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# LARGE SAMPLE BAYESIAN ANALYSIS FOR Geo / *G* / 1 DISCRETE-TIME QUEUEING MODELS[1]

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In this paper, a nonparametric Bayesian analysis of queueing models with geometric input and general service time is performed. In particular, statistical inference for the probability generating function (p.g.f.) of the equilibrium waiting time distribution is considered. The consistency of the posterior distribution for such a p.g.f., as well as the weak convergence to a Gaussian process of a suitable rescaling, are proved. As by-products, results on statistical inference for queueing characteristics are also obtained. Finally, the problem of estimating the probability of a long delay is considered.

**1. Introduction and preliminaries.** Discrete-time queueing models have recently received considerable attention, mainly because of their applications to telecommunication systems based on asynchronous transfer mode (ATM), which is the standard transport vehicle of the broadband integrated services digital network (B-ISDN). In ATM, information is segmented into fixed-size transmission units, called *cells*. The transmission time of a cell is the *cell-time*, and it is naturally taken as a *time slot*. During a time slot, exactly one cell is transmitted, and $0, 1, 2, \dots$ cells can simultaneously enter the system, to have to be transmitted. The system works in discrete time, which is measured in terms of time slots. Cells that cannot be immediately transmitted are stored in a buffer and form a queue. Their transmission is delayed according to a FIFO rule. Multiple arrivals of cells (i.e., *batches*) in a time slot are allowed; they form the *background process* [Roberts, Mocci and Virtamo (1996)]. Although the transmission time of a single cell is deterministic, the time required to transmit a batch of cells is stochastic, because of the stochastic size of the batches.

Let $T_i$ be the r.v. "number of time slots between $(i-1)$th and $i$th batches ($i$th interarrival time), and let $S_i$ be the r.v. $i$th service time (i.e., the size of the $i$th batch). The hypotheses on r.v.'s $T_i$'s and $S_i$'s are:

1. $P(T_i = k \mid \lambda) = \lambda(1-\lambda)^{k-1}$, $k \geq 1$, $0 < \lambda < 1$.

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2. $P(S_i = k|P_b) = b(k)$, $k \geq 1$ ($\sum_{k \geq 1} b(k) = 1$), $P_b$ being the probability measure (concentrated over the positive integers) corresponding to the service time distribution.
3. $\mu = E(S_i|P_b) < \infty$.
4. $(T_i;\ i \geq 1)$ and $(S_i;\ i \geq 1)$ are two sequences of i.i.d. r.v.'s, conditionally on $\lambda$ and $P_b$; r.v.'s $T_i$'s independent of $S_i$'s, conditionally on $\lambda$ and $P_b$.

The corresponding queueing model, with geometric interarrival times, general service times and one service point, is referred to as the Geo$/G/1$ model. It is commonly considered as a realistic model for studying the performance of ATM systems at the cell level; see Louvion, Boyer and Gravey (1988), Gravey, Louvion and Boyer (1990), Roberts, Mocci and Virtamo (1996).

Since $\lambda$ and $b(k)$'s are usually unknown, they have to be estimated on the basis of observed data, consisting in $n$ interarrival times $T_1, \dots, T_n$, and the corresponding service times $S_1, \dots, S_n$.

Statistical problems for queueing models have been considered by many authors, mainly from the classical inference perspective. In particular, Gaver and Jacobs (1988) and Pitts (1994) deal with nonparametric problems. The Bayesian approach has been considered so far only under parametric assumptions on both interarrival and service time distributions. The models considered are continuous time, with exponentially distributed interarrival times and service times; see McGrath and Singpurwalla (1987), the series of papers by Armero (1985, 1994) and Armero and Bayarri (1994, 1996), and the references therein.

This paper is devoted to Bayesian inference for Geo$/G/1$ models. It differs from the Bayesian literature on statistical analysis of queues mainly for two reasons.

1. The queueing model is in discrete time. As mentioned above, this choice is justified by the need to consider models applicable to telecommunication, mainly to ATM systems.
2. No special parametric hypotheses are made on the service time distribution. This makes the methods developed in the present paper suitable for applications to telecommunication ATM systems.

We consider here a nonparametric approach to the estimation of queueing characteristics related to the performance of the system. In particular, we deal with the problem of estimating the probability generating function (p.g.f.) of the equilibrium waiting time distribution. Under the stability condition $\rho = \lambda\mu < 1$, the p.g.f. of the equilibrium waiting time distribution is given by

$$W(z) = \frac{(1-\rho)(1-z)}{1-\lambda-z+\lambda B(z)},$$

where

$$B(z) = \sum_{k=1}^{\infty} z^k b(k)$$

is the p.g.f. of the service time distribution.

The approach based on the p.g.f. is supported by several reasons:

1. Generating functions are the only quantities that can be expressed in a closed form. As a matter of fact, the probability distribution corresponding to $W(z)$, which is of primary interest, is not known in a closed form.
2. All relevant queueing characteristics used to evaluate the performance of the system, such as the waiting time distribution function, the mean waiting time, the variance of the waiting time and many others, can be expressed as functionals of the p.g.f. of the waiting time. Hence, virtually all results in the statistical estimation of queueing characteristics can be obtained as by-products of inference on the p.g.f. of the waiting time distribution.
3. Among the measures of performance, a fundamental role in ATM systems is played by the probability of a "long delay." As specified above, in ATM, batches of cells enter the system and have to be transmitted (one cell per time slot). Cells that cannot be immediately transmitted are stored in a buffer of size $K$, say, and form a queue. The waiting time is equal to the number of cells stored in the buffer. Cells that cannot be either transmitted or stored in the buffer are *lost*. In ATM, the most important measure of performance is the loss probability, which is (approximately) equal to the overflow probability [Roberts, Mocci and Virtamo (1996)], that is, the probability that the waiting time is greater than $K$. Statistical problems related to the overflow probability are dealt with in Section 6. In this case, the use of $W(\cdot)$ is unavoidable.

The p.g.f. $W(\cdot)$ can be viewed as a transform of both interarrival and service time distributions. That is, as a functional that maps the interarrival and service time distributions onto the equilibrium waiting time distribution. The definition of a prior directly for $W(\cdot)$, and its update on the basis of sample data, is a very difficult task. For this reason, it is more natural to work in an "indirect" way. More definitely, our approach consists in (1) constructing priors for interarrival and service time distributions; (2) updating them on the basis of sample data; (3) studying the induced posterior distribution $W(\cdot)$. Unfortunately, the exact posterior law of $W(\cdot)$ cannot be explicitly obtained, and approximations must be used. The attempt to find out such an approximation is the most delicate part of this program. We consider in this paper an asymptotic approximation. Asymptotic approximations play an important role in Bayesian statistics; see, for instance, the discussion in Schervish [(1995), Section 7.4.2] on the Bernstein–von Mises theorem. As Section 5 will make clear, we will prove an infinite-dimensional version of the Bernstein–von Mises theorem, which happens to provide an asymptotic approximation for the posterior law of $W(\cdot)$. In Section 4 the consistency of the posterior law of $W(\cdot)$ is also studied. Another important motivation for the "functional" approach pursued in this paper is provided in Section 6, where the estimate of the probability of a long delay is dealt with.

The functional approach to renewal and queueing processes is fully developed in Grübel (1989) and Grübel and Pitts (1992). Statistical estimation problems, from a non-Bayesian point of view, are considered in Grübel and Pitts (1993) and Pitts (1994).

The results obtained in the present paper are essentially asymptotic results, which can be practically applied when the sample size is "large enough." A discussion of this point is in order. Large samples are sometimes unavailable. However, in the case of ATM systems, which provide the main motivation to the use of Geo/$G$/1 models, measurements of interarrival and service times are taken for sufficiently long periods, so that the corresponding sample size is actually "large": see Section 7.

**2. Prior assignments.** The prior measure for $P_b$ is assumed to be a Dirichlet process with parameter $\beta(\cdot)$ = finite measure with support the set of positive integer numbers. In other words, $(b(1), \dots, b(k))$ has Dirichlet distribution $\mathscr{D}(\beta(1), \dots, \beta(k); \beta - \beta(1) - \cdots - \beta(k))$, for every $k \geq 1$, where

$$\beta = \sum_{k=1}^{\infty} \beta(k) < \infty.$$

We assume further that $\sum k\beta(k) < \infty$.

As a prior distribution for $\lambda$, the natural conjugate Beta distribution $Be(\alpha_1, \alpha_2)$ is used. Furthermore, the prior laws of $\lambda$ and $P_b$ are assumed independent.

