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ON FIXED SIZE CONFIDENCE BANDS FOR THE BUNDLE STRENGTH OF FILAMENTS[1]

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The present paper deals with the asymptotic theory of sequential confidence intervals of prescribed width $2d$ ($d > 0$) and prescribed coverage probability $1 - \alpha$ ($0 < \alpha < 1$) for the (unknown, per unit) strength of bundle of parallel filaments. In this context, certain useful convergence results on the empirical distribution and on the bundle strength of filaments are established and incorporated in the proofs of the main theorems. The results are the sequential counterparts of some fixed sample size results derived in a concurrent paper of Sen, Bhattacharyya and Suh [9].

1. Introduction and summary. For fixed (but large) sample sizes, the distribution theory of bundle strength of filaments has been studied by Daniels [4], Sen, Bhattacharyya and Suh [9], and others. The object of the present investigation is to develop, along the lines of Anscombe [1] and Chow and Robbins [3], the asymptotic theory of bounded length sequential confidence intervals for the bundle strength of filaments.

Let $\{X_i, i \geqq 1\}$ be a sequence of independent and identically distributed (i.i.d.) nonnegative random variables (rv) defined on a probability space $(\Omega, \mathscr{A}, P)$. We assume that X_i has an absolutely continuous cumulative distribution function (cdf) $F(x)$, $x \in [0, \infty)$, with a continuous first derivative (density function) $f(x)$, such that

$$0 < \lambda^2 = \int_0^\infty x^2\, dF(x) < \infty\,, \tag{1.1}$$

and there exists a unique x_0 (a point of continuity of $f(x)$), for which

$$\theta = \sup_{x \geqq 0}\{x[1 - F(x)]\} = x_0[1 - F(x_0)] \quad \text{and} \quad 0 < \pi_0 = F(x_0) < 1\,. \tag{1.2}$$

Note that

$$0 < \theta < \int_0^\infty x\, dF(x) < \lambda\,; \quad 0 < x_0 < \infty\,, \tag{1.3}$$

and the derivative of $x[1 - F(x)]$ vanishes at $x = x_0$, so that $f(x_0) > 0$. Further, we assume that there exist four positive constants C_1, C_2 and $k_1\,(>1)$, $k_2\,(>1)$, such that for sufficiently small δ (>0),

$$\theta - C_1|x - x_0|^{k_1} \leqq x[1 - F(x)] \leqq \theta - C_2|x - x_0|^{k_2} \quad \text{for all } |x - x_0| \leqq \delta\,. \tag{1.4}$$

In fact, if $x[1 - F(x)]$ is twice differentiable in some neighborhood of x_0 and the second derivative does not vanish there, then (1.4) holds with $k_1 = k_2 = 2$.

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Our parameter of interest is θ. For a random sample $X_1, \cdots, X_n$ of size n from the distribution $F(x)$, we denote the ordered random variables by $X_{n,1} \leqq \cdots \leqq X_{n,n}$. Let then

$$(1.5)\qquad D_n = \max_{1\leqq i\leqq n}[(n-i+1)X_{n,i}] = (n-r_n+1)X_{n,r_n} \quad \text{and} \quad Z_n = n^{-1}D_n\,;$$

by virtue of the assumed continuity of $F(x)$ and (1.2), r_n (a random variable assuming only integer values between 1 and n) is unique, with probability one. It is shown in [9] that Z_n converges to θ almost surely (a.s.) as $n \to \infty$. When the X_i represent the breaking stresses of filaments, D_n is equal to the maximum stress which a bundle of n parallel filaments of equal length can stand, and is termed the bundle strength (cf. [4], [9]). We term θ as the *mean (per unit) bundle strength of filaments*. We want to find a confidence interval for θ of prescribed width $2d$, $d > 0$, and prescribed confidence coefficient γ $(0 < \gamma < 1)$. Since, neither $F(x)$ nor the distribution of D_n is explicitly known, no fixed sample size procedure sounds available.

For $n \geqq 1$, we define

$$(1.6)\qquad I_n(d) = [Z_n - d, Z_n + d]\,, \quad d > 0\,, \qquad \text{and} \qquad p_n = (n+1)^{-1}r_n\,.$$

Also, let $\{a_n\}$ be a sequence of known positive constants such that

$$\lim_{n\to\infty} a_n = a \qquad \text{where} \quad (2\pi)^{-\frac{1}{2}}\textstyle\int_{-a}^{a} \exp(-2^{-1}t^2)\,dt = \gamma\,.$$

Finally, let

$$(1.7)\qquad v_n(d) = a_n^2 Z_n^2 p_n/[d^2(1-p_n)] \qquad \text{and} \qquad \nu(d) = a^2\theta^2\pi_0/[d^2(1-\pi_0)]\,.$$

Then, our proposed sequential confidence interval $I_{N(d)}(d) = \{\theta : Z_{N(d)} - d < \theta < Z_{N(d)} + d\}$ is based on the stopping (random) variable $N(d)$, defined by

$$(1.8)\qquad N(d) = \text{first positive integer } n\ (\geqq 2) \text{ for which } v_n(d) \leqq n.$$

The following theorems establish the properties of $N(d)$ and $I_{N(d)}$.

THEOREM 1.1. *If $\lambda < \infty$, then for every $d > 0$, the sequential procedure terminates with probability one. In fact, there exists a t_0 $(=t_0(d) > 0$ for $d > 0)$, such that*

$$(1.9)\qquad E\{\exp[tN(d)]\} < \infty \qquad \textit{for every} \quad -\infty < t \leqq t_0\,.$$

THEOREM 1.2. *Under* (1.1) *through* (1.4), *$N(d)$ is a non-increasing functions of d (>0)*, $\lim_{d\to 0} N(d) = \infty$ a.s., *and* $\lim_{d\to 0} E[N(d)] = \infty$;

$$(1.10)\qquad \lim_{d\to 0}\{N(d)/\nu(d)\} = 1 \quad \text{a.s.}, \qquad \lim_{d\to 0} P\{\theta \in I_{N(d)}(d)\} = \gamma\,,$$

$$(1.11)\qquad \lim_{d\to 0}\{E[N(d)]/\nu(d)\} = 1\,.$$

The proofs of the theorems are postponed to Section 4. It may be noted that the estimator Z_n of θ or $v_n(d)$ of $\nu(d)$ is not linear in X_i or X_i^2 etc., and hence, the results of Anscombe [1] or of Chow and Robbins [3] are not directly applicable. In fact, we need to prove some a.s. convergence results on $\{Z_n\}$ (see

Section 3) for the validation of the condition of 'uniform continuity in probability' (with respect to $n^{-\frac{1}{2}}$) of Anscombe [1] (also implicit in [3]). This, in turn, requires certain convergence results on the empirical cdf which are studied in Section 2. The last section develops sequential fixed percentage error confidence bands for θ.

