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ASYMPTOTIC EQUIVALENCE OF TWO ESTIMATORS FOR AN EXPONENTIAL FAMILY[1]

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It is shown that under certain conditions, the maximum likelihood and the minimum variance unbiased estimators of a positive integral power of the natural parameter in an exponential family have the same asymptotic distribution.

1. Introduction. Let the random variable Z have the probability density $\beta(p)\exp(-pz)r(z)$ with respect to Lebesgue measure, where $r(z)$ is a probability density on (a, b). Here a and b are known and may be infinite. We take the parameter space to be the largest open interval $(-p_2, p_1)$ included in the natural parameter space of the density of Z. Whenever a and b are finite, $p_1 = p_2 = \infty$.

Let $Z_1, \cdots, Z_n$ be independent and identically distributed as Z, then $X_n = \sum_i^n Z_i$ is a complete sufficient statistic and has the probability density $\beta^n(p)\exp(-px_n)r_n(x_n)$, where r_n is the n-fold convolution of r. For the unbiased estimator of p^m, where m is a positive integer, we have the following theorem.

THEOREM 1.1. *Let*

(1.1) *$r(z)$ be positive on (a, b),*

(1.2) *the* mth *derivative $r_n^{(m)}$ of r_n exist in (na, nb).*

Then $r_n^{(m)}(z)/r_n(z)$ is defined for all z in (na, nb).

If we assume also that

(1.3) *$r_n^{(m)}$ is continuous in (na, nb), and*

(1.4) $\lim_{z\to na} r_n^{(j)}(z)\exp(-pz) - \lim_{z\to nb} r_n^{(j)}(z)\exp(-pz) = 0, j \leqq m-1$; *then*

$$E[r_n^{(m)}(X_n)/r_n(X_n)] = p^m .$$

PROOF. The first assertion can be proved by induction and the second by integration by parts.

It may be pointed out that a sufficient condition for (1.2) and (1.3) to hold for a sufficiently large n is that the characteristic function $M(iv)$ of $r(z)$ satisfies

$$(1.5) \qquad M(iv) = O(|v|^{-\delta}) \quad \text{as} \quad |v| \to \infty , \quad \text{for some positive} \quad \delta .$$

This is so because a sufficiently large n implies the integrability of $|v|^m|M^n(iv)|$ which in turn implies that $r_n(z)$ possesses continuous derivatives of orders less

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than or equal to m (see, for example, Wintner [4] page 117). It is clear that, when a and b are finite, (1.5) is sufficient for (1.2), (1.3) and (1.4), for sufficiently large n.

The condition (1.5), for example, is satisfied by normal and gamma densities and also by a polynomial probability density when a and b are finite.

The MLE of p^m can be seen to be $\hat{p}^m \equiv (\Delta^{-1}(\bar{Z}))^m$ when $\Delta[(-p_2, p_1)] = (a, b)$, where $\Delta(p) = E_p(Z)$. Also,

$$(\hat{p}^m - p^m)(-n\Delta'(p))^{\frac{1}{2}}(mp^{m-1})^{-1} \rightarrow_{\mathscr{L}} N(0, 1) \qquad \text{as } n \rightarrow \infty ,$$

where $\Delta'(p)$ is the derivative of $\Delta(p)$ with respect to p, and $\rightarrow_{\mathscr{L}}$ stands for convergence in law. We shall show that the maximum likelihood and the unbiased estimators of p^m have the same asymptotic distribution. This we do by proving a stronger result, namely, that for any positive ε, $n^{1-\varepsilon}$ times the difference in the estimators goes to zero with probability one as n tends to infinity. We first take the case of m equal to one. Section 2 outlines the proof appearing in Section 3. The extension of the result for a positive integer m and a non-trivial application of our result are also given in Section 3.

The convergence in probability of n times the difference in the estimators to a constant would be interesting to investigate.[2] However, we have not been able to prove it, though that the difference is $O_p(n^{-1})$ (following Pratt's [3] notation) is an immediate consequence of (3.14).

