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ON THE CONSTRUCTION OF TREND RESISTANT MIXED LEVEL FACTORIAL RUN ORDERS[1]

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In this paper we consider experimental situations where treatments from a mixed level factorial design are to be applied to experimental units over space or time and where there may be an unknown trend effect which can be approximated by some polynomial function of the order in which the observations are obtained. A method of constructing run orders of treatments for such situations is given which generally yields least squares estimators for main effects that have a higher degree of trend resistance than the least squares estimators of main effects that come from run orders obtainable by other methods of construction previously given in the literature.

1. Introduction. In this paper we consider experimental situations where the treatments that make up a mixed level factorial design are to be applied sequentially to experimental units over space or time and where there may be an unknown or uncontrollable trend effect which is highly correlated with the order in which the observations are obtained. Any ordered application of treatments to experimental units over space or time is called a *run order*. In situations such as just described, the experimenter may prefer to assign treatments to experimental units in such a way that the usual estimates for the factorial effects of interest are not affected by the unknown trend effect. Such run orders are called *trend resistant*. A good deal of work has been done on the construction of trend resistant run orders of factorial designs when all factors have the same number of levels; see Bailey, Cheng and Kipnis (1992) for a summary of much of the work done on this problem. The only work known to the author on the construction of trend resistant mixed level factorial run orders is that done in Coster (1993) and Bailey, Cheng and Kipnis (1992). Coster (1993) used the generalized foldover scheme (GFS) introduced in Coster and Cheng (1988) and pseudofactors to develop some methods of constructing trend resistant mixed level factorial run orders. Bailey, Cheng and Kipnis (1992) give results which unify much of the work previously done on constructing trend free designs and augment some of the results given in Coster (1993). In this paper, we give a new method for constructing mixed level factorial run orders which are trend resistant for main effects. The designs obtained from the method of construction given here generally have a higher degree of trend resistance for main effects than designs obtainable from previously given methods of construction.

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2. Preliminary notation and facts. In this paper we shall consider fractional factorial designs involving n factors, denoted by $A_1, \dots, A_n$, where factor i has s_i levels, $i = 1, \dots, n$. We shall use $\mathbf{d}$ to denote such a design and assume throughout that each s_i is a prime or prime power. We denote the s_i levels of factor A_i by $0, 1, \dots, s_i - 1$. A treatment combination in which factor A_i occurs at level $x_i, i = 1, \dots, n$, is denoted by the n-tuple $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Let G be the set of all possible $s_1 s_2 \cdots s_n$ treatment combinations. Then G is an abelian group under the operation

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (z_1, z_2, \dots, z_n), \quad \text{where } z_i = x_i + y_i \pmod{s_i}. \tag{2.1}$$

The treatment combination $\mathbf{0} = (0, 0, \dots, 0)$ is the additive identity element of G. Recall that if $\mathbf{x} \in G$ and ℓ is a positive integer, then by $\ell\mathbf{x}$ we mean

$$\ell\mathbf{x} = \overbrace{\mathbf{x} + \cdots + \mathbf{x}}^{\ell}.$$

A set of treatment combinations $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is called independent if $\sum_{i=1}^{k} \ell_i \mathbf{x}_i = 0$ implies $\ell_i \mathbf{x}_i = 0$, for $i = 1, \dots, k$. A fractional factorial design $\mathbf{d}$ is usually chosen to be some subgroup of G, and each subgroup of G contains a set of independent generators $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ in the sense that the $\mathbf{x}_i$'s are independent and any other treatment combination $\mathbf{x}$ in the subgroup can be expressed as $\mathbf{x} = \ell_1 \mathbf{x}_1 + \cdots + \ell_k \mathbf{x}_k$, for appropriate integers $\ell_1, \dots, \ell_k$.

The model considered here for analyzing the data from a given fractional factorial design $\mathbf{d}$ is one which utilizes orthogonal polynomials.

DEFINITION 2.1. The system of orthogonal polynomials on m equally spaced points $i = 0, 1, \dots, m - 1$ is the set $\{P_{km}, \ k = 0, \dots, m - 1\}$ of polynomials satisfying

$$\sum_{i=0}^{m-1} P_{km}(i) P_{k'm}(i) = 0 \quad \text{for all } k \neq k', \tag{2.2}$$

where $P_{0m}(i) = 1$, for $i = 0, 1, \dots, m - 1$, and $P_{km}(i)$ is a polynomial of degree k. We assume that each polynomial in the system is scaled so that its values are always integers.

With factor A_i having s_i levels, we associate $s_i - 1$ main effect component parameters A_i^j, $j = 1, \dots, s_i - 1$. A_i^j is called the jth-order main effect of A_i. For a given linearly ordered allocation of the treatments in $\mathbf{d}$ to experimental units, suppose we let $\mathbf{y} = (y_1, \dots, y_N)'$ denote the ordered vector of observations obtained. The model for $\mathbf{d}$ can be written as

$$\mathbf{y} = X\beta + \varepsilon = X_1\beta_1 + X_2\beta_2 + \varepsilon, \tag{2.3}$$

where ε is an $N \times 1$ vector of independent error terms having expectation zero and constant variance σ^2. The parameters in β_1 correspond to the factorial effects defined previously and the parameters in β_2 to possible trend effects. If we let $X = (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_p) = (x_{ij})$, we shall assume that the first column of X, $\mathbf{x}_0$, corresponds to an overall mean effect u and that it is a column of 1's. If $\mathbf{x}_t$ is the column of X_1 corresponding to main effect component A_i^j and factor A_i occurs at level w in run $y_{\underline{\ell}}$, then $x_{\underline{\ell}t} = P_{js_i}(w)$. It is assumed that any trend effect can be represented as a polynomial of the form

$$\text{trend effect} = \alpha_0 + \alpha_1 x + \cdots + \alpha_\theta x^\theta, \qquad x = 1, \ldots, N, \tag{2.4}$$

and the values that x assumes correspond to the positions in which observations are obtained in the run order.