2. Some results on the empirical cdf. Let $c(u)$ be equal to 1 or 0 according as u is > 0 or not. Define the empirical cdf $F_n(x)$ by

$$F_n(x) = n^{-1} \textstyle\sum_{i=1}^n c(x - X_i)\,, \qquad 0 \leqq x < \infty\,. \tag{2.1}$$

By the Glivenko–Cantelli Theorem, we know that

$$\sup_x |F_n(x) - F(x)| \to 0 \quad \text{a.s.,} \qquad \text{as } n \to \infty\,. \tag{2.2}$$

Also, by the well-known results on the Kolmogorov–Smirnov statistics, we know that for every $\varepsilon > 0$, there exists a positive K_ε ($<\infty$), such that for every $n \geqq 1$,

$$P\{\sup_x n^{\frac{1}{2}}|F_n(x) - F(x)| > K_\varepsilon\} < \varepsilon\,; \tag{2.3}$$

in fact, by Lemma 2 of Dvoretzky, Kiefer and Wolfowitz [5], we obtain that for every $n \geqq 1$ and $r \geqq 0$,

$$P\{\sup_x n^{\frac{1}{2}}|F_n(x) - F(x)| > r\} \leqq c\{\exp[-2r^2]\}\,, \qquad \text{where } 0 < c < \infty\,, \tag{2.4}$$

and hence,

$$\sup_x |F_n(x) - F(x)| \leqq n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} \quad \text{a.s.,} \qquad \text{as } n \to \infty\,. \tag{2.5}$$

THEOREM 2.1. *If $\lambda < \infty$, then for every $\varepsilon > 0$, there exist positive constants C ($<\infty$), $\rho(\varepsilon)$: $0 < \rho(\varepsilon) < 1$, and an integer $n_0(\varepsilon)$, such that for $n \geqq n_0(\varepsilon)$,*

$$P\{\sup_{0 \leqq x < \infty} x|F_n(x) - F(x)| > \varepsilon\} \leqq C[\rho(\varepsilon)]^n\,, \tag{2.6}$$

and hence,

$$\sup_{0 \leqq x < \infty} x|F_n(x) - F(x)| \to 0 \quad \text{a.s.,} \qquad \textit{as } n \to \infty\,. \tag{2.7}$$

PROOF. If the range of the cdf F is finite, i.e., $F(a) = 1$ for some $a < \infty$, then noting that $F_n(x) = F(x) = 1$ for all $x \geqq a$, so that

$$\begin{aligned} \sup_{0 \leqq x < \infty} x|F_n(x) - F(x)| &= \sup_{0 \leqq x \leqq a} x|F_n(x) - F(x)| \\ &\leqq a[\sup_x |F_n(x) - F(x)|]\,, \end{aligned} \tag{2.8}$$

the result directly follows from (2.4). Hence, in the sequel, it will be assumed that $F(x) < 1$ for all $x < \infty$.

Note that $\lambda < \infty \Rightarrow x^2[1 - F(x)]$ converges to 0 as $x \to \infty$. Hence, for every $\varepsilon > 0$, there exists a positive K_ε ($x_0 < K_\varepsilon < \infty$), such that

$$\sup_{x \leqq K_\varepsilon} x[1 - F(x)] = \theta\,, \qquad \sup_{x > K_\varepsilon} x[1 - F(x)] \leqq \theta - 2\varepsilon\,; \tag{2.9}$$

$$\inf_{x \geqq K_\varepsilon} \{[1 - F(x) + \varepsilon/2x] \log(1 + \varepsilon/2x[1 - F(x)])\} \geqq 2\varepsilon\,. \tag{2.10}$$

Also,

$$\begin{aligned} &\sup_x \{x|F_n(x) - F(x)|\} \\ &\quad = \max\{\sup_{x \leqq K_\varepsilon} x|F_n(x) - F(x)|, \sup_{x > K_\varepsilon} x|F_n(x) - F(x)|\}\,, \end{aligned} \tag{2.11}$$

Thus, it suffices to show that for $n \geqq n_0(\varepsilon)$,

$$P\{\sup_{x \leqq K_\varepsilon} x|F_n(x) - F(x)| > \varepsilon\} \leqq C_1[\rho(\varepsilon)]^n , \tag{2.12}$$

$$P\{\sup_{x \geqq K_\varepsilon} x|F_n(x) - F(x)| > \varepsilon\} \leqq C_2[\rho(\varepsilon)]^n , \tag{2.13}$$

where C_1 and C_2 are positive (finite) constants. Now, as in (2.8), (2.12) follows from (2.4), so we need only to prove (2.13). For this define

$$b_j^{(n)} = K_\varepsilon + jn^{-1} , \qquad j = 0, \cdots, n^* = [\exp(\varepsilon' n)]; \; 0 < \varepsilon' < \varepsilon/2 . \tag{2.14}$$

Then,

$$P\{\sup_{x \geqq K_\varepsilon} x|F_n(x) - F(x)| > \varepsilon\} \leqq P\{\sup_{K_\varepsilon \leqq x \leqq b_{n^*}^{(n)}} x|F_n(x) - F(x)| > \varepsilon\} + P\{\sup_{x > b_{n^*}^{(n)}} x|F_n(x) - F(x)| > \varepsilon\} , \tag{2.15}$$

where the second term on the right-hand side of (2.15) is bounded from above by

$$P\{\sup_{x > b_{n^*}^{(n)}} x|F_n(x) - F(x)| > \varepsilon, \max_{1 \leqq i \leqq n} X_i \leqq b_{n^*}^{(n)}\} + P\{\max_{1 \leqq i \leqq n} X_i > b_{n^*}^{(n)}\} . \tag{2.16}$$