REMARK 1. The prior distributions specified above are taken mainly for the sake of simplicity and because they allow us to calculate explicitly their posterior means. However, as will become clear later, the results we obtain hold under considerably weaker conditions.

These two priors induce a prior distribution, that is, a stochastic process, for the p.g.f. of the waiting time distribution. As remarked before, because of its complicated expression, we cannot obtain in closed form its finite-dimensional distributions (i.e., its probability law). Some qualitative features are studied in the sequel. Define the quantity

$$\bar{\zeta} = \sup\left\{z \geq 1: \sum_{k=1}^{\infty} \beta(k) z^k < \infty\right\}.$$

Then, the following proposition holds true. Its elementary proof is omitted.

THEOREM 1. *Denote by $z^*$ the greatest positive root of the equation $1 - \lambda - z + \lambda B(z) = 0$. Then, $z^*$ is equal to 1 iff $\bar{\zeta} = 1$, and $z^* > 1$ iff $\bar{\zeta} > 1$. Furthermore, the random function*

$$W(z) = \frac{(1-\rho)(1-z)}{1 - \lambda - z + \lambda B(z)} I_{(\rho < 1)}$$

*is a.s. analytic in $(0, z^*)$, and continuous in $[0, z^*)$ ($[0, 1]$ if $z^* = 1$).*

If the interarrival times $\mathbf{T}_n = (T_1, \dots, T_n)$ and the service times $\mathbf{S}_n = (S_1, \dots, S_n)$ are observed, then, as is well known, the posterior distribution of $(b(1), \dots, b(k))$, given $\mathbf{S}_n$, is $\mathscr{D}(\beta(1) + n\hat{b}(1), \dots, \beta(k) + n\hat{b}(k);\ n + \beta -$

$\sum_{j=1}^{k}\{\beta(j) + n\hat{b}(j)\}$), where

$$\hat{b}(j) = n^{-1}\sum_{i=1}^{n} I_{\{k\}}(S_i), \qquad j = 1, 2, \dots .$$

The posterior distribution of $\lambda$, given $\mathbf{T}_n$ is $Be(n + \alpha_1, n\hat{\lambda}^{-1} + \alpha_2 - n)$, where

$$\hat{\lambda} = n\left(\sum_{i=1}^{n} T_i\right)^{-1}.$$

Finally, the posterior law of the random function $W(\cdot)$, given $\mathbf{S}_n$ and $\mathbf{T}_n$, possesses the properties specified in Theorem 1.

**3. Exact results for the posterior distribution of ρ.** The aim of the present section is to provide the exact posterior distribution of the traffic intensity coefficient $\rho = \lambda\mu$. Such a distribution, in principle, makes it possible to compute the probability that the system reaches equilibrium [Armero and Bayarri (1996)].

In view of assumption 4 in Section 1 and the independence of the prior laws of $\lambda$ and $P_b$, the posterior laws of $\lambda$ and $P_b$ are independent. Furthermore, from Cifarelli and Regazzini (1990) it follows that the posterior d.f. of $\mu = \sum kb(k)$ is given by

$$\begin{aligned} &P(\mu \le x|\mathbf{S}_n) \\ &\quad = \frac{2^{n+\beta-1}}{\pi}(x-1)^{n+\beta}\int_0^{\pi}(\cos(y/2))^{n+\beta-1} \\ &\qquad \times \cos\left(\frac{n+\beta+1}{2}y - \sum_{j=1}^{\infty}\arg\{(x-1)e^{iy} + j - 1\}(\beta(j) + n\hat{b}(j))\right) \\ &\qquad \times \exp\left\{-\sum_{j=1}^{\infty}\log|(x-1)e^{iy} + j - 1|(\beta(j) + n\hat{b}(j))\right\} dy \end{aligned} \tag{3.1}$$

The posterior distribution of $\rho$, given $\mathbf{S}_n$, $\mathbf{T}_n$, is absolutely continuous, and its support is the set $[0, \infty)$. In view of (3.1), its d.f. is of the form

$$\begin{aligned} &P(\rho \le t|\mathbf{S}_n, \mathbf{T}_n) \\ &\quad = \int_0^1 P(b \le t\lambda^{-1}|\mathbf{S}_n)\pi(\lambda|\mathbf{T}_n)\, d\lambda \\ &\quad = \frac{2^{n+\beta-1}}{\pi B(n+\alpha_1, n\lambda^{-1} + \alpha_2 + n)}\int_0^{\min(1,t)}(t-\lambda)^{n+\beta}\int_0^{\pi}(\cos(y/2))^{n+\beta-1} \\ &\qquad \times \cos\left(\frac{n+\beta+1}{2}y - \sum_{j=1}^{\infty}\arg\{(t\lambda^{-1}-1)e^{iy} + j - 1\}(\beta(j) + n\hat{b}(j))\right) \\ &\qquad \times \exp\left\{-\sum_{j=1}^{\infty}\log|(t\lambda^{-1}-1)e^{iy} + j - 1|(\beta(j) + n\hat{b}(j))\right\} dy \\ &\qquad \times \lambda^{\alpha_1-\beta-1}(1-\lambda)^{n/\hat{\lambda}+\alpha_2-n-1}\, d\lambda \end{aligned} \tag{3.2}$$

Finally, from (3.2) it is possible to obtain the posterior probability that the system reaches equilibrium, that is, $P(\rho < 1 | \mathbf{S}_n, \mathbf{T}_n)$.

**4. Posterior consistency results.** The goal of this section is to show the consistency of $W(\cdot)$ as the sample size tends to infinity. See Ghosal, Ghosh and Ramamoorthi (1997) for a nice discussion on the importance of consistency as a "validation of Bayesian methods." In particular, using the same terms as Ghosal, Ghosh and Ramamoorthi (1997), we obtain here $L^\infty$ consistency.

Let $\lambda_0$ and $b_0(k)$, $k \geq 1$, be the "true values" of $\lambda$ and $b(k)$, $k \geq 1$, respectively, and assume that hypotheses (1)–(4) of Section 1 hold true. Note that the $b_0(k)$'s are in the support of the prior because the support of the measure $\beta(\cdot)$ is the set of all positive integers. Furthermore, let $B_0(z) = \sum_{k \geq 1} b_0(k) z^k$ and $\mu_0 = \sum_{k \geq 1} k b_0(k) = B_0'(1)$ be the "true" p.g.f. and the "true" mean value of the service time distribution, respectively, and denote by $\bar{\zeta}_0$ the radius of convergence of $B_0(\cdot)$. Finally, suppose that the "true" traffic intensity coefficient $\rho_0 = \lambda_0 \mu_0$ is (strictly) smaller than 1, so that there exists the equilibrium waiting time distribution whose p.g.f. is given by

$$W_0(z) = \frac{(1 - \rho_0)(1 - z)}{1 - \lambda_0 - z + \lambda_0 B_0(z)}.$$

The main result of the present section is to show that the posterior law of $W(\cdot)$ shrinks towards the true $W_0(\cdot)$ as the sample size increases. In order to prove this, we need some preliminary lemmas. In the sequel, we denote by $P_{b_0}^\infty$ and $P_{\lambda_0}^\infty$ the product probability measures generating data sequences $(S_i; i \geq 1)$ and $(T_i; i \geq 1)$, respectively.

LEMMA 1. *Let* $R = \min(\bar{\zeta}, \overline{\zeta_0})$, *and let $r$ be equal* (i) *to* 1 *if* $R = 1$, (ii) *to a real number greater than* 1 *and* (*strictly*) *smaller than* $R$ *if* $R > 1$. *Then*

$$\lim_{n \to \infty} P\left( \sup_{0 \leq z \leq r} |B(z) - B_0(z)| > \varepsilon \middle| \mathbf{S}_n \right) = 0 \qquad \forall\, \varepsilon > 0,\ a.s.\text{-}P_{b_0}^\infty.$$

LEMMA 2. *Suppose that* $\rho_0 = \lambda_0 \mu_0 < 1$ *and that* $\min(\bar{\zeta}, \overline{\zeta_0}) > 1$. *Then*

$$\lim_{n \to \infty} P\big(|z^* - z_0| > \varepsilon | \mathbf{S}_n, \mathbf{T}_n\big) = 0 \qquad \forall\, \varepsilon > 0,\ a.s.\text{-}P_{b_0}^\infty \times P_{\lambda_0}^\infty. \tag{4.1}$$

REMARK 2. Lemma 1 essentially asserts that the random function $B(\cdot)$, that dwells in the space $C[0, r]$ of continuous functions equipped with the sup-norm, converges in probability [Billingsley (1968), page 24] to $B_0(\cdot)$ as $n$ tends to infinity, a.s.-$P_{b_0}^\infty$.