In this paper we restrict our attention to orthogonal main effects plans. A design $\mathbf{d}$ is said to be an orthogonal main effects plan if the columns of X_1 in model (2.3) corresponding to the main effect parameters form a mutually orthogonal set of vectors. Unless specifically stated otherwise, we shall assume that, for a given design $\mathbf{d}$, all levels of each factor A_i occur the same number of times in the treatment combinations which make up $\mathbf{d}$.

DEFINITION 2.2. Let $\mathbf{d}$ be an orthogonal main effects plan such as previously described. Let $\mathbf{x}_j$ be the column of X_1 in model (2.3) corresponding to the main effect component A_i^k. We say $\mathbf{x}_j$ or A_i^k is t-trend free or t-trend resistant if

$$\sum_{i=1}^{N} x_{ij} i^z = 0 \quad \text{for } z = 0, 1, \ldots, t. \tag{2.5}$$

We say factor A_i is t-trend free if all $s_i - 1$ main effect components of factor A_i are at least t-trend free.

COMMENT. We note that for factor A_i to be t-trend free according to Definition 2.2, the sum of the pth powers of the positions in which the levels of factor A_i occur must all be the same for $p = 0, 1, \ldots, t$.

We now give a lemma which is useful in Sections 3 and 4.

LEMMA 2.3. *Let* $\mathbf{d}$ *be some run order based on* N *experimental units and let* $\mathbf{t}'\mathbf{y} = \sum_{i=1}^N t_i y_i$ *be an estimator for a main effect component of some factor in* $\mathbf{d}$ *which is* p*-trend free. Then*

$$\sum_{i=1}^{N} t_i (ai + b)^z = 0 \quad \textit{for } z = 0, 1, \ldots, p,$$

where $a \geq 0$ *and* b *are constants.*

PROOF. Simply observe that

$$\sum_{i=1}^{N} t_i(ai+b)^z = \sum_{i=1}^{N} t_i \sum_{j=0}^{z} \binom{z}{j} (ai)^j b^{z-j}$$
$$= \sum_{j=0}^{z} \binom{z}{j} a^j b^{z-j} \sum_{i=1}^{N} t_i i^j = 0 \quad \text{for } z = 0, 1, \dots, p,$$

where the last equality follows from the assumption that $\mathbf{t}'\mathbf{y}$ is p-trend free. □

3. Construction of 1-trend free designs. Here we show that by using the results of Phillips (1968) one can easily construct an $s_1 \times s_2 \times \cdots \times s_n$ 1-trend free design.

For $n = 2$, Phillips (1968) has shown that an $s_1 \times s_2$ 1-trend free design can exist only when s_1 and s_2 are both even or both odd. He further describes how such designs can be constructed when s_1 and s_2 are both even or odd by using magic rectangles. We shall use these results as the basis for our construction process. We state our next lemma, which is easy to prove.

LEMMA 3.1. *For $n \geq 3$, an $s_1 \times \cdots \times s_n$ design that is* 1*-trend free for main effects can exist only if* (i) *there exist at least two s_i's which are even or* (ii) *all the s_i's are odd.*

We now show that when the conditions of Lemma 3.1 are satisfied, an $s_1 \times s_2 \times \cdots \times s_n$ design that is 1-trend free for main effects can always be constructed.

THEOREM 3.2. *Let $s_1, s_2, \dots, s_n$ satisfy the conditions of Lemma* 3.1. *Then there exists an $s_1 \times s_2 \times \cdots \times s_n$ design that is* 1*-trend free for main effects.*

PROOF. The proof is by construction. Without loss of generality, assume that the conditions of Lemma 3.1 are satisfied and that s_1 and s_2 are both even if at least two of the s_i's are even or both odd if all the s_i's are odd. Now construct an $s_1 \times s_2$ design $\mathbf{d}_2$ which is 1-trend free in factors A_1 and A_2 using the method of magic rectangles given in Phillips (1968). Denote the ordered treatment combinations in $\mathbf{d}_2$ by $\mathbf{x}_1, \dots, \mathbf{x}_{s_1 s_2}$. By letting the treatment combinations in $\mathbf{d}_2$ serve as blocks and the levels of factor A_3 serve as treatments, one can use the results of Yeh and Bradley (1983) on constructing 1-trend free complete block designs to construct a run order $\mathbf{d}_3$ which is 1-trend free in the main effects of factors A_1, A_2 and A_3. By repeating this same construction process n times, we eventually arrive at a design $\mathbf{d}_n$ which is 1-trend free for the main effects of all factors $A_1, \dots, A_n$. □

COMMENT. Generally speaking, the simplest method of constructing an $s_1 \times s_2 \times \cdots \times s_n$ 1-trend free mixed level factorial experiment is to start with the two smallest odd or even s_i's, say, s_1 and s_2, construct the initial $s_1 \times s_2$ magic rectangle, then extend the initial $s_1 \times s_2$ magic rectangle to the desired $s_1 \times s_2 \times \cdots \times s_n$ design using the methods outlined in Theorem 3.2.