If $X_1, \cdots, X_n$ are all $\leqq b_{n^*}^{(n)}$, for $x \geqq b_{n^*}^{(n)}$, $x|F_n(x) - F(x)| = x[1 - F(x)]$, and as $x^2[1 - F(x)] \to 0$ with $x \to \infty$, for every $\varepsilon > 0$, there exists an $n_0(\varepsilon)$, such that for $n \geqq n_0(\varepsilon)$, $x[1 - F(x)] < \varepsilon$ for every $x > b_{n^*}^{(n)}$. Consequently, the first term of (2.16) vanishes for every $n \geqq n_0(\varepsilon)$. On the other hand, by (2.14) and the fact that as $x \to \infty$, $x^2[1 - F(x)] \to 0$, the second term can be expressed as

$$\begin{aligned} 1 - [F(b_{n^*}^{(n)})]^n &= 1 - \{1 - [1 - F(b_{n^*}^{(n)})]\}^n \\ &\leqq 1 - \{1 - o([b_{n^*}^{(n)}]^{-2})\}^n \\ &= o[\exp(-2\varepsilon' n + \log n)] \\ &= o[\exp(-\varepsilon' n)] , \qquad \text{for } n \geqq n_0(\varepsilon) . \end{aligned} \tag{2.17}$$

Thus, it suffices to show that

$$P\{\sup_{K_\varepsilon \leqq x \leqq b_{n^*}^{(n)}} x|F_n(x) - F(x)| > \varepsilon\} \leqq C_2'[\rho(\varepsilon)]^n ; \qquad 0 < C_2' < \infty . \tag{2.18}$$

In the choice of K_ε in (2.9) and (2.10), we may let (without any loss of generality) $K_\varepsilon \geqq 1$, so that by (2.14), $b_{j+1}^{(n)}/b_j^{(n)} \leqq 1 + n^{-1}$, for every $j \geqq 0$. Then, for $x \in [b_j^{(n)}, b_{j+1}^{(n)}]$,

$$x|F_n(x) - F(x)| \leqq (1 + n^{-1})\{\max[U_{nj}, U_{nj+1}] + b_j^{(n)}[F(b_{j+1}) - F(b_j^{(n)})]\} , \tag{2.19}$$

where

$$U_{nj} = b_j^{(n)}[F_n(b_j^{(n)}) - F(b_j^{(n)})] , \qquad j \geqq 0 . \tag{2.20}$$

Note that for $j \geqq 0$,

$$\begin{aligned} b_j^{(n)}[F(b_{j+1}^{(n)}) - F(b_j^{(n)})] &\leqq \int_{b_j^{(n)}}^{b_{j+1}^{(n)}} x \, dF(x) \\ &\leqq \Big[\int_{b_j^{(n)}}^{b_{j+1}^{(n)}} x^2 \, dF(x)\Big]/b_j^{(n)} \to 0 \qquad \text{as } n \to \infty , \\ &\qquad \text{by (1.1) and (2.14),} \end{aligned} \tag{2.21}$$

and hence, for (2.18), it suffices to show that for every $\varepsilon > 0$, there exist an $n_0(\varepsilon)$ and a $\rho(\varepsilon)$: $0 < \rho(\varepsilon) < 1$, such that for $n \geqq n_0(\varepsilon)$,

$$P\{\max_{0 \leqq j \leqq b_{n^*}} |U_{nj}| > \varepsilon/2\} \leqq C_2'[\rho(\varepsilon)]^n . \tag{2.22}$$

Note that $U_{nj} = (1/n) \sum_{i=1}^n \{b_j^{(n)}[c(b_j^{(n)} - X_i) - F(b_j^{(n)})]\} = n^{-1} \sum_{i=1}^n d_{nij}$, where d_{nij}, $i = 1, \cdots, n$, are i.i.d. bounded rv's with mean 0 and $-b_j^{(n)}F(b_j^{(n)}) \leqq d_{nij} \leqq b_j^{(n)}[1 - F(b_j^{(n)})]$. Thus, by Theorem 3 of Hoeffding [6],

$$\begin{aligned} P\{U_{nj} &< -\varepsilon/2\} \\ &\leqq \{(1 + \varepsilon/2x[1 - F(x)])^{1-F(x)+\varepsilon/2x}(1 - \varepsilon/2xF(x))^{xF(x)-\varepsilon/2}\}^{-n}|_{x=b_j^{(n)}} \\ &\leqq \exp(-\varepsilon n) , \end{aligned} \tag{2.23}$$

as for $x \geqq K_\varepsilon$, by (2.9) and (2.10), $[1 - F(x) + \varepsilon/2x] \log(1 + \varepsilon/2x[1 - F(x)]) + (xF(x) - \varepsilon/2) \log(1 - \varepsilon/2xF(x)) \geqq \varepsilon$. In a similar manner, it follows that

$$P\{U_{nj} > \varepsilon/2\} \leqq \exp(-\varepsilon n) \qquad \text{for} \quad n \geqq n_0(\varepsilon) . \tag{2.24}$$

Consequently, by (2.23), (2.24) and the Bonferroni inequality, for $n \geqq n_0(\varepsilon)$,

$$\begin{aligned} P\{\max_{0 \leqq j \leqq n^*} |U_{nj}| > \varepsilon/2\} &\leqq 2(n^* + 1)[\exp(-\varepsilon n)] \simeq 2[\exp(-\varepsilon n + \varepsilon' n)] \\ &\leqq 2[\exp(-\varepsilon n/2) = 2[\exp(-\varepsilon/2)]^n , \end{aligned} \tag{2.25}$$

and hence, the proof of (2.6) is complete. (2.7) follows directly from (2.6) and the Borel–Cantelli lemma. □

REMARK. When the X_i are real valued rv's, defined over $(-\infty, \infty)$, the theorem readily extends with the range of x as $(-\infty, \infty)$. Further, if the X_i are independent but not necessarily identically distributed, we denote the cdf of X_i by $F_i^*(x)$ and let $\bar{F}_n^* = n^{-1} \sum_{i=1}^n F_i^*$, $n \geqq 1$. If then,

$$\sup_n \int_0^\infty x^2 \, d\bar{F}_n^*(x) < \lambda^2 < \infty , \tag{2.26}$$

the conclusions of Theorem 2.1 hold with F replaced by $\bar{F}_n^*$.

