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CLEAR TWO-FACTOR INTERACTIONS AND MINIMUM ABERRATION

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Wu and Hamada recommend selecting resolution IV designs with the maximum number of clear two-factor interactions (2FIs), called MaxC2 designs. In this paper, we develop a method by using graphical representations, combinatorial and group-theoretic arguments to prove if a given design is a MaxC2 design. In particular, we show that all known minimum aberration designs with resolution IV are MaxC2 designs (except in six cases) and that the second 2^{9-4}, 2^{13-7}, 2^{16-10} and 2^{17-11} designs given in Wu and Hamada are MaxC2 designs. The method also enables us to identify new MaxC2 designs that are too large to be verified by computer search.

1. Introduction. The minimum aberration (MA) criterion is commonly used for selecting optimal 2^{k-p} fractional factorial designs. In some situations, however, other criteria can lead to better designs. A two-factor interaction (2FI) is called clear if it is not aliased with any main effects or other 2FIs. This is explained in Wu and Hamada [(2000), Chapter 4]. They proposed a general rule for selecting 2^{k-p} designs with maximum resolution IV. That is, among resolution IV designs with given k and p, those with the maximum number of clear 2FIs (MaxC2) are the best. Such designs will be called MaxC2 designs. Note that MA designs are not necessarily good according to this criterion. For example, the MA 2^{15-9} design given in Wu and Hamada [(2000), page 197] contains no clear 2FIs, but another 2^{15-9} design of resolution IV given there has 27 clear 2FIs. In this paper, an approach is developed to prove if a given design is a MaxC2 design. The main tools (given in Sections 2 and 4) are a graph representation and classification of length-4 words and some useful identities and bounds on the number of clear 2FIs. These tools can reduce the search for designs to a much smaller set. Together with other combinatorial and group-theoretic arguments, they are effective in proving known and new designs to be MaxC2 designs. A summary of techniques in the proposed approach and concluding remarks are given in Section 5.

A regular 2^{k-p} fractional factorial design is determined by its defining contrast subgroup, which consists of $2^p - 1$ defining words. The vector $W = (A_1, \ldots, A_k)$ is called the word-length pattern, where A_i denotes the number of words of length i in the defining contrast subgroup. The resolution of a design is defined as the

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smallest j with $A_j \geq 1$. To discriminate designs of the same resolution, Fries and Hunter (1980) proposed the following criterion. For two designs d_1 and d_2 with s being the smallest value such that $A_s(d_1) \neq A_s(d_2)$, d_1 is said to have less aberration than d_2 if $A_s(d_1) < A_s(d_2)$. If there is no design with less aberration than d_1, then d_1 has minimum aberration (MA).

It is well known that there exist no designs with resolution at least IV if $k > 2^{k-p-1}$ [see Bose (1947)]. For $2^{k-p-2} + 1 < k \leq 2^{k-p-1}$, the maximum resolution of a 2^{k-p} design is IV, but in this case, Chen and Hedayat (1998) showed that no resolution IV designs can have any clear 2FIs. Let $k_{\max}(q)$ denote the maximum value of k for which there exists a $2^{k-(k-q)}$ design with resolution at least V. Then for $k_{\max}(k-p) < k \leq 2^{k-p-2} + 1$, there exist no designs with resolution at least V, but there exist resolution IV designs with clear 2FIs [see Chen and Hedayat (1998)]. This is the case to which the previous rule of Wu and Hamada should apply. *Throughout this paper, we will focus on this case*. As we will see later, MA designs can be MaxC2 designs in this case, especially when the number of length-4 words is small. However, when k is close to $2^{k-p-2} + 1$, MA designs may have no clear 2FIs while MaxC2 designs can have a large number of clear 2FIs.

For $k_{\max}(k-p) < k \leq 2^{k-p-2} + 1$, MA designs for 32 and 64 runs and for 128 runs with $12 \leq k \leq 14$ have been obtained, either through theoretical derivations or via computer search; see the tables in Wu and Hamada [(2000), Chapter 4]. In Section 3, we use the proposed method to show that, these MA designs are also MaxC2 designs, except for 2^{9-4}, 2^{13-7}, 2^{14-8}, 2^{15-9}, 2^{16-10} and 2^{17-11} designs. We also show in Section 4 that the second 2^{9-4}, 2^{13-7}, 2^{16-10} and 2^{17-11} designs given in Wu and Hamada [(2000), pages 195 and 197] are MaxC2 designs. These designs are obtained by Chen, Sun and Wu (1993) via computer search. The cases of 2^{14-8} and 2^{15-9} designs are still under investigation. The method also enables us to prove that a class of designs due to Tang, Ma, Ingram and Wang (2002) are MaxC2 designs, which cannot be done by computer search.

As the work of Chen (1998) indicated, even for 128-run designs with $k = 13$ and 14, finding MA designs via computer search is not an easy task. For 128-run designs with $15 \leq k \leq 33$, little work has been done for finding either MA or MaxC2 designs. This is in part because there is no complete catalog of 128-run designs. Since in these cases, MaxC2 designs are the best according to the Wu–Hamada rule, it is of particular interest to find designs that have many clear 2FIs. The results in this paper reveal some intrinsic structures of the MaxC2 designs that can potentially be used to construct designs with many clear 2FIs for large run size and to provide useful information for finding MaxC2 designs via computer search. For example, we are able to find a new MaxC2 2^{15-8} design by employing the proposed method to reduce the search to a very small set of designs and use computer enumeration to finish the job.

2. A graphical representation of length-4 words. Let d be a 2^{k-p}_{IV} design, where the subscript IV indicates that d is of resolution IV. Let $A_4(d)$ be the number of its length-4 words, L_4 the set of its length-4 words, $C(d)$ the number of clear 2FIs and

$$U(d) = k(k-1)/2 - C(d) \tag{2.1}$$

the number of unclear 2FIs. Consider a graph consisting of vertices and lines, where each vertex represents an independent word in L_4, and a line connects two vertices if their product is a length-4 word. If the product of three or four vertices is a length-4 word, they are indicated by unfilled circles or boxes, respectively. Such a graph can be used to show relationships among some or all length-4 words.

Let $\mathcal{G}(w_1, \dots, w_m)$ denote the set generated by m independent length-4 words $w_1, \dots, w_m$. Here, "generated" means taking products of $w_1, \dots, w_m$ of various lengths. For example, $\mathcal{G}(w_1, w_2, w_3) = \{w_1, w_2, w_3, w_1w_2, w_1w_3, w_2w_3, w_1w_2w_3\}$. Then

$$G_m = G(w_1, \dots, w_m) = L_4 \cap \mathcal{G}(w_1, \dots, w_m) \tag{2.2}$$

is called the subset of L_4 generated by m independent length-4 words $w_1, \dots, w_m$. A graph is called the graph of d if it represents L_4, and a subgraph of d if it represents G_m for some independent length-4 words $w_1, \dots, w_m$. Figures 1(a)–(h) and 2(a)–(n) give all possible subgraphs of d (i.e., graphs for G_m) for $m \le 4$. Detailed constructions of these graphs are given below. Let l be the number of length-4 words in G_m. Clearly, $m \le l \le 2^m - 1$. *Throughout this section, all length-*4 *words under consideration are in* G_m.