COMMENT. In general, $s_1 \times s_2$ magic rectangles are more difficult to construct when s_1 and s_2 are both odd than when s_1 and s_2 are both even. The methods available for constructing magic rectangles having sides of even order are much more systematic than those methods available for constructing magic rectangles having sides of odd order. Therefore, for convenience and ease of reference, the Appendix gives a listing of some $s_1 \times s_2$ magic rectangles for $s_1, s_2 = 3, 5, 7, 9$, $s_1 \neq s_2$.

4. Main results. In this section we give the main results of this paper. We begin by introducing some additional notation and facts.

Let $\mathbf{d} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ be some mixed level factorial design, and let $\mathbf{y} = (y_1, \ldots, y_n)$, $y_i \in \{0, 1, \ldots, s_i - 1\}$, $i = 1, \ldots, n$, be some treatment combination. Then we define

$$\mathbf{y} \dotplus \mathbf{d} = (\mathbf{y} + \mathbf{x}_1,\ \mathbf{y} + \mathbf{x}_2, \ldots,\ \mathbf{y} + \mathbf{x}_N). \tag{4.1}$$

Also, if $\mathbf{d}_1 = (\mathbf{x}_1, \ldots, \mathbf{x}_M)$ is a run order of an $s_1 \times \cdots \times s_n$ design and $\mathbf{d}_2 = (\mathbf{y}_1, \ldots, \mathbf{y}_n)$ is another run order of an $s_1 \times \cdots \times s_n$ design, then we define

$$\mathbf{d}_2 \dotplus \mathbf{d}_1 = (\mathbf{y}_1 \dotplus \mathbf{d}_1, \mathbf{y}_2 \dotplus \mathbf{d}_1, \ldots, \mathbf{y}_N \dotplus \mathbf{d}_1). \tag{4.2}$$

With (4.1) and (4.2) in mind, we prove the following theorem.

THEOREM 4.1. *Let* $\mathbf{d}_1 = (x_1, \ldots, x_M)$, $x_i \in \{0, 1, \ldots, s_1 - 1\}$, $i = 1, \ldots, M$, *be a run order in a single factor* A_1 *having* s_1 *levels, and let* $\mathbf{d}_2 = (y_1, \ldots, y_N)$, $y_i \in \{0, 1, \ldots, s_1 - 1\}$, $i = 1, \ldots, N$, *be a run order in the same factor* A_1. *If* $\mathbf{d}_1$ *is* q*-trend free and* $\mathbf{d}_2$ *is* p*-trend free, then* $\mathbf{d}_2 \dotplus \mathbf{d}_1 = \mathbf{d}$ *is* $(p + q + 1)$*-trend free.*

PROOF. Throughout the proof, we shall let B_h, $h = 0, 1, \ldots, s_1 - 1$, denote the actual position numbers in which level h of factor A_1 is applied to experimental units in $\mathbf{d}$. Then clearly, $\mathbf{d}$ is $(p + q + 1)$-trend free if, for $h = 0, 1, \ldots, s_1 - 1$ and for the constant $c(z) = \sum_{w=1}^{MN} w^z / s_1$,

$$\sum_{w \in B_h} w^z = c(z) \quad \text{for } z = 0, 1, \ldots, p + q + 1. \tag{4.3}$$

To help show that (4.3) holds for $h = 0, 1, \ldots, s_1 - 1$, we make the following definitions:

$$\begin{aligned} B_{hi} &= \bigl\{k \in \{1, \ldots, M\} \mid (i-1)M + k \in B_h\bigr\} \quad \text{for } i = 1, \ldots, N, \\ &= \text{the set of position numbers } k \text{ in } \mathbf{d}_1 \text{ that produce level } h \text{ in} \\ &\qquad \text{position } (i-1)M + k, \text{ that is, where } y_i + x_k = h \pmod{s_1}. \\ \overline{B}_{hk} &= \bigl\{i \in \{1, \ldots, N\} \mid (i-1)M + k \in B_h\bigr\} \quad \text{for } k = 1, \ldots, M, \\ &= \text{the set of position numbers } i \text{ in } \mathbf{d}, \text{ that produce level } h \text{ in} \\ &\qquad \text{position } (i-1)M + k, \text{ that is, where } y_i + x_k = h \pmod{s_1}. \end{aligned} \tag{4.4}$$

With regard to the definitions given in (4.4), we note the following facts, which follow directly from the definitions and from Lemma 2.3:

(4.5(i))
$$\sum_{w \in B_k} w^z = \sum_{i=1}^{N} \sum_{k \in B_{hi}} \{(i-1)M+k\}^z = \sum_{k=1}^{M} \sum_{i \in \bar{B}_{hk}} \{(i-1)M+k\}^z \quad \text{for } z = 0,1,2,\ldots.$$

For $i = 1,\ldots,N$,

(4.5(ii))
$$\sum_{k \in B_{hi}} \{(i-1)M+k\}^z = C(i,z) \quad \text{for } h = 0,1,\ldots,s_1-1 \text{ and } z = 0,1,\ldots,q,$$

where $C(i,z) = \sum_{j=(i-1)M+1}^{iM} j^z/s_1$.

