values of the observations and can be used to test for symmetry. Parent defined the signed sequential rank (SSR) of

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X_i to be

$$Y_i = \operatorname{sgn}(X_i) R_{ii}^+ .$$

Note that Y_i is the product of $\operatorname{sgn}(X_i)$ and the rank of $|X_i|$ relative to the absolute values of the previous observations only. The value of Y_i does not change as more observations are taken as do the ranks in the Wilcoxon statistic. The range of possible values for Y_i increases with i, so if we define a new variable $Z_i = Y_i/(i+1)$, then $-1 < Z_i < 1$ for all i.

In this paper it is proposed that the statistic

$$Z_n^+ = \sum_{i=1}^n Z_i = \sum_{i=1}^n \frac{1}{i+1} \operatorname{sgn}(X_i) R_{ii}^+$$

be used in a truncated sequential test with linear barriers. The test procedure using the statistic Z_n^+ continues taking observations as long as $Z_n^+ \in (-\alpha, \alpha)$ and $n < N$ where α is a positive constant and N is the truncation point. If $Z_n^+ \notin (-\alpha, \alpha)$ for some $n \leqq N$, then the null hypothesis of symmetry about zero is rejected and if n reaches N without Z_n^+ leaving the interval $(-\alpha, \alpha)$ then the null hypothesis is accepted. It will be shown that a continuous normalized version of $Z_{[Nt]}^+$, $0 \leqq t \leqq 1$, converges to a Brownian motion process on $[0, 1]$ as $N \to \infty$. If N is fairly large, then the probability that Z_n^+ leaves the interval $(-\alpha, \alpha)$ before time N can be approximated by the probability that a Brownian motion process crosses linear barriers by a speciffied time. The Brownian motion approximation is used to determine the critical value α and also the power of the test under certain alternatives. The test relies on the same truncated linear barrier procedure that Miller used for the statistic SR_n/n.

Section 2 of this paper gives necessary and sufficient conditions for the independence of the signed sequential ranks Y_i, or equivalently the Z_i. It is shown that the Z_i are independent under the null hypothesis of symmetry about zero so that Z_n^+ is the sum of independent random variables. Section 3 shows that $Z_{[Nt]}^+$, $0 \leqq t \leqq 1$, when normalized and redefined to be continuous in $[0, 1]$, converges to a Brownian motion process which can be used to approximate the behavior of the statistic Z_n^+. Section 4 gives expressions for the approximate power and expected sample size of the test and compares the SSR statistic with the Wilcoxon statistic for the truncated linear barrier test.

2. Necessary and sufficient conditions for independence of the signed sequential ranks. If $Y_1, Y_2, \cdots$ are the signed sequential ranks corresponding to the sequence $X_1, X_2, \cdots$ of i.i.d. random variables with continuous cdf F, then Parent (1965) has shown that a sufficient condition for the Y_i to be independent is that F satisfy

$$F(-x)[1-F(0)] = F(0)[1-F(x)] , \qquad x \geqq 0 . \tag{2.1}$$

If $F(0) \neq 0$ or 1 and f is the density of F, then (2.1) is equivalent to

$$f(-x) = \frac{F(0)}{1-F(0)} f(x) , \qquad x > 0 ,$$

so that the density at $-x$ is a constant multiple of the density at x. This condition will be satisfied under the null hypothesis of symmetry about zero and also under some alternatives. In this section it will be shown that (2.1) is a necessary condition for the signed sequential ranks to be independent and some additional conditions equivalent to (2.1) will be given.

Consider the n random variables $X_1, X_2, \cdots, X_n$ and the corresponding vector of signed sequential ranks $\mathbf{Y}_n = (Y_1, Y_2, \cdots, Y_n)$. The event $\mathbf{Y}_n = \mathbf{y}_n$ corresponds to a particular ordering of the X's with p of the X's positive and $n - p$ negative, where p is the number of positive components in $\mathbf{y}_n$. The absolute values of the $n - p$ negative X's have a particular ordering among the positive X's. Thus each $\mathbf{y}_n$ corresponds to an event like

$$(0 \leqq \varepsilon_1 X_{j_1} < \varepsilon_2 X_{j_2} < \cdots < \varepsilon_n X_{j_n})$$

where p of the ε_i are 1 and $n - p$ are -1. Let $F_i(\cdot)$ be the cdf of $\varepsilon_i X_{j_i}$, i.e., $F_i(x) = F(x)$ if $\varepsilon_i = 1$ and $F_i(x) = 1 - F(-x)$ if $\varepsilon_i = -1$. Then

$$\begin{aligned} P\{Y_1 = y_1, Y_2 = y_2, \cdots, Y_n = y_n\} &= P\{0 \leqq \varepsilon_1 X_{j_1} \leqq \varepsilon_2 X_{j_2} \leqq \cdots \leqq \varepsilon_n X_{j_n}\} \\ &= \textstyle\int_{u_1=0}^{\infty} \int_{u_2=u_1}^{\infty} \cdots \int_{u_n=u_{n-1}}^{\infty} \prod_{i=1}^{n} dF_i(u_i) . \end{aligned}$$

The following theorem gives necessary and sufficient conditions for the signed sequential ranks $Y_1, Y_2, Y_3, \cdots$ to be independent.

THEOREM 2.1. *If $X_1, X_2, X_3, \cdots$ is a sequence of independent and identically distributed random variables with continuous* cdf *F and $Y_1, Y_2, Y_3, \cdots$ are the corresponding signed sequential ranks, then the following conditions are equivalent:*

(i) *$Y_1, Y_2, Y_3, \cdots$ are independent,*

(ii) $$\begin{aligned} P\{Y_n = y_n\} &= \frac{1}{n}[1 - F(0)] \quad & y_n > 0 \\ &= \frac{1}{n} F(0) & y_n < 0 \end{aligned}$$

for all $n \geqq 1$, where y_n is a nonzero integer in $[-n, n]$,

(iii) $F(-x)[1 - F(0)] = F(0)[1 - F(x)]$, $x \geqq 0$,

(iv) *$|X_1|$ and* sgn (X_1) *are independent,*

(v) *R_{nn}^{+} and* sgn (X_n) *are independent for all $n \geqq 1$.*

PROOF. If $F(0) = 0$, then all of the X's are positive and the signed sequential ranks reduce to sequential ranks (see Parent (1965)) which are independent with $P\{Y_n = y_n\} = 1/n$, $y_n = 1, 2, \cdots, n$. A proof of this is given in Barndorff-Nielsn (1963). In this case sgn $(X_n) = 1$ with probability one so all the conditions are satisfied. By a similar argument all the conditions hold if $F(0) = 1$, so we can assume from now on that $F(0) \neq 0$ or 1.

