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ON THE DISTANCE BETWEEN SMOOTHED EMPIRICAL AND QUANTILE PROCESSES

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We consider Bahadur–Kiefer representations for smoothed quantile processes. We prove that the asymptotics of the distance between smoothed empirical and quantile processes can be completely different from that of the unsmoothed ones. We obtain a complete characterization of the possible limits.

1. Introduction. Let $X_1, X_2, \dots, X_n$ be independent identically distributed random variables with continuous distribution function F. We define the empirical distribution function

$$F_n(x) = \frac{1}{n} \sum_{1 \le i \le n} I\{X_i \le x\} \tag{1.1}$$

and the empirical process $\alpha_n(x) = n^{1/2}(F_n(x) - F(x))$, $-\infty < x < \infty$. The inverses of F and F_n, respectively, are $Q(t) = \inf\{x\colon F(x) \ge t\}$ and $Q_n(t) = \inf\{x\colon F_n(x) \ge t\}$, $0 < t < 1$. Assuming that the density function $f = F'$ exists, we define the quantile process $\rho_n(t) = n^{1/2} f(Q(t))(Q(t) - Q_n(t))$, $0 < t < 1$. If $f'(Q(t))$ exists and is bounded in a neighbourhood of $t_0 \in (0,1)$ and $f(Q(t_0)) > 0$, then it follows from results of Kiefer (1967, 1970) that we have

$$n^{1/4} |\Delta_n(t_0)| \to_{\mathscr{D}} Y, \tag{1.2}$$

where $\Delta_n(t) = \alpha_n(Q(t)) - \rho_n(t)$, and the distribution function of the limiting random variable Y is

$$\begin{aligned} P\{Y \le x\} &= \frac{2}{(t_0(1-t_0))^{1/2}} \\ &\quad \times \int_{-\infty}^{\infty} \Phi\left(\frac{x}{|u|^{1/2}}\right) \phi\left(\frac{u}{(t_0(1-t_0))^{1/2}}\right) du - 1, \end{aligned} \tag{1.3}$$

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with Φ and ϕ standing for the standard normal distribution and density functions, respectively. Kiefer also studied the asymptotics of $\Delta_n(t)$ uniformly in t. Assuming some regularity conditions on F, Kiefer (1970) showed that

$$n^{1/4}(\log n)^{-1/2} \sup_{a \le t \le b} |\Delta_n(t)| \to_{\mathscr{D}} \Big(\sup_{a \le t \le b} |B(t)| \Big)^{1/2}, \tag{1.4}$$

where $\{B(t),\ 0 \le t \le 1\}$ is a Brownian bridge and $0 \le a < b \le 1$.

For a discussion of (1.4) under somewhat milder conditions on F than those of Kiefer, we refer to Csörgő and Révész [(1978), Section 4] or to Csörgő and Révész [(1981), Theorem 5.2.5] or to Csörgő [(1983), Chapter 6]. The finite sample and asymptotic properties of F_n, Q_n and α_n, ρ_n have been extensively studied in mathematical statistics [cf. Shorack and Wellner (1986), Csörgő and Horváth (1993) and references therein]. In case F has a density, it may be more suitable to use a smooth estimator for F rather than the step function F_n. For instance, Efron (1979) suggested that a smooth estimator of F should be more appropriate for generating bootstrap samples.

In this paper we consider smoothing F_n via kernels and inverses of the thus smoothed empiricals. We define smoothed estimators $\tilde{F}_n$ of F by

$$\tilde{F}_n(t) = \frac{1}{h} \int_{-\infty}^{\infty} K\left(\frac{x - t}{h} \right) F_n(x)\, dx, \tag{1.5}$$

where K is a kernel and $h = h(n)$ is the smoothing parameter. Similarly to (1.2), (1.4) and (1.5), we define

$$\tilde{\alpha}_n(x) = n^{1/2}\big(\tilde{F}_n(x) - F(x)\big), \qquad -\infty < x < \infty, \tag{1.6}$$

$$\tilde{Q}_n(t) = \inf\big\{x: \tilde{F}_n(x) \ge t\big\}, \tag{1.7}$$

$$\tilde{\rho}_n(t) = n^{1/2} f(Q(t))\big(Q(t) - \tilde{Q}_n(t)\big), \tag{1.8}$$

$$\tilde{\Delta}_n(t) = \tilde{\alpha}_n(Q(t)) - \tilde{\rho}_n(t), \qquad 0 < t < 1. \tag{1.9}$$

In the next section we study the asymptotics of $\tilde{\Delta}_n(t_0)$, where $t_0 \in (0, 1)$. We show that the limit distribution of $\tilde{\Delta}_n(t_0)$ is not necessarily Y of (1.2). Depending on the choice of the smoothing parameter $h(n)$, we get different limit theorems for $\tilde{\Delta}_n(t_0)$. We study the behaviour of $\sup_{a \le t \le b} |\tilde{\Delta}_n(t)|$ in Section 3. The proofs are presented in Sections 4 and 5.

2. Asymptotics for $\tilde{\Delta}_n(t_0)$. We assume that h and K satisfy the following regularity conditions:

C.1. $h = h(n) \to 0$, $n \to \infty$;
C.2. K is bounded and continuous on $(-\infty, \infty)$;
C.3. $K(u) = 0$ if $u \notin [-A, A]$ for some $0 < A < \infty$;
C.4. $\int_{-\infty}^{\infty} K(u)\, du = 1$;
C.5. $\int_{-\infty}^{\infty} uK(u)\, du = 0$.

If K is a symmetric density function, having compact support, then conditions C.3, C.4 and C.5 hold true.

First we show that, if $h(n)$ is small, then (1.2) holds also for the smoothed processes.

THEOREM 2.1. *We assume that* C.1–C.5 *hold and that*

$$f(Q(t_0)) > 0, \tag{2.1}$$

$$f' \text{ exists and is continuous in a neighbourhood of } Q(t_0), \tag{2.2}$$

$$\limsup_{n\to\infty} n^{1/2}h(n) < \infty. \tag{2.3}$$

Then, as $n \to \infty$, we have

$$n^{1/4}|\tilde{\Delta}_n(t_0)| \to_{\mathscr{D}} Y. \tag{2.4}$$

We note that Mack (1987) proved the Bahadur (1966) type result

$$\tilde{\Delta}_n(t_0) = \mathscr{O}\big(n^{-1/4}(\log n)^{3/4}\big) \quad \text{a.s.}, \tag{2.5}$$

under somewhat stronger conditions than those of Theorem 2.1.