We are now in a position to establish the main result of the present section, that is, the posterior consistency of $W(\cdot)$.

THEOREM 2. *Suppose that* $\rho_0 = \lambda_0 \mu_0 < 1$. *Then*

$$\lim_{n \to \infty} P\left( \sup_{0 \le z \le 1} |W(z) - W_0(z)| > \varepsilon \middle| \mathbf{S}_n, \mathbf{T}_n \right) = 0 \qquad \forall \varepsilon > 0 \tag{4.2}$$

*a.s.*-$P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$. *Under the additional hypothesis* $\min(\bar{\zeta}, \bar{\zeta}_0) = \bar{\zeta}_0 > 1$, *we have the stronger result*

$$\lim_{n \to \infty} P\left( \sup_{0 \le z \le \xi} |W(z) I_{(z^* \le \xi)} - W_0(z)| > \varepsilon \middle| \mathbf{S}_n, \mathbf{T}_n \right) = 0 \qquad \forall\, \varepsilon > 0 \tag{4.3}$$

*for every* $1 < \xi < \bar{\xi}_0$, *a.s.*-$P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$.

COROLLARY 1. *Under the hypotheses of Theorem* 2, *we have*:

(i) $\lim_{n \to \infty} \sup_{0 \le z \le r} |E[B(z)|\mathbf{S}_n] - B_0(z)| = 0$ *a.s.*-$P_{b_0}^{\infty}$.
(ii) $\lim_{n \to \infty} \sup_{0 \le z \le 1} |E[W(z)|\mathbf{T}_n, \mathbf{S}_n] - W_0(z)| = 0$ *a.s.*-$P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$.
(iii) $\lim_{n \to \infty} \sup_{0 \le z \le 1} |\overline{W}_n(z) - W_0(z)| = 0$ *a.s.*-$P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$,

*where*

$$\overline{W}_n(z) = \frac{(1 - E[\lambda|\mathbf{T}_n] E[b|\mathbf{S}_n])(1 - z)}{1 - E[\lambda|\mathbf{T}_n] - z + E[\lambda|\mathbf{T}_n] E[B(z)|\mathbf{S}_n]}$$

*and*

$$E[\lambda|\mathbf{T}_n] = \frac{n + \alpha_1}{n\lambda^{-1} + \alpha_1 + \alpha_2}, \quad E[\mu|\mathbf{S}_n] = \sum_{k=1}^{\infty} k \left\{ \frac{\beta(k)}{n + \beta} + \frac{n\hat{b}(k)}{n + \beta} \right\},$$

$$E[B(z)|\mathbf{S}_n] = \sum_{k=1}^{\infty} z^k \left\{ \frac{\beta(k)}{n + \beta} + \frac{n\hat{b}(k)}{n + \beta} \right\}.$$

REMARK 3. Statement (ii) of Corollary 1 establishes the consistency of the Bayes estimate $E[W(z)|\mathbf{T}_n, \mathbf{S}_n]$. More importantly, statement (iii) provides an approximation for $E[W(z)|\mathbf{T}_n, \mathbf{S}_n]$. This last result is particularly useful, since $E[W(z)|\mathbf{T}_n, \mathbf{S}_n]$ cannot be analytically calculated, while $\overline{W}_n(z)$ can be. In Theorem 4 we will give a more precise evaluation of the goodness of such an approximation.

REMARK. 4. In order to ensure the validity of Theorem 2, the assumptions made in Section 2 on the prior distributions for $\lambda$ and $b(k)$'s are not necessary. From the proofs of Lemma 1, Lemma 2 and Theorem 2, it is apparent that they still hold provided that (i) the prior for $\lambda$ satisfy the conditions of Theorem 7.80 in Schervish (1995); (ii) the series $\sum E[b(k)|\mathbf{S}_n] r^k$ converges for every $n \ge 1$ a.s.-$P_{b_0}^{\infty}$; (iii) the limit $\lim_{n \to \infty} \sum E[b(k)|\mathbf{S}_n] r^k$ exists finite a.s.-$P_{b_0}^{\infty}$ and (iv) $\lim_{n \to \infty} E[|b(k) - b_0(k)|\,|\mathbf{S}_n| = 0$ for every $k \ge 1$, a.s.-$P_{b_0}^{\infty}$.

**5. Bernstein–von Mises-type results.** As remarked before, the exact posterior law of the random function $W(\cdot)$ is not available. A possible way to overcome this problem could consist in resorting to some numerical approximations. However, it is very hard to find out approximations for *all* the finite-dimensional distributions of $W(\cdot)$, that is, for the posterior law of the whole random function $W(\cdot)$. This fact provides a motivation for studying some asymptotic approximations that hold true for a large sample size. The goal of the present section consists in providing a large sample approximation for a suitably rescaled version of $W(\cdot)$,

$$\sqrt{n}\left(W(z) - \overline{W}_n(z)\right) \tag{5.1}$$

In parametric Bayesian statistics, a fundamental role is played by the Bernstein–von Mises theorem [see, e.g., Schervish (1995), pages 435–444], which establishes the asymptotic normality of the posterior law. This result provides, in turn, an asymptotic approximation for the posterior law. In the present section we establish an infinite-dimensional version of the Bernstein–von Mises theorem by showing that the posterior law of (5.1) converges weakly to a centered Gaussian process. Several authors have investigated the asymptotic normality of posterior laws in the infinite dimensional case. See, among others, the paper by Lo (1983), and the more recent works by Cox (1993) and Diaconis and Freedman (1997), who obtain some negative results.

We begin with two lemmas that play a basic role in all subsequent developments.

LEMMA 3. *Let* $R = \min(\bar{\zeta}, \overline{\zeta_0}) = \bar{\zeta}_0$, *and let* $r$ *be equal* (i) *to* 1 *if* $R = 1$, (ii) *to a real number greater than* 1 *and* (*strictly*) *smaller than* $R$ *if* $R > 1$.

*If* $r = 1$, *assume further that* $\sum k^2 \beta(k) < \infty$, $\sum k^2 b_0(k) < \infty$. *Then, the posterior law of*

$$X_n(z) = \sqrt{n}\left(B(z) - E[B(z) | \mathbf{S}_n]\right)$$

*converges weakly in* $C[0, r]$, *equipped with the sup-norm, to a centered Gaussian process with covariance kernel*

$$H(u, v) = B_0(uv) - B_0(u) B_0(v)$$

*as* $n$ *tends to infinity, a.s.-*$P_{b_0}^{\infty}$.

LEMMA 4. *Suppose that* $R = \min(\bar{\zeta}, \bar{\zeta}_0) = \bar{\zeta}_0$ *and that one of the following two hypotheses is fulfilled*:

(a) $R > 1$.
(b) $R = 1$, $\sum_{k=1}^{\infty} k^6 \beta(k) < \infty$, $\sum_{k=1}^{\infty} k^6 b_0(k) < \infty$.

*Furthermore, let* $r = 1$ *if* $R = 1$, *and* $1 \le r < \sqrt{r}$ *if* $R > 1$. *Then, the sequence of stochastic processes* $(Y_n(\cdot);\ n \ge 1)$, *where*

$$Y_n(z) = \sqrt{n}\, \frac{B(z) - E[B(z) | \mathbf{S}_n]}{1 - z}$$

*converges weakly in* $C[0, r]$, *equipped with the sup-norm, to a centered Gaussian process* $Y(\cdot)$ *with covariance kernel*

$$K_0(u,v) = \frac{B_0(uv) - B_0(u)B_0(v)}{(1-u)(1-v)} \tag{5.2}$$

*as n tends to infinity, a.s.*-$P_{b_0}^{\infty}$.