(i) $\Rightarrow$ (ii). Assume that (i) holds. For $n = 1$, $P\{Y_1 = 1\} = 1 - F(0)$ and $P\{Y_1 = -1\} = F(0)$. For $n \geqq 2$ let y be a nonzero integer such that $-(n - 1) \leqq y \leqq n - 1$ and let $s = \operatorname{sgn}(y)$. Using the fact that X_{n-1} and X_n have the same distribution gives

$$
\begin{aligned}
(2.2)\qquad & [\textstyle\prod_{i=1}^{n-2} P\{Y_i = y_i\}]P\{Y_{n-1} = y\}P\{Y_n = y + s\} \\
&= P\{Y_1 = y_1, \cdots, Y_{n-2} = y_{n-2}, Y_{n-1} = y, Y_n = y + s\} \\
&= P\{0 \leqq \varepsilon_1 X_{j_1} \leqq \cdots \leqq \varepsilon_{|y|-1} X_{j_{|y|-1}} \leqq sX_{n-1} \\
&\qquad \leqq sX_n \leqq \varepsilon_{|y|+2} X_{j_{|y|+2}} \leqq \cdots \leqq \varepsilon_n X_{j_n}\} \\
&= P\{0 \leqq \varepsilon_1 X_{j_1} \leqq \cdots \leqq \varepsilon_{|y|-1} X_{j_{|y|-1}} \leqq sX_n \\
&\qquad \leqq sX_{n-1} \leqq \varepsilon_{|y|+2} X_{j_{|y|+2}} \leqq \cdots \leqq \varepsilon_n X_{j_n}\} \\
&= P\{Y_1 = y_1, \cdots, Y_{n-2} = y_{n-2}, Y_{n-1} = y, Y_n = y\} \\
&= [\textstyle\prod_{i=1}^{n-2} P\{Y_i = y_i\}]P\{Y_{n-1} = y\}P\{Y_n = y\}\,.
\end{aligned}
$$

The first and last parts of (2.2) imply that

$$P\{Y_n = y + s\} = P\{Y_n = y\}\,, \qquad y = \pm 1, \cdots, \pm(n-1)\,,$$

and thus $P\{Y_n = y\}$ does not depend on $|y|$. (ii) now follows from the fact that $P\{Y_n > 0\} = P\{X_n > 0\} = 1 - F(0)$ and $P\{Y_n < 0\} = F(0)$.

(ii) $\Rightarrow$ (iii). Let $|X|_k^{n-1}$, $1 \leqq k \leqq n-1$, be the kth order statistic from $\{|X_1|, |X_2|, \cdots, |X_{n-1}|\}$. For $0 \leqq \alpha \leqq 1$ define $|X|_\alpha^-$ and $|X|_\alpha^+$ by

$$
\begin{aligned}
|X|_\alpha^- &= \inf\{u : P(|X| \leqq u) = \alpha\} \\
|X|_\alpha^+ &= \sup\{u : P(|X| \leqq u) = \alpha\}
\end{aligned}
$$

so that if $|X|_\alpha^- = |X|_\alpha^+$ then $|X|_\alpha^-$ is the α percentile point of the distribution of $|X_i|$. Note that since F is assumed to be continuous then $P\{|X| \leqq |X|_\alpha^-\} = P\{|X| \leqq |X|_\alpha^+\} = \alpha$.

Now $Y_n = y_n > 0$ if and only if X_n is positive and is exceeded by exactly $n - y_n$ of the absolute values of the previous $n-1$ observations. Thus

$$P\{0 < Y_n \leqq [n\alpha]\} = P\{0 \leqq X_n \leqq |X|_{[n\alpha]}^{n-1}\} = \int_0^\infty [F(y) - F(0)]\, dF_{|X|_{[n\alpha]}^{n-1}}(y)$$

where $F_{|X|_{[n\alpha]}^{n-1}}(y)$ is the cdf of $|X|_{[n\alpha]}^{n-1}$, $|X|_0^{n-1} = 0$, and $|X|_n^{n-1} = +\infty$.

It is known (see Rao (1965), page 355) that if $|X|_\alpha^- = |X|_\alpha^+$, then $|X|_{[n\alpha]}^{n-1}$ converges to $|X|_\alpha^-$ as $n \to \infty$ with probability 1. A slight generalization of this theorem for the case $|X|_\alpha^- < |X|_\alpha^+$ gives for $\varepsilon > 0$

$$\lim_{m\to\infty} P\{|X|_\alpha^- - \varepsilon \leqq |X|_{[n\alpha]}^{n-1} \leqq |X|_\alpha^+ + \varepsilon \ \text{ for all } \ n > m\} = 1\,.$$

Since $[F(y) - F(0)] \leqq 1$,

$$\int_0^\infty [F(y) - F(0)]\, dF_{|X|_{[n\alpha]}^{n-1}}(y) = \int_{|X|_\alpha^- - \varepsilon}^{|X|_\alpha^+ + \varepsilon} [F(y) - F(0)]\, dF_{|X|_{[n\alpha]}^{n-1}}(y) + o(1)\,.$$

Also,

$$
\begin{aligned}
&[F(|X|_\alpha^- - \varepsilon) - F(0)]P\{|X|_\alpha^- - \varepsilon \leqq |X|_{[n\alpha]}^{n-1} \leqq |X|_\alpha^+ + \varepsilon\} \\
&\qquad \leqq \int_{|X|_\alpha^- - \varepsilon}^{|X|_\alpha^+ + \varepsilon} [F(y) - F(0)]\, dF_{|X|_{[n\alpha]}^{n-1}}(y) \\
&\qquad \leqq [F(|X|_\alpha^+ + \varepsilon) - F(0)]P\{|X|_\alpha^- - \varepsilon \leqq |X|_{[n\alpha]}^{n-1} \leqq |X|_\alpha^+ + \varepsilon\}
\end{aligned}
$$

so that

$$
\begin{aligned}
(2.3)\qquad F(|X|_\alpha^- - \varepsilon) - F(0) &\leqq \liminf_{n\to\infty} P\{0 \leqq X_n \leqq |X|_{[n\alpha]}^{n-1}\} \\
&\leqq \limsup_{n\to\infty} P\{0 \leqq X_n \leqq |X|_{[n\alpha]}^{n-1}\} \\
&\leqq F(|X|_\alpha^+ + \varepsilon) - F(0) .
\end{aligned}
$$

Now (2.3) holds for all $\varepsilon > 0$ and using the fact that $F(|X|_\alpha^-) = F(|X|_\alpha^+)$,

$$
\begin{aligned}
\lim_{n\to\infty} P\{0 < Y_n \leqq [n\alpha]\} &= \lim_{n\to\infty} P\{0 \leqq X_n \leqq |X|_{[n\alpha]}^{n-1}\} \\
&= F(|X|_\alpha^-) - F(0) .
\end{aligned}
$$

A similar argument gives

$$
\begin{aligned}
\lim_{n\to\infty} P\{-[n\alpha] \leqq Y_n < 0\} &= \lim_{n\to\infty} P\{0 < -X_n \leqq |X|_{[n\alpha]}^{n-1}\} \\
&= F(0) - F(-|X|_\alpha^-) .
\end{aligned}
$$

By the hypothesis that (ii) holds

$$\lim_{n\to\infty} P\{0 < Y_n \leqq [n\alpha]\} = \lim_{n\to\infty} \frac{[n\alpha]}{n}[1 - F(0)] = \alpha[1 - F(0)]$$

and

$$\lim_{n\to\infty} P\{-[n\alpha] \leqq Y_n < 0\} = \lim_{n\to\infty} \frac{[n\alpha]}{n} F(0) = \alpha F(0) .$$

It follows that

$$\alpha[1 - F(0)] = F(|X|_\alpha^-) - F(0) \qquad \text{and} \qquad \alpha F(0) = F(0) - F(-|X|_\alpha^-) .$$

Solving for α gives

$$\frac{1}{1 - F(0)}[F(|X|_\alpha^-) - F(0)] = \alpha = \frac{1}{F(0)}[F(0) - F(-|X|_\alpha^-)]$$

and from this it is easy to show that (iii) holds for $x = |X|_\alpha^-$. Since α is an arbitrary number in $[0, 1]$, $F(|X|_\alpha^-) = F(|X|_\alpha^+)$, and $F(-|X|_\alpha^-) = F(-|X|_\alpha^+)$, then (iii) holds for all $x \geqq 0$.