For $i = 1,\ldots,N$,

(4.5(iii))
$$\sum_{k \in B_{hi}} k^z = D(z) \quad \text{for } h = 0,1,\ldots,s_1-1 \text{ and } z = 0,1,\ldots,q,$$

where $D(z) = \sum_{j=1}^{M} j^z/s_1$.

For $k = 1,\ldots,M$,

(4.5(iv))
$$\sum_{i \in \bar{B}_{hk}} \{(i-1)M+k\}^z = \overline{C}(k,z) \quad \text{for } h = 0,1,\ldots,s_1-1 \text{ and } z = 0,1,\ldots,p,$$

where $\overline{C}(k,z) = \sum_{j=1}^{N}\{(j-1)M+k\}^z/s_1$.

For $k = 1,\ldots,M$,

(4.5(v))
$$\sum_{i \in \bar{B}_{hk}} \{(i-1)M\}^z = \overline{D}(z) \quad \text{for } h = 0,1,\ldots,s_1-1 \text{ and } z = 0,1,\ldots,p,$$

where $\overline{D}(z) = \sum_{j=1}^{N}\{(j-1)M\}^z/s_1$.

Using (4.5(i)–(v)), we now proceed to establish (4.3). For $z = 0,1,\ldots,q$, we have

$$\sum_{w \in B_h} w^z = \sum_{i=1}^{N} \sum_{k \in B_{hi}} \{(i-1)M+k\}^z = \sum_{i=1}^{N} C(i,z), \quad \text{for } h = 0,1,\ldots,s_1-1 \quad \left[\text{by } (4.5(\text{ii}))\right].$$

For $z = q + 1, \dots, p + q + 1$, we note that

$$\begin{aligned}
\sum_{w \in B_h} w^z &= \sum_{i=1}^{N} \sum_{k \in B_{hi}} \{(i-1)M + k\}^z \\
&= \sum_{i=1}^{N} \sum_{k \in B_{hi}} \sum_{j=0}^{z} \binom{z}{j} ((i-1)M)^{z-j} k^j \\
&= \sum_{i=1}^{N} \sum_{k \in B_{hi}} \sum_{j=0}^{q} \binom{z}{j} ((i-1)M)^{z-j} k^j + \sum_{i=1}^{N} \sum_{k \in B_{hi}} \sum_{j=q+1}^{z} \binom{z}{j} ((i-1)M)^{z-j} k^j \\
&= \sum_{i=1}^{N} \sum_{j=0}^{q} \binom{z}{j} ((i-1)M)^{z-j} \sum_{k \in B_{hi}} k^j + \sum_{i=1}^{N} \sum_{k \in B_{hi}} \sum_{j=q+1}^{z} \binom{z}{j} ((i-1)M)^{z-j} k^j \\
&= \sum_{i=1}^{N} \sum_{j=0}^{q} \binom{z}{j} ((i-1)M)^{z-j} D(j) + \sum_{i=1}^{N} \sum_{k \in B_{hi}} \sum_{j=q+1}^{z} \binom{z}{j} ((i-1)M)^{z-j} k^j \\
&\qquad\qquad \left[\text{from } (4.5(\text{iii}))\right] \\
&= \sum_{i=1}^{N} \sum_{j=0}^{q} \binom{z}{j} ((i-1)M)^{z-j} D(j) + \sum_{k=1}^{M} \sum_{i \in \bar{B}_{hk}} \sum_{j=q+1}^{z} \binom{z}{j} ((i-1)M)^{z-j} k^j \\
&\qquad\qquad \left[\text{from } (4.5(\text{i}))\right] \\
&= \sum_{i=1}^{N} \sum_{j=0}^{q} \binom{z}{j} ((i-1)M)^{z-j} D(j) + \sum_{k=1}^{M} \sum_{j=q+1}^{z} \binom{z}{j} k^j \sum_{i \in \bar{B}_{hk}} ((i-1)M)^{z-j} \\
&= \sum_{i=1}^{N} \sum_{j=0}^{q} \binom{z}{j} ((i-1)M)^{z-j} D(j) + \sum_{k=1}^{M} \sum_{j=q+1}^{z} \binom{z}{j} k^j \bar{D}(z-j)
\end{aligned}$$

[by (4.5(v)) and the fact that $z - j \le p$ for $q+1 \le z \le p+q+1$ and $j = q+1, \dots, z$].. We note that the last expression given does not depend on h; hence we have the desired result. □

The following generalization of Theorem 4.1 to mixed factorial designs follows directly from Theorem 4.1.

THEOREM 4.2. *Let $\mathbf{d}_1 = (\mathbf{x}_1, \dots, \mathbf{x}_M)$ be a run order in n factors A_i having s_i levels, $i = 1, \dots, n$, and let $\mathbf{d}_2 = (\mathbf{y}_1, \dots, \mathbf{y}_N)$ be a run order in the same factors. If $\mathbf{d}_1$ and $\mathbf{d}_2$ are p_i- and q_i-trend free for the main effects of factor A_i, $i = 1, \dots, n$, respectively, then $\mathbf{d} = \mathbf{d}_2 \dotplus \mathbf{d}_1$ is $(p_i + q_i + 1)$-trend free for the main effects of factor A_i, $i = 1, \dots, n$.*

For the remainder of this section, we shall consider the construction of

$s_1^{n_1} \times s_2^{n_2} \times \cdots \times s_k^{n_k}$ designs, where $n_1 + n_2 + \cdots + n_k = n$. Without loss of generality, we shall assume $n_1 \le n_2 \le \cdots \le n_k$. The construction methods given are based on Theorem 4.2 and the results given in Section 3.