It is obvious that the subgraph of d for $m = 1$ is Figure 1(a). (The only length-4 word w_1 is represented by a vertex.) For $m = 2$, $l = 2$ or 3. If $l = 2$, w_1 and w_2 do not share two common letters, and are represented by two separate vertices in Figure 1(b). For $l = 3$, w_1w_2 is a length-4 word, and any two of w_1, w_2 and w_1w_2 share two common letters and their product gives the third length-4 word.

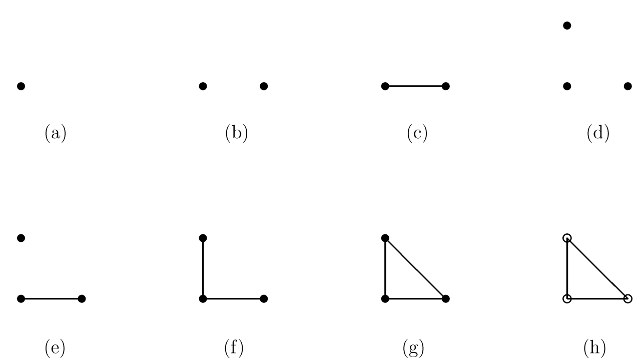

FIG. 1. *Graphs of length-*4 *words*: (a) $m = l = 1$, (b) $m = l = 2$, (c) $m = 2$, $l = 3$, (d) $m = l = 3$, (e) $m = 3$, $l = 4$, (f) $m = 3$, $l = 5$, (g) $m = 3$, $l = 6$, *and* (h) $m = 3$, $l = 7$.

We call these three words a tri-word group. They can be written as *ABCD*, *ABEF* and *CDEF*, and represented by two vertices connected by a line, as shown in Figure 1(c). For $m = 3$ and 4, the following lemma will be used.

LEMMA 2.1. (i) *Suppose that* $v_1, v_2, v_3, u = v_1v_2v_3 \in G_m$. *Then at least one of* v_1v_2, v_1v_3 *and* v_2v_3 *is a length-*4 *word. Thus,* u *is the product of two length-*4 *words* u_1 *and* uu_1 *in* G_m, *where* u_1 *is one of* v_1, v_2 *and* v_3.

(ii) *If* $m = 4$, $l = 5$ *and* $v = w_1w_2w_3w_4 \in G_4$, *any two of* $w_1, \dots, w_4$ *share a common letter and any three of them do not*.

(iii) *If* $m = 4$ *and* $l \geq 6$, *there exist* $l - 4$ *different tri-word groups*.

(iv) *Any two different tri-word groups share at most one common word*.

PROOF. (i) Let a_i be the number of letters that occur i times in v_1, v_2 and v_3. Since v_1, v_2 and v_3 are length-4 words, $a_1 + 2a_2 + 3a_3 = 12$. Since $u = v_1v_2v_3$ is a length-4 word, $a_1 + a_3 = 4$. Thus, $a_2 + a_3 = \{(a_1 + 2a_2 + 3a_3) - (a_1 + a_3)\}/2 = (12 - 4)/2 = 4$; that is, four letters occur at least twice in v_1, v_2 and v_3. This implies that at least one of the three pairs (v_1, v_2), (v_1, v_3) and (v_2, v_3) shares two common letters, say $v_1 = ABCD$ and $v_2 = ABEF$. Then v_1v_2 is a length-4 word and $u = (v_1v_2)v_3$ is the product of two length-4 words v_3 and $uv_3 = v_1v_2$ in G_m.

(ii) Let a_i be the number of letters that occur i times in $w_1, \dots, w_4$. Since $w_1, \dots, w_4$ are length-4 words, $a_1 + 2a_2 + 3a_3 + 4a_4 = 16$. Since $v = w_1w_2w_3w_4$ is a length-4 word, $a_1 + a_3 = 4$. Thus,

$$\begin{aligned} a_2 + a_3 + 2a_4 &= \{(a_1 + 2a_2 + 3a_3 + 4a_4) - (a_1 + a_3)\}/2 \\ &= (16 - 4)/2 = 6. \end{aligned} \tag{2.3}$$

Since $l = 5$ and v is a length-4 word, none of the six pairs (w_i, w_j) $(1 \leq i < j \leq 4)$ share two common letters. Because no pair can share three common letters, any pair can share only one or no letter in common. This implies $a_4 = 0$ or 1. If $a_4 = 1$, $a_2 = a_3 = 0$, which contradicts (2.3). If $a_4 = 0$ and three of $w_1, \dots, w_4$ (say w_1, w_2 and w_3) share a common letter, $a_3 = 1$ and w_1, w_2 and w_3 can be represented by *ABCD*, *AEFG* and *AHJK*, respectively. Then $w_1w_2w_3$ is of length 10, and hence no matter how $w_4 \in G_4$ is chosen, $w_1w_2w_3w_4 \in G_4$ cannot be satisfied. Thus $a_4 = 0$ and $a_3 = 0$, which implies $a_2 = 6$ via (2.3), completing the proof.

(iii) Let $S_1 = \{w_1, \dots, w_4\}$, $S = G_4 \setminus S_1$, $S_2 = S \cap \{w_iw_j : 1 \leq i < j \leq 4\}$ and $S_3 = S \cap \{w_iw_jw_k : 1 \leq i < j < k \leq 4\}$. For $w_iw_j \in S_2$, there is a tri-word group $\{w_i, w_j, w_iw_j\}$. For $u \in S_3$, by part (i), there exists $u_1 \in S_1$ such that $uu_1 \in S_2$. This gives a tri-word group $\{u_1, uu_1, u\}$. We show that these tri-word groups are different from each other. Otherwise, suppose two of them, obtained from $u_2, u_3 \in S_2 \cup S_3$, equal T_0. Then $u_2, u_3 \in T_0$, implying that $T_0 = \{u_2, u_3, u_2u_3\}$. For any $w \in S_2$, its tri-word group T_1 contains two words in S_1. Thus $T_1 \neq T_0$, implying that $u_2, u_3 \notin S_2$. Then $u_2, u_3 \in T_0 \cap S_3$. This is a contradiction since any of the above tri-word groups contains at most one word in S_3.

If $v = w_1w_2w_3w_4 \in G_4$, since $l \geq 6$, there exists a length-4 word $w \in S_2 \cup S_3$. If $w \in S_2$, say, $w = w_1w_2$, v is the product of three length-4 words w, w_3 and w_4. By part (i), v is also the product of two length-4 words v_0 and vv_0. If $w \in S_3$, $v_0 = wv \in S_1$ and $vv_0 = w \in S_3$. Again v is the product of two length-4 words v_0 and vv_0. Thus there is a tri-word group $\{v_0, vv_0, v\}$, which is different from the above tri-word groups obtained from words in $S_2 \cup S_3$, since none of them contain v. In summary, we have obtained $l - 4$ different tri-word groups, one for each length-4 word in S.

