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A CLASS OF NON-PARAMETRIC TESTS FOR HOMOGENEITY AGAINST ORDERED ALTERNATIVES

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In this paper, the c-sample location problem with ordered or restricted alternatives is considered. Linear combinations of Chernoff–Savage type two-sample statistics computed among the $c(c-1)/2$ pairs of samples are proposed as test statistics. It is shown that for each linear combination of two-sample statistics there is another linear combination, using only the $c-1$ two-sample statistics based on adjacent samples as determined by the alternative, which has the same Pitman efficacy. If the ordered alternative is restricted further by specifying the relative spacings in the alternative, then the weighting coefficients can be chosen to maximize the Pitman efficacy over the class of linear combinations. It is also shown that the statistics proposed by Jonckheere [4] and Puri [9] have maximum Pitman efficacy when the alternative specifies equal spacings.

1. Introduction and summary. Let $X_{ik}, k = 1, \cdots n_i, i = 1, \cdots, c$ be random samples from populations with absolutely continuous distribution functions $F_i(x) = F(x - \theta_i), i = 1, \cdots c$. This paper is concerned with testing the null hypothesis $H_0: \theta_1 = \cdots = \theta_c$ against one of the following restricted alternatives: $H_{1A}: \theta_1 \leqq \cdots \leqq \theta_c$ with at least one strict inequality or $H_{1B}: \theta_1 \leqq \cdots \leqq \theta_c$ with at least one strict inequality and $\boldsymbol{\delta}' = (\delta_1, \cdots, \delta_{c-1})$ specified where $\delta_i = (\theta_{i+1} - \theta_i)/(\theta_c - \theta_1)$.

Let $N = \sum_{i=1}^{c} n_i$. For testing H_0 against H_{1A} Terpstra [10] and Jonckheere [4] proposed the statistic

(1) $$J_N = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} M_{ijN}$$

where M_{ijN} is the Mann–Whitney statistic [6] computed from the ith and jth samples for testing the alternative $\theta_j > \theta_i$. The Mann–Whitney statistic is one member of a broad class of two-sample statistics studied by Chernoff and Savage [1]. Puri [9] generalized Jonckheere's statistic by replacing M_{ijN} with any Chernoff–Savage statistic. Following the approach suggested by Hogg [3], Puri's family of statistics is generalized by including weighting coefficients to form arbitrary linear combinations

(2) $$T_N = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} a_{ij} T_{ijN} \qquad a_{ij} \geqq 0\,,$$

where T_{ijN} is any Chernoff–Savage statistic. Denote by Γ the class of statistics

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defined by (2). From the results of Puri [9] it follows that statistics in Γ, when properly standardized, are asymptotically normally distributed.

Let Γ^* be the subclass of Γ consisting of linear combinations of $(T_{12}, T_{23}, \cdots, T_{c-1c})$; then, assuming equal sample sizes, it is proved that for each T_N in Γ there corresponds an "equivalent" statistic T_N^* in Γ^*, where

$$T_N^* = \sum_{k=1}^{c-1} a_k T_{kk+1N} \qquad \text{and} \qquad a_k = \sum_{i=1}^{k} \sum_{j=k+1}^{c} a_{ij} \qquad k = 1, 2, \cdots, c-1 .$$

The "equivalence" refers to the fact that the difference of T_N and T_N^*, when standardized, converges in probability to zero under H_0 and that, for testing H_0 against H_{1A}, the Pitman efficiency of T_N^* with respect to T_N is one. In addition, T_N^* is a much simpler statistic, requiring the computation of $c - 1$ rather than $\binom{c}{2}$ two sample statistics.

If the alternative H_{1B} is considered, the additional information in $\boldsymbol{\delta}$, the vector of relative spacings, can be used to choose the weighting coefficients to maximize the Pitman efficacy of statistics in Γ. It is proved that, if the spacings are equal, Puri's family of statistics and their "equivalents" attain the maximum Pitman efficacy within the class Γ. The method of obtaining the optimum weighting coefficients is given.