THEOREM 4.3. *Consider a complete $s_1^{n_1} \times \cdots \times s_k^{n_k}$ design where the s_i's are all even or all odd and $n_1 = \cdots = n_{k_1} < n_{k_1+1} = \cdots = n_{k_2} < \cdots < n_{k_{t-1}+1} = \cdots = n_{k_t}$ and $n_{k_t} = n_k$. Further, suppose that, for $l = 1, \ldots, t$ and $k_{l-1} + 1 \le i \le k_l$, there exists an ordered set of generators $\mathbf{x}_j^{(i)}$, $j = 1, \ldots, n_i$, for a complete $s_i^{n_i}$ factorial design such that all levels of each factor having s_i levels occur in $\{0\mathbf{x}_1^{(i)}, \mathbf{x}_1^{(i)}, \ldots, (s_i - 1)\mathbf{x}_1^{(i)}\}$; for each factor A_p having s_i levels, let $n_{k_l}(p)$ be the number of generators $\mathbf{x}_j^{(i)}$, $j = 1, \ldots, n_{k_l}$, in which all levels of factor A_p occur in $\{0\mathbf{x}_j^{(i)}, \mathbf{x}_j^{(i)}, 2\mathbf{x}_j^{(i)}, \ldots, (s_i - 1)\mathbf{x}_j^{(i)}\}$. Then there exists a run order $\mathbf{d}$ such that the following hold:*

(i) *$\mathbf{d}$ is $(2n_{k_l}(p) - 1)$-trend free in all factors A_p having s_i levels, $k_{l-1}+1 \le i \le k_l$, $l = 1, \ldots, t - 1$;*
(ii) *$\mathbf{d}$ is $(2n_{k_t}(p) - 1)$-trend free in all factors A_p having s_i levels, $i \ge k_{t-1} + 1$, when $k_t - k_{t-1} > 1$;*
(iii) *$\mathbf{d}$ is $(n_{k_t}(p) + n_{k_{t-1}}(p) - 1)$-trend free in all factors A_p having s_i levels, $i = k_t$, when $k_t - k_{t-1} = 1$.*

PROOF. The proof is by construction. Since the proof is essentially the same for both the even and odd cases, we shall only give it for the situation where all the s_i's are odd. To begin with, use the methods of Section 3 to obtain a series of $n_k = n_{k_t}$ run orders $\overline{\mathbf{d}}_1, \overline{\mathbf{d}}_2, \ldots, \overline{\mathbf{d}}_{n_{k_t}}$ in factors $\overline{A}_1, \overline{A}_2, \ldots, \overline{A}_k$, where factor $\overline{A}_i$ has s_i levels, $0, 1, \ldots, s_i - 1$ and where the $\overline{\mathbf{d}}_j$ have the following properties:

(4.6)
(i) If $n_{k_{l-1}} + 1 \le j \le n_{k_l}$, $l = 1, \ldots, t - 1$, $k_0 = 0$, $\overline{\mathbf{d}}_j$ is an $s_{k_{l-1}+1} \times s_{k_{l-1}+2} \times \cdots \times s_k$ design which is 1-trend free in factors $\overline{A}_{k_{l-1}+1}, \ldots, \overline{A}_k$ and has 0 entries for factors $\overline{A}_1, \ldots, \overline{A}_{k_{l-1}}$.
(ii) If $k_t - k_{t-1} > 1$ and $n_{k_{t-1}} + 1 \le j \le n_{k_t}$, $\overline{\mathbf{d}}_j$ is an $s_{k_{t-1}+1} \times \cdots \times s_k$ design which is 1-trend free in factors $\overline{A}_{k_{t-1}+1}, \ldots, \overline{A}_k$ and has 0 entries for factors $\overline{A}_1, \ldots, \overline{A}_{k_{t-1}}$.
(iii) If $k_t - k_{t-1} > 1$ and $j = k_t$, $\overline{\mathbf{d}}_j$ is an s_k design which is 0-trend free in factor $\overline{A}_k$ and has 0 entries for factors $\overline{A}_1, \ldots, \overline{A}_{k-1}$.

Now, from the $\overline{\mathbf{d}}_j$, we construct run orders $\mathbf{d}_j$ in factors $A_1, \ldots, A_n$ according to the following rule:

(4.7) If a treatment combination in $\overline{\mathbf{d}}_j$ has factor $\overline{A}_i$ occurring at level p, $p \in \{0, 1, \ldots, s_{i-1}\}$, replace p by $p\mathbf{x}_j^{(i)}$.