(iv) If two tri-word groups share two common words u_1 and u_2, they must equal $\{u_1, u_2, u_1u_2\}$, contradicting the assumption that they are different. □

PROPOSITION 2.1. *For any 2_{IV}^{k-p} design d, all the possible subgraphs of d for $m = 3$ and 4 are given by Figures* 1(d)–(h) *and* 2(a)–(n), *respectively.*

REMARK. The condition $m \leq 4$ is not as restrictive as it may appear because these results can apply to designs whose graphs contain more than four independent length-4 words. Working on the subgraphs of these designs for $m = 4$ can often reduce the search for MaxC2 designs to a much smaller set; see, for example, the MaxC2 2^{12-6} and 2^{15-8} designs given in Sections 3 and 4.

PROOF OF PROPOSITION 2.1. For $m = 3$, if none of w_1w_2, w_1w_3 and w_2w_3 are length-4 words, it follows from Lemma 2.1(i) that $w_1w_2w_3$ is not a length-4 word. Thus $l = 3$ and w_1, w_2 and w_3 are represented by three separate vertices, as in Figure 1(d). If at least one of w_1w_2, w_1w_3 and w_2w_3 is a length-4 word, say $w_1w_2 \in G_3$, w_1 and w_2 share two common letters and can be written as $w_1 = ABCD$ and $w_2 = ABEF$. Thus $w_1w_2 = CDEF$. Suppose w_3 contains t letters from $A, \ldots, F$. For $t = 0$ or 1, w_3 does not share two common letters with any of w_1, w_2 or w_1w_2, as represented by Figure 1(e), with vertices w_1, w_2 and w_3. For $t = 2$, since the pairs *AB*, *CD* and *EF* are symmetric and two letters within each pair are symmetric, it suffices to consider $w_3 = ABGH$ or *ACGH*. If $w_3 = ACGH$, we have Figure 1(f), with vertices *ABCD*, *ABEF* and *ACGH*. If $w_3 = ABGH$, we have Figure 1(g), with vertices *ABCD*, *ABEF* and *ABGH*, which generate six length-4 words, called a six-word loop. For $t \geq 3$, any of *AB*, *CD* or *EF* cannot occur in w_3. Otherwise, say *AB* and *C* occur in w_3, then w_3ABCD would be a length-2 word, contradicting the assumption that d is of resolution IV. Thus it suffices to consider $w_3 = ACEG$, since two letters within each pair *AB*, *CD* or *EF* are symmetric. Then we have Figure 1(h), with vertices *ABCD*, *ABEF* and *ACEG*, which generate seven length-4 words, called a seven-word group. Note that the vertices are unfilled circles, indicating that the product of three vertices is a length-4 word.

For $m = 4$, many more cases need to be considered. Clearly, for $l = 4$, we have Figure 2(a). For $l = 5$, if one of w_iw_j $(1 \leq i < j \leq 4)$ is a length-4 word, we have Figure 2(c), with vertices $w_1, \ldots, w_4$. If none of w_iw_j $(1 \leq i < j \leq 4)$ are

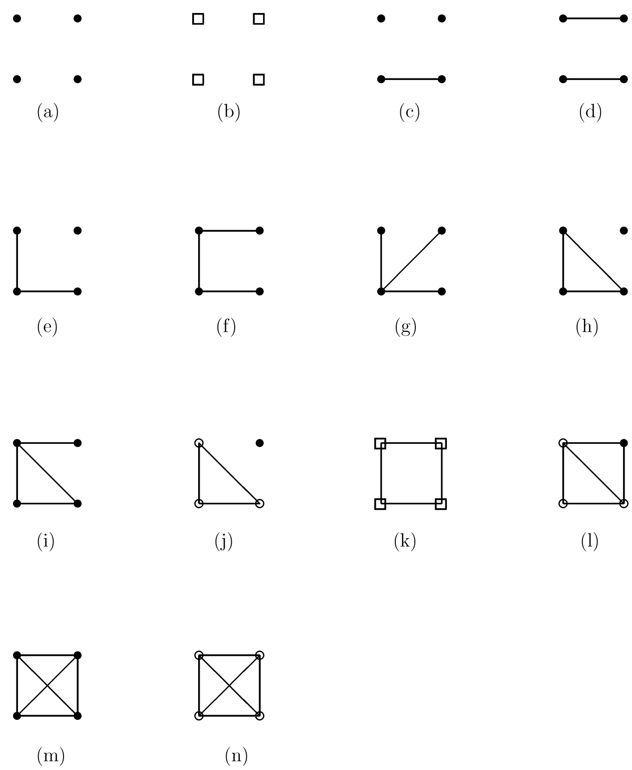

FIG. 2. *Graphs of length-4 words for $m = 4$*: (a) $l = 4$, (b) $l = 5$ (*without a line*), (c) $l = 5$ (*with a line*), (d), (e) $l = 6$, (f), (g) $l = 7$ (*without a six-word loop*), (h) $l = 7$ (*with a six-word loop*), (i) $l = 8$ (*with a six-word loop*), (j) $l = 8$ (*with a seven-word group*), (k) $l = 9$, (l), (m) $l = 10$, (n) $l = 14$.

length-4 words, it follows from Lemma 2.1(i) that none of $w_i w_j w_k$ ($1 \le i < j < k \le 4$) are length-4 words. Hence $w_1 w_2 w_3 w_4 \in G_4$ and we have Figure 2(b). By Lemma 2.1(ii), the four vertices can be written as *ABCD*, *AEFG*, *BEHJ* and *CFHK*. Since the product of the four vertices is a length-4 word, they are represented by unfilled boxes.

For $l = 6$, by Lemma 2.1(iii) and (iv), there exist two different tri-word groups with at most one common word. If they share no common word, they must contain four independent length-4 words and we have Figure 2(d). (If they contain only three independent length-4 words, $m = 3$ and one of the graphs in Figure 1 would contain two disjoint tri-word groups, which is not the case.) Otherwise, their common word is represented by a vertex with two lines connected to it and another independent length-4 word not in the groups is represented by a separate vertex, as in Figure 2(e). For $l = 7$, by Lemma 2.1(iii) and (iv), there exist three different tri-word groups with at most one common word between any two of them. If the three groups share one common word, it can be represented by a vertex with three lines connected to it, as in Figure 2(g). If two of the three groups do not share a common word, they contain six length-4 words. Since $l = 7$, the third group contains only one word not in the two groups and thus has a common word with each of them. This can be represented by Figure 2(f). If any two groups share a common word not in the third group, the three groups contain a six-word loop, which is generated by the three common words. Thus there is another independent length-4 word. This can be represented by Figure 2(h). It remains to consider $m = 4$ and $l \geq 8$.