Roughly speaking, (2.4) holds if $\tilde{F}_n(t_0) - E\tilde{F}_n(t_0)$ dominates the bias term $E\tilde{F}_n(t_0) - F(t_0)$. Condition (2.3) amounts to saying that, in this case, $\tilde{F}_n$ is rightly smoothed in the sense that the bias becomes negligible. If (2.3) fails to hold, then the bias term also plays a crucial role in the limit.

We will also make use of assumptions C.6 and C.7.

C.6. There is an integer $r \geq 1$ such that

$$\int_{-\infty}^{\infty} u^i K(u)\,du = \begin{cases} 0, & \text{if } 1 \leq i \leq r-1, \\ r!\,u_r \neq 0, & \text{if } i = r. \end{cases} \tag{2.6}$$

That is, in addition to C.5, the number u_r is defined by this assumption; in addition to C.2 and C.3, we have the following:

C.7. K' and K'' exist and are continuous on $[-A, A]$.

As is usual, $f^{(0)}$ means f in the sequel.

THEOREM 2.2. *We assume that* C1–C.7 *and* (2.1) *hold and that*

$$f^{(2r)} \text{ exists and is continuous in a neighbourhood of } Q(t_0), \tag{2.7}$$

$$n^{1/2}h(n) \to \infty, \qquad n \to \infty. \tag{2.8}$$

If

$$n^{1/2}h^r(n) \to c_0, \qquad n \to \infty, \tag{2.9}$$

where $0 \leq c_0 < \infty$. Then, as $n \to \infty$, we have

$$\begin{aligned}(nh(n))^{1/2}|\tilde{\Delta}_n(t_0)| \\ \to_{\mathscr{D}} \Bigg| &\Big\{N_1(t_0(1-t_0))^{1/2} + c_0 u_r f^{(r-1)}(Q(t_0))\Big\} \\ &\times N_2\left(\frac{1}{f(Q(t_0))}\int_{-\infty}^{\infty} K^2(u)\,du\right)^{1/2}\Bigg|,\end{aligned} \tag{2.10}$$

where N_1 and N_2 are independent standard normal random variables and u_r is defined by condition C.6.

It is interesting to note that if we have a high-order kernel, then the difference between the smoothed empirical and quantile processes is smaller than in the unsmoothed case. Condition (2.8) means that $h(n)$ cannot be too small, that is, we should not undersmooth.

The next result covers the case when the limit is determined by the bias term.

THEOREM 2.3. *We assume that* C.1–C.7 *and* (2.1) *hold,*

$$f^{(3r)} \text{ exists and is continuous in a neighbourhood of } Q(t_0) \tag{2.11}$$

and

$$n^{1/2}h^r(n) \to \infty, \qquad n \to \infty. \tag{2.12}$$

(i) *If*

$$n^{1/2}h^{r+1/2}(n) \to c_0, \qquad n \to \infty, \tag{2.13}$$

where $0 \le c_0 < \infty$, then, as $n \to \infty$, we have

$$\begin{aligned} h^{-r+1/2}(n)\left|\tilde{\Delta}_n(t_0)\right| \\ \to_{\mathscr{D}} \Bigg| & N\frac{u_r f^{(r-1)}(Q(t_0))}{f^{1/2}(Q(t_0))}\left(t_0(1-t_0)\int_{-\infty}^{\infty}K^2(u)\,du\right)^{1/2} \\ & + \frac{c_0 u_r^2 f^{(r)}(Q(t_0)) f^{(r-1)}(Q(t_0))}{f(Q(t_0))} \\ & - \frac{1}{2}\frac{f'(Q(t_0))}{f^2(Q(t_0))}c_0\left(u_r f^{(r-1)}(Q(t_0))\right)^2\Bigg|, \end{aligned} \tag{2.14}$$

where N is a standard normal random variable.

(ii) *If*

$$n^{1/2}h^{r+1/2}(n) \to \infty \quad \text{and} \quad \frac{n^{1/2}h^r(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty, \tag{2.15}$$

then, as $n \to \infty$, we have

$$\begin{aligned} n^{-1/2}h^{-2r}(n)\left|\tilde{\Delta}_n(t_0)\right| \\ \to_P \Bigg| & \frac{u_r^2 f^{(r)}(Q(t_0)) f^{(r-1)}(Q(t_0))}{f(Q(t_0))} \\ & - \frac{1}{2}\frac{f'(Q(t_0))}{f^2(Q(t_0))}\left(u_r f^{(r-1)}(Q(t_0))\right)^2\Bigg|. \end{aligned} \tag{2.16}$$

If we take $n^{-1/2r} < h(n) < n^{-1/(2r+1)}$ with $r \geq 2$, then $\tilde{\Delta}_n(t_0)$ in (2.14) is smaller than $\Delta_n(t_0)$. However, if $n^{-3/(8r)} < h(n) < 1/\log n$, then (1.2) and (2.16) imply that $\tilde{\Delta}_n(t_0)$ is much larger than its unsmoothed version.

3. Asymptotics for $\sup_{a \leq t \leq b} |\tilde{\Delta}_n(t)|$. Next we consider uniform versions of the results in Theorem 2.1–2.3. These uniform versions will require uniform versions of the former conditions. Let $0 < c < a < b < d < 1$.

THEOREM 3.1. *We assume that* C.1–C.5 *hold, and the following*:

$$\inf_{c \leq t \leq d} f(Q(t)) > 0; \tag{3.1}$$

$$f'(Q(t)) \text{ exists and is continuous on } [c, d]; \tag{3.2}$$

$$\text{there is a } \nu > 0 \text{ such that } n^{-\nu} \leq h(n); \tag{3.3}$$

$$n^{1/2} h(n) \to 0, \qquad n \to \infty. \tag{3.4}$$

Then, as $n \to \infty$, we have

$$n^{1/4} (\log n)^{-1/2} \sup_{a \leq t \leq b} |\tilde{\Delta}_n(t)| \to_{\mathscr{D}} \left\{ \sup_{a \leq t \leq b} |B(t)| \right\}^{1/2}, \tag{3.5}$$

where $\{B(t),\ 0 \leq t \leq 1\}$ is a Brownian bridge.

The next result shows that the limit distribution of the difference between the smooth empirical and quantile processes is not necessarily like that of their nonsmoothed originals.