We note that $\mathbf{d}_1$ obtained from $\overline{\mathbf{d}}_1$ in (4.7) is 1-trend free for the main effects of all factors since each factor having s_i levels has all of its levels occurring in

$\{0\mathbf{x}_1^{(i)}, \mathbf{x}_1^{(i)}, \dots, (s_i - 1)\mathbf{x}_1^{(i)}\}$, $i = 1, \dots, k$. We also note that, for $j \geq 2$, $\mathbf{d}_j$ obtained from $\overline{\mathbf{d}}_j$ in (4.7) is 1-trend free for the main effects of any factor A_p which has all of its levels occurring in $\overline{\mathbf{d}}_j$. Finally, we obtain the desired run order $\mathbf{d}$ by sequentially applying Theorem 4.2 to $\mathbf{d}_1, \dots, \mathbf{d}_{n_k}$, that is,

$$\mathbf{d} = \left(\mathbf{d}_{n_k} \dotplus \cdots \dotplus (\mathbf{d}_3 \dotplus (\mathbf{d}_2 \dotplus \mathbf{d}_1)) \cdots \right).$$

The result now follows from Theorem 4.2. □

COMMENT. Bailey, Cheng and Kipnis (1992) give several methods for constructing mixed level factorial run orders. Some of the run orders they produce do, under certain conditions, produce various contrasts for main effects that have degrees of trend resistance analogous to those given in Theorem 4.3. However, the designs produced by Theorem 4.3 seem to produce mixed level designs with higher degrees of trend resistance for all main effect contrasts than do the methods of construction given in Bailey, Cheng and Kipnis (1992).

EXAMPLE 4.4. Consider the construction of a trend resistant $3^2 \times 5^3$ design. Following the construction process outlined in the proof of Theorem 4.3, we begin by obtaining the designs $\overline{\mathbf{d}}_1$, $\overline{\mathbf{d}}_2$ and $\overline{\mathbf{d}}_3$ in factors $\overline{A}_1$ and $\overline{A}_2$ given by

$$\begin{aligned}\overline{\mathbf{d}}_1 = \overline{\mathbf{d}}_2 = \{&(1,4),(2,0)(1,3)(0,2)(0,1)(2,1)(0,0)(1,2)(2,4)\\ &(0,3)(2,3)(2,2)(1,1)(0,4)(1,0)\}\end{aligned}$$

and

$$\overline{\mathbf{d}}_3 = \{(0,4)(0,0)(0,3)(0,2)(0,1)\}.$$

Now, it is easily seen that $\mathbf{x}_1^{(1)} = (1,1)$ and $\mathbf{x}_2^{(1)} = (1,2)$ will generate a complete 3^2 design and that $\mathbf{x}_1^{(2)} = (1,1,1)$, $\mathbf{x}_2^{(2)} = (1,2,1)$ and $\mathbf{x}_3^{(2)} = (1,1,2)$ will generate a complete 5^3 design. Thus the designs $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$ that we construct from $\overline{\mathbf{d}}_1$, $\overline{\mathbf{d}}_2$ and $\overline{\mathbf{d}}_3$ using these generators are given by

$$\mathbf{d}_1 = \left\{\begin{array}{l}(1,1,4,4,4)(2,2,0,0,0)(1,1,3,3,3)(0,0,2,2,2)(0,0,1,1,1)\\ (2,2,1,1,1)(0,0,0,0,0)(1,1,2,2,2)(2,2,4,4,4)(0,0,3,3,3)\\ (2,2,3,3,3)(2,2,2,2,2)(1,1,1,1,1)(0,0,4,4,4)(1,1,0,0,0)\end{array}\right\},$$

$$\mathbf{d}_2 = \left\{\begin{array}{l}(1,2,4,3,4)(2,1,0,0,0)(1,2,3,1,3)(0,0,2,4,2)\\ (0,0,1,2,1)(2,1,1,2,1)(0,0,0,0,0)(1,2,2,4,2)\\ (2,1,4,3,4)(0,0,3,1,3)(2,1,3,1,3)(2,1,2,4,2)\\ (1,2,1,2,1)(0,0,4,3,4)(1,2,0,0,0)\end{array}\right\},$$

$$\mathbf{d}_3 = \{(0,0,4,4,3)(0,0,0,0,0)(0,0,3,3,1)(0,0,2,2,4)(0,0,1,1,2)\}.$$

Finally, the desired design is $\mathbf{d} = \mathbf{d}_3 \dotplus (\mathbf{d}_2 \dotplus \mathbf{d}_1)$. In $\mathbf{d}$, by Theorem 4.3, the main effects for all 3-level factors are 3-trend free whereas the main effects for all 5-level factors are all 4-trend free.

EXAMPLE 4.5. Consider the construction of a trend resistant $2^2 \times 4^2$ design. Following the construction process outlined in the proof of Theorem 4.3, we begin by obtaining the designs $\overline{\mathbf{d}}_1$ and $\overline{\mathbf{d}}_2$ in factors $\overline{A}_1$ and $\overline{A}_2$ given by

$$\overline{\mathbf{d}}_1 = \overline{\mathbf{d}}_2 = \{(0,0)(1,1)(1,2)(0,3)(0,3)(1,2)(1,1)(0,0)\}.$$

Now it is easily seen that $\mathbf{x}_1^{(1)} = (1,1)$ and $\mathbf{x}_2^{(1)} = (1,0)$ will generate a complete 2^2 design and that $\mathbf{x}_1^{(2)} = (1,1)$ and $\mathbf{x}_2^{(2)} = (1,2)$ will generate a complete 4^2 design. However, since $\mathbf{x}_2^{(1)} = (1,0)$ and $\mathbf{x}_2^{(2)} = (1,2)$, $2\mathbf{x}_2^{(2)} = (2,0)$ and $3\mathbf{x}_2^{(2)} = (3,2)$, we see that not all levels of factor A_2 occur in $\mathbf{x}_2^{(1)}$ and not all levels of factor A_4 occur in $0\mathbf{x}_2^{(2)}, \mathbf{x}_2^{(2)}, 2\mathbf{x}_2^{(2)}$ and $3\mathbf{x}_2^{(2)}$. Now, the designs $\mathbf{d}_1$ and $\mathbf{d}_2$ that we construct from $\overline{\mathbf{d}}_1$ and $\overline{\mathbf{d}}_2$ using these generators are given by

$$\mathbf{d}_1 = \left\{ \begin{matrix} (0,0,0,0)(1,1,1,1)(1,1,2,2)(0,0,3,3) \\ (0,0,3,3)(1,1,2,2)(1,1,1,1)(0,0,0,0) \end{matrix} \right\},$$

$$\mathbf{d}_2 = \left\{ \begin{matrix} (0,0,0,0)(1,0,1,2)(1,0,2,0)(0,0,3,2) \\ (0,0,3,2)(1,0,2,0)(1,0,1,2)(0,0,0,0) \end{matrix} \right\}.$$