The equivalence of (iii) and (iv) was proved by Parent by showing that the joint distribution of sgn (X_n) and $|X_n|$ will factor into the product of the marginals if and only if (iii) holds. (iv) $\rightarrow$ (v) follows from the fact that the X's are independent, and (v) $\rightarrow$ (ii) follows from the distribution of R_{nn}^+ and of sgn (X_n).

(ii) $\Rightarrow$ (i). (Parent). If (ii) holds then

$$\prod_{i=1}^n P\{Y_i = y_i\} = \frac{1}{n!} F(0)^{n-p}[1 - F(0)]^p$$

where p is the number of positive components in $\mathbf{y}_n$. From what we have already proved (ii) implies (iii) so if $\varepsilon_i = -1$ then

$$dF_i(x) = d[1 - F(-x)] = \frac{F(0)}{1 - F(0)} dF(x) .$$

Hence

$$
\begin{aligned}
P\{\mathbf{Y}_n = \mathbf{y}_n) &= \textstyle\int_{u_1=0}^{\infty} \int_{u_2=u_1}^{\infty}, \cdots, \int_{u_n=u_{n-1}}^{\infty} \prod_{i=1}^{n} dF_i(u_i) \\
&= \left[\frac{F(0)}{1-F(0)}\right]^{n-p} \textstyle\int_{u_1=0}^{\infty} \int_{u_2=u_1}^{\infty}, \cdots, \int_{u_n=u_{n-1}}^{\infty} \prod_{i=1}^{n} dF(u_i) \\
&= \left[\frac{F(0)}{1-F(0)}\right]^{n-p} P\{0 \leqq X_{j_1} \leqq X_{j_2} \leqq \cdots \leqq X_{j_n}\} \\
&= \frac{1}{n!} F(0)^{n-p}[1-F(0)]^p .
\end{aligned}
$$

Thus $P\{\mathbf{Y}_n = \mathbf{y}_n\} = \prod_{i=1}^{n} P\{Y_i = y_i\}$ so $Y_1, Y_2, \cdots, Y_n$ are independent. Since this is true for every $n \geqq 1$, $Y_1, Y_2, Y_3, \cdots$ are independent.

3. Convergence of the signed sequential rank statistic to Brownian motion.

3.1. *Introduction.* The statistic that was proposed for testing the hypothesis of symmetry was

$$
Z_n^+ = \textstyle\sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} \frac{Y_i}{i+1} = \sum_{i=1}^{n} \frac{1}{i+1} \operatorname{sgn}(X_i) R_{ii}^+ .
$$

Let $Z_0^+ = 0$ and define

$$
Z_N^+(t) = Z_{[Nt]}^+ + (Nt - [Nt]) Z_{[Nt]+1} , \qquad 0 \leqq t \leqq 1 .
$$

For each value of N, $Z_N^+(t)$ is in $C[0, 1]$, the space of continuous functions on $[0, 1]$ where C is given the uniform topology induced by the metric

$$
\rho(g, h) = \sup_{0 \leqq t \leqq 1} |g(t) - h(t)| , \qquad g, h \in C[0, 1] .
$$

Using the theory of weak convergence of probability measures on $C[0, 1]$ it will be shown that a process $X_N(t)$ which is asymptotically equivalent to

$$
\frac{Z_N^+(t) - EZ_N^+(t)}{\operatorname{Var}(Z_N^+(t))^{\frac{1}{2}}} t^{\frac{1}{2}}
$$

converges in distribution to a Brownian motion process on $[0, 1]$ with mean zero and variance t as $N \to \infty$. The proof requires that the original observations be independent and identically distributed but does not require that the signed sequential ranks Y_i, or equivalently the Z_i, be independent.

3.2. *Preliminary calculations.* Before proving the convergence to Brownian motion it is necessary to compute the mean and variance of Z_n^+. To facilitate the computations Y_n can be rewritten as

$$
Y_n = \textstyle\sum_{i=1}^{n} \phi_{in}
$$

where

$$
\begin{aligned}
\phi_{in} = \phi_{in}(X_i, X_n) &= 1 \qquad |X_i| \leqq X_n \\
&= 0 \qquad -|X_i| < X_n < |X_i| \qquad (3.1) \\
&= -1 \quad\; X_n \leqq -|X_i| .
\end{aligned}
$$

Thus

$$Z_n^+ = \sum_{j=1}^n Z_j = \sum_{j=1}^n \frac{1}{j+1} \sum_{i=1}^j \phi_{ij} .$$

The mean and variance of Z_n^+ can thus be expressed in terms of the means, variances, and covariances of $\{\phi_{ij}\}$.

Let

$$\xi = E[\phi_{ij}] = \tfrac{1}{2} - \textstyle\int_{-\infty}^{\infty} F(-x)\, dF(x) , \qquad i \neq j ,$$

$$\theta = E[\phi_{jj}] = 1 - 2F(0) ,$$

and

$$\gamma = E[\phi_{ij}\phi_{jk}] = \tfrac{1}{6} - \textstyle\int_{-\infty}^{\infty} F(-x)\, dF(x) + \int_{-\infty}^{\infty} F(-x)^2\, dF(x) .$$

Then

$$EZ_j = \frac{1}{j+1} \sum_{i=1}^j E[\phi_{ij}] = \frac{j-1}{j+1}\xi + \frac{1}{j+1}\theta$$

and

$$EZ_n^+ = n\xi + (\theta - 2\xi)\log(n+1) + O(1) .$$

Also

$$\operatorname{Var}(Z_j) = \frac{1}{(j+1)^2}[(j-1)^2(\tfrac{1}{3} - \xi^2) + (j-1)(\tfrac{7}{6} - 2\xi\theta) + 1 - \theta^2] ,$$

and for $j < k$

$$\operatorname{Cov}(Z_j, Z_k) = \frac{1}{(j+1)(k+1)}[(j-1)(3\gamma - 2\xi^2) + \tfrac{1}{2}\theta^2 - \theta\xi] .$$

Combining the variance and covariance terms for the Z's gives

$$\begin{aligned} &\operatorname{Var}(Z_n^+ - Z_m^+) \\ &\quad = \sum_{j=m+1}^n \frac{1}{(j+1)^2}[(j-1)^2(\tfrac{1}{3} - \xi^2) + (j-1)(\tfrac{7}{6} - 2\xi\theta) + 1 - \theta^2] \\ &\qquad + 2\sum_{j=m+1}^{n-1} \sum_{k=j+1}^n \frac{1}{(j+1)(k+1)} \\ &\qquad \times [(j-1)(3\gamma - 2\xi^2) + \tfrac{1}{2}\theta^2 - \theta\xi] \qquad (m < n) \end{aligned} \tag{3.2}$$

and

$$\operatorname{Var}(Z_n^+) = n\tau^2 + o(n)$$

where $\tau^2 = [\tfrac{1}{3} + 6\gamma - 5\xi^2]$.

In proving convergence it will be convenient to use the statistic

$$A_n = \sum_{j=1}^n E[Z_n^+ \mid X_j] - (n-1)EZ_n^+$$

as an approximation to Z_n^+. To get a continuous version of A_n corresponding to the continuous function $Z_N^+(t)$ define

$$\begin{aligned} A_N(t) &= A_{[Nt]} + (Nt - [Nt])(A_{[Nt]+1} - A_{[Nt]}) \\ &= \sum_{j=1}^{[Nt]+1} E[Z_N^+(t) \mid X_j] - [Nt]EZ_N^+(t) \end{aligned}$$

for $0 \leqq t \leqq 1$. Note that if the X_j are independent then $A_N(t)$ is constructed from partial sums of independent random variables so that the functional central limit theorem can be applied to the sequence $\{A_N(t)\}$.