Suppose first that G_4 does not contain any six-word loop or seven-word group. Then Figure 1(f) is a subgraph of d and its three vertices can be written as $w_1 = ABCD$, $w_2 = ABEF$ and $w_3 = ACGH$. Hence $w_1w_2 = CDEF$, $w_1w_3 = BDGH$, $w_2w_3 = BCEFGH$ and $w_1w_2w_3 = ADEFGH$. Suppose another independent length-4 word w_4 contains t letters from $A, \ldots, H$. Since $l \geq 8$, $t \geq 2$. [Otherwise, the graph for G_4 would be Figure 2(e) and then $l = 6$.] If $t = 2$, $w_2w_3w_4$ and $w_1w_2w_3w_4$ cannot be length-4 words. Since $l \geq 8$, there are two more length-4 words of the form w_4u and w_4v, where u and v belong to $\{w_1, w_2, w_3, w_1w_2, w_1w_3\}$. Denote the two letters in w_4 that are from $A, \ldots, H$ by x and y. Then both u and v contain x and y. Since u and v cannot share three letters in common, they must share two letters in common. Then w_4, u and v would lead to a six-word loop, contradicting the assumption that G_4 does not contain any six-word loop. Thus $t \geq 3$. If w_4 contains EF (or GH) for which d is of resolution IV, w_4, w_2 and w_1w_2 (or w_4, w_3 and w_1w_3) would lead to a six-word loop, again a contradiction. If w_4 contains three of $A, \ldots, F$ (or three of A, B, C, D, G and H) for which d is of resolution IV, w_4, w_1 and w_2 (or w_4, w_1 and w_3) would lead to a seven-word group, contradicting the assumption that G_4 does not contain any seven-word group. In summary, w_4 contains exactly one letter from each of $ABCD$, EF and GH. Thus w_4 can be written as $AEGJ$ since two letters within each pair EF or GH are symmetric. (It is easy to verify that choosing w_4 to be $AEGJ$, $BEGJ$, $CEGJ$ or $DEGJ$ leads to the same graph.) Then w_2w_4, w_3w_4 and $w_1w_2w_3w_4$ are also length-4 words. Hence $l = 9$ and the graph for G_4 can be represented by Figure 2(k).

Suppose now that G_4 contains no seven-word group, but it has a six-word loop, which can be written as $ABCD$, $ABEF$, $ABGH$, $CDEF$, $CDGH$ and $EFGH$. Suppose another independent length-4 word w contains t letters from $A, \ldots, H$. Since $l \geq 8$, $t \geq 2$. [Otherwise, the graph for G_4 would be Figure 2(h) and then $l = 7$.] If $t \geq 3$, any of AB, CD, EF or GH cannot occur in w. Otherwise, say AB and C occur in w, then $wABCD$ would be a length-2 word, contradicting the assumption that d is of resolution IV. Since the pairs AB, CD, EF and GH are symmetric and two letters within each pair are symmetric, we only need to consider $w = ACEG$ or $ACEJ$. Then $ABCD$, $ABEF$ and w would lead to a seven-word group, contradicting the assumption that G_4 contains no seven-word group. Thus $t = 2$. If w contains one of AB, CD, EF and GH, say, AB, we have Figure 2(m), and the four vertices can be chosen as $ABCD$, $ABEF$, $ABGH$ and $ABJK$. If w contains one of the other 24 2FIs consisting of $A, \ldots, H$, say, AC, we have Figure 2(i).

Finally, suppose that G_4 contains a seven-word group, which can be written as $ABCD$, $ABEF$, $ACEG$, $CDEF$, $BDEG$, $BCFG$ and $ADFG$. Since all two-letter combinations of $A, \ldots, G$ occur twice in the group, if another length-4 word w contains, for example, AB, w cannot contain any of $C, \ldots, F$. Thus, w can have only $t \leq 3$ letters from $A, \ldots, G$. If $t = 0$ or 1, we have Figure 2(j). If $t = 2$, we have Figure 2(l), with four vertices $ABCD$, $ABEF$, $ACEG$ and $ABHJ$. If $t = 3$, we have Figure 2(n), with vertices $ABCD$, $ABEF$, $ACEG$ and $BCEH$. $\square$

From the above graphs, it is easy to obtain the number of unclear 2FIs, $U(d)$, when $G_m = L_4$ and d has $m \leq 4$ independent length-4 words. In this case, $l = A_4(d)$. If a graph does not contain any three-vertex loop (a six-word loop or a seven-word group) or unfilled boxes, $U(d) = 6m + 3(l - m) = 3(m + l)$, where m is the number of vertices and $l - m$ is the number of lines. This includes Figures 1(a)–(f), 2(a) and 2(c)–(g). If a graph contains only one three-filled-vertex loop (a six-word loop), $U(d) = 6m + 3(l - m) + 1 = 3(m + l) + 1$. This includes Figures 1(g), 2(h) and 2(i), and correspondingly, $U(d) = 28, 34$ and 37. Figures 1(h), 2(n), 2(k) and 2(m) contain 7, 8, 9 and 10 letters, respectively. Since any 2FIs involving these letters are unclear, then correspondingly, $U(d) = 21, 28, 36$ and 45. Finally, by the above descriptions of Figures 2(b), (j) and (l), we have $U(d) = 30, 27$ and 34, respectively.

3. MA and MaxC2 designs. Based on the graphs constructed in the last section, the following propositions are established. They identify some cases in which an MA 2^{k-p} design denoted as $d_{\mathrm{MA}} = d_{\mathrm{MA}}(k, p)$ is also a MaxC2 design.

PROPOSITION 3.1. *If $A_4(d_{\mathrm{MA}}) = 1$ or 2, d_{MA} is a MaxC2 design and its number of clear 2FIs is $k(k-1)/2 - 6A_4(d_{\mathrm{MA}})$.*

PROOF. Clearly the proposition holds for $A_4(d_{\mathrm{MA}}) = 1$. For $A_4(d_{\mathrm{MA}}) = 2$, d_{MA} has 12 unclear 2FIs. For any 2^{k-p}_{IV} design d, $A_4(d) \geq A_4(d_{\mathrm{MA}}) = 2$. Thus either Figure 1(b) or 1(c) is a subgraph of d, which has at least 12 unclear 2FIs. This proves the proposition. $\square$

PROPOSITION 3.2. *Let d be a 2^{k-p}_{IV} design. If $A_4(d) \geq 3$, d has at least 15 unclear 2FIs. If $A_4(d) \geq 4$, d has at least 21 unclear 2FIs.*

PROOF. For $A_4(d) \geq 3$, either Figure 1(c) or 1(d) is a subgraph of d, which has 15 or 18 unclear 2FIs, respectively. For $A_4(d) \geq 4$, if any two length-4 words do not share two common letters, Figure 2(a) or 2(b) is a subgraph of d and hence $U(d) \geq 24$. Otherwise, there exist three independent length-4 words with at least two of them sharing two common letters. In this case, one of Figures 1(e)–(h) is a subgraph of d, and correspondingly, $U(d) \geq 21, 24, 28$ or 21. $\square$