Now, by (2.3) and (2.5), for every K_ε $(<\omega)$,

$$\sup_{0 \leqq x \leqq K_\varepsilon} n^{\frac{1}{2}}|x[F_n(x) - F(x)]| = O_p(1) , \tag{2.27}$$

$$\sup_{0 \leqq x \leqq K_\varepsilon} n^{\frac{1}{2}}|x[F_n(x) - F(x)]| = O((\log n)^{\frac{1}{2}}) \quad \text{a.s.}, \qquad \text{as } n \to \infty . \tag{2.28}$$

Let us then write

$$a_{n,\alpha} = n^{-\alpha} \log n , \qquad A(n, \alpha) = [x_0 - a_{n,\alpha}, x_0 + a_{n,\alpha}] , \tag{2.29}$$

$$G_n(x_0, \alpha) = \sup\{n^{\frac{1}{2}}|[F_n(x) - F(x)] - [F_n(x_0) - F(x_0)]| : x \in A(n, \alpha)\} . \tag{2.30}$$

THEOREM 2.2. *For any* α: $0 < \alpha < 1$, *as* $n \to \infty$

$$G_n(x_0, \alpha) = O(n^{-\alpha/2} \log n) , \qquad \textit{with probability } 1. \tag{2.31}$$

PROOF. We consider $2n^\beta$ $(\beta \leqq \frac{1}{2})$ equidistant points on $A(n, \alpha)$ and apply the same technique as in Lemma 1 of Bahadur [2], who considered the special case $\alpha = \frac{1}{2}$; for brevity, the details are omitted. Further, if the distributions are not all the same, we can proceed as in Sen [8] and derive the same result (on replacing F by $\bar{F}_n^*$).

Let us now define

$$(2.32)\qquad M_n = n^{\frac{1}{2}}\{x_0[F_n(x_0) - F(x_0)]\} \qquad \text{and} \qquad \zeta^2 = x_0^2\pi_0(1 - \pi_0)\,.$$

THEOREM 2.3. *For every positive* ε *and* η, *there exists a* δ (>0), *such that for all* $n \geqq n_0(\varepsilon, \eta)$ *and* $N \leqq \delta n$

$$(2.23)\qquad P\{\sup_{1\leqq j\leqq N} |M_n - M_{n+j}| > \varepsilon\zeta\} < \eta\,.$$

PROOF. From (2.1) and (2.32), it follows that $M_n^* = n^{\frac{1}{2}}M_n$ forms a martingale sequence, and hence, by the well-known Kolmogorov inequality

$$(2.34)\qquad P\{\sup_{1\leqq j\leqq N} |M_{n+j}^*| > t_1\} \leqq t_1^{-2}E\{M_{n+N}^*\}^2 \leqq n(1+\delta)(\zeta/t_1)^2\,,$$

for all $N \leqq \delta n$. Also, for all $1 \leqq j \leqq N \leqq \delta n$,

$$(2.35)\qquad \begin{aligned} |M_{n+j} - M_n| &= |(n+j)^{-\frac{1}{2}}M_{n+j}^* - n^{-\frac{1}{2}}M_n^*| \\ &\leqq n^{-\frac{1}{2}}\{|M_{n+j}^* - M_n^*| + [\{(1+\delta)^{\frac{1}{2}} - 1\}/(1+\delta)^{\frac{1}{2}}]|M_n^*|\}\,. \end{aligned}$$

Hence, it follows from (2.34) and (2.35) that we are only to show that for every positive ε' $(<\varepsilon)$ and η' $(<\eta)$, there exists a δ (>0) such that

$$(2.36)\qquad P\{\sup_{1\leqq j\leqq N} |M_{n+j}^* - M_n^*| > n^{\frac{1}{2}}\zeta\varepsilon'\} < \eta' \qquad \text{for all} \quad N \leqq \delta n$$

and this readily follows from the Kolmogorov inequality on martingales (cf. [7], page 386).

3. Convergence of Z_n and p_n. We are mainly interested in the following theorems.

THEOREM 3.1. *If* $\lambda < \infty$, *then for every* $\varepsilon > 0$, *there exist a positive integer* $n_0(\varepsilon)$ *and a* $\rho(\varepsilon)$: $0 < \rho(\varepsilon) < 1$, *such that for* $n \geqq n_0(\varepsilon)$,

$$(3.1)\qquad P\{|Z_n - \theta| > \varepsilon\} \leqq [\rho(\varepsilon)]^n\,.$$

PROOF. By (1.5) and (2.1), we have

$$(3.2)\qquad Z_n = [1 - F_n(X_{n,r_n})] = \sup_{x\geqq 0}\{x[1 - F_n(x)]\}\,,$$

and by assumption,

$$(3.3)\qquad \theta = x_0[1 - F(x_0)] = \sup_{x\geqq 0}\{x[1 - F(x)]\}\,.$$

Also, (3.2), (3.3) and the event that $\sup_x |x[F_n(x) - F(x)]| < \varepsilon$ imply that

$$|Z_n - \theta| = |\sup_x x[1 - F(x)] - \sup_x x[1 - F_n(x)]| \leqq \sup_x |x[F_n(x) - F(x)]| \leqq \varepsilon\,.$$

Hence, the proof directly follows from Theorem 2.1. □

Note that (3.1) implies that

$$(3.4)\qquad [\lambda < \infty] \Rightarrow Z_n \to \theta \quad \text{a.s.,} \qquad \text{as} \quad n \to \infty\,.$$

For our purposes, we require some relatively stronger results, stated below.