*We are now in a position to prove the main result of the present section. To be precise, let R be defined as in Lemma* 4, *and let* $\delta = 1$ *if* $R = 1$, $1 \leq \delta < \min(\sqrt{R}, z_0)$ *if* $R > 1$. *The following result holds.*

THEOREM 3. *Under the hypotheses of Lemma* 4, *the sequence of stochastic processes*

$$T_n(z) = \sqrt{n}\left(W(z)I_{(z^* > \delta)} - \overline{W}_n(z)\right)$$

*converges weakly in* $C[0, \delta]$, *equipped with the sup-norm, to a centered Gaussian process with covariance kernel*

$$\begin{aligned} &L_0(u,v) \\ &\quad = (1-\rho_0)^{-2}\big\{\sigma_{\lambda_0}^2 C_0(u)C_0(v) + \lambda_0^2\sigma_{B_0}^2 W_0(u)W_0(v) \\ &\qquad + \lambda_0^2 W_0(u)W_0(v)^2 K_0(u,v) + \lambda_0^2\big(D_0(u,v) + D_0(v,u)\big)\big\}, \end{aligned} \tag{5.3}$$

*where* $K_0(u,v)$ *is given by* (5.2) *and*

$$\begin{aligned} C_0(u) &= \frac{1}{1-u}(1 - B_0(u))W_0(u)^2 - \mu_0 W_0(u), \\ D_0(u,v) &= \frac{W_0(u)W_0(v)^2(vB_0'(v) - B_0(v))}{1-v}, \\ \sigma_{\lambda_0}^2 &= \lambda_0^2(1-\lambda_0), \\ \sigma_{B_0}^2 &= B_0''(1) + B_0'(1) - B_0'(1)^2 \end{aligned}$$

*as n goes to infinity a.s.*-$P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$.

Observing that $z^* \geq 1$ whenever $\rho < 1$, from Theorem 3 we easily obtain the following corollary.

COROLLARY 2. *Under the assumptions of Theorem* 3, *the sequence of stochastic processes* $(\sqrt{n}(W(z) - \overline{W}_n(z))$; $n \geq 1)$ *converges weakly in* $C[0,1]$, *equipped with the sup-norm, to a centered Gaussian process with limiting covariance kernel* (5.3).

Theorem 3 clarifies the large sample behavior of the posterior law of $W(\cdot)$. The limiting covariance kernel depends on the "true" $\lambda_0$, $\mu_0$, $B_0(\cdot)$, $W_0(\cdot)$. However, in view of the results in Section 4, $L_0(u,v)$ can be approximated by

$\overline{L}_n(u,v)$, which is defined as $L_0(u,v)$ except that $\lambda_0$, $b_0$, $B_0(\cdot)$, $W_0(\cdot)$ are replaced by $E[\lambda \mid \mathbf{T}_n]$, $E[b \mid \mathbf{S}_n]$, $E[B(\cdot) \mid \mathbf{S}_n]$, $\overline{W}_n(\cdot)$, respectively. Such an approximation works since, as a consequence of the results in Section 4, $\overline{L}_n(\cdot,\cdot)$ converges *uniformly* to $L_0(\cdot,\cdot)$ as $n$ tends to infinity, a.s.-$P^{\infty}_{\lambda_0} \times P^{\infty}_{b_0}$. The technique used to prove Theorem 3 is also useful to show the accuracy of the approximation of the Bayes estimate $E[W(z)|\mathbf{T}_n, \mathbf{S}_n]$ by $\overline{W}_n(z)$. Corollary 1 tells us that $\overline{W}_n(z)$ tends uniformly a.s. to $E[W(z)|\mathbf{T}_n, \mathbf{S}_n]$. A Taylor expansion similar to that of Theorem 3 shows that the following stronger result holds true.

THEOREM 4. *Under the assumptions of Theorem* 3, *we have*

$$\lim_{n\to\infty} \sqrt{n} \sup_{0\le z\le 1} |E[W(z)|\mathbf{T}_n, \mathbf{S}_n] - \overline{W}_n(z)| = 0 \qquad a.s.\text{-}P^{\infty}_{\lambda_0} \times P^{\infty}_{b_0}.$$

REMARK 5. Theorem 4 improves considerably statement (iii) of Corollary 1, since it shows that the difference between the Bayes estimate $E[W(z)|\mathbf{T}_n, \mathbf{S}_n]$ and its approximation $\overline{W}_n(z)$ tends (uniformly, a.s.) to 0 at a rate faster than $n^{-1/2}$.

REMARK 6. Lemmas 3 and 4 still hold under assumptions on the priors weaker than those made in Section 2. It is enough to assume that (i) the r.v.'s $b(1),\dots,b(k)$, for every $k \ge 1$ satisfy the conditions of Theorem 3 in Freedman (1963), (ii) the series $\sum \mathrm{Var}[b(k)|\mathbf{S}_n]k^2r^{2k}$ converges a.s.-$P^{\infty}_{b_0}$, (iii) the limit $\lim_{n\to\infty} n\sum \mathrm{Var}[b(k)|\mathbf{S}_n]k^2r^{2k}$ exists finite a.s.-$P^{\infty}_{b_0}$. In order that Theorem 3 hold true, the posterior distribution of $\lambda$ must converge in law to a normal distribution, a.s.-$P^{\infty}_{\lambda_0}$. This follows, for instance, from Fraser and McDonnough (1984).

**6. Some applications of the previous results.** As remarked in Section 1, the asymptotic results obtained so far are interesting not only in themselves, but also because they can be applied to estimating queueing characteristics, that is, functionals of $W(\cdot)$. We consider here the problem of estimating the probability that the equilibrium waiting time is greater than $K$, $K$ being a nonnegative integer. As a further application, one could consider the estimate of the moments of the waiting time distribution. Define

$$\begin{aligned}\mathscr{W}(k) &= \textit{probability that the equilibrium waiting time is equal to } k\\ &= k!\,\frac{d^kW(z)}{dz^k}\bigg|_{z=0}.\end{aligned}$$

From Theorems 3 and 4 we derive the following corollary, which provides a Bernstein–von Mises-type result for the posterior distribution of $\mathscr{W}(k)$.

COROLLARY 3. *Under the assumptions of Theorem* 3, *we have*

(i) $$\sqrt{n}\,\bar{\sigma}(k)^{-1}\big(\mathscr{W}(k) - \overline{\mathscr{W}}_n(k)\big) \xrightarrow{d} N(0,1) \qquad a.s.\text{-}P^{\infty}_{\lambda_0} \times P^{\infty}_{b_0}$$

*as n tends to infinity, where*

$$\overline{\mathscr{W}}_n(k) = k!\frac{d^k \overline{W}_n(z)}{dz^k}\bigg|_{z=0}, \ \bar{\sigma}(k)^2 = (k!)^2\left(\frac{d^{2k}}{du^k\,dv^k}\overline{L}(u,v)|_{u=0,\,v=0}\right)$$

(ii) $$\lim_{n\to\infty}\sqrt{n}\,\big|E\big[\overline{\mathscr{W}}(k)\big|\mathbf{T}_n, \mathbf{S}_n\big] - \overline{\mathscr{W}}_n(k)\big| = 0 \qquad a.s.\text{-}P^{\infty}_{\lambda_0} \times P^{\infty}_{b_0}.$$

To prove claims (i)–(ii), it is enough to take into account that $W(\cdot)$ is analytic, and that it can be written as a power series with radius of convergence strictly greater than 1.

In principle, the same kind of result holds also for

$$G(K) = \sum_{k>K} \mathscr{W}(k)$$

and for the moments of the equilibrium waiting time distribution. In view of its importance in ATM applications, where $K$ is the buffer size and $G(K)$ is the overflow probability, we give here a few more results on the Bayes estimation of (a suitable approximation of) $G(K)$. In view of Theorem 4, the Bayes estimate $E[G(K)|\mathbf{T}_n, \mathbf{S}_n]$ can be conveniently approximated by

(6.1) $$\overline{G}_n(K) = \sum_{k>K} \overline{\mathscr{W}}_n(k) = 1 - \sum_{k=1}^{K} k!\left(\frac{d^k}{dz^k}\overline{W}_n(z)\bigg|_{z=0}\right)$$

so that

$$\lim_{n\to\infty}\sqrt{n}\,|E[G(K)|\mathbf{T}_n,\mathbf{S}_n] - \overline{G}_n(K)| = 0 \qquad \text{a.s.-}P^{\infty}_{\lambda_0} \times P^{\infty}_{b_0}.$$

However, except when $K$ is small, the numerical computation of (6.1) is hard, or even impossible. Cases of typical interest for ATM applications are $K \geq 100$ (small buffers) or $K \geq 500$ (large buffers).