COROLLARY 3.1. *For $A_4(d_{\mathrm{MA}}) = 3$ or 4, if there exist two length-4 words with two common letters, d_{MA} is a MaxC2 design and its number of clear 2FIs is $k(k-1)/2 - 6A_4(d_{\mathrm{MA}}) + 3$.*

PROOF. In this case, the graph of d_{MA} is Figure 1(c) or 1(e), and thus $U(d_{\mathrm{MA}}) = 15$ or 21, respectively. The corollary then follows from Proposition 3.2. $\square$

PROPOSITION 3.3. *If $A_4(d_{\mathrm{MA}}) = 3$ and $A_4(d_{\mathrm{MA}}(k-1, p-1)) = 2$, d_{MA} is a MaxC2 design and its number of clear 2FIs is $k(k-1)/2 - 18$.*

PROOF. It suffices to show that $U(d) \geq 18$ for any 2_{IV}^{k-p} design d. Note that $A_4(d) \geq A_4(d_{\mathrm{MA}}) = 3$. If $A_4(d) \geq 4$, by Proposition 3.2, $U(d) \geq 21$. If $A_4(d) = 3$, it suffices to show that any two length-4 words of d do not share any common letter, say A. Otherwise, all the words of d that do not contain letter A would form the defining contrast subgroup of a $2_{\mathrm{IV}}^{(k-1)-(p-1)}$ design d_1 with $A_4(d_1) \leq 1$. This would imply that $A_4(d_{\mathrm{MA}}(k-1, p-1)) \leq 1$, which contradicts the assumption. Thus, each of the three length-4 words of d has 6 unclear 2FIs and none of them overlap, implying that $U(d) = 18$. $\square$

Recall that, for $k_{\max}(k-p) < k \leq 2^{k-p-2} + 1$, there exist resolution IV designs with clear 2FIs. So far, MA designs with $k_{\max}(k-p) < k \leq 2^{k-p-2} + 1$ have been obtained only for 15 cases: 32- and 64-run designs, and 128-run designs with $k = 12$, 13 and 14. These are given in Tables 4A.3, 4A.5 and 4A.7 of Wu and Hamada (2000). It follows from the above results that the following MA designs are also MaxC2 designs: 2^{7-2}, 2^{9-3}, 2^{10-4}, 2^{12-5}, 2^{13-6} (by Proposition 3.1); 2^{8-3}, 2^{11-5} (by Corollary 3.1); and 2^{14-7} (by Proposition 3.3). Proposition 3.4 below shows that the MA 2^{12-6} design is also a MaxC2 design. None of the other six MA designs (for 2^{9-4}, 2^{13-7}, 2^{14-8}, 2^{15-9}, 2^{16-10} and 2^{17-11}) are MaxC2 designs. First we prove the following lemmas.

LEMMA 3.1. (i) *For any 2_{IV}^{12-6} design d, if a letter, say A, occurs in a length-4 word, there are at least four length-4 words that do not contain letter A.*
(ii) *For any 2_{IV}^{12-6} design d, we have $A_4(d) \geq 6$.*

PROOF. (i) All the words of d that do not contain letter A form the defining contrast subgroup of a 2_{IV}^{11-5} design d_1. Since $A_4(d_1) \geq A_4(d_{\mathrm{MA}}(11, 5)) = 4$ [see Chen (1992)], d_1 has at least four length-4 words, which are also the length-4 words of d that do not contain letter A.

(ii) By part (i), $A_4(d) \geq 5$. Since d has 12 factors (letters), then at least two length-4 words share a common letter, say A. By part (i), there are at least four length-4 words without letter A. Hence $A_4(d) \geq 6$. $\square$

LEMMA 3.2. *Figure 2(j) cannot be the graph of any 2_{IV}^{12-6} design d.*

PROOF. Suppose otherwise that the graph of a 2_{IV}^{12-6} design d is Figure 2(j), which has three connected vertices $w_1 = ABCD$, $w_2 = ABEF$ and $w_3 = ACEG$.

Then $w_4 = w_1w_2 = CDEF$, $w_5 = w_1w_3 = BDEG$, $w_6 = w_2w_3 = BCFG$ and $w_7 = w_1w_2w_3 = ADFG$. If the isolated vertex w_8 contains any letter from $A, \dots, G$, say, A, only three length-4 words (w_4, w_5 and w_6) do not contain letter A, contradicting Lemma 3.1(i). Thus w_8 must contain four letters from H, J, K, L, M, say $w_8 = HJKL$. Since $p = 6$, there are two other independent words, w_9 and w_{10}, at least one of which (say w_9) does not contain letter M. (Otherwise, we can replace w_9 by w_9w_{10}.) Furthermore, we can assume w_9 contains at most two letters from H, J, K, L. (Otherwise, we can replace w_9 by w_8w_9.)

Let $S_1 = \{w_1, \dots, w_7\}$, $S_2 = \{w_9, w_9w_1, \dots, w_9w_7\}$ and $S = S_1 \cup S_2$. Note that $A, \dots, G$ occur $7 \times 8 = 56$ times in S and $7 \times 4 = 28$ times in S_1. Thus they occur $56 - 28 = 28$ times in S_2. Then there exists $w_* \in S_2$ that contains at most three letters from $A, \dots, G$. (Otherwise, $A, \dots, G$ would occur at least $8 \times 4 = 32$ times in S_2.) Since all words in S_2 contain at most two letters from H, J, K, L and w_* has length at least 5, w_* must contain three letters from $A, \dots, G$ and two letters from H, J, K, L. If w_* contains any of the 28 three-letter combinations of $A, \dots, G$ other than ABG, ACF, BCE, ADE, BDF, CDG and EFG, multiplying w_* with one of $w_1, \dots, w_7$ reduces the length of w_* by 2, leading to a length-3 word. This contradicts the assumption that d is of resolution IV. Thus w_* can be written as $ABGHJ$. Then $S_3 = \{w_*, w_1w_*, \dots, w_7w_*\} \setminus \{w_4w_*\} = \{ABGHJ, CDGHJ, EFGHJ, BCEHJ, ADEHJ, ACFHJ, BDFHJ\}$.

Consider another independent word w_{10} with t letters from $A, \dots, G$. Note that all 21 two-letter combinations of $A, \dots, G$ occur in S_3. If $t \geq 2$, multiplying w_{10} with one of the words in S_3 gives a word w_{10}^* which reduces t by at least 1. Thus we only need to consider $t \leq 1$. Furthermore, w_{10} contains at most three letters from H, J, K, L, M. (Otherwise, we can replace w_{10} by w_8w_{10}.) Then w_{10} is of length at most 4, which contradicts the assumptions that d is of resolution IV and there are only four independent length-4 words (w_1, w_2, w_3 and w_8). Thus Figure 2(j) cannot be the graph of any 2_{IV}^{12-6} design d. $\square$

PROPOSITION 3.4. *The MA* 2^{12-6} *design* $d_{\mathrm{MA}} = d_{\mathrm{MA}}(12, 6)$ [*see page* 197 *in Wu and Hamada* (2000)] *is a MaxC*2 *design and its number of clear* 2*FIs is* 36.