THEOREM 3.2. *Under* (1.1)—(1.4), $[n^{\frac{1}{2}}(Z_n - \theta) + M_n] \to 0$, *with probability one, as* $n \to \infty$.

PROOF. Since $x[1-F(x)]$ has a unique maximum θ at x_0, for every (small) $\varepsilon > 0$, there exists a δ (>0), such that

$$x[1-F(x)] \leqq \theta - 2\varepsilon \qquad \text{for all} \quad |x - x_0| > \delta . \tag{3.5}$$

Thus, by (2.7), (3.3) and (3.5),

$$\sup\{x[1-F_n(x)]: |x-x_0| > \delta\} \leqq \theta - \varepsilon , \qquad \text{with probability 1, as } n \to \infty . \tag{3.6}$$

With the definition of k_2 in (1.4), we let $\alpha = 1/2k_2$ in (2.29), and let

$$\tilde{A}(n, \alpha) = [x_0 - \delta, x_0 + \delta] - A(n, \alpha) . \tag{3.7}$$

Then, for all $x \in A(n, \alpha)$

$$\begin{aligned} &\sup\{|n^{\frac{1}{2}}\{x[1-F_n(x)] - x[1-F(x)]\} + M_n| : x \in A(n, \alpha)\} \\ &\quad = \sup\{|(x_0 - x)n^{\frac{1}{2}}[F_n(x) - F(x)] + x_0 G_n(x, \alpha)| : x \in A(n, \alpha)\} + o(n^{-\frac{1}{2}}) \\ &\quad = O(n^{-\alpha/2}[\log n]) , \qquad \text{with probability 1}, \qquad \text{as } n \to \infty , \end{aligned} \tag{3.8}$$

by virtue of Theorem 2.2, (2.27) and (2.29). Consequently,

$$n^{\frac{1}{2}}[\sup[x\{1-F_n(x)\}: x \in A(n, \alpha)] - \theta\} + M_n = O(n^{-\alpha/2}(\log n)) , \tag{3.9}$$

with probability 1, as $n \to \infty$. Finally, by (2.28), as $n \to \infty$,

$$\sup\{|x[F_n(x) - F(x)]| : x \in \tilde{A}(n, \alpha)\} = O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}) , \tag{3.10}$$

with probability 1, and by (1.4) and the value of $\alpha = 1/(2k_2)$,

$$\sup\{x[1-F(x)]: x \in \tilde{A}(n, \alpha)\} \leqq \theta - O([n^{-\frac{1}{2}}(\log n)^{k_2}]) . \tag{3.11}$$

Upon noting that for large n, $O(n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}) = o(1)O(n^{-\frac{1}{2}}(\log n)^{k})$ for all $k \geqq 1$, it follows from (3.10) and (3.11) that

$$\sup\{x[1-F_n(x)]: x \in \tilde{A}(n, \alpha)\} \leqq \theta - O([n^{-\frac{1}{2}}(\log n)^{k_2}]) , \qquad \text{with probability 1.} \tag{3.12}$$

Since, by (2.27) and (2.32), $|M_n| \leqq x_0(\log n)^{\frac{1}{2}}$, with probability 1 (as $n \to \infty$), it follows from (3.6), (3.9) and (3.12) that as $n \to \infty$,

$$W_n = n^{\frac{1}{2}}(Z_n - \theta) + M_n \to 0 , \qquad \text{with probability 1.} \ \square \tag{3.13}$$

Now, from (3.13), it follows that for everv $\varepsilon > 0$, there exists an $\eta > 0$, such that for all $n \geqq n_0(\varepsilon, \eta)$

$$P\{|W_{n+j}| > \varepsilon \ \text{ for at least one } \ j = 1, \cdots, N, \cdots\} < \eta . \tag{3.14}$$

Hence, from Theorem 2.3, Theorem 3.2 and (3.14), we arrive at the following.

THEOREM 3.3. *For every positive ε and η there exists a $\delta > 0$, such that for all $n \geqq n_0(\varepsilon, \eta)$ and $N \leqq \delta n$*

$$P\{|Z_{n+j} - Z_n| < n^{-\frac{1}{2}}\zeta\varepsilon \ \textit{for all} \ j = 1, \cdots, N\} \geqq 1 - \eta . \tag{3.15}$$

Further, we state the following result already proved in [4], [9]: for every t,

$$\lim_{n\to\infty} P\{n^{\frac{1}{2}}(Z_n - \theta)/\zeta \leqq t\} = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{t} \exp(-\tfrac{1}{2}x^2)\,dx\,, \quad -\infty < t < \infty\,. \tag{3.16}$$

THEOREM 3.4. *If* $\lambda < \infty$, *for every* $\varepsilon > 0$, *there exist an* $n_0(\varepsilon)$ *and a* $\rho(\varepsilon): 0 < \rho(\varepsilon) < 1$, *such that for* $n \geqq n_0(\varepsilon)$,

$$P\{p_n < \pi_0 - \varepsilon\} \leqq 2[\rho(\varepsilon)]^n \quad \textit{and} \quad P\{p_n > \pi_0 + \varepsilon\} \leqq 2[\rho(\varepsilon)]^n\,. \tag{3.17}$$

PROOF. We only prove the result for $p_n > \pi_0 + \varepsilon$, as the same proof holds for the other case. Corresponding to every $\varepsilon > 0$, we can find an η (>0) such that $F(x_0 + \eta) = \pi_0 + \varepsilon/2$. Then, we have

$$\begin{aligned} P\{p_n > \pi_0 + \varepsilon\} & \\ &= P\left\{\frac{n}{n+1} F_n(X_{n,r_n}) > \pi_0 + \varepsilon\right\} \\ &\leqq P\{F_n(X_{n,r_n}) > \pi_0 + \varepsilon\} \\ &\leqq P\{F_n(X_{n,r_n}) > \pi_0 + \varepsilon,\ X_{n,r_n} \leqq x_0 + \eta\} + P\{X_{n,r_n} > x_0 + \eta\} \\ &\leqq P\{F_n(x_0 + \eta) > \pi_0 + \varepsilon\} + P\{X_{n,r_n} > x_0 + \eta\}\,. \end{aligned} \tag{3.18}$$