2. Outline of the proof. When $\bar{Z} = c$, the unbiased estimate $T_n(c)$ (say) of p is

$$r_n^{(1)}(nc)/r_n(nc) = n^{-1}s_n'(c)/s_n(c) ,$$

where $s_n(\bar{z})$ is the probability density of $\bar{Z}$ when Z has probability density $r(z)$ and s_n' is its derivative. To prove that, for any positive ε,

$$n^{1-\varepsilon}(\Delta^{-1}(\bar{Z}) - T_n(\bar{Z})) \rightarrow 0 \qquad \text{with probability } 1 \qquad \text{as } n \rightarrow \infty ,$$

it suffices to show that (see Section 3)

$$T_n(c) - \Delta^{-1}(c) = O(n^{-1}) \qquad \text{as } n \rightarrow \infty, \text{ uniformly in } c \in C , \tag{2.1}$$

where C is a compact set contained in (a, b). We get asymptotic expansions for $s_n(c)$ and $s_n'(c)$ uniform in $c \in C$ to prove (2.1). We shall outline how it is done for $s_n(c)$; the procedure for $s_n'(c)$ is similar.

Let

$$\begin{aligned} M(u + iv) &= \int \exp((u + iv)z)r(z)\, dz \\ &= \exp(K(u + iv)) , \end{aligned}$$

where u and v are real, $i = (-1)^{\frac{1}{2}}$, and $K(u + iv)$ is say the principal branch of $\log M(u + iv)$. If $|M^n(u + iv)|$ is integrable with respect to v, where u belongs to the interval $(-p_1, p_2)$, then

$$s_n(c) = (2\pi)^{-1}n \int_{-\infty}^{\infty} \exp(n[K(u + iv) - (u + iv)c]\, dv . \tag{2.2}$$

[2] Suggested to the author by Peter J. Bickel.

To get an asymptotic expansion of $s_n(c)$, we must choose a proper u in (2.2). The choice of u involves the saddle-point method. We take u equal to u_0, a saddle-point of $K(u+iv)-(u+iv)c$. Assumption B (Section 3) ensures the existence of such a u_0, $u_0 \in (-p_1, p_2)$. This is how Daniels [1] has expressed $s_n(c)$ as an integral with $u = u_0$ in (2.2). To get the asymptotic power series of $s_n(c)$, which is uniform in $c \in C$, we use Lemma 3.2. Our method is the same as that given in Gnedenko and Kolmogorov ([2], page 228). Gnedenko and Kolmogorov show that it suffices to consider a small neighborhood of $v = 0$ in the integral (2.2). More explicitly, using the fact that

(2.3) the characteristic function of an absolutely continuous distribution is less than a number less than one, in $|v| > v_1$, with v_1 positive,

they show that the contribution of the remaining path is exponentially small relative to that of the neighborhood. For our problem, where we want the asymptotic power series to be uniform in $c \in C$, we choose a neighborhood of $v = 0$ which is independent of $c \in C$. To apply the argument of Gnedenko and Kolmogorov, we need an analogue of (2.3) involving a family of distributions, in fact, we need uniform (in $c \in C$) convergence to zero of the characteristic function as $|v|$ goes to infinity. Assumption A (Section 3) ensures this convergence.

3. Proof of asymptotic equivalence and an application. We state and prove the result in the following theorem.

THEOREM 3.1. *Let the random variable Z have the probability density $\beta(p)\exp(-pz)r(z)$ with respect to Lebesgue measure, where $r(z)$ is positive on (a, b), with a and b possibly infinite. Let the parameter space of $\beta(p)\exp(-pz)r(z)$ be a non-degenerate interval $(-p_2, p_1)$, the largest open interval contained in the natural parameter space. Let*

$$\Delta(p) = \beta(p) \int z \exp(-pz)r(z)\, dz$$

and $M(u+iv) = \int \exp((u+iv)z)r(z)\, dz$, where u and v are real and $i = (-1)^{\frac{1}{2}}$. Also let the following assumptions hold.

ASSUMPTION A. *For any compact set U contained in $(-p_1, p_2)$, there exist $\delta > 0$, A and v_0 (δ, A, v_0 may depend on U), such that*

$$|v| > v_0 \quad \textit{implies} \quad \sup_{u \in U} |M(u+iv)| < A|v|^{-\delta}\,.$$

ASSUMPTION B.

$$\lim_{p \to -p_2} \Delta(p) = b\,, \qquad \lim_{p \to p_1} \Delta(p) = a\,.$$

Then, for any positive ε and any $p \in (-p_2, p_1)$,

$$n^{1-\varepsilon}(\hat{p} - \tilde{p}) \to 0 \quad \textit{with probability } 1 \qquad \textit{as } n \to \infty\,, \tag{3.1}$$

where $\hat{p}$ and $\tilde{p}$ are the MLE *and the minimum variance unbiased estimator of p respectively, based on a sample of size n.*