THEOREM 3.2. *We assume that* C.1–C.7, (2.8) *and* (3.1) *hold and that*

$$f^{(2r)}(Q(t)) \text{ exists and is continuous on } [c, d]. \tag{3.6}$$

If

$$n^{1/2} h^r(n) \to c_0, \qquad n \to \infty, \tag{3.7}$$

where $0 \leq c_0 < \infty$, then, as $n \to \infty$, we have

$$\begin{aligned} &\left(\frac{nh(n)}{2 \log 1/h(n)} \right)^{1/2} \sup_{a \leq t \leq b} |\tilde{\Delta}_n(t)| \\ &\quad \to_{\mathscr{D}} \sup_{a \leq t \leq b} \left| \frac{B(t) + c_0 u^r f^{(r-1)}(Q(t))}{f^{1/2}(Q(t))} \right| \left(\int_{-\infty}^{\infty} K^2(u)\, du \right)^{1/2}, \end{aligned} \tag{3.8}$$

where $\{B(t),\ 0 \leq t \leq 1\}$ is a Brownian bridge.

Our last theorem is on the case when the uniform limit is completely determined by the bias term.

THEOREM 3.3. *We assume that* C.1–C.7, (2.8) *and* (3.1) *hold and that*

$$f^{(3r)}(Q(t)) \text{ exists and is continuous on } [c,d]. \tag{3.9}$$

(i) *If*

$$\frac{n^{1/2}h^{r+1/2}(n)}{\log 1/h(n)} \to 0, \qquad n \to \infty, \tag{3.10}$$

then, as $n \to \infty$, *we have*

$$\begin{aligned} &\frac{h^{-r+1/2}(n)}{(2\log 1/h(n))^{1/2}} \sup_{a \le t \le b} |\tilde{\Delta}_n(t)| \\ &\quad \to_P \left(\int_{-\infty}^{\infty} K^2(u)\,du\right)^{1/2} \sup_{a\le t\le b}\left|\frac{u_r f^{(r-1)}(Q(t))}{f^{1/2}(Q(t))}\right|. \end{aligned} \tag{3.11}$$

(ii) *If*

$$\frac{n^{1/2}h^{r+1/2}(n)}{\log 1/h(n)} \to \infty \quad \text{and} \quad \frac{n^{1/2}h^{r}(n)}{(\log n)^{1/2}} \to \infty, \qquad n \to \infty, \tag{3.12}$$

then, as $n \to \infty$, *we have*

$$\begin{aligned} &n^{-1/2}h^{-2r}(n) \sup_{a\le t\le b} |\tilde{\Delta}_n(t)| \\ &\quad \to_P \sup_{a\le t\le b}\left|\frac{u_r^2 f^{(r)}(Q(t)) f^{(r-1)}(Q(t))}{f(Q(t))} \right. \\ &\qquad\qquad \left. - \frac{1}{2}\frac{f'(Q(t))}{f^2(Q(t))}\left(u_r f^{(r-1)}(Q(t))\right)^2\right|. \end{aligned} \tag{3.13}$$

4. Proofs of Theorems 2.1–2.3.

First we note that

$$\tilde{F}_n\big(\tilde{Q}_n(t)\big) = t \quad \text{for all } 0 < t < 1 \tag{4.1}$$

and

$$\begin{aligned} \tilde{\alpha}_n(x) &= \int_{-\infty}^{\infty} K(u)\alpha_n(x+uh)\,du \\ &\quad + n^{1/2}\int_{-\infty}^{\infty} K(u)(F(x+uh)-F(x))\,du. \end{aligned} \tag{4.2}$$

Let $\tilde{E}_n(t) = \tilde{F}_n(Q(t))$, $\tilde{e}_n(t) = n^{1/2}(\tilde{E}_n(t) - t)$ and $\tilde{U}_n(t) = F(\tilde{Q}_n(t))$, $0 < t < 1$. It is easy to see that $\tilde{U}_n$ is the inverse of $\tilde{E}_n$.

The following lemma helps us to study the last term in (4.2).

LEMMA 4.1. *We assume that* C.1–C.6 *hold,* $\inf_{\gamma \le t \le \delta} f(Q(t)) > 0$ *and* $f^{(j)}(Q(t))$ *exists and is continuous on* $[\gamma, \delta]$, $r \le j$. *Then for all* $\gamma < \gamma' < \delta' < \delta$ *we have*

$$\sup_{Q(\gamma') \le x \le Q(\delta')} \left| \int_{-\infty}^{\infty} K(u)\{F(x+uh) - F(x)\}\,du - \sum_{r \le i \le j} u_i h^i f^{(i-1)}(x) \right| = o(h^{2j}), \tag{4.3}$$

where

$$u_i = \frac{1}{i!}\int_{-\infty}^{\infty} u^i K(u)\,du.$$

PROOF. Taylor expansion implies (4.3) immediately. □

PROOF OF THEOREM 2.1. The weak convergence of $\alpha_n(t)$ yields

$$\sup_{-\infty < x < \infty} \left| \int_{-\infty}^{\infty} K(u)\,\alpha_n(x+uh)\,du \right| = \mathscr{O}_P(1). \tag{4.4}$$

By (4.2)–(4.4) we have

$$\sup_{x \in \Lambda_0} |\tilde{\alpha}_n(x)| = \mathscr{O}_P(1), \tag{4.5}$$

where $\Lambda_0 = [x_0 - \varepsilon, x_0 + \varepsilon]$ for some $\varepsilon > 0$ and $x_0 = Q(t_0)$. Hence we can find a neighbourhood λ_0 of t_0 such that

$$\sup_{t \in \lambda_0} \left|\tilde{U}_n(t) - t\right| = \mathscr{O}_P(n^{-1/2}). \tag{4.6}$$

The mean value theorem, (2.2) and (4.6) yield

$$R_n(t_0) = \tilde{\rho}_n(t_0) - n^{1/2}\big(t_0 - \tilde{U}_n(t_0)\big) = \mathscr{O}_P(1)\,n^{1/2}\big(\tilde{U}_n(t_0) - t_0\big)^2. \tag{4.7}$$

By (4.2) we get

$$\begin{aligned} \tilde{\Delta}_n(t) &= \tilde{e}(t) - n^{1/2}\big(t - F(\tilde{Q}_n(t))\big) - R_n(t) \\ &= \tilde{e}_n(t) - \tilde{e}_n\big(\tilde{U}_n(t)\big) - R_n(t). \end{aligned} \tag{4.8}$$

Komlós, Major and Tusnády (1975, 1976) constructed a sequence of Brownian bridges $\{B_n(t),\ 0 \le t \le 1\}$ such that

$$\sup_{-\infty < x < \infty} |\alpha_n(x) - B_n(F(x))| = \mathscr{O}(n^{-1/2}\log n) \quad \text{a.s.} \tag{4.9}$$