Now, by Theorem 1 of Hoeffding [6], we have upon noting that $F(x_0+\eta) = \pi_0 + \frac{1}{2}\varepsilon$,

$$P\{F_n(x_0 + \eta) > \pi_0 + \varepsilon\} \leqq \exp(-\tfrac{1}{2}n\varepsilon^2) = [\rho_1(\varepsilon)]^n\,, \tag{3.19}$$

where $\rho_1(\varepsilon) < 1$. Also, $[X_{n,r_n} > x_0 + \eta] \Rightarrow X_{n,r_n}[1 - F(X_{n,r_n})] < \theta - \delta$, where $\delta > 0$. Therefore

$$\begin{aligned} P\{X_{n,r_n} > x_0 + \eta\} & \\ &\leqq P\{X_{n,r_n}[1 - F(X_{n,r_n})] < \theta - \delta\} \\ &\leqq P\{X_{n,r_n}[1 - F(X_{n,r_n})] < \theta - \delta,\ |X_{n,r_n}[F_n(X_{n,r_n}) \\ &\qquad - F(X_{n,r_n})]| \leqq \delta/2\} \\ &\qquad + P\{|X_{n,r_n}[F_n(X_{n,r_n}) - F(X_{n,r_n})]| > \delta/2\} \\ &\leqq P\{X_{n,r_n}[1 - F_n(X_{n,r_n})] < \theta - \delta/2\} \\ &\qquad + P\{|X_{n,r_n}[F_n(X_{n,r_n}) - F(X_{n,r_n})]| > \delta/2\} \\ &\leqq P\{Z_n < \theta - \delta/2\} + P\{\sup_x |x[F_n(x) - F(x)]| > \delta/2\}\,. \end{aligned} \tag{3.20}$$

Hence, (3.17) follows from (3.18), (3.19), (3.20) and Theorems 2.1 and 3.1. □

Note that (3.17) implies that

$$[\lambda < \infty] \Rightarrow p_n \to \pi_0 \quad \text{a.s.,} \qquad \text{as } n \to \infty\,. \tag{3.21}$$

4. Proofs of Theorems 1.1 and 1.2. Since (1.9) implies that $P\{N(d) < \infty\} = 1$, we need to prove only (1.9) for Theorem 1.1. Define

$$Q_n(d) = P\{N(d) > n\}\,, \qquad n = 0, 1, \cdots;\ Q_0(d) = 1\,. \tag{4.1}$$

Then, upon noting that for every $t > 0$,

$$E[\exp(tN(d))] \leqq 1 + |e^t - 1|\{\textstyle\sum_{n=1}^{\infty} [\exp(nt)]Q_n(d)\}\,, \tag{4.2}$$

it suffices to show that there exists a positive t_0, such that

$$[\exp(nt_0)]Q_n(d) \to 0 \quad \text{as} \quad n \to \infty, \qquad \text{for every (fixed) } d > 0. \tag{4.3}$$

Now, by (1.7) and (1.8),

$$\begin{aligned} Q_n(d) &= P\{N(d) > n\} = P\{v_r(d) > r, r = 1, \cdots, n\} \\ &\leqq P\{v_n(d) > n\} = P\{Z_n^2 > (nd^2/a_n^2)(1/p_n - 1)\} \\ &\leqq P\{Z_n^2 > (nd^2/a_n^2)(p_n^{-1} - 1), p_n \leqq \pi_0 + \beta\} + P\{p_n > \pi_0 + \beta\}, \end{aligned} \tag{4.4}$$

where $\beta > 0$. Since $p_n \leqq \pi_0 + \beta \Rightarrow p_n^{-1} - 1 \geqq \beta^* > \pi_0^{-1} - 1$, we have

$$Q_n \leqq P\{Z_n > (n\beta^*)^{\frac{1}{2}} d/a_n\} + P\{p_n > \pi + \beta\}. \tag{4.5}$$

Now, $a_n \to a$ (>0), as $n \to \infty$. Hence, for every d, there exists an $n_0(d)$ such that for $n \geqq n_0(d)$, $(n\beta^*)^{\frac{1}{2}} d/a_n > \theta + \varepsilon$, where $\varepsilon > 0$. Hence, from (4.5), we have for all $n \geqq n_0(d)$,

$$Q_n \leqq P\{Z_n > \theta + \varepsilon\} + P\{p_n > \pi + \beta\}. \tag{4.6}$$

Thus, (4.3) directly follows from (4.6), Theorem 3.1 and Theorem 3.4, and the proof of (1.9) is complete.

To prove Theorem 1.2, we note that by (1.7), $v_n(d) \uparrow \infty$ as $d \to 0$, and hence, by (1.8), $N(d)$ is non-increasing in d (>0). From (1.7), (3.4) and (3.21), it follows that as $n \to \infty$,

$$Y_n = v_n(d)/\nu(d) \to 1 \quad \text{a.s.}, \qquad \text{uniformly in } d\ (>0). \tag{4.7}$$

Hence, $\lim_{d\to 0} v_n(d) = \infty$ a.s., which by (1.8), leads to $\lim_{d \to 0} N(d) = \infty$ a.s. Finally, by Theorem 1.1 and the monotone convergence theorem (cf. [7], page 124), it follows that $\lim_{d\to 0} EN(d) = \infty$.

Now, upon writing $f(n) = na^2/a_n^2$ and $t = a^2\nu(d)$, it follows from (4.7) that the conditions of Lemma 1 of Chow and Robbins [3] are all satisfied. Also, by virtue of (3.15) and (3.16), the conditions (C1) and (C2) of Anscombe [1] hold for $\{Z_n\}$ (these conditions are implicit in [3]). Hence, the proof of (1.10) follows along the lines of Lemma 1 of Chow and Robbins [3] (together with the proofs of their (4) and (5)).