The basic idea consists, using an obvious notation, in (1) finding out a suitable approximation for $G_0(K)$, $G_0^*(K)$, say, that holds for "large" values of $K$, (2) estimating $G_0^*(K)$ instead of the original $G_0(K)$. This approach was first used by Gaver and Jacobs (1988) for estimating the probability of a long delay in a continuous-time $M/G/1$ model with *known* intensity parameter of the (Poisson) input stream. Define

$$G_0^*(K) = \frac{1-\rho_0}{\lambda_0 B_0'(z_0) - 1} z_0^{-(K+1)}.$$

It can be shown [see Bruneel and Kim (1993), or, for a different proof, Conti (1997)] that the *relative* error of approximation tends to zero as $K$ increases; that is,

$$\lim_{K\to\infty}\frac{G_0(K) - G_0^*(K)}{G_0(K)} = 0.$$

Define

$$G^*(K) = \frac{1-\rho}{\lambda B'(z^*) - 1}(z^*)^{-(K+1)} I_{(\rho<1)}$$

and

$$\overline{G}_n^*(K) = \frac{1 - E[\rho|\mathbf{T}_n, \mathbf{S}_n]}{E[\lambda|\mathbf{T}_n]\overline{B}'(\bar{z}) - 1}\bar{z}^{-(K+1)},$$

where $\overline{B}(z) = E[B(z)|\mathbf{S}_n]$, and (1) $\bar{z} = 1$ if $E[\rho|\mathbf{T}_n, \mathbf{S}_n] \geq 1$, (2) $\bar{z}$ is the unique solution greater than 1 of the equation $\overline{W}_n(z) = 0$, otherwise.

As a consequence of the results in Section 4, we have the following proposition.

THEOREM 5. *Suppose that $R$, defined as in Theorem 2, is greater than 1 and that $\rho_0 < 1$. Then*

$$\lim_{n\to\infty} P\big(|G^*(K) - G_0^*(K)| > \varepsilon|\mathbf{T}_n, \mathbf{S}_n\big) = 0 \qquad \forall\, \varepsilon > 0 \ a.s.\text{-}P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty},$$

$$\lim_{n\to\infty} \left|\overline{G}_n^*(K) - G_0^*(K)\right| = 0 \qquad a.s.\text{-}P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}.$$

**7. Application to ATM teletraffic data.** In this section we apply to real data the techniques developed so far. The data we are using are a part of the experimental measurements made in 1996 by Telecom Italia (the Italian telephone company), as a member of the European JAMES (Joint ATM Experiment on European Services) project. More information on the JAMES project and related activities can be found, for instance, in Gnetti (1997).

The measurements refer to the traffic generated by a video-conference. Interarrival times (i.e., number of time slots) between consecutive cells were measured by an analyser HP Broadband Series Test-System HP75000. The time-slot length is $1.247\ 10^{-5}$ sec. Since only *one* traffic source was considered, only one cell per time slot can "arrive," and hence there is no queue. In this way, there is no buffer occupancy, and no cells are lost. We stress that interarrival times due to lost cells cannot be either measured or observed, so that traffic measurements taken in the presence of lost cells tend to be longer than they really are. In order to observe "genuine" interarrival times, no cell loss can be allowed.

In order to produce traffic coming from different sources that are simultaneously transmitting cells, the traffic generated by our single source was divided into streams, each of them composed of 85048 time slots. Each stream virtually corresponds to a real traffic path generated by a (virtual) source. Superposition of streams produces traffic virtually generated by different sources.

The traffic intensity obviously depends on the number of sources that are simultaneously connected to the ATM node. In order to evaluate the average impact of the traffic intensity parameter, $\rho$, on the cell loss probability, the traffic corresponding to 40 streams was superimposed. The prior distributions for $\lambda$ and $b(k)$'s are:

*Prior for* $\lambda$: $\lambda \sim Be(1000, 1000)$.

*Prior for* $b(\cdot)$: $(b(1), \dots, b(k)) \sim \mathscr{D}(\beta(1), \dots, \beta(k);\ \beta - \beta(1) - \cdots - \beta(k))$, for every $k \geq 1$, where $\beta(k) = 7^k/k!$ for every $k \geq 1$.

The measure $\beta(\cdot)$ is chosen proportional to a Poisson probability measure $\beta(k) = \beta e^{-\theta}\theta^k/k!$. This choice is mainly supported by mathematical convenience, since the radius of convergence of the series $\sum \beta(k)z^k$ is infinite, and the assumptions of Theorem 2 are fulfilled. This prior choice implies that $E[b(k)] = e^{-\theta}\theta^k/k!$. The parameter $\theta$ was "estimated" by fitting the prior expected values $E[b(k)]$ to the relative frequencies obtained from a "training sample" of 2018 measurements made in an experimental network [Gnetti (1997)]. On the basis of this procedure, the value $\theta = 6.97 \simeq 7$ was obtained. Finally, the total mass $\beta$ was chosen equal to $e^7 = 1096.6$, which is very close to the sample size 2018 of the training sample. A similar approach was also used to construct the prior for $\lambda$.

The total number of time slots is 85048; the number of time slots with arrivals, that is, the sample size, is $n = 34071$. The total mass $\beta = e^7$ is small if compared to the sample size $n$, and hence the prior does not have a great influence on the posterior. From the sample data, the values $\hat{\lambda} = 0.40061$ and $\hat{b}(k)$'s displayed in Table 1 are obtained [note that $\hat{b}(k) = 0$ for every $k \geq 16$].

The posterior means of $\lambda$, $B(z)$, $b$ and $\rho$, as well as $\overline{W}_n(z)$, are

$$E[\lambda | \mathbf{T}_n] = 0.40289269,$$

$$E[B(z) | \mathbf{S}_n] = 0.00002844(e^{7z} - 1) + 0.96884453 \sum_{k=1}^{15} z^k \hat{b}(k),$$

$$E[b | \mathbf{S}_n] = 1.91509682, \qquad E[\rho | \mathbf{T}_n, \mathbf{S}_n] = 0.77157851,$$

$$\overline{W}_n(z) = \frac{0.22842149(1 - z)}{0.59710731 - z + 0.40289269 E[B(z) | \mathbf{S}_n]}.$$

From $\overline{W}_n(z)$ the approximate Bayes estimates of the waiting time probabilities $\overline{\mathscr{W}}_n(k)$, as well as the value $\bar{z} = 0.183938$, are obtained. In Table 2 the values $\overline{\mathscr{W}}_n(k)$ (a), $\overline{G}_n(k)$ (b) and $\overline{G}^*_n(k)$ (c), for $k = 0, 15\ (1)$, are displayed. All

TABLE 1

*Sample $\hat{b}(k)$'s values*

| $k =$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\hat{b}(k) =$ | 0.54938220 | 0.32115290 | 0.06292742 | 0.02568167 | 0.01455783 |
| $k =$ | 6 | 7 | 8 | 9 | 10 |
| $\hat{b}(k) =$ | 0.01039007 | 0.00736697 | 0.00369819 | 0.00120337 | 0.00111532 |
| $k =$ | 11 | 12 | 13 | 14 | 15 |
| $\hat{b}(k) =$ | 0.00137947 | 0.00044026 | 0.00049896 | 0.00014675 | 0.00005870 |

TABLE 2
*Approximated Bayes estimates of the waiting time probabilities and cumulative probabilities*

| $k$ | $\overline{\mathscr{W}}_n(k)$ | $\overline{G}_n(k)$ | $\overline{G}_n^*(k)$ | $k$ | $\overline{\mathscr{W}}_n(k)$ | $\overline{G}_n(k)$ | $\overline{G}_n^*(k)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.38255 | 0.61745 | 0.60878 | 8 | 0.02865 | 0.15813 | 0.15770 |
| 1 | 0.12068 | 0.49677 | 0.51420 | 9 | 0.02449 | 0.13363 | 0.13320 |
| 2 | 0.10784 | 0.41852 | 0.43432 | 10 | 0.02087 | 0.11276 | 0.11251 |
| 3 | 0.06140 | 0.35712 | 0.36684 | 11 | 0.01760 | 0.09517 | 0.09503 |
| 4 | 0.05204 | 0.30508 | 0.30985 | 12 | 0.01485 | 0.08031 | 0.08026 |
| 5 | 0.04531 | 0.25977 | 0.26171 | 13 | 0.01249 | 0.06782 | 0.06780 |
| 6 | 0.03933 | 0.22044 | 0.22105 | 14 | 0.01053 | 0.05729 | 0.05726 |
| 7 | 0.03366 | 0.18678 | 0.18671 | 15 | 0.00888 | 0.04842 | 0.04837 |

computations were performed by Mathematica. The results show that, even for small values of $k$, the approximation $\overline{G}_n(k) \simeq \overline{G}_n^*(k)$ is satisfactory.