PROOF. First we show that any 2_{IV}^{12-6} design d has at least four independent defining words of length 4. Otherwise, since $A_4(d) \geq 6$ by Lemma 3.1(ii), the graph of d is Figure 1(g) or 1(h). Let $ABCD$ be a length-4 word of d. Then from the description of these two graphs in Section 2, it can be shown that only three length-4 words of d do not contain letter A. This contradicts Lemma 3.1(i).

Noting that $U(d) = 66 - C(d)$, we need to prove $U(d) \geq 30$ for any 2_{IV}^{12-6} design d. Suppose that L_4 contains s different tri-word groups. If $s = 0$, $U(d) = 6A_4(d) \geq 6 \times 6 = 36$. If $s = 1$, $U(d) = 6A_4(d) - 3 \geq 6 \times 6 - 3 = 33$. If $s \geq 2$, one of Figures 2(d)–(n) is a subgraph of d. If one of Figures 2(d)–(i) and 2(k)–(m) is a subgraph of d, by the discussion in the last paragraph of Section 2, $U(d) \geq 30$.

If Figure 2(n) is a subgraph of d, the four vertices can be denoted by $ABCD$, $ABEF$, $ACEG$ and $BCEH$. Denote the 12 factors by $A, \dots, H, J, K, L, M$. Since $p = 6$, there are two other independent defining words, w_9 and w_{10}, one of which (say w_9) contains at most two letters from J, K, L, M. (Otherwise, we can replace w_9 by $w_9 w_{10}$.) Since all 56 three-letter combinations of $A, \dots, H$ occur in the 14 length-4 words of the subgraph, a product of w_9 and some of these words can always lead to a word w' with at most two letters from $A, \dots, H$. Then w' is a length-4 word with two letters from J, K, L, M, say $w' = ABJK$ (since d is a resolution IV design). This leads to $JK = AB = CD = EF = GH$, which implies that any 2FIs consisting of $A, \dots, H$, J, K are unclear. Thus $U(d) \geq 45$.

It remains to consider Figure 2(j) being a subgraph of d, which has a seven-word group $w_1, \dots w_7$, as written in the proof of Lemma 3.2. This gives 21 unclear 2FIs consisting of $A, \dots, G$. By Lemma 3.2, d has two other length-4 words, v_1 and v_2 (in addition to $w_1, \dots, w_7$). If v_1 or v_2 shares two common letters with any of $w_1, \dots, w_7$, since Figure 2(j) is a subgraph of d, the graph of d contains Figure 2(l) or 2(n) and a separate vertex, whose $U(d) \geq 34 + 6 = 40$ or $28 + 6 = 34$ as shown in Section 2. Otherwise, the 21 2FIs and the 12 2FIs for v_1 and v_2 (or 15 2FIs for v_1, v_2 and $v_1 v_2$ if $v_1 v_2$ is a length-4 word) do not overlap, and thus $U(d) \geq 21 + 12 = 33$. □

4. MaxC2 designs. The graph representation and classification of length-4 words developed in Section 2 allows us to obtain bounds for the number of clear 2FIs and to reduce the search for designs to a much smaller set. The reduction is usually substantial enough to identify a large MaxC2 design with a complete computer search over the much smaller set. An example is the search for and proof of a MaxC2 2^{15-8} design given below. In this section, we will also develop some useful combinatorial identities and inequalities and group-theoretic arguments to reduce the number of candidate designs.

LEMMA 4.1. (i) *For any* 2_{IV}^{14-7} *design* d_1, $A_4(d_1) \geq 3$. *If* $A_4(d_1) = 3$, *any two of the three length*-4 *words of* d_1 *cannot share a common letter*.

(ii) *For any* 2_{IV}^{15-8} *design* d, *we have* $A_4(d) \geq 5$.

(iii) *Any* 2_{IV}^{15-8} *design* d *has at least four independent defining words of length*-4.

Part (i) follows from Chen [(1998), page 1269] and is used to prove parts (ii) and (iii). Their proofs are based on the same methods for proving Lemma 3.1 and the first statement in the proof of Proposition 3.4 and are thus omitted.

LEMMA 4.2. *Figure* 2(c) *cannot be the graph of any* 2_{IV}^{15-8} *design* d.

PROOF. For design d, let $1, \dots, 7$ denote the seven independent columns and let $A, \dots, H$ denote the other eight columns. Assume that Figure 2(c) is the graph of d. Then $A_4(d) = 5$ and the tri-word group in the graph can be written as $123A$,

$124B$ and $34AB$. Let w_1 and w_2 denote the other two length-4 words. All the words of d that do not contain letter A form the defining contrast subgroup of a 2_{IV}^{14-7} design d_1, which contains three length-4 words: $124B$, w_1 and w_2. By Lemma 4.1(i), w_1 and w_2 do not share a common letter and neither of them contains any of $1, 2, 4, B$. Similarly, we can show that none of $1, 2, 3, A$ occurs in w_1 or w_2. Thus, the five length-4 words of d can be written as $123A$, $124B$, $34AB$, $567C$ and $EFGH$. Thus $A = 123$, $B = 124$, $C = 567$ and $H = EFG$, and we only need to look for the other 4 columns D, E, F and G. The only feasible columns that can be used to define D, E, F or G are those consisting of (1) two of 1, 2, 3, 4 and two of 5, 6, 7; (2) one of 134 and 234 and at least one of 5, 6, 7 and (3) 1234 and two of $5, 6, 7$. We thus end up with choosing 4 out of only 35 columns. This makes a complete computer search an easy task. A program written for this purpose verified that none of the choices can lead to a resolution IV design with the graph of Figure 2(c). $\square$

LEMMA 4.3. *Figure* 2(j) *cannot be the graph of any* 2_{IV}^{15-8} *design* d.

The proof of Lemma 4.3 is similar to that of Lemma 4.2 (and also with the help of a computer search) and is thus omitted.

PROPOSITION 4.1. *The* 2_{IV}^{15-8} *design with* $A = 123$, $B = 124$, $C = 134$, $D = 234$, $E = 1256$, $F = 1357$, $G = 1467$ *and* $H = 1234567$ *is a MaxC*2 *design with* 77 *clear* 2*FIs*.