PROOF. Let c belong to (a, b) and C be any compact set in (a, b) containing c. Because of the continuity of Δ^{-1}, $\Delta^{-1}C$ is compact. With an obvious meaning for $-\Delta^{-1}C \equiv U$, U is also compact. From Assumption A, $|M^n(u + iv)|$ is integrable with respect to v for a sufficiently large n, and the choice of n is independent of $u \in U$. So,

$$s_n(c) = \int_{-\infty}^{\infty} (2\pi)^{-1} n \exp(n[K(u + iv) - (u + iv)c])\, dv \tag{3.2}$$

for any $u \in U$, where $K(u + iv)$ is the principal branch of $\log M(u + iv)$. From Assumption B, there exists a u_0 in U such that $K'(u_0) = \Delta(-u_0) = c$, where K' is the derivative of K. Thus when $\bar{Z} = c$, the negative of the saddle-point u_0 is the MLE of p.

In (3.2), we take $u = u_0$ to get

$$\begin{aligned} h^{-1}(s_n(c + h) - s_n(c)) \\ &= (2\pi)^{-1} n \int_{-\infty}^{\infty} \exp(n[K(u_0 + iv) - (u_0 + iv)c]) h^{-1} \\ &\quad \times [\exp(-n(u_0 + iv)h) - 1]\, dv\,. \end{aligned} \tag{3.3}$$

For a sufficiently large n, using Assumption A we apply the Lebesgue Dominated Convergence Theorem to get

$$s_n'(c) = (2\pi)^{-1} n \int_{-\infty}^{\infty} \exp(n[K(u_0 + iv) - (u_0 + iv)c])[-n(u_0 + iv)]\, dv\,.$$

so that the unbiased estimate is

$$-u_0 - i\,\frac{\int_{-\infty}^{\infty} \exp(n[K(u_0 + iv) - (u_0 + iv)c]) v\, dv}{\int_{-\infty}^{\infty} \exp(n[K(u_0 + iv) - (u_0 + iv)c])\, dv}\,. \tag{3.4}$$

We shall develop asymptotic power series for the integrals in (3.4), which in turn will give the asymptotic power series for $T_n(c)$.

Let $s_1^{(c)}(\zeta)$ be the density

$$\beta(-u_0) \exp(u_0(c + \zeta)) s_1(c + \zeta) \tag{3.5}$$

and $g_c(v)$ be its characteristic function. Denoting the second derivative of K by K'', we have the denominator of (3.4) equal to

$$\begin{aligned} \exp(n[K(u_0) - u_0 c])[nK''(u_0)]^{-\frac{1}{2}} \int_{-\infty}^{\infty} g_c^{\,n}(v[nK''(u_0)]^{-\frac{1}{2}})\, dv \\ = \exp(n[K(u_0) - u_0 c])[nK''(u_0)]^{-\frac{1}{2}} 2\pi s_n^{(c)}(0)\,, \end{aligned}$$

where $s_n^{(c)}$ is the normalized n-fold convolution of $s_1^{(c)}$, $|g_c^{\,n}(v)|$ being integrable because of Assumption A. We shall show that the asymptotic power series we obtain is uniform in $c \in C$, a compact set contained in (a, b). The argument is the same as that given in Gnedenko and Kolmogorov ([2], page 228). We first state a lemma (Gnedenko and Kolmogorov [2], page 204).

LEMMA 3.2. *Let ζ_i, $i = 1, 2, \cdots$, be* i.i.d. *with mean zero and variance σ^2, and $g(v)$ be the characteristic function of ζ_1. Also, let β_j be the* jth *absolute moment of ζ_1, γ_j be the* jth *cumulant of ζ_1, $\lambda_j = \gamma_j \sigma^{-j}$, $\rho_j = \beta_j \sigma^{-j}$, and Θ_j denote a quantity bounded by a constant dependent on j only. Suppose ζ_1 has a finite* kth *moment*