Using the conditional expectations of $\{\phi_{ij}\}$, A_n can be written as

$$A_n = \sum_{j=1}^{n}\left[\frac{j-1}{j+1}(F^+(X_j) - \xi) + \frac{1}{j+1}(\operatorname{sgn} X_j - \theta) + \left(\sum_{k=j+1}^{n}\frac{1}{k+1}\right)(\hat{F}(X_j) - \xi)\right] + EZ_n^+ \tag{3.3}$$

and the variance of $A_n - A_m$ is given by

$$\begin{aligned}\operatorname{Var}(A_n - A_m) = \sum_{j=m+1}^{n} &\frac{1}{(j+1)^2}[(j-1)^2(\tfrac{1}{3} - \xi^2) \\ &+ (j-1)(1 - 2\xi\theta + 2\gamma - \xi^2) + 1 - \theta^2] \\ &+ 2\sum_{j=m+1}^{n}\sum_{k=j+1}^{n}\frac{1}{(j+1)(k+1)} \\ &\times [(j-1)(3\gamma - 2\xi^2) + (\tfrac{1}{2}\theta^2 - \xi\theta)]\,.\end{aligned} \tag{3.4}$$

Thus

$$\begin{aligned}\operatorname{Var}(A_N(t)) &= \operatorname{Var}(A_{[Nt]}) + (Nt - [Nt])^2 \operatorname{Var}(A_{[Nt]+1} - A_{[Nt]}) \\ &= [Nt]\tau^2 + o([Nt])\,.\end{aligned} \tag{3.5}$$

In order to use A_n as an approximation to Z_n^+ it is necessary to show that A_n is in some sense close to Z_n^+. We will use $E[Z_n^+ - A_n]^2$ as a measure of closeness. If $g(X_1, \cdots, X_n)$ is a function of independent random variables $X_1, \cdots, X_n$ and if $\sum_{j=1}^{n} E[g(X_1, \cdots, X_n) \mid X_j] - (n-1)Eg(X_1, \cdots, X_n)$ is used as an approximation to $g(X_1, \cdots, X_n)$, then from Lemma 4.1 in Hájek (1968)

$$E[g - \sum_{j=1}^{n} E(g \mid X_j) + (n-1)Eg]^2 = \operatorname{Var}(g) - \operatorname{Var}(\sum_{j=1}^{n} E(g \mid X_j))\,. \tag{3.6}$$

Using (3.2), (3.4) and (3.6), we have

$$\begin{aligned}E[(Z_n^+ - Z_m^+) - (A_n - A_m)]^2 &= \operatorname{Var}(Z_n^+ - Z_m^+) - \operatorname{Var}(A_n - A_m) \\ &= \sum_{j=m+1}^{n}\frac{j-1}{(j+1)^2}(\tfrac{1}{6} - 2\gamma + \xi^2) \\ &= \log\left(\frac{n}{m+1}\right)(\tfrac{1}{6} - 2\gamma + \xi^2) + O(1)\,, \\ &\qquad n > m \geqq 0\,,\end{aligned} \tag{3.7}$$

where $O(1)$ is for $n \to \infty$.

3.3. *A functional central limit theorem.* We can now state and prove the theorem on the convergence of the sequence $\{X_N(t)\}$. Note that $X_N(t)$ is a random function in the space $C[0, 1]$ of continuous functions on $[0, 1]$.

THEOREM 3.1. *If $X_1, X_2, \cdots$ is a sequence of* i.i.d. *random variables with continuous* cdf F, *and $Z_N^+(t)$ is defined by*

$$Z_N^+(t) = \sum_{i=1}^{[Nt]} Z_i + (Nt - [Nt])Z_{[Nt]+1}\,,$$

where

$$Z_i = \frac{1}{i+1}\operatorname{sgn}(X_i)R_{ii}^+\,,$$

then the function

$$X_N(t) = \frac{1}{N^{\frac{1}{2}}\tau}(Z_N{}^+(t) - Nt\xi), \qquad 0 \leqq t \leqq 1,$$

where $EZ_N{}^+(t) = Nt\xi + o(Nt)$ *and* $\operatorname{Var}(Z_N{}^+(t)) = Nt\tau^2 + o(Nt)$, *converges in distribution to a Brownian motion process on* $[0, 1]$ *with mean zero and variance* t.

PROOF. To prove that $X_N(t)$ converges to a Brownian motion process it is sufficient (Theorem 8.1, Billingsley (1968)) to verify the convergence of the finite dimensional distributions and the tightness of $\{X_N(t)\}$.

For $t \in [0, 1]$, $X_N(t)$ can be written

$$\begin{aligned} X_N(t) = \left(\frac{[Nt]}{N}\right)^{\frac{1}{2}} \frac{A_N(t) - EA_N(t)}{\operatorname{Var}(A_N(t))^{\frac{1}{2}}} \frac{\operatorname{Var}(A_N(t))^{\frac{1}{2}}}{([Nt])^{\frac{1}{2}}\tau} &+ \frac{1}{N^{\frac{1}{2}}\tau}(Z^+_{[Nt]} - A_{[Nt]}) \\ + \frac{1}{N^{\frac{1}{2}}\tau}(EA_{[Nt]} - [Nt]\xi) &+ \frac{1}{N^{\frac{1}{2}}\tau}(Nt - [Nt])(Z_{[Nt]+1} - \xi) \\ + \frac{1}{N^{\frac{1}{2}}\tau}(Nt - [Nt])(A_{[Nt]+1} - EA_{[Nt]+1} &- A_{[Nt]} + EA_{[Nt]}). \end{aligned} \tag{3.8}$$

As $N \to \infty$, $([Nt]/N)^{\frac{1}{2}} \to t^{\frac{1}{2}}$ and $1/([Nt]\tau)^{\frac{1}{2}} \operatorname{Var}(A_N(t))^{\frac{1}{2}} \to 1$. The second term on the right in (3.8) converges in probability to zero by Chebychev's inequality and (3.7), and the third term converges to zero since $EA_{[Nt]} - [Nt]\xi = (\theta - 2\xi)\log([Nt] + 1) + O(1)$. The fourth term also converges in probability to zero since $(Nt - [Nt])(Z_{[Nt]+1} - \xi) \leqq 2$. To show that the fifth term converges in probability to zero use (3.4) and Chebychev's inequality. From this it follows that

$$X_N(t) - t^{\frac{1}{2}} \frac{A_N(t) - EA_N(t)}{\operatorname{Var}(A_N(t))^{\frac{1}{2}}} \to_P 0. \tag{3.9}$$

Let $t_1, t_2, \cdots, t_M$ be M points in $[0, 1]$. Using (3.9) we can conclude (see e.g. Theorem 4.1, Billingsley (1968)) that

$$(X_N(t_1), X_N(t_2), \cdots, X_N(t_M))$$

has the same limiting joint distribution as

$$\left(t_1^{\frac{1}{2}} \frac{A_N(t_1) - EA_N(t_1)}{\operatorname{Var}(A_N(t_1))^{\frac{1}{2}}}, t_2^{\frac{1}{2}} \frac{A_N(t_2) - EA_N(t_2)}{\operatorname{Var}(A_N(t_2))^{\frac{1}{2}}}, \cdots, t_M^{\frac{1}{2}} \frac{A_N(t_M) - EA_N(t_M)}{\operatorname{Var}(A_N(t_M))^{\frac{1}{2}}}\right). \tag{3.10}$$