## APPENDIX

PROOF OF LEMMA 1. Take $0 < \eta < \varepsilon$. Then there exists $m$ such that

$$\sum_{k>m} b_0(k) r^k < \frac{\eta^2}{2}, \qquad \sum_{k>m} \beta(k) r^k < \frac{\eta^2}{2}.$$

Now we can first write

$$\begin{aligned}
&P\left( \sup_{0\le z\le r} |B(z) - B_0(z)| > \varepsilon \,\middle|\, \mathbf{S}_n \right)\\
&\quad = P\left( \sup_{0\le z\le r} |B(z) - B_0(z)| > \varepsilon,\ \sum_{k>m} (b(k) + b_0(k)) r^k > \varepsilon/2 \,\middle|\, \mathbf{S}_n \right)\\
&\qquad + P\left( \sup_{0\le z\le r} |B(z) - B_0(z)| > \varepsilon,\ \sum_{k>m} (b(k) + b_0(k)) r^k \le \varepsilon/2 \,\middle|\, \mathbf{S}_n \right)\\
&\quad \le P\left( \sum_{k>m} (b(k) + b_0(k)) r^k > \eta/2 \,\middle|\, \mathbf{S}_n \right)\\
&\qquad + P\left( \sum_{k=1}^{m} |b(k) - b_0(k)| r^k > \varepsilon - \sum_{k>m} (b(k) + b_0(k)) r^k, \right.\\
&\qquad\qquad\qquad \left. \sum_{k>m} (b(k) + b_0(k)) r^k \le \varepsilon/2 \,\middle|\, \mathbf{S}_n \right)\\
&\quad \le P\left( \sum_{k>m} b(k) r^k \ge \eta(1-\eta)/2 \,\middle|\, \mathbf{S}_n \right)\\
&\qquad + P\left( \sum_{k=1}^{m} |b(k) - b_0(k)| r^k \ge \varepsilon/2 \,\middle|\, \mathbf{S}_n \right).
\end{aligned}$$

In the second place, from Markov's inequality and the SLLN we obtain

$$
\begin{aligned}
&P\left(\sum_{k>m} b(k)r^k \geq \eta(1-\eta)/2 \middle| \mathbf{S}_n\right) \\
&\quad \leq \frac{2}{\eta(1-\eta)} \sum_{k>m} E[b(k)|\mathbf{S}_n] r^k \\
&\quad = \frac{2}{\eta(1-\eta)} \sum_{k>m} \left(\frac{1}{n+\beta}\beta(k) + \frac{n}{n+\beta}\hat{b}(k)\right) r^k \\
&\quad < \frac{2}{\eta(1-\eta)} \left\{\frac{1}{n+\beta}\frac{\eta^2}{2} + \frac{n}{n+\beta}\sum_{k>m}\hat{b}(k)r^k\right\} \\
&\quad \to \frac{2}{\eta(1-\eta)} \sum_{k>m} b_0(k) r^k \\
&\quad < \frac{\eta}{1-\varepsilon} \qquad \forall\, \eta \in (0,\varepsilon),\ \text{a.s.-}P_{b_0}^{\infty}
\end{aligned}
\tag{A.1}
$$

as $n$ tends to infinity.

Furthermore, it is easily seen that

$$
\begin{aligned}
&P\left(\sum_{k=1}^{m} |b(k) - b_0(k)| r^k > \varepsilon/2 \middle\| \right) \\
&\quad \leq \frac{2}{\varepsilon} \sum_{k=1}^{m} E\big[|b(k) - b_0(k)|\,|\mathbf{S}_n\big] r^k \\
&\quad \leq \frac{2}{\varepsilon} \sum_{k=1}^{m} \left\{\left(E[b(k)|\mathbf{S}_n] - b_0(k)\right)^2 + \mathrm{Var}[b(k)|\mathbf{S}_n]\right\}^{1/2} r^k \to 0
\end{aligned}
\tag{A.2}
$$

as $n$ tends to infinity, a.s.-$P_{b_0}^{\infty}$. The statement of the lemma now follows from (A.1) and (A.2). $\square$

PROOF OF LEMMA 2. For the sake of brevity, define the functions $G(z) = 1 - \lambda - z + \lambda B(z)$, $G_0(z) = 1 - \lambda_0 - z + \lambda_0 B_0(z)$ [observe that $G_0(z) < 0$ $(< 0)$ as $z > z_0$ $(z < z_0)$]. The relationship $|z^* - z_0| < \varepsilon$ holds iff $G(z_0 - \varepsilon) < 0$ and $G(z_0 + \varepsilon) > 0$. Furthermore, from a well-known result [see, e.g., Schervish (1995), page 430], the probability $(P(|\lambda - \lambda_0| > \varepsilon|\mathbf{T}_n)$ tends to zero as $n$ approaches infinity, for every $\varepsilon > 0$, a.s.-$P_{\lambda_0}^{\infty}$. Taking $\eta > 0$ small, such that

$$
\eta < \max\left(\left|\frac{G_0(z_0 - \varepsilon)}{2(1 - B_0(z_0 - \varepsilon))}\right|, \left|\frac{G_0(z_0 + \varepsilon)}{2(1 - B_0(z_0 + \varepsilon))}\right|\right)
$$

we get the following chain of inequalities:

$$\begin{aligned}
&P(|z^* - z_0| > \varepsilon|\mathbf{S}_n, \mathbf{T}_n)\\
&\quad\le P(G(z_0 - \varepsilon) - G_0(z_0 - \varepsilon) \ge -G_0(z_0 - \varepsilon)|\mathbf{S}_n, \mathbf{T}_n)\\
&\qquad + P(G(z_0 + \varepsilon) - G_0(z_0 + \varepsilon) \le -G_0(z_0 + \varepsilon)|\mathbf{S}_n, \mathbf{T}_n)\\
&\quad\le P\big(\lambda(1 - B(z_0 - \varepsilon)) - \lambda_0(1 - B_0(z_0 - \varepsilon))\\
&\qquad\qquad \ge -G_0(z_0 - \varepsilon), |\lambda - \lambda_0| \le \eta|\mathbf{S}_n, \mathbf{T}_n\big)\\
&\qquad + P\big(\lambda(1 - B(z_0 + \varepsilon)) - \lambda_0(1 - B_0(z_0 + \varepsilon))\\
&\qquad\qquad \le -G_0(z_0 + \varepsilon), |\lambda - \lambda_0| \le \eta|\mathbf{S}_n, \mathbf{T}_n\big)\\
&\qquad + 2P(|\lambda - \lambda_0| > \eta|\mathbf{S}_n, \mathbf{T}_n)\\
&\quad\le P\Big(|B(z_0 - \varepsilon) - B_0(z_0 - \varepsilon)| > -G_0(z_0 - \varepsilon)\big(2(\lambda_0 + \eta)\big)^{-1}|\mathbf{S}_n, \mathbf{T}_n\Big)\\
&\qquad + P\Big(|B(z_0 + \varepsilon) - B_0(z_0 + \varepsilon)| > G_0(z_0 + \varepsilon)\big(2(\lambda_0 + \eta)\big)^{-1}|\mathbf{S}_n, \mathbf{T}_n\Big)\\
&\qquad + 2P(|\lambda - \lambda_0| > \eta|\mathbf{S}_n, \mathbf{T}_n).
\end{aligned}$$

From Lemma 1, conclusion (4.1) easily follows. □

PROOF OF THEOREM 2. Relationship (4.2) is a consequence of Lemma 1 and the continuous mapping theorem [Billingsley (1968), page 30], since the mapping $f(B(\cdot)) = (1 - \rho)(1 - z)/(1 - \lambda - z + \lambda B(z))$ is continuous w.r.t. the sup-norm.

Relationship (4.3) follows from Lemmas 1 and 2 and the continuous mapping theorem.

PROOF OF COROLLARY 1. Relationship (i) is proved by direct inspection of $E[B(z)|\mathbf{S}_n]$. To prove relationship (ii), note first that the posterior mean $E[W(z)|\mathbf{S}_n, \mathbf{T}_n]$ does exist, since $0 \le W(z) \le 1$ if $0 \le z \le 1$. Take now a positive $\varepsilon$. From the inequality,

$$\begin{aligned}
\sup_{0 \le z \le 1} |E[W(z)|\mathbf{S}_n, \mathbf{T}_n - W_0(z)| &\le \varepsilon P\Big(\sup_{0 \le z \le 1} |W(z) - W_0(z)| < \varepsilon\Big|\mathbf{S}_n, \mathbf{T}_n\Big)\\
&\quad + P\Big(\sup_{0 \le z \le 1} |W(z) - W_0(z)| \ge \varepsilon\Big|\mathbf{S}_n, \mathbf{T}_n\Big)
\end{aligned}$$

and Theorem 2, relationship (ii) is easily verified. The proof of relationship (iii) is similar. □

PROOF OF LEMMA 3. In order to prove the lemma, we have to show that (i) the (posterior) finite-dimensional distributions of $X_n(\cdot)$ converge in law to multivariate normal distribution, and (ii) the sequence of random functions $(X_n(\cdot);\ n \ge 1)$ is tight. For the sake of clarity, the proof is split into steps.