The alternative H_{1B} provides a practical alternative to the assumption of equal spacing which is generally made by default in applying Jonckheere's statistic. This alternative requires no more justification than the assumption of equal spacings. Furthermore, the methods developed in this paper provide the means to study the robustness of the Pitman efficacy of the statistics to errors in the choice of relative spacings.

Haller [2] has considered a different class of statistics consisting of a linear combination of the c two-sample statistics formed by comparing each sample against the combined sample. The two-sample statistic used may be of the Chernoff–Savage type. He derives the weighting coefficients which maximize the Pitman efficiency for a specified relative spacing in the ordered alternative. Although the class of statistics considered by Haller and our class Γ are disjoint, Haller has proved that, for equal spacings, the efficiency of Puri's statistic with respect to the optimal statistic in his class is one. It follows that the statistics in Γ^* "equivalent" to Puri's statistics are as efficient in the Pitman sense as Haller's optimal statistic.

2. Equivalent linear combinations of Chernoff–Savage statistics. Let $N_{ij} = n_i + n_j$, $N = \sum_{i=1}^{c} n_i$ and let $F_{iN}(x)$ be the empirical distribution function of the sample from the ith population. Let $\gamma_{ij} = (n_j/n_i)^{\frac{1}{2}}$ and $\lambda_{ij} = n_j/(n_i + n_j)$.

It is assumed that γ_{ij} is a constant, independent of N and not equal to 0 for any pair i, j. That is, the relative proportions of sample sizes are held constant as N tends to infinity. Let $H_{ijN}(x) = (1 - \lambda_{ij})F_{iN}(x) + \lambda_{ij}F_{jN}(x)$ be the empirical distribution function of the combined ith and jth samples. Similarly, let

$H_{ij}(x) = (1 - \lambda_{ij})F_i(x) + \lambda_{ij}F_j(x)$ be the combined population distribution function for the ith and jth populations. Define

$$T_{ijN} = \int_{-\infty}^{\infty} J_{N_{ij}}[H_{ijN}(x)]\, dF_{jN}(x) \tag{3}$$

where $J_{N_{ij}}(h)$ is constant on the intervals $(k/N_{ij}, (k+1)/N_{ij}]$, $k = 0, 1, \cdots, N_{ij} - 1$ and depends on i and j only through n_i and n_j. This implies that all of the $\binom{c}{2}$ two-sample statistics are of the same type: Mann-Whitney, Normal Scores, etc.

Let $\boldsymbol{\tau}_N' = (\tau_{12N}, \tau_{13N}, \cdots, \tau_{1cN}, \tau_{23N}, \cdots, \tau_{2cN}, \cdots, \tau_{c-1cN})$ be the $\binom{c}{2}$ dimensional random vector with elements τ_{ijN}, $1 \leqq i < j < c$, where

$$\tau_{ijN} = n_j^{\frac{1}{2}}(1 - \lambda_{ij})^{-1}\{T_{ijN} - \int_{-\infty}^{\infty} J[H_{ij}(x)]\, dF_j(x)\} \tag{4}$$

and suppose $J(h) = \lim_{N\to\infty} J_{N_{ij}}(h)$ exists for $0 < h < 1$.

Suppose that for each pair i, j, $1 \leqq i \leqq j \leqq c$ the four conditions of the Chernoff-Savage [1] Theorem 1 hold. Puri [9] in the proof of his Theorem 4.1 proves that for any fixed $F_i(x)$, $i = 1, 2, \cdots, c$ the random vector $\boldsymbol{\tau}_N$ converges in law to a random vector having a multivariate normal distribution with null mean vector and covariance matrix H.