To prove (1.11), we require to show as in Lemma 2 of [3] that $E(\sup_n Y_n) < \infty$ or verify the conditions of their Lemma 3. Now, by definition,

$$Y_n = (a_n^2/a^2)([1 - \pi_0]/\pi_0\theta^2)[Z_n^2 p_n/(1 - p_n)], \tag{4.8}$$

and hence, it suffices to show that $E(\sup_n Z_n^2 p_n/(1 - p_n)) < \infty$. But,

$$\begin{aligned} U_n = Z_n^2 p_n/(1 - p_n) &= n^{-2}(n - r_n + 1) r_n X_{n,r_n}^2 \\ &\leqq (n^{-1} r_n)(n^{-1} \textstyle\sum_{i=r_n}^n X_{n,i}^2) \leqq n^{-1} \sum_{i=1}^n X_i^2 \quad \text{a.e.}, \end{aligned} \tag{4.9}$$

and hence, it suffices to show that $E(\sup_n n^{-1} \sum_{i=1}^n X_i^2) < \infty$. A sufficient condition for this is, of course, that $E(X^4) < \infty$. However, as in Lemma 3 of [3], we prove (1.11) without unnecessarily assuming that $E(X^4) < \infty$. For this, consider first the following lemma.

LEMMA 4.1. *$U_n/U_{n-1} \geqq (1 - n^{-1})^2$ for all $n \geqq 2$.*

PROOF. Given the sample of size $n-1$ i.e., $X_{n-1,1} \leqq \cdots \leqq X_{n-1,n-1}$, X_n can belong to one of the n intervals: $X_{n-1,i-1} \leqq x \leqq X_{n-1,i}$, $i = 1, \cdots, n$, where $X_{n-1,0} = 0$ and $X_{n-1,n} = \infty$. If $X_{n-1,i-1} \leqq X_n \leqq X_{n-1,i}$, we have the following two arrays corresponding to the samples of sizes $n-1$ and n respectively:

$$(n-1)X_{n-1,1}, \cdots, (n-i+1)X_{n-1,i-1}, (n-i)X_{n-1,i}, \cdots, X_{n-1,n-1};$$
$$nX_{n-1,1}, \cdots, (n-i+2)X_{n-1,i-1}, (n-i+1)X_n, (n-i)X_{n-1,i}, \cdots, X_{n-1,n-1},$$

where $(n-r_{n-1})X_{n-1,r_{n-1}}$ and $(n-r_n+1)X_{n,r_n}$ are the maximum values within the first and the second rows. Thus, (i) if $r_{n-1} \geqq i$, we may have either $r_n = r_{n-1}+1 > i$, or $r_n = r_{n-1} = i$. In the first case, $X_{n,r_n} = X_{n-1,r_{n-1}}$ and hence, $r_n(n-r_n+1)X^2_{n,r_n} \geqq r_{n-1}(n-r_{n-1})X^2_{n-1,r_{n-1}}$. In the second case, $X_{n,r_n} = X_n$, and hence $(n-r_n+1)X_n \geqq (n-r_n)X_{n-1,r_n} = (n-r_n)X_{n-1,i}$. Thus, $r_n(n-r_n+1)X^2_{n,r_n} = r_{n-1}(n-r_{n-1}+1)X_{n,r_n} \cdot X_{n,r_n} \geqq r_n(n-r_{n-1})X^2_{n-1,r_{n-1}}$. Hence, in either case,

$$r_n(n-r_n+1)X^2_{n,r_n} \geqq r_{n-1}(n-r_{n-1})X^2_{n-1,r_{n-1}}. \tag{4.10}$$

(ii) If $r_{n-1} \leqq i-1$, it is quite evident that $r_n \geqq r_{n-1}$, $X_{n,r_n} \geqq X_{n-1,r_{n-1}}$ and $(n-r_n+1)X_{n,r_n} \geqq (n-r_{n-1})X_{n-1,r_{n-1}}$. Consequently, (4.12) holds again. Thus,

$$\frac{U_n}{U_{n-1}} = \left(\frac{n-1}{n}\right)^2 \left[\frac{r_n(n-r_n+1)X^2_{n,r_n}}{r_{n-1}(n-r_{n-1})X^2_{n-1,r_{n-1}}}\right]^2 \geqq \left(1-\frac{1}{n}\right)^2 \quad \text{a.e.} \tag{4.11}$$

Hence the lemma.

Using now Lemma 4.1, (4.8) and (4.9), the proof of (1.11) follows exactly as in [3, pages 630–631]. Alternatively, a proof similar to the one in (5.10) through (5.14) can be sketched. Hence, the proof of Theorem 1.2 is complete.

5. Fixed percentage error confidence bands for θ. For every $d > 0$, we define $d_1 = \exp(-d)$ and $d_2 = \exp(d)$, so that $0 < d_1 < 1 < d_2 < \infty$. We intend to find a confidence interval $I_n(d) = \{\theta : d_1 Z_n \leqq \theta \leqq d_2 Z_n\}$ such that

$$p\{\theta \in I_n(d)\} = \gamma \quad (0 < \gamma < 1). \tag{5.1}$$

Our desired sequential confidence interval $I_{N(d)}(d) = \{Z_{N(d)}d_1 \leqq \theta \leqq Z_{N(d)}d_2\}$ is based on the stopping (random) variable $N(d)$, defined by

$$N(d) = \text{smallest integer } n\ (\geqq 1) \text{ for which } p_n \leqq nd^2\{nd^2 + a_n{}^2\}^{-1}, \tag{5.2}$$

where p_n and a_n are defined as in Section 1. We also define π_0 as in (1.2), and let

$$\nu(d) = a^2\pi_0\{(1-\pi_0)d^2\}^{-1}. \tag{5.3}$$

THEOREM 5.1. *The results of Theorems* 1.1 *and* 1.2 *also hold for the sequential procedure in* (5.2), *with $\nu(d)$ defined by* (5.3).