If we can show that the normalized approximation statistic

$$t^{\frac{1}{2}} \frac{A_N(t) - EA_N(t)}{\operatorname{Var}(A_N(t))^{\frac{1}{2}}}, \qquad 0 \leqq t \leqq 1, \tag{3.11}$$

converges to a Brownian motion process, then (3.10) must converge to the proper multivariate normal distribution. First we check that Lindeberg's condition holds when applied to

$$\frac{A_N - EA_N}{\operatorname{Var}(A_N)^{\frac{1}{2}}} = \frac{\sum_{j=1}^N [E(Z_N{}^+ \mid X_j) - EZ_N{}^+]}{\operatorname{Var}(A_N)^{\frac{1}{2}}} . \tag{3.12}$$

Now by (3.3) and (3.5)

$$|E(Z_N{}^+ \mid X_j) - EZ_N{}^+| \leqq 2\left[\frac{j}{j+1} + \sum_{k=j+1}^N \frac{1}{k+1}\right] = o(\operatorname{Var}(A_N)^{\frac{1}{2}}) .$$

Applying Theorem 3.1 in Prohorov (1956), (3.11) converges to a Brownian motion process. This proves the convergence of the finite dimensional distributions.

To prove tightness of $\{X_N(t)\}$ it is sufficient to prove (see Theorem 8.2, Billingsley (1968)) that for every $\varepsilon > 0$ and $\eta > 0$ there exists a $\delta \in (0, 1)$ and an integer N_0 such that

$$P\{\sup_{|t-s| \leqq \delta} |X_N(t) - X_N(s)| > \varepsilon\} \leqq \eta \quad \text{when} \quad N > N_0 . \tag{3.13}$$

The probability in (3.13) can be written as

$$\begin{aligned} P\{&\sup_{|t-s|<\delta} |X_N(t) - X_N(s)| > \varepsilon\} \\ &\leqq P\left\{\sup_{|t-s|<\delta} \frac{1}{N^{\frac{1}{2}}\tau} |(A_N(t) - EA_N(t)) - (A_N(s) - EA_N(s))| > \varepsilon\right\} \\ &\quad + P\left\{\sup_{|t-s|<\delta} \frac{1}{N^{\frac{1}{2}}\tau} |(Z_N{}^+(t) - A_N(t)) - (Z_N{}^+(s) - A_N(s))| > \varepsilon\right\} \\ &\quad + P\left\{\sup_{|t-s|<\delta} \frac{1}{N^{\frac{1}{2}}\tau} |(EA_N(t) - Nt\xi) - (EA_N(s) - Ns\xi)| > \varepsilon\right\} . \end{aligned} \tag{3.14}$$

The first term on the right in (3.14) can be made arbitrarily small by taking δ small enough and N large enough, since (3.11) converges to Brownian motion. The last term on the right also converges to zero since

$$(EA_N(t) - Nt\xi) - (EA_N(s) - Ns\xi) = (\theta - 2\xi)\log\left(\frac{[Nt]+1}{[Ns]+1}\right) + O(1) .$$

For the middle term use the corollary to Theorem 8.3 in Billingsley (1968) to get

$$\begin{aligned} P&\left\{\sup_{|t-s|<\delta} \frac{1}{N^{\frac{1}{2}}\tau} |(Z_N{}^+(t) - A_N(t)) - (Z_N{}^+(s) - A_N(s))| > \varepsilon\right\} \\ &\leqq \sum_{n=0}^{N-1} P\left\{\sup_{n/N \leqq t \leqq (n+1/N)} \frac{1}{N^{\frac{1}{2}}\tau} \left|(Z_N{}^+(t) - A_N(t)) - \left(Z_N{}^+\left(\frac{n}{N}\right) - A_N\left(\frac{n}{N}\right)\right)\right| > \frac{\varepsilon}{3}\right\} \\ &= \sum_{n=0}^{N-1} P\left\{\frac{1}{(N\tau)^{\frac{1}{2}}} \left|\left(Z_N{}^+\left(\frac{n+1}{N}\right) - Z^+\left(\frac{n}{N}\right)\right) - \left(A_N\left(\frac{n+1}{N}\right) - A_N\left(\frac{n}{N}\right)\right)\right| > \frac{\varepsilon}{3}\right\} . \end{aligned}$$

Chebychev's inequality and (3.7) can now be used to show that this term converges to zero. This establishes the tightness of $\{X_N(t)\}$ and the proof is complete.

4. A truncated test for symmetry.

4.1. *Description of the test.* It has been shown that if $Y_1, Y_2, \cdots$ are the signed sequential ranks of $X_1, X_2, \cdots$ from a distribution with continuous cdf F and $Z_n{}^+ = \sum_{i=1}^n Y_i/(i+1)$ then

$$E[Z_n{}^+] = n\xi + o(n) \qquad \text{and} \qquad \operatorname{Var}(Z_n{}^+) = n\tau^2 + o(n)\,.$$

If f is the density of F and f is symmetric about zero, then $EZ_n{}^+ = 0$ and $\operatorname{Var}(Z_n{}^+) = \frac{1}{3}n + o(n)$. If f is symmetric about some positive constant or if f has median zero but is skewed to the right, then the expected value of $Z_n{}^+$ is positive. This suggests that $Z_n{}^+$ could be used to test the hypothesis that f is symmetric about zero against a general class of alternatives that includes symmetric distributions with shifted means and skewed distributions with zero medians. The null hypothesis would be rejected if $Z_n{}^+$ or $|Z_n{}^+|$ is too large. It is anticipated that most alternatives of interest will be shift alternatives so the problem can be stated formally as testing the null hypothesis

$$H_0\colon\ f \quad \text{is symmetric about} \quad 0$$

against the alternative hypothesis

$$H_\delta\colon\ f \quad \text{is symmetric about some} \quad \delta \neq 0\,.$$

The test procedure is a sequential procedure in which an upper bound N is placed on the number of observations. At each stage n ($\leqq N$) an observation X_n is taken and the value of $Z_n{}^+$ is computed. If, for some $n \leqq N$, $Z_n{}^+$ does not fall in some fixed interval $(\mathscr{b}, \mathscr{a})$, $\mathscr{b} \leqq 0 \leqq \mathscr{a}$, then sampling stops and H_0 is rejected. If $Z_n{}^+ \in (\mathscr{b}, \mathscr{a})$ and $n < N$, then another observation is taken and the value of Z_{n+1}^+ is computed. If $n = N$ and $Z_n{}^+ \in (\mathscr{b}, \mathscr{a})$, then sampling stops and H_0 is accepted. In cases where only one-sided alternatives are considered there is only one rejection boundary for $Z_n{}^+$, i.e., either $\mathscr{b} = -\infty$ or $\mathscr{a} = \infty$.

If the value of the test statistic is close to zero in the two-sided test, then as n approaches N a point is reached from which it is not possible to reach the rejection boundary no matter what the value of the remaining Z's. This fact leads to the use of an inner acceptance boundary that permits early acceptance of H_0. At any point $n_1 \leqq N$ the maximum amount that $Z_n{}^+$ can increase or decrease while taking the remaining $N - n_1$ observations is $\sum_{i=n_1+1}^N (i/i+1)$. Thus if for any $n \leqq N$, $Z_n{}^+$ satisfies

$$\mathscr{b} + \sum_{i=n+1}^N \frac{i}{i+1} < Z_n{}^+ < \mathscr{a} - \sum_{i=n+1}^N \frac{i}{i+1}\,,$$

then H_0 can be accepted since it will not be possible to reject at a later time.