Under the null hypothesis $F_i(x) = F(x)$, $i = 1, 2, \cdots c$, so that the elements of H are

$$\begin{aligned}
&h_{ij,lk} = 0 && \text{all subscripts different}\\
&h_{ij,ij} = \sigma^2(1 + \gamma_{ij}^2) && i < j\\
&h_{il,jl} = \sigma^2 && i < l, \quad j < l, \quad i \neq j\\
&h_{li,lj} = \gamma_{li}\gamma_{lj}\sigma^2 && l < i, \quad l < j, \quad i \neq j\\
&h_{il,lj} = h_{lj,il} = -\gamma_{lj}\sigma^2 && i < l < j
\end{aligned} \tag{5}$$

where σ^2 depends on the particular Chernoff-Savage statistic in question.

In the remainder of the paper we will assume that the sample sizes are equal. An examination of the non-full rank covariance matrix H for equal sample sizes under H_0 shows that the linear combination $\tau_{ijN} - \sum_{k=1}^{j-1} \tau_{k\ k+1N}$ is asymptotically degenerate. This suggests using $\sum_{k=i}^{j-1} \tau_{k\ k+1N}$ as a replacement for τ_{ijN}. Thus, define for

$$L_N = \mathbf{A}'\boldsymbol{\tau}_N = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} a_{ij} T_{ijN} \qquad a_{ij} \geqq 0 \tag{6}$$

the random variable L_N^* where

$$L_N^* = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} a_{ij} \sum_{k=i}^{j-1} \tau_{k\ k+1N} \qquad a_{ij} \geqq 0 \tag{7}$$

and τ_{ijN} is given by (4).

THEOREM 2.1. *An equivalent expression for L_N^* given in* (7) *is*:

$$L_N^* = \sum_{k=1}^{c-1} a_k \tau_{k\ k+1N} \qquad \textit{where} \quad a_k = \sum_{i=1}^{k} \sum_{j=k+1}^{c} a_{ij}\,. \tag{8}$$

PROOF. The triple summation in (7) is over all triples (i, j, k) such that $1 \leqq i \leqq k$, $i < j \leqq c$ and $i \leqq k < j$ or equivalently $1 \leqq i \leqq k < j \leqq c$. From (8) the triple sum is over all triples (i, j, k) such that $1 \leqq k < j$, $1 \leqq i \leqq k$ and $k < j \leqq c$ or equivalently $1 \leqq i \leqq k < j \leqq c$. Thus the summations are identical.

THEOREM 2.2. *Suppose the sample sizes are equal.* L_N *and* L_N^* *converge in law to* L *and* L^* *respectively having univariate normal distributions with zero means. Furthermore, under* H_0, $L_N - L_N^*$ *converges in probability to zero and hence* $\text{Var}(L) = \text{Var}(L^*)$.

PROOF. Let $\mathbf{A}^{*\prime} = (a_{12}^*, a_{13}^*, \cdots)$ be the $\binom{c}{2}$ dimensional vector with elements $a_{ij}^* = 0$ if $j \neq i + 1$ and $a_{k,k+1}^* = a_k$ defined by (8). From Puri's theorem and the continuity theorem it follows that $L_N = \mathbf{A}'\boldsymbol{\tau}_N$ and $L_N^* = \mathbf{A}^{*\prime}\boldsymbol{\tau}_N$ converge in law to L and L^* having univariate normal distributions with zero means and variances $\mathbf{A}'H\mathbf{A}$ and $\mathbf{A}^{*\prime}H\mathbf{A}^*$, respectively. Similarly, $L_N - L_N^* = (\mathbf{A} - \mathbf{A}^*)'\boldsymbol{\tau}_N$ is asymptotically normally distributed with zero mean and variance $(\mathbf{A} - \mathbf{A}^*)'H(\mathbf{A} - A^*)$. Note that $L_N - L_N^* = \mathbf{A}'\mathbf{S}$ where $S_{ij} = \tau_{ijN} - \sum_{k=i}^{j-1} \tau_{k\,k+1N}$ $1 \leqq i < j \leqq c$. Similarly, $\mathbf{S} = B\boldsymbol{\tau}_N$ where B is a $\binom{c}{2}$ by $\binom{c}{2}$ matrix of zeros and plus or minus ones. Thus, $(\mathbf{A} - \mathbf{A}^*)'H = \mathbf{A}'BH$. From the structure of H under H_0 it can be shown that BH is the null matrix. Thus, under H_0, for equal sample sizes $L_N - L_N^*$ converges in law to a random variable degenerate at zero.

