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ON SUFFICIENCY AND INVARIANCE

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Let $\mathscr{P}$ be a family of probability measures defined on a σ-field $\mathscr{A}$ on X and G be a group of transformations on X such that $Pg^{-1} \in \mathscr{P}$ for all $P \in \mathscr{P}$, $g \in G$. Let $\mathscr{A}_I$ be the σ-field of G-invariant sets of $\mathscr{A}$ and $\mathscr{A}_{I^*}$ the σ-field of $\mathscr{P}$-almost G-invariant sets of $\mathscr{A}$. Let $\mathscr{A}_S$ be a sufficient σ-field for $\mathscr{P}|\mathscr{A}$. Hall, Wijsman and Ghosh proved that $\mathscr{A}_S \cap \mathscr{A}_I$ is sufficient for $\mathscr{P}|\mathscr{A}_I$ if $g\mathscr{A}_S = \mathscr{A}_S$ for each $g \in G$ and $\mathscr{A}_S \cap \mathscr{A}_I \sim \mathscr{A}_S \cap \mathscr{A}_{I^*}(\mathscr{P})$. They posed the question whether the first condition alone suffices to prove this result. An example shows that the answer is no. For dominated families we show that $\mathscr{A}_S \cap \mathscr{A}_{I^*}$ is always sufficient for $\mathscr{P}|\mathscr{A}_{I^*}$, a result which is not true any more for undominated families.

Let $\mathscr{P}$ be a family of probability measures (p-measures) defined on a σ-field $\mathscr{A}$ over a basic set X. If $\mathscr{B}, \mathscr{C} \subset \mathscr{A}$ are σ-fields we write $\mathscr{B} \subset \mathscr{C}(\mathscr{P})$ iff for every $B \in \mathscr{B}$ there exists $C \in \mathscr{C}$ with $P(B \triangle C) = 0$ for all $P \in \mathscr{P}$, and $\mathscr{B} \sim \mathscr{C}(\mathscr{P})$ iff $\mathscr{B} \subset \mathscr{C}(\mathscr{P})$ and $\mathscr{C} \subset \mathscr{B}(\mathscr{P})$. Let G be a group of bijective $\mathscr{A}, \mathscr{A}$-measurable transformations on X, $\mathscr{A}_I := \{A \in \mathscr{A} : gA = A \text{ for all } g \in G\}$ the σ-field of G-invariant sets and $\mathscr{A}_{I^*} := \{A \in \mathscr{A} : P(A \triangle gA) = 0 \text{ for all } P \in \mathscr{P}, g \in G\}$ the σ-field of $\mathscr{P}$-almost G-invariant sets.

Assume that $\{Pg^{-1} : P \in \mathscr{P}, g \in G\} = \mathscr{P}$, where Pg^{-1} denotes the p-measure on $\mathscr{A}$, defined by $Pg^{-1}(A) := P(g^{-1}A)$, $A \in \mathscr{A}$. Let $\mathscr{A}_S$ be a sub-σ-field of $\mathscr{A}$ which is sufficient for $\mathscr{P}|\mathscr{A}$. Starting from problems in the domain of applied statistics Hall, Wijsman and Ghosh [3] investigate the question under which conditions the σ-field $\mathscr{A}_S \cap \mathscr{A}_I$ is sufficient for $\mathscr{P}|\mathscr{A}_I$. They show as their main result (Theorem 3.1) that the conditions A (i): "$g\mathscr{A}_S = \mathscr{A}_S$ for all $g \in G$", and A (ii): "$\mathscr{A}_S \cap \mathscr{A}_I \sim \mathscr{A}_S \cap \mathscr{A}_{I^*}(\mathscr{P})$" ensure that $\mathscr{A}_S \cap \mathscr{A}_I$ is sufficient for $\mathscr{P}|\mathscr{A}_I$. They pose the question (see page 596) whether condition A (i) alone suffices to prove this result. The following example shows that the answer is no, even if $\mathscr{A}_I \sim \mathscr{A}_{I^*}(\mathscr{P})$ and $\mathscr{P}|\mathscr{A}$ is dominated.