4.2. *Power of the tests.* In Theorem 3.1 it was shown that the function

$$X_N(t) = \frac{1}{N^{\frac{1}{2}}\tau}(Z_N{}^+(t) - Nt\xi)\,, \qquad 0 \leqq t \leqq 1\,,$$

converges in distribution to a normalized Brownian motion process $X(t)$, $0 \leqq t \leqq 1$, as $N \to \infty$. $Z_N{}^+(t)$ is just a continuous version of $Z^+_{[Nt]}$ so $N^{\frac{1}{2}}\tau(Z^+_{[Nt]} - [Nt]\xi)$ has approximately the same distributions as $X(t)$ for large N. Under the null hypothesis $\xi = 0$ and $\tau = 1/3^{\frac{1}{2}}$ so in this case $(3/N)^{\frac{1}{2}}Z^+_{[Nt]}$ has approximately the same distribution as $X(t)$. It is well known that

$$P\{\sup\nolimits_{0 \leqq t \leqq T} X(t) \geqq c\} = 2\Phi\left(-\frac{c}{T^{\frac{1}{2}}}\right)$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. Thus for the one-sided test under the null hypothesis

$$\begin{aligned} P\{\text{reject } H_0 \mid H_0\} &= P\{Z_n{}^+ \geqq \alpha \text{ for some } n = 1, 2, \cdots, N\} \\ &= P\left\{\sup\nolimits_{0 \leqq t \leqq 1}\left(\frac{3}{N}\right)^{\frac{1}{2}} Z^+_{[Nt]} \geqq \left(\frac{3}{N}\right)^{\frac{1}{2}}\alpha = c\right\} \\ &\approx P\{\sup\nolimits_{0 \leqq t \leqq 1} X(t) \geqq c\} = 2\Phi(-c) \end{aligned}$$

where $c = (3/N)^{\frac{1}{2}}\alpha$. It will be convenient to use $(3/N)^{\frac{1}{2}}Z_n{}^+$ as the test statistic since in this case the critical value c will not (under the Brownian motion approximation) depend on N.

Under the alternative hypothesis $(3/N)^{\frac{1}{2}}Z^+_{[Nt]}$ has approximately the same distribution as a Brownian motion process with mean $(3N)^{\frac{1}{2}}\xi t$ and variance $3\tau^2 t$. If $X^*(t)$ is a Brownian motion process with mean μt and variance $\sigma^2 t$ then Dinges (1962) proved that

$$\begin{aligned} (4.1) \qquad & P\{\sup\nolimits_{0 \leqq t \leqq T} X^*(t) \geqq c\} \\ &= \exp\left(\frac{2\mu c}{\sigma^2}\right)\Phi\left(\frac{-\mu T - c}{\sigma T^{\frac{1}{2}}}\right) + \Phi\left(\frac{\mu T - c}{\sigma T^{\frac{1}{2}}}\right). \end{aligned}$$

Using (4.1), the probability that the test rejects by a given time $M (M \leqq N)$ is given by

$$\begin{aligned} (4.2) \qquad P\{\max\nolimits_{1 \leqq n \leqq M} Z_n{}^+ \geqq \alpha\} &\approx P\left\{\sup\nolimits_{0 \leqq t \leqq M/N} X^*t) \geqq \left(\frac{3}{N}\right)^{\frac{1}{2}}\alpha\right\} \\ &= \exp\left(\frac{2\xi}{\tau^2}\right)\Phi\left(\frac{-\xi M - \alpha}{\tau M^{\frac{1}{2}}}\right) + \Phi\left(\frac{\xi M - \alpha}{\tau M^{\frac{1}{2}}}\right). \end{aligned}$$

Setting $M = N$ in (4.2) gives the power of the test.

For two-sided tests we will consider only symmetric tests, although the unsymmetric case could be treated with some increase in the complexity of the formulas. Anderson has shown (1960) that if $X^*(t)$ is a Brownian motion process with mean μt and variance $\sigma^2 t$ then

$$
\begin{aligned}
&P\{\sup_{0\leq t\leq T} |X^*(t)| \geq c\} \\
(4.3) \qquad &= \textstyle\sum_{s=0}^{\infty} (-1)^s \\
&\times \left\{\Phi\left(\frac{\mu T - (2s+1)c}{\sigma T^{\frac{1}{2}}}\right) \exp\left(\frac{-2sc\mu}{\sigma^2}\right)\left(1 + \exp\left(\frac{-2c\mu}{\sigma^2}\right)\right)\right. \\
&\left. + \Phi\left(\frac{-\mu T - (2s+1)c}{\sigma T^{\frac{1}{2}}}\right) \exp\left(\frac{2sc\mu}{\sigma^2}\right)\left(1 + \exp\left(\frac{2c\mu}{\sigma^2}\right)\right)\right\}.
\end{aligned}
$$

The first term in the series (4.3) above is just

$$
(4.4) \qquad P\{\sup_{0\leq t\leq T} X^*(t) \geq c\} + P\{\inf_{0\leq t\leq T} X^*(t) \leq -c\}.
$$

The series is alternating and if the first term is used as an approximation to the whole expression the error is bounded by the second term which is small for values of μ, σ and c in our case. Thus for the two-sided test the critical value c can be determined by

$$
\begin{aligned}
P\{\text{reject } H_0 \mid H_0\} &= P\left\{\max_{1\leq n\leq N} \left|\left(\frac{3}{N}\right)^{\frac{1}{2}} Z_n^+\right| \geq c\right\} \\
&\approx P\{\sup_{0\leq t\leq 1} |X(t)| \geq c\} \\
&\approx 4\Phi(-c).
\end{aligned}
$$

Under the alternative hypothesis

$$
\begin{aligned}
&P\left\{\max_{1\leq n\leq M} \left|\left(\frac{3}{N}\right)^{\frac{1}{2}} Z_n^+\right| \geq c\right\} \\
&\qquad \approx P\{\sup_{0\leq t\leq M/N} |X^*(t)| \geq c\} \\
(4.5) \qquad &\qquad \approx P\{\sup_{0\leq t\leq M/N} X^*(t) \geq c\} \\
&\qquad\qquad + P\{\inf_{0\leq t\leq M/N} X^*(t) \leq -c\} \\
&\qquad = \exp\left(\frac{2a\xi}{\tau^2}\right)\Phi\left(\frac{-\xi M - a}{\tau M^{\frac{1}{2}}}\right) + \Phi\left(\frac{\xi M - a}{\tau M^{\frac{1}{2}}}\right) \\
&\qquad\qquad + \exp\left(\frac{-2a\xi}{\tau^2}\right)\Phi\left(\frac{\xi M - a}{\tau M^{\frac{1}{2}}}\right) + \Phi\left(\frac{-\xi M - a}{\tau M^{\frac{1}{2}}}\right).
\end{aligned}
$$

Setting $M = N$ in (4.5) gives the approximate power of the test.

4.4. *Expected sample sizes.* The Brownian motion approximation can be used to compute the expected sample size for the one-sided and two-sided tests. If t^* is the time until a Brownian motion process $X^*(t)$ either exits from (b, a) or t reaches T, then

$$
Et^* = \textstyle\int_0^T s\left[\frac{d}{ds} P\{t^* \leq s\}\right] ds + T \cdot P\{t^* = T\}.
$$

For a one-sided test $P\{t^* \leq s\} = P\{\sup_{0\leq t\leq s} X^*(t) \geq c\}$ for $s < T$, and this probability is given by (4.1). Anderson (1960) has proved the following lemma.