PROOF. Note that

$$p_n \leqq nd^2(nd^2 + a_n{}^2)^{-1} \Leftrightarrow p_n(1-p_n)^{-1} \leqq nd^2 a_n{}^{-2}. \tag{5.4}$$

Hence, if we define $Q_n(d)$ as in (4.1), to prove the results parallel to those in Theorem 1.1, we are only to show that (4.3) holds for the stopping variable $N(d)$ defined by (5.2). Now, by (5.2) and (5.4),

$$(5.5)\qquad \begin{aligned} Q_n(d) &= P\{N(d) > n\} = P\{p_r(1-p_r)^{-1} > rd^2a_r^{-2},\ r = 1, \cdots, n\} \\ &\leqq P\{p_n(1-p_n)^{-1} > nd^2a_n^{-2}\} = P\{p_n > nd^2(nd^2 + a_n^2)^{-1}\}\,. \end{aligned}$$

Now, for every $d > 0$, there exists an n_0, such that $nd^2/(nd^2 + a_n^2) \geqq \pi_0 + \varepsilon$, for all $n \geqq n_0$, where $\varepsilon > 0$. Hence, for $n \geqq n_0$,

$$(5.6)\qquad Q_n(d) \leqq P\{p_n > \pi_0 + \varepsilon\}\,,$$

and hence, (4.3) follows directly from Theorem 3.4.

To prove the results parallel to those in Theorem 1.2, we use (3.16) and some standard results on transformation of statistics, and obtain that

$$(5.7)\qquad \mathscr{L}(n^{\frac{1}{2}} \log [Z_n/\theta]) \to \mathscr{N}(0, \pi_0(1-\pi_0)^{-1})\,.$$

Also, by (1.1), θ is strictly positive, and hence, from (3.4) and Theorem 3.3 it follows that for every positive ε and η, there exists a $\delta > 0$, such that for all $n \geqq n_0(\varepsilon, \eta)$ and $N \leqq \delta n$,

$$(5.8)\qquad P\{n^{\frac{1}{2}}|\log Z_{n+j} - \log Z_n| < \varepsilon\{\pi_0/(1-\pi_0)\}^{\frac{1}{2}},\ j = 1, \cdots, N\} > 1 - \eta\,.$$

Hence, $\{\log Z_n\}$ satisfies both the conditions (C1) and (C2) of Anscombe (1952). We also note that by virtue of (3.21)

$$(5.9)\qquad y_n = p_n(1-\pi_0)/(1-p_n)\pi_0 \to 1 \quad \text{a.s.,} \qquad \text{as } n \to \infty\,,$$

and hence, using (5.2), (5.4), (5.7)—(5.9) and Lemma 1 of [3], it follows that (1.9) holds for the stopping rule (5.2). To prove (1.11), we proceed as follows.

Choose some arbitrarily small ε (>0), and defining $\nu(d)$ as in (5.3), let

$$(5.10)\qquad n_1(d) = [\nu(d)(1-\varepsilon)] \qquad \text{and} \qquad n_2(d) = [\nu(d)(1+\varepsilon) + 1]\,.$$

Then,

$$(5.11)\qquad E\{N(d)/\nu(d)\} = [\nu(d)]^{-1}[\textstyle\sum_1 + \sum_2 + \sum_3 nP\{N(d) = n\}]\,,$$

where the summation $\sum_1$ extends over $n \leqq n_1(d)$, $\sum_2$ over $n_1(d) < n \leqq n_2(d)$ and $\sum_3$ over $n > n_2(d)$. Since $\lim_{d\to 0} \nu(d) = \infty$ and $N(d)/\nu(d) \to 1$ a.s., as $d \to 0$, for every $\varepsilon > 0$, we can find a d_0 (>0), such that for $0 < d \leqq d_0$, $P\{n_1(d) < N(d) \leqq n_2(d)\} = P\{1 - \varepsilon \leqq N(d)/\nu(d) \leqq 1 + \varepsilon\} > 1 - \eta$, where η (>0) is a preassigned small number. Then, for $d \leqq d_0$, the first sum on the right-hand side of (5.11) is less than

$$(5.12)\qquad (1-\varepsilon)P\{N(d) < n_1(d)\} \leqq \eta(1-\varepsilon) \leqq \eta\,.$$

The second sum is equal to $1 + R(d)$, where

$$(5.13)\qquad \begin{aligned} |R(d)| &\leqq \varepsilon P\{n_1(d) < N(d) \leqq n_2(d)\} \\ &\quad + [1 - P\{n_1(d) < N(d) \leqq n_2(d)\}] \leqq \varepsilon + \eta\,. \end{aligned}$$

Finally, on using (5.5), we have on noting that $Q_{n_2(d)}(d) = P\{N(d)/\nu(d) \geqq 1 + \varepsilon\}$ that

$$
\begin{aligned}
&[\nu(d)]^{-1} \textstyle\sum_3 nP\{N(d) = n\} \\
&\quad = [\nu(d)]^{-1}\{\textstyle\sum_{n_2(d)+1}^{\infty} Q_n(d) + [n_2(d) + 1]Q_{n_2(d)}(d)\} \\
&\quad \leqq [\nu(d)]^{-1}[\textstyle\sum_{n_2(d)}^{\infty} P\{p_n > nd^2/(nd^2 + a_n^2)\}] \\
(5.14) &\qquad\quad + \{[n_2(d) + 1]/\nu(d)\}Q_{n_2(d)}(d) \\
&\quad \leqq [\nu(d)]^{-1} \textstyle\sum_{n_2(d)}^{\infty} P\{p_n > \pi_0(1 + \varepsilon)[(1 + \varepsilon)\pi_0 + (1 - \pi_0)a_n^2/a^2]^{-1}\} \\
&\qquad\quad + \{[n_2(d) + 1]/\nu(d)\}\eta \\
&\quad \leqq [\nu(d)]^{-1} \textstyle\sum_{n_2(d)}^{\infty} P\{p_n > \pi_0 + \varepsilon'\} + \{\eta[n_2(d) + 1]/\nu(d)\}; \quad \varepsilon' > 0 .
\end{aligned}
$$

Hence, by Theorem 3.4 and (5.10), it follows that the right-hand side of (5.14) can be made arbitrarily small by letting d be small. Hence, (1.11) follows from (5.11) through (5.14). □

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