*Step* 1 (Convergence of finite-dimensional distributions). We confine ourselves to one-dimensional distributions, since the same reasoning applies also to the multidimensional case. From Theorem 3 in Freedman (1963), it follows that, a.s.-$P_{b_0}^{\infty}$, for every $K \geq 1$,

$$\sqrt{n} \sum_{k=1}^{K} \left( b(j) - E[b(j) | \mathbf{S}_n] \right) z^j$$

converges in law, as $n$ tends to infinity, to a normal distribution with mean 0 and variance

$$\sigma_K^2(z) = \sum_{k=1}^{K} b_0(k)^2 z^{2k} - \left( \sum_{k=1}^{K} b_0(k) z^k \right)^2.$$

For every $\varepsilon > 0$ and $K \geq 1$, we have

$$\begin{aligned} & P\left( \sqrt{n} \sum_{k=1}^{\infty} \left( b(k) - E[b(k) | \mathbf{S}_n] \right) z^k \leq t \,\middle|\, \mathbf{S}_n \right) \\ \text{(A.3)} \qquad & \leq P\left( \sqrt{n} \sum_{k=1}^{K} \left( b(k) - E[b(k) | \mathbf{S}_n] \right) z^k \leq t + \varepsilon \,\middle|\, \mathbf{S}_n \right) \\ & \quad + P\left( \left| \sqrt{n} \sum_{k>K} \left( b(k) - E[b(k) | \mathbf{S}_n] \right) z^k \right| \geq \varepsilon \,\middle|\, \mathbf{S}_n \right) \end{aligned}$$

Then, taking a positive $\gamma$ such that $\tau = r + \gamma < \sqrt{R}$, when $K$ is large enough so that $\sum_{k>K} (r/\tau)^k \leq 1$, the following chain of inequalities holds true:

$$\begin{aligned} & P\left( \left| \sqrt{n} \sum_{k>K} \left( b(k) - E[b(k) | \mathbf{S}_n] \right) z^k \right| \geq \varepsilon \,\middle|\, \mathbf{S}_n \right) \\ & \leq P\left( \sqrt{n} \sum_{k>K} \left| b(k) - E[b(k) | \mathbf{S}_n] \right| z^k \geq \varepsilon \sum_{k>K} \frac{z^k}{\tau} \,\middle|\, \mathbf{S}_n \right) \\ & \leq P\left( \bigcup_{k \geq 1} \left\{ \sqrt{n} \left| b(k) - E[b(k) | \mathbf{S}_n] \right| z^k \geq \varepsilon \left( \frac{z}{\tau} \right)^k \right\} \,\middle|\, \mathbf{S}_n \right) \\ \text{(A.4)} \qquad & \leq \sum_{k>K} P\left( \sqrt{n} \left| b(k) - E[b(k) | \mathbf{S}_n] \right| \tau^k \geq \varepsilon | \right) \\ & \leq \frac{n}{\varepsilon^2} \sum_{k>K} \left\{ \frac{\left( \beta(k) + n\hat{b}(k) \right)\left( n + \beta - n\hat{b}(k) \right)}{(n+\beta)^2 (n + \beta + 1)} \right\} \tau^{2k} \\ & \leq \frac{1}{\varepsilon^2} \sum_{k>K} \left( \hat{b}(k) + n^{-1} \beta(k) \right) \end{aligned}$$

Now, let $n$ go to infinity. Taking (with obvious symbols) $\Phi(x) = P(N(0,1) \leq x)$, the first term in (A.3) tends to $\Phi((t + \varepsilon)/\sigma_K(z))$, a.s.-$P_{b_0}^{\infty}$. The term (A.4) tends to

$$\frac{1}{\varepsilon^2}\sum_{k>K} b_0(k)\tau^k \tag{A.5}$$

a.s.-$P_{b_0}^{\infty}$. Hence

$$\begin{aligned}
&\lim_{n\to\infty} P\left(\sqrt{n}\sum_{k=1}^{\infty}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k \le t \,\middle|\, \mathbf{S}_n\right)\\
&\qquad \le \Phi\big((t+\varepsilon)/\sigma_K(z)\big) + \frac{1}{\varepsilon}\sum_{k>K} b_0(k)\tau^{2k}.
\end{aligned}$$

Letting $K$ tend to infinity and $\varepsilon$ to zero in such fashion that (A.5) tends to zero, we finally obtain

$$\begin{aligned}
&\lim_{n\to\infty} P\left(\sqrt{n}\sum_{k=1}^{\infty}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k \le t \,\middle|\, \mathbf{S}_n\right)\\
&\qquad \le \Phi\Big(t/\sqrt{H(z,z)}\Big) \qquad \text{a.s.-}P_{b_0}^{\infty}.
\end{aligned} \tag{A.6}$$

To prove the reverse inequality, it is enough to start from

$$\begin{aligned}
&P\left(\sqrt{n}\sum_{k=1}^{\infty}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k \le t \,\middle|\, \mathbf{S}_n\right)\\
&\quad \ge P\Bigg(\sqrt{n}\sum_{k=1}^{K}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k\\
&\qquad\qquad \le t-\sqrt{n}\sum_{k>K}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k,\\
&\qquad\qquad \left|\sqrt{n}\sum_{k>K}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k\right| \le \varepsilon \,\Bigg|\, \mathbf{S}_n\Bigg)\\
&\quad \ge P\left(\sqrt{n}\sum_{k=1}^{K}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k \le t-\varepsilon \,\middle|\, \mathbf{S}_n\right)\\
&\qquad + P\left(\sqrt{n}\left|\sum_{k>K}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k\right| \le \varepsilon \,\middle|\, \mathbf{S}_n\right) - 1\\
&\quad \ge P\left(\sqrt{n}\sum_{k=1}^{K}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k \le t-\varepsilon \,\middle|\, \mathbf{S}_n\right)\\
&\qquad - P\left(\sqrt{n}\left|\sum_{k>K}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k\right| > \varepsilon \,\middle|\, \mathbf{S}_n\right).
\end{aligned} \tag{A.7}$$

As before, it is now easy to prove that

$$\lim_{n\to\infty} P\left(\sqrt{n}\sum_{k=1}^{\infty}\big(b(k)-E[b(k)|\mathbf{S}_n]\big)z^k \le t \,\middle|\, \mathbf{S}_n\right) \ge \Phi\Big(t/\sqrt{H(z,z)}\Big) \tag{A.8}$$

a.s.-$P_{b_0}^{\infty}$, and from (A.7) and (A.8) the relationship

$$\lim_{n\to\infty} P\left(\sqrt{n}\sum_{k=1}^{\infty}\left(b(k)-E[b(k)|\mathbf{S}_n]\right)z^k \le t\middle|\mathbf{S}_n\right) = \Phi\left(t/\sqrt{H(z,z)}\right)$$

a.s.-$P_{b_0}^{\infty}$ follows. The same technique applies, with small changes, to prove the convergence of all finite-dimensional distributions of $X_n(\cdot)$, and this concludes Step 1.

