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TWO EXAMPLES OF INVARIANT BAYES PROCEDURES

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The construction of proper Bayes tests which are invariant under non-compact groups has been treated in Kiefer and Schwartz (1965) and Schwartz (1966, 1969). These papers are limited to *testing* problems concerning *exponential* families' pdf's. The present note shows, by means of two simple examples, that the same methods can sometimes be applied more generally. One example considers estimation of the ratio of the mean to the variance for the normal distribution. The second example considers testing problems in which the pdf's have an exponential factor as well as a non-exponential factor.

1. Estimating μ/σ^2. Let $X_1, X_2, \cdots, X_N$ be i.i.d., each normally distributed with mean μ and variance σ^2. The problem is to estimate μ/σ^2 with loss function $|\theta - \mu/\sigma^2|^p|\mu/\sigma^2|^{-p}$ where $p > 0$. This problem is invariant under the multiplicative group of nonzero real numbers. An estimator T is scale invariant if $T(aX_1, \cdots, aX_N) = a^*T(X_1, \cdots, X_N)$ where $a \to a^*$ is a homomorphism.

Construct an a priori distribution on the parameter space as follows: Let $1 + \eta^2 = 1/\sigma^2$ and $\eta = \mu/\sigma^2$ and let η have (unnormalized) a priori density $|\eta|^k(1 + \eta^2)^{N/2} \exp\{N\eta^2/2(1 + \eta^2)\}$ where $N - k > 1$. For integrability of (1) below $k - p > -1$ is also required.

Then the proper Bayes estimator is the value of θ which minimizes

$$\int_{-\infty}^{\infty} |\theta - \eta|^p|\eta|^{k-p} \exp\{\tfrac{1}{2}(1 + \eta^2) \sum -x_i^2 + \eta \sum x_i\}\, d\eta$$

which (and this is the point at which the exponential structure of the normal density is used) is equivalent to minimizing

$$\int_{-\infty}^{\infty} |\theta - \eta|^p|\eta|^{k-p} \exp\{-\tfrac{1}{2}\eta^2 \sum x_i^2 + \eta \sum x_i\}\, d\eta\,. \tag{1}$$

It is obvious that the effect of a scale change $(x_1, \cdots, x_N) \to (ax_1, \cdots, ax_N)$ is to multiply the minimizing value by $1/a$. (Simply substitute $\gamma = a\eta$ in (1).)

For $p = 2$ and k even, the Bayes estimator has the form EY^{k-1}/EY^{k-2} where Y is a normal rv with mean $(\sum x_i/\sum x_i^2)$ and variance $(\sum x_i^2)^{-1}$. Taking $k = 2$ yields the estimator $\sum x_i/\sum x_i^2$ and taking $k = 4$ results in the estimator

$$\sum X_i(3 \sum X_i^2 + (\sum X_i)^2)(\sum X_i^2)^{-1}(\sum X_i^2 + (\sum X_i)^2)^{-1}\,.$$

Remark 1. The Bayes estimators are unique and therefore admissible.

Remark 2. If, more generally, the loss function is $L((\theta\sigma^2/\mu) - 1)$, one gets $(\theta\eta^{-1} - 1)|\eta^k|$ multiplying the exponential in (1). Assuming integrability, the same argument shows the invariance of the Bayes estimator.

Remark 3. The multivariate analog of this example, estimating $\sum^{-1} \mu$, would seem straightforward.

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2. pdf's containing the factor $\exp(-\frac{1}{2}x^2/\sigma^2)$. Let $X_1, \cdots, X_n$ be i.i.d. with common pdf of the form

$$1/\sigma K(a/\sigma)\exp\{-\tfrac{1}{2}(x^2/\sigma^2)\}f(ax/\sigma^2)$$

where f is a specified function which satisfies the condition that $1/K(y) = \int_{-\infty}^{\infty}\exp(-\frac{1}{2}x^2)f(xy)\,dx$ is uniformly bounded on $0 < y < 1$. Consider the problem of testing $H_0: a = 0$ vs $H_1: a \neq 0$ and construct an a priori density as follows:

Let $1 + \eta^2 = 1/\sigma^2$ and under H_1, $\eta = a/\sigma^2$.

Let η have (unnormalized) a priori density $(1+\eta^2)^{-N/2}$ under H_0 and $((1+\eta^2)^{-\frac{1}{2}}K(\eta(1+\eta^2)^{-\frac{1}{2}}))^N$ under H_1. Then the corresponding Bayes test has a critical region of the form

$$\frac{\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}(1+\eta^2)\sum x_i^2\}\prod_{i=1}^{N}f(\eta x_i)\,d\eta}{\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}(1+\eta^2)\sum x_i^2\}\,d\eta} > C\,.$$

Cancelling the term $\exp(-\frac{1}{2}\sum x_i^2)$ in both numerator and denominator, the resulting expression is again seen to be invariant under $(x_1, \cdots, x_n) \to (bx_1, \cdots, bx_n)$.

Note that taking $f(ax/\sigma^2) = p_1 + p_2\exp\{-ax/\sigma^2\}$ corresponds to a mixture of two normal distributions under H_1 one of which has mean zero. In this case the critical region is

$$\prod_1^N (p_1 + p_2\exp(X_i^2/\sum X_i^2)) > C\,.$$

In a similar way examples involving the mixture of several normal distributions can also be constructed.

Remark. There is no difficulty in constructing analogous multivariate examples in the manner of Kiefer and Schwartz (1965).

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