LEMMA.

$$\int_0^T t\left[\frac{d}{dt}\Phi\left(\frac{At+B}{t^{\frac{1}{2}}}\right)\right]dt$$
$$= -\frac{2AB-1}{2A^2}\phi\left(\frac{AT+B}{T^{\frac{1}{2}}}\right) - \frac{1}{2A^2}e^{-2AB}\Phi\left(\frac{-AT+B}{T^{\frac{1}{2}}}\right)$$
$$-\frac{T^{\frac{1}{2}}}{A}\frac{1}{(2\pi)^{\frac{1}{2}}}\exp\left[-(AT+B)^2/2T\right], \qquad B<0.$$

Using the lemma it is easy to verify that

$$\int_0^T s\left[\frac{d}{ds}P\{t^* \leqq s\}\right]ds = \frac{c}{\mu}\left[\Phi\left(\frac{\mu T-c}{\sigma T^{\frac{1}{2}}}\right) - \exp\left(\frac{2c\mu}{\sigma^2}\right)\Phi\left(\frac{-\mu T-c}{\sigma T^{\frac{1}{2}}}\right)\right],$$

so that

$$Et^* = \Phi\left(\frac{\mu T-c}{\sigma T^{\frac{1}{2}}}\right)\left(\frac{c}{\mu}-T\right) - \Phi\left(\frac{-\mu T-c}{\sigma T^{\frac{1}{2}}}\right)\exp\left(\frac{2c\mu}{\sigma^2}\right)\left(\frac{c}{\mu}+T\right)+T, \qquad \mu \neq 0. \tag{4.6}$$

Now $Et^* \approx E(n^*/N)$, where n^* is the sample size of the test, so substituting $\mu = (3N)^{\frac{1}{2}}\xi$, $\sigma^2 = 3\tau^2$ and $T = 1$ into (4.6) gives

$$En^* \approx \Phi\left(\frac{N\xi-\alpha}{\tau N^{\frac{1}{2}}}\right)\left(\frac{\alpha}{\xi}-N\right) - \Phi\left(\frac{-N\xi-\alpha}{\tau N^{\frac{1}{2}}}\right)\exp\left(\frac{2\alpha\xi}{\sigma^2}\right)\left(\frac{\alpha}{\xi}+N\right)+N.$$

For the two-sided test Anderson (1960) has shown that

$$\begin{aligned}\int_0^T s\frac{d}{ds}P\{t^* \leqq s\}\,ds &= \frac{c}{\mu}\sum_{s=0}^{\infty}(-1)^s(2s+1)\left\{\Phi\left(\frac{\mu T-(2s+1)c}{\sigma T^{\frac{1}{2}}}\right)\right.\\ &\quad\times\exp\left(\frac{-2sc\mu}{\sigma^2}\right)\left(1+\exp\left(\frac{-2c\mu}{\sigma^2}\right)\right)\\ &\quad\left.-\Phi\left(\frac{-\mu T-(2s+1)c}{\sigma T^{\frac{1}{2}}}\right)\exp\left(\frac{2sc\mu}{\sigma^2}\right)\left(1+\exp\left(\frac{2c\mu}{\sigma^2}\right)\right)\right\}.\end{aligned} \tag{4.7}$$

The first term in (4.7) above is

$$\int_0^T s\frac{d}{ds}P\{\sup_{0\leqq t\leqq s}X^*(t) \geqq c\}\,ds + \int_0^T s\frac{d}{ds}P\{\inf_{0\leqq t\leqq s}X^*(t) \leqq -c\}\,ds. \tag{4.8}$$

The first term on the right in (4.8) above is given by (4.6) and the second term by substituting $-\mu$ into (4.6). The series (4.7) is an alternating series and a bound for the error involved in using only the first term (4.8) is given by the second term, which is small for values of μ, σ and c in our case. Thus

$$(4.9)\qquad Et^* \approx \Phi\left(\frac{\mu T - c}{\sigma T^{\frac{1}{2}}}\right)\left(\frac{c}{\mu} - T\right)\left(1 + \exp\left(\frac{-2c\mu}{\sigma^2}\right)\right)$$
$$- \Phi\left(\frac{-\mu T - c}{\sigma T^{\frac{1}{2}}}\right)\left(\frac{c}{\mu} + T\right)\left(1 + \exp\left(\frac{2c\mu}{\sigma^2}\right)\right) + 2T\,.$$

Now $Et^* \approx E(n^*/N)$ and letting $\mu = (3N)^{\frac{1}{2}}\xi$, $\sigma^2 = 3\tau^2$, and $T = 1$ in (4.9) gives

$$En^* \approx \Phi\left(\frac{N\xi - \alpha}{\tau N^{\frac{1}{2}}}\right)\left(\frac{\alpha}{\xi} - N\right)\left(1 + \exp\left(\frac{-2\alpha\xi}{2}\right)\right)$$
$$- \left(\frac{-N\xi - \alpha}{\tau N^{\frac{1}{2}}}\right)\left(1 + \exp\left(\frac{2\alpha\xi}{\tau^2}\right)\right) + 2N\,.$$

4.5. *Comparison of the* SSR *and Wilcoxon statistics for truncated linear barrier tests.* Miller's truncated sequential rank test for symmetry about zero using the Wilcoxon signed-rank statistic was described in the Introduction. If R_{in}^+ is the rank of $|X_i|$ in $(|X_1|, |X_2|, \cdots, |X_n|)$ and $U_i^n = \operatorname{sgn}(X_i) \cdot R_{in}^+$, then the Wilcoxon signed-rank statistic can be expressed as

$$\mathrm{SR}_n = \textstyle\sum_{i=1}^n \operatorname{sgn}(X_i) R_{in}^+ = \sum_{i=1}^n U_i^n\,.$$

Note that U_i^n, unlike $Y_i = \operatorname{sgn}(X_i)R_{ii}^+$, depends on all of the observations $X_1, X_2, \cdots, X_n$ so that the rank of X_i changes after each new observation is taken. Although SR_n can be written in such a way as to make the computation fairly simple, the U_i^n are not independent even under the null hypothesis so that SR_n is not the sum of independent signed ranks.

Miller and Sen (1972) have shown that a continuous version of

$$\frac{\mathrm{SR}_{[Nt]} - E(\mathrm{SR}_{[Nt]})}{\operatorname{Var}(\mathrm{SR}_{[Nt]})^{\frac{1}{2}}}\, t^{\frac{1}{2}}\,, \qquad 0 \leqq t \leqq 1\,,$$

converges in distribution to normalized Brownian motion as $N \to \infty$. If $(3/N)^{\frac{1}{2}}\mathrm{SR}_n/n$ is used as the test statistic then Miller's test is, under the null hypothesis, approximately equivalent to normalized Brownian motion crossing a linear barrier.