Using Theorem 2.1, the weighting coefficients for the "equivalent" form of Jonckheere's statistic J_N defined by (1) are $a_k = k(c - k)$, $k = 1, \cdots, c - 1$ so that $J_N^* = \sum_{k=1}^{c-1} k(c - k)M_{k\,k+1N}$.

Another statistic is defined by $\bar{R}_N = \sum_{i=1}^{c} i\bar{R}_{iN}$. This is Spearman's rho statistic [5], used as a test of correlation between the index i and $\bar{R}_{iN}$, the average of the ranks of the items in the ith sample. Now

$$\bar{R}_N = \frac{1}{n} \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} (j - i)M_{ijN} + \frac{c(n + 1)(c + 1)}{4}$$

and hence

$$\bar{R}_N' = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} (j - i)M_{ijN}$$

may be considered. For this statistic $a_k = \sum_{i=1}^{k} \sum_{j=k+1}^{c} (j - i) = (c/2)k(c - k)$; thus, the "equivalent" form for $\bar{R}_N'$ is $(c/2)J_N^*$. This shows that the correspondence between a statistic and its "equivalent" is not one to one.

Note that so far the location parameters have been surpressed; the distribution theory is valid for any set of absolutely continuous distribution functions $F_i(x)$, $i = 1, 2, \cdots, c$, satisfying the conditions of Puri's theorem. The random vector $\boldsymbol{\tau}_N$ is not a statistic since it depends on the unknown distribution functions. In the following, the location parameter family is considered and the dependence of the random variables, their moments and the covariance matrix H on $\boldsymbol{\theta}$, the vector of locations, is clearly specified.

Only equal sample sizes have been considered because in general the elements of H and thus the weighting coefficients needed for an "equivalent" statistic depend on the $\binom{c}{2}$ ratios of sample sizes γ_{ij}.

3. Efficiency properties. Let $\Omega = \{\boldsymbol{\theta} = (\theta_1, \cdots, \theta_c) : \theta_1 \leqq \cdots \leqq \theta_c\}$ be the parameter space and let $\omega = \{\boldsymbol{\theta} : \theta_1 = \cdots = \theta_c\}$; then the hypotheses to be considered are $H_0 : \boldsymbol{\theta} \in \omega$ against $H_{1A} : \boldsymbol{\theta} \in \Omega - \omega$.

Fix $\boldsymbol{\theta} \in \Omega - \omega$ and $\boldsymbol{\theta}_0 \in \omega$ and define $\boldsymbol{\theta}_N = \boldsymbol{\theta}_0 + N^{-\frac{1}{2}}\boldsymbol{\theta}$. Note that when $\boldsymbol{\theta}$ is chosen in $\Omega - \omega$, the relative spacings, $\boldsymbol{\delta}$, are fixed and remain constant as N increases.

Since $F_i(x) = F(x - \theta_i)$, $i = 1, \cdots, c$ and assuming equal sample sizes, from (6) we can write $L_N = \mathbf{A}'\boldsymbol{\tau}_N = 2(N/c)^{\frac{1}{2}}(T_N - \mu(\boldsymbol{\theta}))$ where $\mu(\boldsymbol{\theta}) = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} a_{ij}\mu_{ij}(\boldsymbol{\theta})$ and

$$\mu_{ij}(\boldsymbol{\theta}) = \int_{-\infty}^{\infty} J\left(\frac{F(x) + F(x + \theta_j - \theta_i)}{2}\right) dF(x) . \tag{9}$$

Now, L_N is asymptotically normally distributed with mean 0 and variance $\eta^2(\boldsymbol{\theta}) = \mathbf{A}'H(\boldsymbol{\theta})\mathbf{A}$. The Pitman efficacy for such statistics is discussed by Puri [8]. The reader is referred to his paper for definitions.