$(k \geqq 3)$, *then for* $|v| \leqq n^{\frac{1}{2}}(8k)^{-1}(\rho_k)^{-3/k} \equiv V_{kn}$, *we have*

$$|g^n([\sigma n^{\frac{1}{2}}]^{-1}) - \exp(-v^2/2)[1 + \textstyle\sum_{j=1}^{k-2} P_j(iv)n^{-j/2}]| \leqq \Theta_1(k)\Theta_2(n)V_{kn}^{-(k-2)}(|v|^k + |v|^{3(k-2)})\exp(-v^2/4)\,,$$

where $\Theta_1(k)$ *depends on* k *only*, $\Theta_2(n)$ *depends on* n *only and tends to zero as* n *tends to infinity*, $P_j(iv) = \sum_{m=1}^{j} c_{mj}(iv)^{j+2m}$ *is a polynomial in* (iv) $(i = (-1)^{\frac{1}{2}})$ *of degree* $3j$, *the coefficient* c_{mj} *is a polynomial in* $\lambda_3, \cdots, \lambda_{j-m+3}$ *with numerical coefficients and* $c_{mj} = \Theta_k \rho_j^{(j+2m)/k}$.

We shall indicate the dependence on c of the cumulants, moments, the polynomial $P_j(iv)$ and other characteristics of the density $s_1^{(c)}(\zeta)$ by using the notations $\gamma_j(c)$, $\beta_j(c)$, $P_j^{(c)}(iv)$, etc. It is clear that moment of any order of $s_1^{(c)}(\zeta)$ exists, also

$$\sup_{c\in C}\beta_j(c) \leqq \beta_j^0 \quad (\text{say}) < \infty\,,$$

and

$$\inf_{c\in C}\sigma(c) = \inf_{c\in C}[K''(u_0)]^{\frac{1}{2}} \equiv \sigma_0 > 0\,,$$

because $K''(u_0)$ is a continuous function of c. Thus,

$$\inf_{c\in C} V_{kn}(c) \geqq V_{kn}^0 \quad (\text{say}) > 0\,,$$
$$\sup_{c\in C}\lambda_j(c) \leqq \lambda_j^0 \quad (\text{say}) < \infty\,, \qquad \text{and}$$
$$\sup_{c\in C}\rho_j(c) \leqq \rho_j^0 \quad (\text{say}) < \infty\,.$$

Let

$$h_c(v) = \exp(-v^2/2)[1 + \textstyle\sum_{j=1}^{k-2} P_j^{(c)}(iv)n^{-j/2}] \qquad \text{and}$$
$$2\pi p_j^{(c)} = \int \exp(-v^2/2)P_j^{(c)}(iv)\,dv\,,$$

then

$$
\begin{aligned}
&|\textstyle\int_{-\infty}^{\infty} g_c^n(v[nK''(u_0)]^{-\frac{1}{2}})\,dv - \int_{-\infty}^{\infty} h_c(v)\,dv| \\
(3.6)\qquad &= 2\pi|s_n^{(c)}(0) - [(2\pi)^{-\frac{1}{2}} + \textstyle\sum_{j=1}^{k-2} p_j^{(c)}n^{-j/2}]| \\
&= 2\pi|s_n^{(c)}(0) - [(2\pi)^{-\frac{1}{2}} + \textstyle\sum_{j=1}^{[(k-2)/2]} p_{2j}^{(c)}n^{-j}]|\,.
\end{aligned}
$$

The last equality in (3.6) follows from that $P_j^{(c)}(iv)$ is a polynomial in (iv) involving only odd powers of v or only even powers of v according as j is odd or even, so that $p_j^{(c)}$ is zero for odd j; $[(k-2)/2]$ involved is the greatest integer contained in $(k-2)/2$. Now

$$
\begin{aligned}
(3.6) &\leqq \textstyle\int_{-V_{kn}^0}^{V_{kn}^0} |g_c^n(v[nK''(u_0)]^{-\frac{1}{2}}) - h_c(v)|\,dv + \int_{|v|>V_{kn}^0} |h_c(v)|\,dv \\
(3.7)\qquad &\quad + \textstyle\int_{|v|>V_{kn}^0} |g_c^n(v[nK''(u_0)]^{-\frac{1}{2}})|\,dv \\
&\equiv I_1(c) + I_2(c) + I_3(c)\,,
\end{aligned}
$$

where $I_1(c)$, $I_2(c)$, and $I_3(c)$, are the integrals involved on the right side of (3.7) in that order. From Lemma 3.2,

$$(3.8)\qquad \sup_{c\in C} I_1(c) = o(n^{-(k-2)/2}) \qquad \text{as } n \text{ tends to infinity.}$$