Putting together (4.2), (4.3) and (4.9), we obtain

$$\begin{aligned}\sup_{x\in\Lambda_0}&\left|\tilde{\alpha}_n(x)-\int_{-\infty}^{\infty}K(u)B_n(F(x+uh))\,du\right| \\ &=\mathscr{O}_P(n^{-1/2}\log n)+\mathscr{O}(h^2n^{1/2}) \\ &=o_P(n^{-1/4}).\end{aligned} \tag{4.10}$$

The modulus of continuity of a Brownian bridge [cf. Csörgő and Révész (1981), page 42] gives

$$\begin{aligned}\sup_{x\in\Lambda_0}\ &\sup_{-A\le u\le A}|B_n(F(x+uh))-B_n(F(x))| \\ &=\mathscr{O}_P\big((h\log 1/h)^{1/2}\big),\end{aligned} \tag{4.11}$$

and, therefore, we have

$$\sup_{t\in\lambda_0}|\tilde{e}_n(t)-B_n(t)|=o_P(n^{-1/4})+\mathscr{O}_P\big((h\log 1/h)^{1/2}\big). \tag{4.12}$$

By (2.1) we have

$$n^{1/2}\big(t-\tilde{U}_n(t)\big)=\tilde{e}_n\big(\tilde{U}_n(t)\big). \tag{4.13}$$

Using (4.6) and (4.11)–(4.13), we get

$$\begin{aligned}n^{1/2}\big(t_0-\tilde{U}_n(t_0)\big)&=B_n(t_0)+\mathscr{O}_P\big((h\log 1/h)^{1/2}\big) \\ &\quad+\mathscr{O}_P\big(n^{-1/4}(\log n)^{1/2}\big).\end{aligned} \tag{4.14}$$

Applying (4.12)–(4.14) and using again the modulus of continuity of a Brownian bridge, we conclude

$$\begin{aligned}&\left|\tilde{e}_n\big(\tilde{U}_n(t_0)\big)-\int_{-\infty}^{\infty}K(u)B_n\big(F\big(Q(t_0-n^{-1/2}B_n(t_0))+uh\big)\big)\,du\right| \\ &\quad=o_P(n^{-1/4})+\mathscr{O}_P\big(n^{-3/8}(\log n)^{3/4}\big) \\ &\qquad+\mathscr{O}_P\big(n^{-1/4}(h\log 1/h)^{1/2}(\log n+\log 1/h)^{1/2}\big) \\ &\quad=o_P(n^{-1/4}).\end{aligned} \tag{4.15}$$

Since the distribution of B_n does not depend on n, it is enough to prove that

$$\begin{aligned}\xi_n=n^{1/4}\bigg|\int_{-\infty}^{\infty}K(u)\big\{&B(F(Q(t_0)+uh)) \\ &-B\big(F\big(Q(t_0-n^{-1/2}B(t_0))+uh\big)\big)\big\}\,du\bigg| \\ \to_{\mathscr{D}}\ Y,&\end{aligned} \tag{4.16}$$

where $\{B(t),\ 0 \le t \le 1\}$ is a Brownian bridge. We can find a Wiener process $\{W(t),\ 0 \le t \le 1\}$ such that

$$B(t) = W(t) - tW(1). \tag{4.17}$$

It is easy to verify that we have

$$\xi_n = n^{1/4}\left|\int_{-\infty}^{\infty} K(u)\{W(F(Q(t_0) + uh)) - W\big(F\big(Q\big(t_0 - n^{-1/2}B(t_0)\big) + uh\big)\big)\}\,du\right| + \mathscr{O}_P(n^{-1/4}). \tag{4.18}$$

Let

$$\tau_n(y) = n^{1/4}\int_{-\infty}^{\infty} K(u)\{W(F(Q(t_0) + uh)) - W\big(F\big(Q\big(t_0 - n^{-1/2}y\big) + uh\big)\big)\}\,du,$$

$-\infty < y < \infty$. The joint distribution of $\tau_n(y)$ and $B(t_0)$ is bivariate normal. Elementary calculations give

$$E\tau_n(y) = 0, \qquad EB(t_0) = 0, \qquad \operatorname{var} B(t_0) = t_0(1 - t_0),$$
$$\lim_{n\to\infty} \operatorname{var}\tau_n(y) = |y| \quad \text{and} \tag{4.19}$$
$$\lim_{n\to\infty} \operatorname{cov}(\tau_n(y), B(t_0)) = 0 \quad \text{for all } y.$$

Applying (4.19), we get

$$\begin{aligned}\lim_{n\to\infty} P\{\xi_n \le x\} &= \lim_{n\to\infty}\int_{-\infty}^{\infty} P\{|\tau_n(y)| \le x | B(t_0) = y\} \\ &\quad\times \frac{1}{(t_0(1 - t_0))^{1/2}}\phi\left(\frac{y}{(t_0(1 - t_0))^{1/2}}\right)dy \\ &= \int_{-\infty}^{\infty}\left(2\Phi\left(\frac{x}{|y|^{1/2}}\right) - 1\right)\frac{1}{(t_0(1 - t_0))^{1/2}}\phi\left(\frac{y}{(t_0(1 - t_0))^{1/2}}\right)dy,\end{aligned} \tag{4.20}$$

which completes the proof of (2.4). □

Proof of Theorem 2.2. First we note that (4.2), (4.3) and (4.9) yield

$$\begin{aligned}\sup_{x\in\Lambda_0}\Bigg|\tilde{\alpha}_n(x) - \Bigg(\int_{-\infty}^{\infty} K(u)B_n(F(x + uh))\,du \\ + n^{1/2}\sum_{r\le i\le 2r} u_i h^i f^{(i-1)}(x)\Bigg)\Bigg| \\ = \mathscr{O}_P(n^{-1/2}\log n) + o(h^{2r}n^{1/2}).\end{aligned} \tag{4.21}$$