Finally, the desired design is $\mathbf{d} = \mathbf{d}_2 \dotplus \mathbf{d}_1$. In $\mathbf{d}$, by Theorem 4.3, the main effects of factors A_1 and A_3 are 3-trend free and the main effects of factors A_2 and A_4 are 1-trend free.

DEFINITION 4.6. Let $\mathbf{d}_1 = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{N_1})$ be a factorial design involving n_1 factors $A_1, \ldots, A_{n_1}$ and let $\mathbf{d}_2 = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{N_2})$ be a factorial design involving n_2 factors $\widehat{A}_1, \ldots, \widehat{A}_{n_2}$. Now let $\mathbf{d} = \mathbf{d}_1 \boxplus \mathbf{d}_2$ be that design involving $n_1 + n_2$ factors and having $N_1 \times N_2$ runs which is defined by

$$\begin{aligned} \mathbf{d} = \mathbf{d}_1 \boxplus \mathbf{d}_2 = \{&(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_1, \mathbf{y}_2), \ldots, (\mathbf{x}_1, \mathbf{y}_{N_2}) \\ &(\mathbf{x}_2, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2), \ldots, (\mathbf{x}_2, \mathbf{y}_{N_2}) \\ &\vdots \\ &(\mathbf{x}_{N_1}, \mathbf{y}_1), (\mathbf{x}_{N_1}, \mathbf{y}_2), \ldots, (\mathbf{x}_{N_1}, \mathbf{y}_{N_2})\}, \end{aligned} \tag{4.8}$$

where factors $A_1, \ldots, A_{n_1}$ of $\mathbf{d}$ correspond to factors $A_1, \ldots, A_{n_1}$ of $\mathbf{d}_1$, and factors $A_{n_1+1}, \ldots, A_{n_1+n_2}$ of $\mathbf{d}$ correspond to factors $\widehat{A}_1, \ldots, \widehat{A}_{n_2}$ of $\mathbf{d}_2$.

LEMMA 4.7. *Let $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}$ be designs defined as in Definition 4.6 and assume that $\mathbf{d}_1$ is p_i-trend free for the main effects of factor A_i, $i = 1, \ldots, n_1$, and $\mathbf{d}_2$ is q_j-trend free for the main effects of factor $\widehat{A}_j$, $j = 1, \ldots, n_2$. If $\mathbf{d} = \mathbf{d}_1 \boxplus \mathbf{d}_2$,* *then* $\mathbf{d}$ *is* p_i*-trend free for the main effects of factors* A_i, $i = 1, \dots, n_1$ *and* q_j*-trend free for the main effects of factor* A_{n_1+j}, $j = 1, \dots, n_2$.

PROOF. This follows from examining the run order given in (4.8) and applying Lemma 2.3. □

With (4.8) and Lemma 4.7, it is now easy to construct mixed factorial designs having factors with even and odd numbers of levels. In particular, suppose you want to construct a trend free $s_1^{n_1} \times \cdots \times s_p^{n_p} \times s_{p+1}^{n_{p+1}} \times \cdots \times s_k^{n_k}$ design $\mathbf{d}$, where s_i, $i = 1, \dots, p$, is even and s_i, $i = p+1, \dots, k$, is odd. To accomplish this, simply construct run orders $\mathbf{d}_1$ and $\mathbf{d}_2$ out of the factors having odd and even numbers of levels, respectively, using the methods described in Theorem 4.3, and then construct the run order $\mathbf{d} = \mathbf{d}_1 \boxplus \mathbf{d}_2$. The run order $\mathbf{d}$ will have by Lemma 4.7 the same trend free properties as the component designs $\mathbf{d}_1$ and $\mathbf{d}_2$ have. The following example illustrates the construction process.

EXAMPLE 4.8. Consider the construction of a trend resistant $3^2 \times 5^3 \times 2^2 \times 4^2$ design. Using the construction process previously outlined, let $\mathbf{d}_1$ denote the trend resistant $3^2 \times 5^3$ design obtained in Example 4.4, and let $\mathbf{d}_2$ denote the trend resistant $2^2 \times 4^2$ design obtained in Example 4.5. Then $\mathbf{d} = \mathbf{d}_1 \boxplus \mathbf{d}_2$ is, by Theorem 4.3 and Lemma 4.7, a $3^2 \times 5^3 \times 2^2 \times 4^2$ run order which is 4-trend free in the main effects of factors A_3, A_4, A_5, 3-trend free in the main effects of factors A_1, A_2, A_6, A_8 and 1-trend free in the main effects of factors A_7 and A_9.