Next, since $\sup_{c\in C}|c_{mj}| < \infty$,

$$
\begin{aligned}
(3.9)\qquad \sup_{c\in C} I_2(c) &\leqq \sup_{c\in C} \textstyle\int_{|v|>V_{kn}^0} \exp(-v^2/2)[1 + \sum_{j=1}^{k-2}|P_j^{(c)}(iv)|n^{-j/2}]\,dv \\
&= o(n^{-m}) \qquad \text{for any positive } m, \text{ as } n \text{ tends to infinity.}
\end{aligned}
$$

Also,

$$I_3(c) \leqq \sigma_1 n^{\frac{1}{2}} \int_{|v| \geqq b_0/\sigma_1} \sup_{c \in C} |g_c^n(v)| \, dv \,, \tag{3.10}$$

where $b_0 n^{\frac{1}{2}} = V_{kn}^0$, and $\sigma_1 = \sup_{c \in C} [K''(u_0)]^{\frac{1}{2}}$.

From Assumption A, we can choose a v_1 greater than v_0 such that

$$\begin{aligned} \sup_{c \in C} \sup_{|v| > v_1} |g_c(v)| &= \sup_{|v| > v_1} \sup_{c \in C} |M(u_0 + iv)/M(u_0)| \\ &\leqq \exp(-d) \qquad \text{(say) with } \; d \; \text{ positive.} \end{aligned}$$

Also, by the continuity of $|M(u_0 + iv)/M(u_0)|$ in c and v,

$$\sup_{c \in C} \sup_{b_0/\sigma_1 \leqq |v| \leqq v_1} |M(u_0 + iv)/M(u_0)| \leqq \exp(-f) \quad \text{(say)}$$

with f positive. Thus

$$\sup_{c \in C} \sup_{|v| \geqq b_0/\sigma_1} |g_c(v)| \leqq \exp(-d) \qquad \text{(say) with } \; d \; \text{ positive,}$$

which implies that

$$\int_{|v| \geqq b_0/\sigma_1} \sup_{c \in C} |g_c^n(v)| \, dv \leqq \text{const. } n^{\frac{1}{2}} \exp(-(n - m_0)d) \,,$$

where m_0 is large enough to make $\int_{|v| > v_1} |v|^{-\delta m_0} \, dv$ finite.

Hence from (3.10), we get

$$\sup_{c \in C} I_3(c) = o(n^{-m}) \qquad \text{for any } \; m > 0 \,, \qquad \text{as } \; n \to \infty \,. \tag{3.11}$$

The relations (3.7)—(3.9) and (3.11) imply that

$$s_n^{(c)}(0) = (2\pi)^{-\frac{1}{2}} + \sum_{j=1}^{[(k-2)/2]} p_{2j}^{(c)} n^{-j} + o(n^{-(k-2)/2}) \qquad \text{as } \; n \to \infty \,, \tag{3.12}$$

the order being uniform in $c \in C$.

Proceeding for the integral in the numerator of (3.4) in the same way as above, we get

$$\begin{aligned} \int_{-\infty}^{\infty} & \exp(n[K(u_0 + iv) - (u_0 + iv)c]) v \, dv \\ &= 2\pi [K''(u_0)]^{-1} n^{-\frac{1}{2}} \exp(n[K(u_0) - u_0 c]) \\ &\quad \times \left[\sum_{j=1}^{[(k-1)/2]} q_{2j-1}^{(c)} n^{-j} + o(n^{-(k-1)/2})\right] \end{aligned} \tag{3.13}$$

as n tends to infinity, the order being uniform in $c \in C$, where $[(k-1)/2]$ above is the greatest integer contained in $(k-1)/2$, and

$$2\pi q_j^{(c)} = \int_{-\infty}^{\infty} v P_j^{(c)}(iv) \exp(-v^2/2) \, dv \,.$$

From (3.4), (3.12) and (3.13) it is clear that

$$\sup_{c \in C} |T_n(c) + u_0| = \sup_{c \in C} |T_n(c) - \Delta^{-1}(c)| = O(n^{-1}) \qquad \text{as } \; n \to \infty \,. \tag{3.14}$$

To prove (3.1), we proceed as follows:

Let $E_p(\bar{Z}) = \Delta(p) = c_0$ (say). Now $\bar{Z}$ tends to c_0 with probability one as n tends to infinity, that is,

$$\lim_{j \to \infty} P(|\bar{Z} - c_0| \leqq \eta \; \text{ for all } \; n \geqq j) = 1 \qquad \text{for all } \; \eta > 0 \,.$$