Let

$$\Gamma(t) = \int_{-\infty}^{\infty} K(u) B(F(Q(t) + uh))\, du$$
$$(4.22) \qquad = \frac{1}{h} \int_{-\infty}^{\infty} K\left(\frac{Q(u) - Q(t)}{h}\right) B(u)\, du,$$

where $\{B(t),\ 0 \le t \le 1\}$ is a Brownian bridge. Next we estimate the increments of $\Gamma(t)$. By C.4, a two-term Taylor expansion gives

$$\Gamma(t) - \Gamma(s)$$
$$= \frac{1}{h} \int_{-\infty}^{\infty} \left\{ K\left(\frac{Q(u) - Q(t)}{h}\right) - K\left(\frac{Q(u) - Q(s)}{h}\right)\right\} B(u)\, du$$
$$(4.23) \qquad = \frac{Q(s) - Q(t)}{h} \int_{-\infty}^{\infty} K'(u)\{B(F(Q(s) + uh)) - B(s)\}\, du$$
$$+ \frac{1}{2}\left(\frac{Q(s) - Q(t)}{h}\right)^2 \int_{-\infty}^{\infty} K''(\xi)\{B(F(Q(s) + uh)) - B(s)\}\, du,$$

where $|\xi - u| \le |Q(s) - Q(t)|/h$. It is easy to check that, for all $0 < s < 1$, we have

$$(4.24) \qquad \sup_{-A \le u \le A} |B(F(Q(s) + uh)) - B(s)| = \mathscr{O}_P(h^{1/2}).$$

Thus, for all $C > 0$, we get

$$(4.25) \qquad \sup_{|t - t_0| \le C/n^{1/2}} \Bigg| \Gamma(t) - \Gamma(t_0) - \frac{Q(t_0) - Q(t)}{h} \times \int_{-\infty}^{\infty} K'(u)\{B(F(Q(t_0) + uh)) - B(t_0)\}\, du \Bigg|$$
$$= \mathscr{O}_P(n^{-1}h^{-3/2}) = o_P\big((nh)^{-1/2}\big).$$

By (4.8) and (4.21) we obtain

$$(4.26) \qquad (nh)^{1/2}\tilde{\Delta}_n(t_0) = (nh)^{1/2} L_{n,1}(t_0) + (nh)^{1/2} L_{n,2}(t_0) - (nh)^{1/2} R_n(t_0) + o_P(1),$$

where

$$(4.27) \qquad L_{n,1}(t_0) = \int_{-\infty}^{\infty} K(u)\Big\{B_n(F(Q(t_0) + uh)) - B_n F\big(Q(\tilde{U}_n(t_0)) + uh\big)\Big)\Big\}\, du$$

and

$$L_{n,2}(t_0) = n^{1/2} \sum_{r \le i \le 2r} u_i h^i \{ f^{(i-1)}(Q(t_0)) - f^{(i-1)}(Q(\tilde{U}_n(t_0))) \}. \tag{4.28}$$

By (4.21) it is clear that the bias term is not necessarily small. Using (4.13) and (4.21), we get

$$\begin{aligned} n^{1/2}\big(t_0 - \tilde{U}_n(t_0)\big) &\\ = \int_{-\infty}^{\infty} K(u) B_n\Big(F\big(Q\big(\tilde{U}_n(t_0)\big) + uh\big)\Big)\, du &\\ + n^{1/2} u_r h^r f^{(r-1)}\Big(Q\big(\tilde{U}_n(t_0)\big)\Big) &\\ + \mathscr{O}_P\big(n^{-1/2} \log n\big) + \mathscr{O}\big(n^{1/2} h^{r+1}\big), & \end{aligned} \tag{4.29}$$

and, therefore, (4.6) holds. The continuity of B_n and $f^{(r-1)}$ imply

$$n^{1/2}\big(t_0 - \tilde{U}_n(t_0)\big) = B_n(t_0) + n^{1/2} h^r u_r f^{(r-1)}(Q(t_0)) + o_P(1). \tag{4.30}$$

Since $|\tilde{U}_n(t_0) - t_0| = \mathscr{O}_P(n^{-1/2})$, (4.25) yields

$$\begin{aligned} L_{n,1}(t_0) &\\ = \frac{Q\big(\tilde{U}_n(t_0)\big) - Q(t_0)}{h} &\\ \times \left(\int_{-\infty}^{\infty} K'(u) \{ B_n(F(Q(t_0) + uh)) - B_n(t_0) \}\, du + o_P(nh)^{-1/2} \right). & \end{aligned} \tag{4.31}$$

Similarly, the mean value theorem gives

$$\begin{aligned} L_{n,2}(t_0) = n^{1/2} u_r h^r \frac{f^{(r)}(Q(t_0))}{f(Q(t_0))} \big(t_0 - \tilde{U}_n(t_0)\big) &\\ + \big|t_0 - \tilde{U}_n(t_0)\big| o_P(n^{1/2} h^r). & \end{aligned} \tag{4.32}$$

By condition (2.9), and by (4.6) and (4.32), we get

$$(nh)^{1/2} L_{n,2}(t_0) = o_P(1). \tag{4.33}$$

Putting together (4.31), (4.24) and (4.30), we obtain

$$\begin{aligned} (nh)^{1/2} L_{n,1}(t_0) &\\ = -\frac{B_n(t_0) + n^{1/2} h^r u_r f^{(r-1)}(Q(t_0))}{f(Q(t_0))} &\\ \times \frac{1}{h^{1/2}} \int_{-\infty}^{\infty} K'(u) \{ B_n(F(Q(t_0) + uh)) - B_n(t_0) \}\, du + o_P(1). & \end{aligned} \tag{4.34}$$

Let

$$\eta_n = h^{-1/2} \int_{-\infty}^{\infty} \big(B_n(F(Q(t_0) + uh)) - B_n(t_0)\big)\, dK(u). \tag{4.35}$$

The joint distribution of $(\eta_n, B_n(t_0))$ is bivariate normal for each n. Elementary calculations yield

$$E\eta_n = 0, \qquad EB_n(t_0) = 0, \qquad \operatorname{var} B_n(t_0) = t_0(1-t_0),$$
$$\lim_{n\to\infty} \operatorname{var}\eta_n = f(Q(t_0))\int_{-\infty}^{\infty} K^2(u)\,du \quad \text{and} \quad \lim_{n\to\infty} E\eta_n B_n(t_0) = 0. \tag{4.36}$$

Theorem 2.2 now follows immediately from (4.26) and (4.33)–(4.36). □

Proof of Theorem 2.3. By (2.11) it follows from (4.9), (4.13) and (4.3) that

$$t_0 - \tilde{U}_n(t_0) = u_r h^r f^{(r-1)}(Q(t_0)) + o_P(h^r). \tag{4.37}$$

Now (4.7) gives

$$R_n(t_0) = \frac{n^{1/2}}{2}\frac{f'(Q(t_0))}{f^2(Q(t_0))}\left(t_0 - \tilde{U}_n(t_0)\right)^2 + \mathscr{O}_P(n^{1/2}h^{3r}). \tag{4.38}$$