PROOF. For any 2_{IV}^{15-8} design d, by Lemma 4.1, $A_4(d) \geq 5$ and d has at least four independent length-4 words. By Lemmas 4.2 and 4.3, neither Figure 2(c) or 2(j) can be the graph of d. Thus one of Figures 2(b), 2(d)–2(i) and 2(k)–2(n) is a subgraph of d. By the discussion in the last paragraph of Section 2, $U(d) \geq 28$. Note that $U(d) = 28$ if and only if Figure 2(n) is the graph of d. The design given in the proposition is a 2_{IV}^{15-8} design with 28 unclear 2FIs and is thus a MaxC2 design. [It is obtained by first assuming that Figure 2(n) is the graph of a 2_{IV}^{15-8} design and going through the same process as for Figures 2(c) and 2(j). The design given above is then obtained.] $\square$

We now develop some useful combinatorial identities and inequalities. These results, together with some group-theoretic arguments, are used to show that a class of designs due to Tang, Ma, Ingram and Wang (2002) and the second 2^{9-4}, 2^{13-7}, 2^{16-10} and 2^{17-11} designs given in Wu and Hamada [(2000), pages 195 and 197] are MaxC2 designs. Some notation from Cheng, Steinberg and Sun (1999) will be used. For a 2_{IV}^{k-p} design d, let $f = 2^{k-p} - 1 - k$ and $m_j(d)$ be the number of 2FIs in the jth alias set not containing main effects, where $j = 1, \dots, f$. Furthermore, we define $N_i = \#\{1 \leq j \leq f : m_j(d) = i\}$ for $i \geq 0$. Then N_i $(i \geq 1)$ is the number of alias sets that contain i 2FIs and $N_1 = C(d)$ is the number of clear 2FIs. Let $I(d)$ denote the number of 2FIs of d. Then $U(d) = I(d) - C(d)$.

Note that *any 2FIs in the same alias set do not share a common letter*. Thus $m_j(d) \le k/2$ and $N_i = 0$ for $i > r$, where r is the integral part of $k/2$. Then

$$I(d) = k(k-1)/2 = \sum_{i=1}^{r} iN_i = C(d) + \sum_{i=2}^{r} iN_i \tag{4.1}$$

and

$$f = 2^{k-p} - k - 1 = \sum_{i=0}^{r} N_i = N_0 + C(d) + \sum_{i=2}^{r} N_i. \tag{4.2}$$

It follows from Cheng, Steinberg and Sun (1999) that

$$A_4(d) = \frac{1}{6}\left[\sum_{j=1}^{f} m_j(d)\{m_j(d)-1\}\right] = \frac{1}{6}\left\{\sum_{i=2}^{r} i(i-1)N_i\right\}. \tag{4.3}$$

We have the following lemmas.

LEMMA 4.4. *If $N_i = 0$ for $i > 4$, N_2 is a multiple of* 3.

PROOF. If $N_i = 0$ for $i > 4$, it follows from (4.3) that

$$A_4(d) = \frac{1}{6}(2N_2 + 6N_3 + 12N_4) = \frac{1}{3}N_2 + N_3 + 2N_4.$$

Thus $N_2 = 3\{A_4(d) - N_3 - 2N_4\}$, implying that N_2 is a multiple of 3. $\square$

LEMMA 4.5. *If $N_i > 0$ for some i, where $2 \le i \le r$, $U(d) \ge i(2i-1)$.*

PROOF. If $N_i > 0$, there exists an alias set with i 2FIs. These 2FIs contain $2i$ letters, any two of which form an unclear 2FI. Thus $U(d) \ge \binom{2i}{2} = i(2i-1)$. $\square$

LEMMA 4.6. (i) *If $N_i = 0$ for $i = j+1, \dots, r$, where $2 < j < r$, then*

$$C(d) \le \{(j-1)f + N_j - I(d)\}/(j-2).$$

(ii) *If $N_i = 0$ for $i = j, \dots, r$, where $2 < j \le r$, then*

$$C(d) \le \{(j-1)f - I(d)\}/(j-2).$$

PROOF. It follows from (4.1) and (4.2) that

$$\begin{aligned} I(d) &= C(d) + jN_j + \sum_{i=2}^{j-1} iN_i \\ &\le C(d) + jN_j + (j-1)\sum_{i=2}^{j-1} N_i \\ &= C(d) + jN_j + (j-1)\{f - N_0 - C(d) - N_j\} \\ &= C(d) + N_j + (j-1)\{f - N_0 - C(d)\} \\ &\le C(d) + N_j + (j-1)\{f - C(d)\}. \end{aligned} \tag{4.4}$$

Thus, $C(d) \le \{(j-1)f + N_j - I(d)\}/(j-2)$. This proves part (i). Part (ii) follows from part (i) by noting that $N_j = 0$. $\square$

PROPOSITION 4.2. *For any* 2_{IV}^{k-p} *design* d *with* $k = 2^{k-p-2} + 1$, *we have* $C(d) \le 2k-3$. *If* $C(d) = 2k-3$, $N_{r-1} = k-2$ *and* $N_i = 0$ *for* $i \ne 1, r-1$.

PROOF. Note that $r = (k-1)/2 > 2$ and $f = 2^{k-p} - 1 - k = 3k - 5$. If $N_r > 0$, by Lemma 4.5, $U(d) \ge r(2r-1) = (k-1)(k-2)/2$. Thus $C(d) = I(d) - U(d) = k(k-1)/2 - U(d) \le k-1$. If $N_r = 0$, by Lemma 4.6(ii),

$$\begin{aligned} C(d) &\le \frac{(r-1)f - I(d)}{r-2} = \frac{\{(k-1)/2 - 1\}(3k-5) - k(k-1)/2}{(k-1)/2 - 2} \\ &= 2k - 3 > k - 1. \end{aligned}$$

That is, $C(d) \le 2k-3$. If $C(d) = 2k-3$, $C(d) = \{(r-1)f - I(d)\}/(r-2)$, and hence $I(d) = C(d) + (r-1)\{f - C(d)\}$. Together with the inequalities in (4.4) (with $j = r$), this implies that $N_{r-1} = k-2$ and $N_i = 0$ for $i \ne 1, r-1$. $\square$

COROLLARY 4.1. *The second* 2_{IV}^{9-4} *and* 2_{IV}^{17-11} *designs given in Wu and Hamada* [(2000), *pages* 195 *and* 197] *are MaxC*2 *designs*.

PROOF. These two designs have 15 and 31 clear 2FIs (see the tables there), respectively. The corollary then follows from Proposition 4.2, which implies that, for any 2_{IV}^{9-4} design d_1 and 2_{IV}^{17-11} design d_2, $C(d_1) \le 2 \times 9 - 3 = 15$ and $C(d_2) \le 2 \times 17 - 3 = 31$. $\square$

COROLLARY 4.2. *The* 2_{IV}^{k-p} *designs* d *with* $k = 2^{k-p-2} + 1$ *given by Tang*, *Ma*, *Ingram and Wang* (2002) *are MaxC*2 *designs*.

PROOF. It follows from Proposition 4.2 by noting that these designs have $C(d) = 2k-3$ as shown by their authors. $\square$

PROPOSITION 4.3. *The second* 2_{IV}^{13-7} *design given in Wu and Hamada* [(2000), *page* 197] *is a MaxC*2 *design with* 36 *clear* 2*FIs*.