Given any $\eta_0 > 0$, choose η such that $0 < \eta < \eta_0$ and $[c_0 - \eta, c_0 + \eta]$ is contained in (a, b). Then for sufficiently large j and any positive ε,

$$P(|n^{1-\varepsilon}(T_n(\bar{Z}) - \Delta^{-1}(\bar{Z}))| \leqq \eta_0 \; \text{ for all } \; n \geqq j) \geqq P(|\bar{Z} - c_0| \leqq \eta \; \text{ for all } \; n \geqq j) \,.$$

The above inequality follows from the relationship (3.14) for $C = [c_0 - \eta, c_0 + \eta]$. Thus the theorem is proved.

In Theorem 3.1, we have considered the case $m = 1$ only, we indicate below that Assumptions A and B are sufficient for

(3.15) $n^{1-\varepsilon}$ ($\hat{p}^m$ — the minimum variance unbiased estimator of p^m) tends to zero with probability one as n tends to infinity for any positive ε and any positive integer m.

To prove (3.15), we note that the unbiased estimator of p^m is $T_n^{(m)}(\bar{Z}) = n^{-m}(s_n(\bar{Z}))^{-1}(\partial^m/\partial\bar{Z}^m)s_n(\bar{Z})$, using Assumption A and applying the Lebesgue Dominated Convergence Theorem m times, we get for sufficiently large n,

$$T_n^{(m)}(c) = \frac{\int_{-\infty}^{\infty} \exp(n[K(u_0 + iv) - (u_0 + iv)c])(-(u_0 + iv))^m \, dv}{\int_{-\infty}^{\infty} \exp(n[K(u_0 + iv) - (u_0 + iv)c]) \, dv} .$$

Expanding $(u_0 + iv)^m$ and then obtaining asymptotic series for

$$\int_{-\infty}^{\infty} \exp(n[K(u_0 + iv) - (u_0 + iv)c])v^j \, dv , \qquad j = 0, 1, \cdots, m ,$$

gives us

$$\sup_{c \in C} |T_n^{(m)}(c) - (-u_0)^m| = O(n^{-1}) \qquad \text{as } n \to \infty , \tag{3.16}$$

where C is a compact set contained in (a, b). We note that $(-u_0)^m$ is the MLE of p^m, when $\bar{Z} = c$. Reasoning as in Theorem 3.1, we conclude (3.15) from (3.16).

Remark 1. When a and b are finite and the density r is a polynomial, Assumption A is true. A particular case is the uniform density. The assumption is also satisfied, for example, for r a normal or a gamma density.

Remark 2. Assumption B holds when $(-p_2, p_1)$ is the natural parameter space of $\beta(p)\exp(-pz)r(z)$ and also when $p_1 = p_2 = \infty$ (Daniels [1]). Particular cases of the latter are when a and b are finite, while those of the former are r a normal or a gamma density. So, Assumptions A and B being satisfied, (3.15) is true when r is normal or gamma. For the normal r, the two estimators are the same when $m = 1$ and (3.1) is trivially true, while for a gamma r (3.1) can be easily seen to hold without making use of our result. We give below a nontrivial application of our result.

Example. Let Z have the probability density

$$p[1 - \exp(-p)]^{-1} \exp(-pz) I_{(0,1)}(z) ,$$

where I is the indicator function, with respect to Lebesgue measure. The MLE of p^m, m a positive integer, is $(\Delta^{-1}(\bar{Z}))^m$ where $\Delta(p) = p^{-1} - [\exp(p) - 1]^{-1}$, and the minimum variance unbiased estimator of p^m is

$$\frac{\partial^m}{\partial(n\bar{Z})^m} r_n(n\bar{Z})/r_n(n\bar{Z}) , \quad n \geqq 2m + 1 ,$$

where

$$r_n(n\bar{Z}) = (\Gamma(n))^{-1} \sum_{k=0}^{k_0} (-1)^k \binom{n}{k} (n\bar{Z} - k)^{n-1}, \quad k_0 < n\bar{Z} \leqq k_0 + 1, \quad k_0 = 0, 1, \cdots, n-1.$$

Since r is here a uniform density, both Assumptions A and B hold, and so extending Theorem 3.1 implies that both the estimators are asymptotically normal with mean p^m and variance $m^2 p^{2(m-1)}(-n\Delta'(p))^{-1}$.

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