THEOREM 3.1. *If the statistic T_N in Γ and its "equivalent" T_N^* in Γ^* satisfy the conditions for Pitman efficiency, then their relative Pitman efficiency is* 1.

PROOF. Let

$$\eta^*(\boldsymbol{\theta}) = \mathbf{A}^{*\prime}H(\boldsymbol{\theta})\mathbf{A}^* ,$$
$$b_l^* = (c^{\frac{1}{2}}\eta^*(\boldsymbol{\theta}_0))^{-1} \int_{-\infty}^{\infty} J'(F(z))F'(z)\, dF(z)[a_{l-1} - a_l] \quad \text{and}$$
$$b_l = (c^{\frac{1}{2}}\eta(\boldsymbol{\theta}_0))^{-1} \int_{-\infty}^{\infty} J'(F(z))F'(z)\, dF(z)[\textstyle\sum_{i=1}^{l-1} a_{il} = \sum_{j=l+1}^{c} a_{lj}]$$

where, for convenience, $a_0 = a_c = 0$ and $a_{11} = a_{cc+1} = 0$. The Pitman efficiency of L_N^* with respect to L_N is $e_p(L_N^*, L_N) = [\sum_{l=1}^{c} b_l^*\theta_l / \sum_{l=1}^{c} b_l\theta_l]^2$. However

$$\textstyle\sum_{l=1}^{c} \theta_l(a_{l-1} - a_l) = \sum_{l=1}^{c-1} a_l(\theta_{l+1} - \theta_l) \quad \text{and}$$
$$\textstyle\sum_{l=1}^{c} \theta_l[\sum_{i=1}^{l-1} a_{il} - \sum_{j=l+1}^{c} a_{lj}] = \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} a_{ij}(\theta_j - \theta_i) .$$

From Theorem 2.2., since $\eta(\boldsymbol{\theta}_0) = \eta^*(\boldsymbol{\theta}_0)$,

$$e_p(L_N^*, L_N) = [\textstyle\sum_{l=1}^{c-1} a_l(\theta_{l+1} - \theta_l) / \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} a_{ij}(\theta_j - \theta_i)]^2 .$$

Recall that $a_l = \sum_{i=1}^{l} \sum_{j=l+1}^{c} a_{ij}$ and observe that $(\theta_j - \theta_i) = \sum_{l=i}^{j-1} (\theta_{l+1} - \theta_l)$. Making these substitutions yields

$$e_p(L_N^*, L_N) = \textstyle\sum_{l=1}^{c-1} \sum_{i=1}^{l} \sum_{j=l+1}^{c} a_{ij}(\theta_{l+1} - \theta_l) / \sum_{i=1}^{c-1} \sum_{j=i+1}^{c} \sum_{l=i}^{j-1} a_{ij}(\theta_{l+1} - \theta_l)^2 .$$

In Theorem 2.1 it was proved that these summations are equal. Thus $e_p(L_N^*, L_N) = 1$ independent of $\boldsymbol{\theta}$.

We now consider the alternative hypothesis H_{1B}. Recall that H_{1B} specifies $\boldsymbol{\theta} \in \Omega$ and $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_{c-1})$, where $\delta_j = (\theta_{j+1} - \theta_j)/(\theta_c - \theta_1)$, is assumed to be known. It has been shown that for any linear combination of Chernoff–Savage statistics L_N, in Γ there is an "equivalent" linear combination, L_N^*, in Γ^*. The statistic L_N^* has Pitman efficiency one with respect to L_N. For this reason, only the class of statistics Γ^* need be considered in deriving the weighting coefficients a_k, $k = 1, 2, \cdots, c - 1$, which give maximum Pitman efficacy for testing alternatives H_{1B}.