Under the alternative hypothesis $E(\mathrm{SR}_n) = n^2\xi + o(n^2)$ and $\operatorname{Var}(\mathrm{SR}_n) = 4n^3\zeta^2 + o(n^3)$ where $\zeta^2 = \int_{-\infty}^{\infty} F(-x)^2\,dF(x) - F^{(2)}(0)^2$. Then

$$E\left[\left(\frac{3}{N}\right)^{\frac{1}{2}} \frac{\mathrm{SR}_n}{n}\right] \approx (3N)^{\frac{1}{2}}\xi\,\frac{n}{N}$$

and

$$\operatorname{Var}\left(\left(\frac{3}{N}\right)^{\frac{1}{2}} \frac{\mathrm{SR}_n}{n}\right) \approx 12\zeta^2\,\frac{n}{N}\,.$$

Thus the test statistic $(3/N)^{\frac{1}{2}}\mathrm{SR}_n/n$ can be approximated by $X^*(n/N)$ where $X^*(t)$ is a Brownian motion process with mean $(3N)^{\frac{1}{2}}\xi t$ and variance $12\zeta^2 t$. Miller has shown that the approximations are very good for $N = 20$ and $N = 50$

where the alternative hypothesis is that the observations are from the double exponential distribution with density

$$f(x) = \tfrac{1}{2} \exp[-|x - \Delta|] , \qquad -\infty < x < \infty .$$

The mean of this distribution is Δ. If the Brownian motion approximation is a reasonable approximation to the SSR statistic, then it should be possible to compare Miller's test using the Wilcoxon statistic with the test using the SSR statistic.

In order to check on the accuracy of the Brownian motion approximation to the power of the SSR test, several simulations were run for the null distribution and for alternatives in the double exponential distribution with mean Δ. Various values of c and Δ were selected for $N = 10, 20, 50$. A summary of the results is given in Table 1. For $N = 50$ the same 2000 sequences were used to calculate the probability of rejection under the null hypothesis for the three given values of c. Under the null hypothesis ($\Delta = 0$) the Z's are independent and

$$P\left\{Z_n = \frac{i}{n+1}\right\} = \frac{1}{2n} , \qquad i = \pm 1, \pm 2, \cdots, \pm n .$$

In this case the Z's can be generated directly from uniform random variables without having to do any ranking. Under most alternative hypotheses the Z's will not be independent, so random variables from the specified alternative have to be generated and the Z's obtained from the observations by ranking.

Comparing, in Table 1, the simulation power with the Brownian motion power shows that Brownian motion approximation over estimates the probability of rejection for all values of N, Δ and c considered. Although the power of the tests is lower than the Brownian motion approximation predicted, the tests are not necessarily worse than expected because the probability of accepting the null hypothesis when it is true is greater than the value assumed by the approximation.

It appears that in order to get the desired significance level α, the c-value obtained from $4\Phi(-c) = \alpha$ should be reduced slightly so that the significance level is actually α and the power is increased. Since it is difficult to adjust the c-value in the simulation results the c-value in the approximation formulas was increased to a value, say c', such that the significance level predicted by the formulas was equal to the value actually obtained in the simulation. When this was done the Brownian motion approximation to the power was very close to the power estimated by the simulation. The results for the adjusted Brownian motion approximation are given in the last two columns of Table 1.

For purposes of comparison, the power of one-sided tests using the SSR and Wilcoxon statistics was computed for shift alternatives for obervations from the double exponential distribution centered at Δ. The choice of the double exponential distribution permits the easy calculation of $\operatorname{Var}(Z_n^+)$ and $\operatorname{Var}(SR_n)$ which require that $\int_{-\infty}^{\infty} F(-x)^2\, dF(x)$ be evaluated. The power of the two-sided

TABLE 1
Power of two-sided SSR *test for* $N = 10, 20, 50$

N	C	Δ	Number of Sequences	Number of Rejections	Estimated Power	Brownian Motion Power	Adjusted Brownian Motion Power	$\frac{c'}{c}$
10	2.241	0	1000	26	.026	.050	.026	1.11
20	1.960	0	2000	122	.061	.100	.061	1.104
		.5	500	205	.410	.494	.400	1.104
		1.0	500	475	.850	.908	.852	1.104
		1.5	500	490	.980	.993	.983	1.104
50	1.960	0	2000	160	.080	.100	.08	1.048
	2.241	0	2000	72	.036	.050	.036	1.055
		.75	500	483	.966	.974	.964	1.055
	2.807	0	2000	17	.0085	.01	.0085	1.019

TABLE 2
Approximate power and expected sample size for 5 % *one-sided tests with* $N = 20$ *for* SSR *and Wilcoxon statistics*

$N = 20$	SSR Statistic		Wilcoxon Statistic	
	Power	$E[n]$	Power	$E[n]$
$\Delta = 0$	.0500	19.74	.0500	19.74
.5	.4940	17.18	.4888	17.27
1.0	.9078	13.58	.9273	13.68
1.5	.9932	11.55	.9995	11.56

TABLE 3
Approximate power and expected sample size for 5 % *one-sided tests with* $N = 50$ *for* SSR *and Wilcoxon statistics*

$N = 50$	SSR Statistic		Wilcoxon Statistic	
	Power	$E[n]$	Power	$E[n]$
$\Delta = 0$	.0500	49.35	.0500	49.35
.25	.4057	44.15	.4037	44.22
.50	.8564	34.06	.8604	34.19
.75	.9894	26.20	.9925	26.22
1.00	.99977	21.94	.99995	21.94

test for all but near alternatives is almost exactly the same as the power of a one-sided test with the same c-value.

Table 2 gives the power and expected sample size from the Brownian motion approximation of a 5 % one-sided test using both the SSR and Wilcoxon statistics for $N = 20$ and $\Delta = 0.5, 1.0, 1.5$. Table 3 gives the power and expected sample size for $N = 50$ and $\Delta = .25, .50, .75, 1.00$.

Examination of Tables 2 and 3 reveals that the tests using the SSR and Wilcoxon statistics are almost equivalent for the range of N and Δ considered. The only area where the two tests differ significantly is for large Δ where the power is very high for both tests.

It appears that for extreme alternatives, the power of the test using the Wilcoxon statistic converges to one faster than the power of the SSR test as $N \to \infty$. Using only first order terms, the means of the Brownian motion approximations are $(3N)^{\frac{1}{2}}\xi t$ for both the SSR and Wilcoxon test statistics, but the variances are $3\tau^2 t$ for the SSR statistic and $12\zeta^2 t$ for the Wilcoxon where $\zeta^2 = E[F(-X)^2] - E[F(-X)]^2$. For location shift alternatives $\xi > 0$, so $(3N)^{\frac{1}{2}}\xi t \to \infty$ as $N \to \infty$. If x is large then $\Phi(x) \approx 1 - (1/x)\phi(x)$ so that for N large enough that $\mu \gg c$, the power of the one-sided test is

$$P\{\text{reject } H_0\} \approx 1 - \frac{2\sigma c}{\mu^2 - c^2}\phi\left(\frac{\mu - c}{\sigma}\right)$$

where $\mu = (3N)^{\frac{1}{2}}\xi$ and σ^2 is the variance of the particular test statistic being used.

Since μ is the same for both tests the power is determined by the values of σ^2; the smaller the value of σ^2 the higher the power. Under H_0, $\sigma^2 = 1$ for both test statistics but as $F(0) \to 0$ (corresponding to far alternatives) the variance of the Wilcoxon statistic $\to 0$ while the variance of the SSR statistic $\to \frac{1}{4}$. Thus the Wilcoxon statistic should be more powerful for far alternatives and large N, but in this case the power of both tests is very high.

In conclusion we can say that for reasonable values of N and alternatives that are not too far from the null hypothesis the Wilcoxon and SSR statistics are essentially equivalent for the truncated linear barrier test. Even in the case of extreme shifts and large N the power of both tests is so high that any difference in power might very well be unimportant in practice. The SSR statistic should be slightly easier to use than the Wilcoxon statistic, but its main advantage over the Wilcoxon statistic is the independence of the Z's under the null hypothesis. For small values of N it should be possible to compute the exact null distribution using a computer, and even in cases where simulation is necessary, the independence makes the sequence of Z's easy to generate.

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