Then, similarly to (4.21), we have

$$\sup_{x\in\Lambda_0}\left|\tilde{\alpha}_n(x) - \left\{\int_{-\infty}^{\infty} K(u)B_n(F(x+uh))\,du + n^{1/2}\sum_{r\le i\le 3r} u_i h^i f^{(i-1)}(x)\right\}\right|$$
$$= \mathscr{O}_P(n^{-1/2}\log n) + o(n^{1/2}h^{3r}).$$

Thus, we obtain

$$\begin{aligned}\tilde{\Delta}_n &= L_{n,1}(t_0) + L_{n,3}(t_0) - \frac{1}{2}\frac{f'(Q(t_0))}{f^2(Q(t_0))}n^{1/2}\left(t_0 - \tilde{U}_n(t_0)\right)^2 \\ &\quad + \mathscr{O}_P(n^{-1/2}\log n) + \mathscr{O}_P(n^{1/2}h^{3r}),\end{aligned} \tag{4.39}$$

where

$$L_{n,3}(t_0) = n^{1/2}\sum_{r\le i\le 3r} u_i h^i\left\{f^{(i-1)}(Q(t_0)) - f^{(i-1)}\left(Q(\tilde{U}_n(t_0))\right)\right\}.$$

Applying (4.23) and (4.37), we get

$$L_{n,1}(t_0) = \frac{u_r h^{r-1/2} f^{(r-1)}(Q(t_0))}{f(Q(t_0))}\eta_n + o_P(h^{r+1/2}), \tag{4.40}$$

where η_n is defined in (4.35). The mean value theorem and (4.37) yield

$$L_{n,3}(t_0) = n^{1/2}h^{2r}u_r^2\frac{f^{(r)}(Q(t_0))f^{(r-1)}(Q(t_0))}{f(Q(t_0))} + o_P(n^{1/2}h^{2r}). \tag{4.41}$$

Since η_n is normal for each n, Theorem 2.3 follows from (4.36) and (4.39)–(4.41). □

5. Proofs of Theorems 3.1–3.3. Let $\Lambda = [c', d']$, where $c < c' < a < b < d' < d$.

PROOF OF THEOREM 3.1. A two-term Taylor expansion gives

$$\sup_{t\in\Lambda}\left|R_n(t) - n^{1/2}\frac{f'(Q(t))}{f^2(Q(t))}\big(t - \tilde{U}_n(t)\big)^2\right| = \mathscr{O}_P(1)n^{1/2}\sup_{t\in\Lambda}\left|t - \tilde{U}_n(t)\right|^3, \tag{5.1}$$

where

$$R_n(t) = \tilde{\rho}(t) - n^{1/2}\big(t - \tilde{U}_n(t)\big). \tag{5.2}$$

By (4.2) and (4.3) we have

$$\sup_{t\in\Lambda}|\tilde{\alpha}_n(Q(t))| = \mathscr{O}_P(1). \tag{5.3}$$

Hence we get

$$\sup_{t\in\Lambda}\left|t - \tilde{U}_n(t)\right| = \mathscr{O}_P(n^{-1/2}). \tag{5.4}$$

Applying (4.9), we obtain

$$\sup_{t\in\Lambda}\left|\tilde{e}_n(t) - \int_{-\infty}^{\infty}K(u)B_n(F(Q(t) + uh))\,du\right| = o_P(n^{-1/4}). \tag{5.5}$$

The modulus of continuity of a Brownian bridge [cf. Csörgő and Révész (1981), page 42] yields

$$\sup_{t\in\Lambda}\sup_{|u|\le A}|B_n(F(Q(t) + uh)) - B_n(t)| = \mathscr{O}_P\big((h\log 1/h)^{1/2}\big), \tag{5.6}$$

and, therefore, (5.5) gives

$$\sup_{t\in\Lambda}|\tilde{e}_n(t) - B_n(t)| = o_P(n^{-1/4}) + \mathscr{O}_P\big((h\log 1/h)^{1/2}\big). \tag{5.7}$$

Using again the modulus of continuity of B_n, we get

$$\sup_{t\in\Lambda}\left|n^{1/2}\big(t - \tilde{U}_n(t)\big) - B_n(t)\right| = o_P(n^{1/4}) + \mathscr{O}_P\big((h\log 1/h)^{1/4}\big) \tag{5.8}$$

and

$$\begin{aligned}\sup_{t\in\Lambda}&\left|\tilde{e}_n\big(\tilde{U}_n(t)\big) - B_n\big(t - n^{-1/2}B_n(t)\big)\right|\\ &= o_P(n^{-1/4}) + \mathscr{O}_P\big((h\log 1/h)^{1/2}\big)\\ &\quad + \mathscr{O}_P\big(n^{-1/4}(h\log 1/h)^{1/2}(\log n + \log 1/h)^{1/2}\big).\end{aligned} \tag{5.9}$$

Now (4.8), (5.2), (5.4), (5.7) and (5.9) yield

$$\begin{aligned}
&n^{1/4}(\log n)^{-1/2} \sup_{a \le t \le b} |\tilde{\Delta}_n(t)| \\
(5.10) \quad &= n^{1/4}(\log n)^{-1/2} \sup_{a \le t \le b} |B_n(t) - B_n(t - n^{-1/2}B_n(t))| + o_P(1).
\end{aligned}$$

We now show that, for any Brownian bridge $\{B(t), 0 \le t \le 1\}$, we have

$$n^{1/4}(\log n)^{-1/2} \sup_{a \le t \le b} |B(t) - B(t - n^{-1/2}B(t))| \to_{\mathscr{D}} \Big(\sup_{a \le t \le b} |B(t)| \Big)^{1/2},$$

which, via (5.10), will also complete the proof of (3.5).