The Pitman efficiency of one statistic with respect to another is the ratio of two efficacies. The Pitman efficacy of L_N^* is $e(L_N^*) = (\sum_{l=1}^{c} \theta_l b_l^*)^2$ where b_l^* is given in Theorem 3.1. The efficacy of L_N^* will be considered as a function of the vector $\mathbf{A}' = (a_1, a_2, \cdots, a_{c-1})$ and maximized by selecting the appropriate vector $\hat{\mathbf{A}}$. From the proof of Theorem 3.1 it follows that

$$\begin{aligned} e(L_N^*) &= \frac{1}{c\mathbf{A}'H^*\mathbf{A}} (\textstyle\int_{-\infty}^{\infty} J'(F(z))F'(z)\,dF(z))^2[\sum_{l=1}^{c-1} a_l(\theta_{l+1} - \theta_l)]^2 \\ &= \frac{(\theta_c - \theta_1)^2}{c\mathbf{A}'H^*\mathbf{A}} (\textstyle\int_{-\infty}^{\infty} J'(F(z))F'(z)\,dF(z))^2 \mathbf{A}'\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{A} \\ &= R\,\frac{\mathbf{A}'\boldsymbol{\delta}\boldsymbol{\delta}'\mathbf{A}}{\mathbf{A}'H^*\mathbf{A}} \end{aligned}$$

where R is a constant independent of $\mathbf{A}$, H^* is the full rank asymptotic covariance matrix of $(\tau'_{12N}, \cdots, \tau'_{c-1cN})$ and $\boldsymbol{\delta} = (\delta_1, \cdots, \delta_{c-1})$ the known vector of relative spacings specified by H_{1B}.

THEOREM 3.2. *A vector* $\hat{\mathbf{A}}$ *which maximizes* $e(L_N^*)$ *for a given vector* $\boldsymbol{\delta}$ *is* $\hat{\mathbf{A}} = H^{*-1}\boldsymbol{\delta}$.

PROOF. This result is a standard result in matrix theory.

THEOREM 3.3. *When* H_0 *is true and when the sample sizes are equal* αH^{*-1} *is given by*

$$\begin{aligned} h_{ij}^{-1} &= 2i(c-j)/c \qquad i \leqq j \\ &= 2j(c-i)/c \qquad i \geqq j \qquad \text{where } \alpha \text{ is a constant.} \end{aligned}$$

For $c = 3$ the solution for an arbitrary vector $\boldsymbol{\delta}$ can be easily obtained. Let $\boldsymbol{\delta}' = (\gamma, 1-\gamma)$, $0 < \gamma < 1$. Note that $\sum_{i=1}^{c-1} \delta_i = 1$. However, it is convenient to normalize $\boldsymbol{\delta}$ to place a 1 in the first position to obtain $\boldsymbol{\delta}' = (1, \alpha)$, $0 < \alpha < \infty$. Thus, $\hat{\mathbf{A}}' = ((4 + 2a)/3, (2 + 4a)/3)$. Note that for equally spaced alternatives $a = 1$ and $\hat{\mathbf{A}}' = (2, 2)$. This is the test equivalent to Puri's or Jonckheere's statistics where each of the 3 two-sample statistics carried equal weight.

In the special case of equal spacings, $\boldsymbol{\delta}' = (1, 1, \cdots, 1)$, the vector $\hat{\mathbf{A}}$ can be determined.

THEOREM 3.4. *When* $\boldsymbol{\delta}' = (1, 1, \cdots, 1)$, *the vector* $\hat{\mathbf{A}}$ *has elements* $a_k = k(c-k)$, $k = 1, 2, \cdots, c-1$.