EXAMPLE 1. Let $X = \{0, 1\}^{\mathbb{R}}$ and $\mathscr{A}$ be the power set of X. Let G be the group of all coordinate permutations of the product space $\{0, 1\}^{\mathbb{R}}$. Let $P_0|\mathscr{A}[P_1|\mathscr{A}]$ be the p-measure concentrated at the point $x_0 \in X[x_1 \in X]$ with coordinates identical 0 [identical 1]. Let $\mathscr{P} := \{P_0, P_1\}$. Since $P_0 g^{-1} = P_0$, $P_1 g^{-1} = P_1$ for all $g \in G$ we have $\mathscr{P}g^{-1} = \mathscr{P}$ for all $g \in G$. Let $\mathscr{A}_S$ be the product σ-field of $\mathscr{A}_r$, $r \in \mathbb{R}$, where $\mathscr{A}_r$ denotes for each $r \in \mathbb{R}$ the power set of $\{0, 1\}$. Obviously $g\mathscr{A}_S = \mathscr{A}_S$ for each $g \in G$ and $\mathscr{A}_I \sim \mathscr{A}_{I^*}(\mathscr{P})$. Since there exists $A \in \mathscr{A}_S$ with $x_0 \in A$ and $x_1 \notin A$, $\mathscr{A}_S$ is sufficient for $\mathscr{P}|\mathscr{A}$. As $\varnothing$ and X are

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the only G-invariant sets contained in $\mathscr{A}_S$, $\mathscr{A}_S \cap \mathscr{A}_I$ is not sufficient for $\mathscr{P}|\mathscr{A}_I$.

If $\mathscr{A}_S \cap \mathscr{A}_{I^*}$ is sufficient for $\mathscr{P}|\mathscr{A}_{I^*}$ then condition A(ii) guarantees in a trivial manner that also $\mathscr{A}_S \cap \mathscr{A}_I$ is sufficient for $\mathscr{P}|\mathscr{A}_{I^*}$ and hence for $\mathscr{P}|\mathscr{A}_I$. Therefore it is desirable to give conditions under which $\mathscr{A}_S \cap \mathscr{A}_{I^*}$ is sufficient for $\mathscr{P}|\mathscr{A}_{I^*}$. Berk ([2], Lemma 3(i)) proves that "$g\mathscr{A}_S \sim \mathscr{A}_S(\mathscr{P})$ for each $g \in G$" is such a condition. Since each minimal sufficient σ-field fulfills this condition (see Lemma 2 of [2]), since each dominated family admits a minimal sufficient σ-field and since for dominated families each σ-field containing a sufficient σ-field is sufficient itself we obtain

REMARK 2. If $\mathscr{P}|\mathscr{A}$ is dominated and $\mathscr{A}_S$ is sufficient for $\mathscr{P}|\mathscr{A}$ then

(1) $\mathscr{A}_S \cap \mathscr{A}_{I^*}$ is sufficient for $\mathscr{P}|\mathscr{A}_{I^*}$,

(2) $\mathscr{A}_S \cap \mathscr{A}_I \sim \mathscr{A}_S \cap \mathscr{A}_{I^*}(\mathscr{P})$ implies that $\mathscr{A}_S \cap \mathscr{A}_I$ is sufficient for $\mathscr{P}|\mathscr{A}_I$.

The following example shows that the assertions of the preceding remark are not true in general for undominated families $\mathscr{P}|\mathscr{A}$, even if $\mathscr{P}|\mathscr{A}_{I^*}$ is dominated.

EXAMPLE. 3. Let $X = \mathbb{R} - \{0\}$, $\mathscr{A}$ the Borel-field on X and G the group of all transformations $x \to \alpha x$ for $\alpha > 0$. Let $P|\mathscr{A}$, $Q|\mathscr{A}$ be the p-measure defined by $P(-1) = P(1) = \frac{1}{2}$, $Q(A) := \lambda(A \cap (0, 1))$, $A \in \mathscr{A}$, where λ denotes the Lebesgue-measure. Let $\mathscr{P} := \{Pg^{-1}, Qg^{-1} : g \in G\}$. Then $\mathscr{P}g^{-1} = \mathscr{P}$ for each $g \in G$. Since the empty set is the only $\mathscr{P}|\mathscr{A}$-null set, we have

$$\mathscr{A}_{I^*} = \mathscr{A}_I = \{\emptyset; (-\infty, 0); (0, \infty); X\}.$$

The σ-field $\mathscr{A}_S := \{A \in \mathscr{A} : \{-1, 1\} \subset A \text{ or } \{-1, 1\} \subset \bar{A}\}$ is sufficient for $\mathscr{P}|\mathscr{A}$ but $\mathscr{A}_S \cap \mathscr{A}_{I^*} = \{\emptyset, X\}$ is not sufficient for $\mathscr{P}|\mathscr{A}_{I^*}$ because $P|\mathscr{A}_{I^*} \neq Q|\mathscr{A}_{I^*}$.

Example 1 shows also that the second assertion of Remark 2 is not true in general if the condition $\mathscr{A}_S \cap \mathscr{A}_I \sim \mathscr{A}_S \cap \mathscr{A}_{I^*}(\mathscr{P})$ is replaced by $\mathscr{A}_I \sim \mathscr{A}_{I^*}(\mathscr{P})$.

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