To save space, the proof of Proposition 4.3 is omitted here. A detailed proof is given in Wu and Wu (2000).

LEMMA 4.7. *For any* 2_{IV}^{k-p} *design* d, *if there exist two alias sets with six* 2*FIs containing the same* 12 *letters*, *there are exactly three alias sets with six* 2*FIs with these* 12 *letters*.

PROOF. Denote an alias set with six 2FIs by $AB = CD = EF = GH = JK = LM$. It gives 30 additional pairs of aliased 2FIs, that is, $AC = BD$, $AD = BC, \dots,$ $JL = KM$, $JM = KL$. Another alias set with six 2FIs containing these 12 letters must consist of three of these pairs, say, $AC = BD = EG = FH = JL = KM$. Then there are seven independent length-4 defining words (*ABCD*, *ABEF*, *ABGH*, *ABJK*, *ABLM*, *ACEG* and *ACJL*) with the 12 letters, which represent 12 columns of d. Since d is of resolution IV, four independent columns $(1, 2, 3$ and $4)$ can define at most four other columns (123, 124, 134 and 234), which gives at most eight columns of d. Thus, at least five of the 12 letters correspond to independent columns of d, implying that there are at most seven independent length-4 defining words with the 12 letters. Hence there is no other independent length-4 word with four of the 12 letters. In this case, it can be verified (by writing out all alias patterns generated by the seven length-4 defining words) that there are exactly three alias sets with six 2FIs containing the 12 letters. □

PROPOSITION 4.4. *The second* 2^{16-10}_{IV} *design given in Wu and Hamada* [(2000), *page* 197] *is a MaxC*2 *design with* 29 *clear* 2*FIs*.

PROOF. Consider any 2^{16-10}_{IV} design d. Then $f = 47$, $I(d) = 120$ and $r = 8$. It suffices to show that $C(d) \leq 29$ or $U(d) \geq 91$. If $N_8 > 0$, by Lemma 4.5, $U(d) \geq 120$. If $N_8 = 0$ and $N_7 > 0$, again by Lemma 4.5, $U(d) \geq 91$. If $N_8 = N_7 = 0$, by Lemma 4.6(i), $C(d) \leq \{5f + N_6 - I(d)\}/4 = (5 \times 47 + N_6 - 120)/4 = 29.75 + (N_6 - 4)/4$. Thus $C(d) \leq 29$ for $N_6 \leq 4$.

It remains to consider the case in which $N_8 = N_7 = 0$ and $N_6 \geq 5$. Denote an alias set with six 2FIs by $AB = CD = EF = GH = JK = LM$. Since $N_6 \geq 5$, by Lemma 4.7, there exists another alias set with six 2FIs and $t \geq 2$ new letters (other than $A, \dots, H, J, K, L, M$). (It cannot contain only one new letter since the 60 additional aliased 2FIs, $AC = BD$, $AD = BC, \dots, JL = KM$, $JM = KL$, occur pairwise in an alias set.) If $t \geq 3$, $U(d) \geq 66 + t(t-1)/2 + t(12 - t) \geq 96$. Otherwise, without loss of generality, we have $AC = BD = EG = FH = JP = LQ$. (Choosing $JP = KQ$ gives $AB = CD = EF = GH = JK = LM = PQ$, contradicting the assumption that $N_8 = N_7 = 0$.) Then $KM = JL = PQ$ and the 91 2FIs consisting of $A, \dots, H, J, K, L, M, P, Q$ are unclear. Thus $U(d) \geq 91$, proving the proposition. □

5. Summary and concluding remarks. Proving whether a design is a MaxC2 design is technically challenging. Unlike the MA criterion which is defined in terms of the word-length pattern, the number of clear 2FIs is a complicated mathematical function of the defining contrast subgroup. This technical difficulty may explain why there has been no general method for tackling the problem. In this paper, we give a method for proving whether a resolution IV design is a MaxC2 design. First we develop a graph representation and classification of length-4 words. This representation of designs allows us to obtain bounds for the number of clear 2FIs and to reduce the search for designs to a much smaller set. We also develop some useful combinatorial identities and inequalities (as given in Section 4) and group-theoretic arguments (as used in Sections 3 and 4) to reduce the number of candidate designs. If theoretical arguments alone cannot do the job, the reduction is usually substantial enough for computer search to finish the job. An example is the search for and proof of a MaxC2 2^{15-8} design in Section 4.

In addition to proving the MaxC2 property of many existing designs, we are able to find new MaxC2 designs such as a class of designs due to Tang, Ma, Ingram and Wang (2002) and the 2^{15-8} design mentioned above. It demonstrates that the tools can be effectively used to find new MaxC2 designs. We have given lengthy proofs for several designs (known to be MaxC2 designs by computer search) in order to show how the approach can be implemented. Only through the demonstration in these proofs will the potential users be able to use and improve the tools to find designs for new situations such as 128- and 256-run designs, three-level designs, two-level designs in blocks or with control and noise factors for parameter design application. It is our hope that this work is a modest start of a new direction of research in fractional factorial designs.

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REFERENCES

BOSE, R. C. (1947). Mathematical theory of the symmetrical factorial design. *Sankhyā* **8** 107–166.

CHEN, H. and HEDAYAT, A. S. (1998). 2^{n-m} designs with resolution III or IV containing clear two-factor interactions. *J. Statist. Plann. Inference* **75** 147–158.

CHEN, J. (1992). Some results on 2^{n-k} fractional factorial designs and search for minimum aberration designs. *Ann. Statist.* **20** 2124–2141.

CHEN, J. (1998). Intelligent search for 2^{13-6} and 2^{14-7} minimum aberration designs. *Statist. Sinica* **8** 1265–1270.

CHEN, J., SUN, D. X. and WU, C. F. J. (1993). A catalogue of two-level and three-level fractional factorial designs with small runs. *Internat. Statist. Rev.* **61** 131–145.

CHENG, C.-S., STEINBERG, D. M. and SUN, D. X. (1999). Minimum aberration and model robustness for two-level fractional factorial designs. *J. Roy. Statist. Soc. Ser. B* **61** 85–93.

FRIES, A. and HUNTER, W. G. (1980). Minimum aberration 2^{k-p} designs. *Technometrics* **22** 601–608.

TANG, B., MA, F., INGRAM, D. and WANG, H. (2002). Bounds on the maximum number of clear two-factor interactions for 2^{m-p} designs of resolution III and IV. *Canad. J. Statist.* **30** 127–136.

WU, C. F. J. and HAMADA, M. (2000). *Experiments*: *Planning*, *Analysis*, *and Parameter Design Optimization*. Wiley, New York.

WU, H. and WU, C. F. J. (2000). Clear two-factor interactions and minimum aberration. Technical Report 363, Dept. Statistics, Univ. Michigan, Ann Arbor.

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