COROLLARY. *In the class* Γ *of statistics, linear combinations with equal weightings proposed by Jonckheere* [3] *and more generally by Puri* [8] *for testing* H_0 *against* H_{1A} *with equal spacings have maximum Pitman efficacy for the Chernoff–Savage statistic used.*

Note that the statistic with maximum Pitman efficacy in Γ^* is unique up to a multiplicative constant but is not unique in Γ. For example, if linear combinations of Mann–Whitney statistics are considered then J_N, J_N^*, and $\bar{R}_N'$ defined at the end of Section 3 have the maximum Pitman efficacy in Γ for equal spacings. Note also that the class Γ is defined for an arbitrary, but fixed, type of parent Chernoff–Savage statistic; however, the optimum weighting coefficients are independent of the two sample statistics used.

Let $\boldsymbol{\delta}' = (0, 1, 0)$. That is, it is hypothesized that for $c = 4$ the first two locations are equal and the last two are both equal but greater than the first pair. The optimum weighting coefficients are $\mathbf{A} = (1, 2, 1)$ which specifies weighting the statistic τ_{23N} twice as heavily as τ_{12N} and τ_{34N} which are weighted equally. In this case an intuitive approach is interesting. The hypothesis suggests pooling the samples from the first two and last two pairs of samples and computing a single two-sample statistic. If this is done for the Mann–Whitney statistic, the result is $M_{1+2,3+4} = M_{13} + M_{23} + M_{14} + M_{24}$ which, in the latter form, is in Γ and has the "equivalent" form with weights (2, 4, 2) which, when normalized, corresponds to the previous result. The Pitman efficiency of the optimum statistic in Γ^* with respect to Puri's statistic is 1.25.

As an example of optimum weights and the resulting increased efficiency Table 1 shows the weights for 5 different alternative spacings and the Pitman efficiencies relative to Puri's statistic (optimum for equal spacings) for values of $c = 4, 5, 6, 7$. The alternative spacings considered are: (A) equal spacings,

TABLE 1

Optimal weighting coefficients

(the number in parenthesis is the Pitman efficiency of the optimal test to the test of Puri)

	A	B	C	D	E
$c = 4$	1.00	1.00	1.00	1.00	1.00
	1.33	1.50	1.38	1.60	1.64
	1.00	1.00	1.06	1.40	1.55
	(1.000)	(1.020)	(1.002)	(1.050)	(1.087)
$c = 5$	1.00	1.00	1.00	1.00	1.00
	1.50	1.58	1.55	1.75	1.81
	1.50	1.75	1.59	2.00	2.23
	1.00	1.08	1.09	1.50	1.88
	(1.000)	(1.018)	(1.003)	(1.056)	(1.148)
$c = 6$	1.00	1.00	1.00	1.00	1.00
	1.60	1.67	1.65	1.83	1.89
	1.80	2.00	1.91	2.31	2.58
	1.60	1.67	1.74	2.29	2.84
	1.00	1.00	1.12	1.57	2.26
	(1.000)	(1.012)	(1.004)	(1.059)	(1.219)
$c = 7$	1.00	1.00	1.00	1.00	1.00
	1.67	1.71	1.71	1.88	1.94
	2.00	2.13	2.11	2.50	2.77
	2.00	2.25	2.17	2.75	3.36
	1.67	1.79	1.86	2.50	3.48
	1.00	1.04	1.14	1.63	2.68
	(1.000)	(1.010)	(1.005)	(1.061)	(1.298)

$\delta_i = 1$ $i = 1, 2, \cdots, c-1$; (B) $\delta_{c/2} = 2$ if c is even or $\delta_{(c+1)/2} = 2$ if c is odd and $\delta_i = 1$ otherwise; (C) $\delta_i = 1 + .1(i-1)$ $i = 1, 2, \cdots, c-1$; (D) $\delta_i = 1 + (i-1)$ $i = 1, 2, \cdots, c-1$; (E) $\delta_1 = 1$, $\delta_i = 2\delta_{i-1}$ $i = 2, 3, \cdots, c-1$. The spacings (B) represent the occurrence of a missing sample interior to an otherwise equal spacing situation. The alternatives (C), (D) and (E) represent increasingly more severe examples of increasing the relative spacings.