It is well known [cf., e.g., Csörgő and Horváth (1993), page 164] that

$$\begin{aligned}
&\sup_{0 \le t \le 1} |\alpha_n(Q(t)) - n^{1/2}(t - F(Q_n(t)))| \\
&\quad = \sup_{0 \le t \le 1} |\alpha_n(Q_n(t)) - \alpha_n(Q(t))| + \mathscr{O}(n^{-1/2}).
\end{aligned}$$

A two-term Taylor expansion gives

$$\sup_{a \le t \le b} |\Delta_n(t)| = \sup_{a \le t \le b} |\alpha_n(Q(t)) - n^{1/2}(t - F(Q_n(t)))| + \mathscr{O}_P(n^{-1/2}).$$

Consequently, by (4.9) combined with these two statements, we obtain

$$\begin{aligned}
&\sup_{a \le t \le b} |\Delta_n(t)| \\
&\quad = \sup_{a \le t \le b} |B_n(F(Q_n(t))) - B_n(t)| + \mathscr{O}_P(n^{-1/2} \log n) \\
&\quad = \sup_{a \le t \le b} \Big| B_n\big(t - n^{-1/2}\alpha_n(Q(t)) + n^{-1/2}\alpha_n(Q(t)) \\
&\qquad\qquad - (t - F(Q_n(t)))\big) - B_n(t) \Big| + \mathscr{O}_P(n^{-1/2} \log n).
\end{aligned}$$

This, in turn, via (4.9), Theorem 2 of Kiefer (1970) and the modulus of continuity of a Brownian bridge yields

$$\begin{aligned}
\sup_{a \le t \le b} |\Delta_n(t)| &= \sup_{a \le t \le b} |B_n(t - n^{-1/2}B_n(t)) - B_n(t)| \\
&\quad + \mathscr{O}_P\big(n^{-3/8}(\log\log n)^{1/8}(\log n)^{1/4}\big).
\end{aligned}$$

Using now the result (1.4) of Kiefer (1970) in combination with the last statement and (5.10), we obtain (3.5). □

The proofs of Theorems 3.2 and 3.3 are based on the following lemma, in which $\{B(t), 0 \le t \le 1\}$ denotes a Brownian bridge.

LEMMA 5.1. *We assume that* C.1–C.7, (3.1) *and* (3.2) *hold and that*

$$\lim_{n\to\infty}\sup_{c\le t\le d}|l_n(t)-l(t)|=0, \tag{5.11}$$

where $l(t)$ *is a continuous function on* $[c,d]$. *Then we have*

$$\left(\frac{1}{2\log 1/h}\right)^{1/2}\sup_{a\le t\le b}\left|\frac{\nu B(t)+l_n(t)}{f^{1/2}(Q(t))}\frac{1}{(hf(Q(t)))^{1/2}}\times\int_{-\infty}^{\infty}K'(u)\{B(F(Q(t)+uh))-B(t)\}\,du\right|$$

$$\to_{\mathscr{D}}\sup_{a\le t\le b}\left|\frac{\nu B(t)+l(t)}{f^{1/2}(Q(t))}\right|\left(\int_{-\infty}^{\infty}K^2(u)\,du\right)^{1/2},$$

for all $-\infty<\nu<\infty$.

PROOF. First we show that

$$\left(\frac{1}{2h\log 1/h}\right)^{1/2}\sup_{\alpha\le t\le\beta}\frac{1}{f^{1/2}(Q(t))}\times\left|\int_{-\infty}^{\infty}K'(u)\{B(F(Q(t)+uh))-B(t)\}\,du\right|\to_P\left(\int_{-\infty}^{\infty}K^2(u)\,du\right)^{1/2}, \tag{5.12}$$

for all $c<\alpha<\beta<d$. By (4.17) we have

$$\sup_{\alpha\le t\le\beta}\left|\int_{-\infty}^{\infty}K'(u)\{B(F(Q(t)+uh))-B(t)\}\,du-\int_{-\infty}^{\infty}K'(u)\{W(t+f(Q(t))uh)-W(t)\}\,du\right|=o_P\left((h\log 1/h)^{1/2}\right).$$

Let $C>0$ and define $G(t)=\int_{-\infty}^{\infty}K'(u)\{W(t+u)-W(t)\}\,du$. It is easy to check that $G(t)$ is a stationary Gaussian process with $EG(t)=0$, $EG(t)G(s)=\int_{-\infty}^{\infty}K(u)K(t+u)\,du$ and

$$\sup_{\alpha'\le t\le\beta'}\left|\frac{1}{h^{1/2}}\int_{-\infty}^{\infty}K'(u)\{W(t+Cuh)-W(t)\}\,du\right|$$

$$=_{\mathscr{D}}C^{1/2}\sup_{\alpha'/(Ch)\le t\le\beta'/(Ch)}|G(t)|$$

$$=_{\mathscr{D}}C^{1/2}\sup_{0\le t\le(\beta'-\alpha')/(Ch)}|G(t)|,$$

for all $\alpha \le \alpha' < \beta' \le \beta$. Since $EG(t)G(0) = 0$ if $|t| > 2A$ and

$$EG(t)G(0) = \int_{-\infty}^{\infty} K^2(u)\,du - \frac{|t|}{2}(K^2(A) + K^2(-A)) + \frac{t^2}{2}\int_{-\infty}^{\infty} K(u)K''(u)\,du + o(t^2)$$

as $t \to 0$, we can use Pickands [(1969), Theorem 2.1] and get

$$\left(\frac{1}{2\log 1/h}\right)^{1/2} \sup_{0\le t\le(\beta'-\alpha')/(Ch)} |G(t)| \to_P \left(\int_{-\infty}^{\infty} K^2(u)\,du\right)^{1/2}.$$

For any $\varepsilon > 0$ we can find an integer M such that with $\alpha = t_1 < t_2 < \cdots < t_M = \beta$ we have

$$\max_{1<i\le M} \sup_{t_{i-1}\le t\le t_i} |f(Q(t)) - f(Q(t_i))| \le \varepsilon/(2A).$$

The modulus of continuity of a Wiener process [cf. Csörgő and Révész (1981), page 28] gives

$$\limsup_{n\to\infty} \left(\frac{1}{2h\log 1/h}\right)^{1/2} \times \max_{1<i\le M} \sup_{t_{i-1}\le t\le t_i} \left| \int_{-\infty}^{\infty} K'(u)\{W(t + f(Q(t))uh) - W(t + f(Q(t_i))uh)\}\,du \right| \le \varepsilon^{1/2} \sup_{-\infty<u<\infty} |K'(u)| \quad \text{a.s.}$$

Since ε can be taken arbitrarily small, we have established (5.12).