5. Fractional factorial designs. In a regular $(1/s^p)$ fraction of a complete s^n design, there are n factors occurring at s levels and a total of s^{n-p} experimental runs. Usually $n-p$ of the factors are referred to as the basic factors and the remaining factors as added factors. The levels of the added factors are usually determined from those of the basic factors by some set of defining relations. If one is interested in constructing a run order of an $s_1^{n_1-p_1} \times s_2^{n_2-p_2} \times \cdots \times s_k^{n_k-p_k}$ mixed level fractional factorial experiment where each $s_i^{n_i-p_i}$, $i = 1, \dots, k$, denotes a $(1/p_i)$ fraction of a complete $s_i^{n_i}$ design, one can follow the methods of construction outlined in Section 4 after selecting appropriate sets of generators $\mathbf{x}_j^{(i)}$, $j = 1, \dots, n_i - p_i$. Of course, in selecting the generators for the $s_i^{n_i-p_i}$ design, $i = 1, \dots, k$, using the corresponding relations, one needs to make sure that the main effect components of all factors in the designs are adequately free from confounding with interactions. Once this is done, the methods of construction given in Section 4 are straightforward to apply. The following example illustrates the process.

EXAMPLE 5.1. Consider the construction of a trend resistant regular $3^{3-1} \times 5^{4-2}$ design. Following the construction process outlined in the proof of Theorem 4.3, we begin by obtaining the designs $\overline{\mathbf{d}}_1$ and $\overline{\mathbf{d}}_2$ in factors $\overline{A}_1$ and $\overline{A}_2$ given by

$$\overline{\mathbf{d}}_1 = \overline{\mathbf{d}}_2 = \{(1,4)(2,0)(1,3)(0,2)(0,1)(2,1)(0,0)(1,2)\\ (2,4)(0,3)(2,3)(2,2)(1,1)(0,4)(1,0)\}.$$

Now, it is easily seen that $\mathbf{x}_1^{(1)} = (1,1,2)$ and $\mathbf{x}_2^{(1)} = (1,2,1)$ will generate an orthogonal 3^{3-1} main effects design and that $\mathbf{x}_1^{(2)} = (1,1,1,1)$ and $\mathbf{x}_2^{(2)} = (1,2,3,4)$ will generate an orthogonal 5^{4-2} main effects design. Thus the designs $\mathbf{d}_1$ and $\mathbf{d}_2$ that we construct from $\overline{\mathbf{d}}_1$ and $\overline{\mathbf{d}}_2$ using these generators are given by

$$\mathbf{d}_1 = \left\{ \begin{array}{l} (1,1,2,4,4,4,4)(2,2,1,0,0,0,0)(1,1,2,3,3,3,3)(0,0,0,2,2,2,2)(0,0,0,1,1,1,1) \\ (2,2,1,1,1,1,1)(0,0,0,0,0,0,0)(1,1,2,2,2,2,2)(2,2,1,4,4,4,4)(0,0,0,3,3,3,3) \\ (2,2,1,3,3,3,3)(2,2,1,2,2,2,2)(1,1,2,1,1,1,1)(0,0,0,4,4,4,4)(1,1,2,0,0,0,0) \end{array} \right\},$$

$$\mathbf{d}_2 = \left\{ \begin{array}{l} (1,2,1,4,3,2,1)(2,1,2,0,0,0,0)(1,2,1,3,1,4,2)(0,0,0,2,4,1,3)(0,0,0,1,2,3,4) \\ (2,1,2,1,2,3,4)(0,0,0,0,0,0,0)(1,2,1,2,4,1,3)(2,1,2,4,3,2,1)(0,0,0,3,1,4,2) \\ (2,1,2,3,1,4,2)(2,1,2,2,4,1,3)(1,2,1,1,2,3,4)(0,0,0,4,3,2,1)(1,2,1,0,0,0,0) \end{array} \right\},$$

Finally, the desired design $\mathbf{d} = \mathbf{d}_2 \dotplus \mathbf{d}_1$. In $\mathbf{d}$, by Theorem 4.3, the main effects of all 3- and 5-level factors are 3-trend free.

APPENDIX

	0	1	2	3	4
0	7	5	4	10	14
1	15	13	8	3	1
2	2	6	12	11	9

	0	1	2	3	4	5	6
0	10	19	4	20	1	17	6
1	7	9	8	11	14	13	15
2	19	12	4	18	32	24	17

	0	1	2	3	4	5	6	7	8
0	13	4	22	10	1	19	16	7	25
1	2	20	11	8	26	17	5	23	14
2	27	18	9	24	15	6	21	12	3

	0	1	2	3	4	5	6
0	9	31	34	7	30	14	1
1	16	15	13	28	3	26	25
2	19	12	4	18	32	24	17
3	11	10	33	8	23	21	20
4	35	22	6	29	2	5	27

	0	1	2	3	4	5	6	7	8
0	22	7	37	30	15	45	17	2	32
1	5	35	20	13	43	28	6	36	21
2	34	19	4	38	23	8	42	27	12
3	40	25	10	33	18	3	41	26	11
4	14	29	44	1	16	31	9	24	39

	0	1	2	3	4	5	6	7	8
0	31	10	52	28	7	49	37	16	58
1	19	61	40	9	51	30	5	47	26
2	46	25	4	50	29	8	63	42	21
3	62	41	20	53	32	11	44	23	2
4	43	22	1	56	35	14	60	39	18
5	17	38	59	13	34	55	3	24	45
6	16	37	58	15	36	57	12	33	54

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