For ease in comparison the coefficients have been normalized so that the first weight is unity. It is noted that, with the exception of the case mentioned earlier, the optimal weights defy intuition.

It is further noted that Puri's statistic seems quite robust against the violation of equal spacings. Only for alternative (E), where the spacings are doubling, is the optimal statistic a significant improvement.

In conclusion, the use of statistics in Γ^* equivalent to Puri's family are suggested for simplicity of calculation. Further adjustment of weighting coefficients need only be considered if the assumption of equal spacings is grossly inadequate.

4. The exact and approximate null distribution of J^*. Under the null hypothesis, the means, variances and covariances of the Mann-Whitney statistics M_{ij}, $i = 1, 2, \cdots, c-1$, $j = i+1, \cdots, c$ computed from the c samples are $EM_{ij} = n_i n_j/2$, $\operatorname{Var}(M_{ij}) = n_i n_j(n_i + n_j + 1)/12$, $\operatorname{Cov}(M_{ij}, M_{kl}) = 0$ if all i, j, k, l are different and if i, j, k are all different

$$\operatorname{Cov}(M_{ij}, M_{ik}) = \operatorname{Cov}(M_{ji}, M_{ki}) = n_i n_j n_k/12$$
$$\operatorname{Cov}(M_{ij}, M_{ki}) = \operatorname{Cov}(M_{ji}, M_{ik}) = -n_i n_j n_k/12 .$$

Tryon [11] has given an elementary derivation of the covariances. It is now easy to calculate the mean and variance of J^*. These are listed in Table 2 for $c = 3, 4, 5, 6$ assuming equal sample sizes along with the necessary weights a_k, to construct J^*. For equal sample sizes, under the null hypothesis $EJ^* = n^2(c^3 - c)/12$ and $\operatorname{Var} J^* = n^2(c^3 - c)(10n + c^2 + 1)/360$.

Hence using the standard normal table and a continuity correction the observed significance level for J^* can be approximated. The exact distribution of J^* under H_0 has been calculated by R. E. Odeh (in a personal communication) for various combinations of c and n. Exact probabilities nearest .05 and .01 are given in Table 3 along with the normal approximation. The approximation is useful for very small values of c and n. From Table 3 it appears that for J^*, under most circumstances, the asymptotic distribution is adequate for constructing tests.

TABLE 2

c	Weights for J^*	$E(J^*)$	$V(J^*)$
3	2, 2	$2n^2$	$n^2(8n + 8)/12$
4	3, 4, 3	$5n^2$	$n^2(20n + 34)/12$
5	4, 6, 6, 4	[illegible]	[illegible]

TABLE 3
Exact values for $P(J^* \geqq k)$ under H_0
[the number in brackets is the normal approximation]

	c			
n	k			
	3	4	5	6
3	28(.0345)	59(.0525)	112(.0516)	189(.0488)
	[.0329]	[.0537]	[.0485]	[.0495]
	32(.0048)	66(.0094)	122(.0097)	203(.0094)
	[.0041]	[.0073]	[.0075]	[.0082]
4	46(.0384)	101(.0471)	192(.0489)	324(.0518)
	[.0375]	[.0485]	[.0465]	[.0526]
	50(0.115)	110(.0098)	206(.0094)	344(.0101)
	[.0096]	[.0084]	[.0075]	[.0089]
5	68(.0449)	153(.0492)	292(.0510)	—
	[.0446]	[.0495]	[.0495]	—
	76(.0076)	165(.0104)	310(.0107)	—
	[.0062]	[.0091]	[.0091]	—
6	94(.0524)	216(.0488)	414(.0491)	—
	[.0526]	[.0495]	[.0485]	—
	104(.0098)	232(.0095)	438(.0093)	—
	[.0084]	[.0082]	.[0079]	—

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