Let $q_n(t) = \{\nu B(t) + l_n(t)\}/f^{1/2}(Q(t))$. By the almost-sure continuity of B and (3.11), for each $\varepsilon > 0$ and $\delta > 0$ we can find $a = s_1 < s_2 < \cdots < s_R = b$ such that

$$\limsup_{n\to\infty} P\left\{\max_{1<i\le R} \sup_{s_{i-1}\le s\le s_i} |q_n(s) - q_n(s_i)| \ge \varepsilon\right\} \le \delta. \tag{5.13}$$

Applying (5.12) with $\alpha = s_{i-1}$, $\beta = s_i$, $1 < i \le R$, by (5.13) we immediately get Lemma 5.1. □

PROOF OF THEOREM 3.2. Lemma 4.1 and (4.9) imply

$$\sup_{t\in\Lambda} \left| \tilde{e}_n(t) - \left\{\int_{-\infty}^{\infty} K(u)B_n(F(Q(t) + uh))\,du + n^{1/2} \sum_{r\le i\le 2r} u_i h^i f^{(i-1)}(Q(t))\right\} \right| = \mathscr{O}_P(n^{-1/2}\log n) + o(n^{1/2}h^{2r}). \tag{5.14}$$

Thus we get

$$\sup_{t\in\Lambda} |\tilde{e}_n(t)| = \mathscr{O}_P(1) + \mathscr{O}_P(n^{1/2}h^r),$$

which yields

$$\sup_{t\in\Lambda} \left|t - \tilde{U}_n(t)\right| = \mathscr{O}_P(n^{-1/2}) + \mathscr{O}_P(h^r). \tag{5.15}$$

We write again

$$\begin{aligned}\tilde{\Delta}_n(t) &= L_{n,1}(t) + L_{n,2}(t) - n^{1/2}\frac{f'(Q(t))}{f^2(Q(t))}\left(t - \tilde{U}_n(t)\right)^2 \\ &\quad + \mathscr{O}_P\left(n^{-1/2}\log n\right) + \mathscr{O}_P(n^{1/2}h^{2r}),\end{aligned} \tag{5.16}$$

where $L_{n,1}$ and $L_{n,2}$ are defined in (4.27) and (4.28). Using (5.6) and (4.23), we obtain

$$\begin{aligned}\sup_{t\in\Lambda} \Bigg| L_{n,1}(t) &- \frac{Q\left(\tilde{U}_n(t)\right) - Q(t)}{h} \\ &\times \int_{-\infty}^{\infty} K'(u)\{B_n(F(Q(t) + uh)) - B(t)\}\,du \Bigg| \\ &= \mathscr{O}_P\left(h^{-1/2}(\log h)^{1/2} \sup_{t\in\Lambda} \left|t - \tilde{U}_n(t)\right|^2\right).\end{aligned} \tag{5.17}$$

We note that (4.30) is true uniformly in t, that is, we have

$$\begin{aligned}\sup_{t\in\Lambda} &\left|n^{1/2}\left(t - \tilde{U}_n(t)\right) - \left(B_n(t) + n^{1/2}h^r u_r f^{(r-1)}(Q(t))\right)\right| \\ &= o_P(1).\end{aligned} \tag{5.18}$$

Putting together (5.17) and (5.18), we get

$$\begin{aligned}\sup_{t\in\Lambda} \Bigg| \left(\frac{nh}{\log 1/h}\right)^{1/2} & L_{n,1}(t) \\ &+ \frac{B_n(t) + n^{1/2}u_r h^r f^{(r-1)}(Q(t))}{f(Q(t))} \frac{1}{(h\log 1/h)^{1/2}} \\ &\times \int_{-\infty}^{\infty} K'(u)\{B_n(F(Q(t) + uh)) - B_n(t)\}\,du \Bigg| = o_P(1).\end{aligned} \tag{5.19}$$

The mean value theorem and (5.18) yield

$$\left(\frac{nh}{\log 1/h}\right)^{1/2} \sup_{t \in \Lambda} |L_{n,2}(t)| = o_P(1). \tag{5.20}$$

Now Theorem 3.2 follows immediately from (5.16), (5.17), (5.19), (5.20) and Lemma 5.1. □

PROOF OF THEOREM 3.3. We follow the proof of Theorem 2.3. Similarly to (4.39), it is enough to determine the limit distribution of

$$\sup_{a \le t \le b} \left| \frac{u_r h^{r-1/2} f^{(r-1)}(Q(t))}{f^{1/2}(Q(t))} \left(\frac{1}{hf(Q(t))}\right)^{1/2} \right.$$
$$\times \int_{-\infty}^{\infty} K'(u)\{B_n(F(Q(t)+uh)) - B_n(t)\}\,du$$
$$+ n^{1/2} h^{2r} u_r^2 \frac{f^{(r)}(Q(t)) f^{(r-1)}(Q(t))}{f(Q(t))}$$
$$\left. - \frac{1}{2} n^{1/2} h^{2r} \frac{f'(Q(t))}{f^2(Q(t))} \left(u_r f^{(r-1)}(Q(t))\right)^2 \right|.$$

Applying Lemma 5.1, we get (3.11) and (3.13) immediately. □

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REFERENCES

BAHADUR, R. R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577–580.

CSÖRGŐ, M. (1983). *Quantile Processes with Statistical Applications*. SIAM, Philadelphia.

CSÖRGŐ, M. and HORVÁTH, L. (1993). *Weighted Approximations in Probability and Statistics*. Wiley, New York.

CSÖRGŐ, M. and RÉVÉSZ, P. (1978). Strong approximations of the quantile process. *Ann. Statist.* **6** 882–894.

CSÖRGŐ, M. and RÉVÉSZ, P. (1981). *Strong Approximations in Probability and Statistics*. Academic Press, New York.

EFRON, B. (1979). Bootstrap methods: another look at the jackknife. *Ann. Statist.* **7** 1–26.

KEIFER, J. (1967). On Bahadur's reprsentation of sample quantiles. *Ann. Math. Statist.* **38** 1323–1342.

KEIFER, J. (1970). Deviations between the sample quantile process and the sample D.F. In *Nonparametric Techniques in Statistical Inference* (M. L. Puri, ed.) 299–319. Cambridge Univ. Press.

KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1975). An approximation of partial sums of independent R.V.'s and sample DF. I. *Z. Wahrsch. Verw. Gebiete* **32** 111–131.

KOMLÓS, J., MAJOR, P. and TUSNÁDY, G. (1976). An approximation of partial sums of independent R.V.'s and the sample DF. II. *Z. Wahrsch. Verw. Gebiete* **34** 33–58.

MACK, Y. P. (1987). Bahadur's representation of sample quantiles based on smoothed estimates of a distribution function. *Probability and Mathematical Statistics* (Wroclaw) **8** 138–189.

PICKANDS, J., III (1969). Asymptotic properties of the maximum in a stationary Gaussian process. *Trans. Amer. Math. Soc.* **145** 75–86.

SHORACK, G. R. and WELLNER, J. A. (1986). *Empirical Processes with Applications to Statistics.* Wiley, New York.

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