*Step* 2 (Tightness). Taking into account that $X_n(0) = 0$ a.s., and using Theorem 8.3 in Billingsley (1968), we only have to show that for each positive $\varepsilon$ and $\eta$, there exists a positive $\delta$ and an integer $n_0$ (that may depend on the data sequence) such that

$$\text{(A.9)}\qquad \frac{1}{\delta}P\left(\sup_{z\le s\le z+\delta}|X_n(s)-X_n(z)| \ge \varepsilon|\mathbf{S}_n\right) \le \eta \qquad \forall\, n \ge n_0$$

a.s.-$P_{b_0}^{\infty}$. We develop the proof only in the case $R > 1$, since the case $R = 1$ can be dealt with in a similar way, with just minor changes. Taking again a positive $\gamma$ such that $\tau = r + \gamma < \sqrt{R}$, we have first

$$\begin{aligned}
&\frac{1}{\delta}P\left(\sup_{z\le s\le z+\delta}|X_n(s)-X_n(z)|\ge\varepsilon\middle|\mathbf{S}_n\right)\\
&\quad\le \frac{1}{\delta}P\left(\sup_{z\le s\le z+\delta}\sqrt{n}\sum_{k=1}^{\infty}\left|\left(b(k)-E[b(k)|\mathbf{S}_n]\right)\right|(s^k-z^k)\ge\varepsilon\middle|\mathbf{S}_n\right)\\
&\quad\le \frac{1}{\delta}P\Bigg(\delta\sqrt{n}\sum_{k=1}^{\infty}\left|\left(b(k)-E[b(k)|\mathbf{S}_n]\right)\right|k(z+\delta)^{k-1}\\
\text{(A.10)}\qquad&\qquad\qquad \ge\left(1-\frac{z+\delta}{\tau}\right)\varepsilon\sum_{k=1}^{\infty}\left(\frac{z+\delta}{\tau}\right)^{k-1}\Bigg|\mathbf{S}_n\Bigg)\\
&\quad\le \frac{1}{\delta}\sum_{k=1}^{\infty}P\Bigg(\left|\left(b(k)-E[b(k)|\mathbf{S}_n]\right)\right|\tau^{k-1}\\
&\qquad\qquad \ge \frac{\varepsilon}{\delta k\sqrt{n}}(1-r\tau^{-1})\Bigg|\mathbf{S}_n\Bigg)\\
&\quad\le \frac{n\delta\tau^2}{(\varepsilon\gamma)^2}\sum_{k=1}^{\infty}k^2\operatorname{Var}[b(k)|\mathbf{S}_n]\tau^{2(k-1)}\\
&\quad= \frac{n\delta\tau^2}{(\varepsilon\gamma)^2}\sum_{k=1}^{\infty}k^2\frac{\left(\beta(k)+n\hat{b}(k)\right)\left(n+\beta-n\hat{b}(k)\right)}{(n+\beta)^2(n+\beta+1)}\tau^{2(k-1)}\\
&\quad\le \frac{\delta\tau^2}{(\varepsilon\gamma)^2}\sum_{k=1}^{\infty}k^2\left(\frac{\beta(k)}{n}+\hat{b}(k)\right)\tau^{2(k-1)}.
\end{aligned}$$

From the SLLN, it is easily seen that the series $\sum k^2(n^{-1}\beta(k) + \hat{b}(k))\tau^{2(k-1)}$ converges to $\sum k^2 b_0(k)\tau^{2(k-1)}$, a.s.-$P_{b_0}^{\infty}$. Hence, for every positive $\varepsilon$, there exists $n_0$ [depending on $(S_i;\ i \geq 1)$] for which the inequality

$$\sum_{k=1}^{\infty} k^2 \left( \frac{\beta(k)}{n} + \hat{b}(k) \right) \tau^{2(k-1)} \leq \sum_{k=1}^{\infty} k^2 b_0(k) \tau^{2(k-1)} + \varepsilon \tag{A.11}$$

holds true for every $n \geq n_0$. From (A.10) and (A.11), taking $\delta < (\eta\varepsilon^2\gamma^2)\tau^{-2}(\varepsilon + \sum k^2 b_0(k)\tau^{2(k-1)})^{-1}$, relationship (A.9) is obtained. $\square$

PROOF OF LEMMA 4. It is enough to observe that, because of the properties of $B(\cdot)$, under either (a) or (b), the mapping $f(X_n(\cdot)) = (1-z)^{-1}X_n(z)$ is continuous in $C[0, r]$ w.r.t. the sup-norm, and then to apply the continuous mapping theorem [Billingsley (1968), page 30].

PROOF OF THEOREM 3. To begin the proof, observe first that

$$T_n(z) = T_{1n}(z) + T_{2n}(z) + T_{3n}(z) + \sqrt{n}\,\overline{W}_n(z)\{I_{(\rho \geq 1)} + I_{(\rho < 1)}I_{(z^+ \leq \delta)}\}$$

where

$$T_{1n}(z) = -(1-\rho)^{-1}W(z)\{\sqrt{n}\,(\rho - E[\rho|\mathbf{S}_n])\}I_{(\rho<1)}I_{(z^*>\delta)},$$

$$T_{2n}(z) = (1 - E[\rho|\mathbf{T}_n, \mathbf{S}_n])(1-z)\sqrt{n}\left( \frac{1}{1-\lambda-z+\lambda B(z)} - \frac{1}{1-E[\lambda|\mathbf{T}_n] - z + E[\lambda|\mathbf{T}_n]B(z)} \right) I_{(\rho<1)}I_{(z^*>\delta)},$$

$$T_{3n}(z) = (1 - E[\rho|\mathbf{T}_n, \mathbf{S}_n])(1-z)\sqrt{n}\left( \frac{1}{1-E[\lambda|\mathbf{T}_n] - z + E[\lambda|\mathbf{T}_n]B(z)} - \frac{1}{1-E[\lambda|\mathbf{T}_n] - z + E[\lambda|\mathbf{T}_n]E[B(z)|\mathbf{S}_n]} \right) I_{(\rho<1)}I_{(z^*>\delta)}.$$

Now, as a consequence of Theorem 7.80 in Schervish (1995), it is easy to see that $P(\rho \geq 1|\mathbf{S}_n, \mathbf{T}_n)$ converges to zero at an exponential rate, so that $\sqrt{n}\,P(\rho \geq 1|\mathbf{S}_n, \mathbf{T}_n)$ tends to zero as $n$ tends to infinity, a.s.-$P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$. On the other hand, from Lemma 2 we derive $\sqrt{n}\,P(\rho < 1,\ z^* \leq \delta|\mathbf{S}_n,\ \mathbf{T}_n) \leq \sqrt{n}\,P(G(\delta) \leq 0|\mathbf{T}_n,\ \mathbf{S}_n) \to 0$ as $n$ goes to infinity, a.s.-$P_{\lambda_0} \times P_{b_0}$. Hence, using Corollary 1 we have

$$\lim_{n\to\infty} \sqrt{n}\left( \sup_{0 \leq z \leq \delta} \overline{W}_n(z) \right)\{I_{(\rho\geq 1)} + I_{(\rho<1)}I_{(z^* \leq \delta)}\} = 0 \tag{A.12}$$

a.s.-$P_{\lambda_0} \times P_{b_0}$. In the second place, taking into account that $\lambda$ is consistent [see, e.g., Schervish (1995), pages 430–432), and using Theorems 1 and 4.1 in Billingsley [(1968), page 25], it follows that, a.s.-$P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$, $T_{1n}(z)$ possesses the same asymptotic distribution as

$$T_{1n}^*(z) = -(1-\rho_0)^{-1} W_0(z)\{\sqrt{n}\,\lambda_0(b - E[b|\mathbf{S}_n]) + \sqrt{n}\,b_0(\lambda - E[\lambda|\mathbf{T}_n])\} I_{(\rho<1)} I_{(z^*>\delta)}. \tag{A.13}$$

A Taylor expansion of the term $T_{2n}$ shows that

$$T_{2n}(z) = \frac{1-B(z)}{(1-E[\rho|\mathbf{T}_n,\mathbf{S}_n])(1-z)} \left\{\frac{(1-E[\rho|\mathbf{T}_n,\mathbf{S}_n])(1-z)}{1-\lambda^*-z+\lambda^* B(z)}\right\}^2 \times \sqrt{n}\,(\lambda - E[\lambda|\mathbf{T}_n]) I_{(\rho<1)} I_{(z^*>\delta)},$$

where $\lambda^*$ lies in the interval having extremes $\lambda$ and $E[\lambda|\mathbf{T}_n]$. Using Lemmas 2 and 4, Theorems 2 and 4.1 in Billingsley (1968), it turns out that $T_{2n}$ possesses the same asymptotic distribution as

$$T_{2n}^*(z) = \frac{1-B_0(z)}{(1-\rho_0)(1-z)} W_0(z)^2 \sqrt{n}\,(\lambda - E[\lambda|\mathbf{T}_n]) \qquad \text{a.s.-}P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}. \tag{A.14}$$

The same technique can be applied to the term $T_{3n}$, showing that, as $n$ tends to infinity, it possesses the same asymptotic distribution as

$$T_{3n}^*(z) = -\frac{\lambda_0}{1-\rho_0} \frac{W_0(z)^2}{1-z} \sqrt{n}\,(B(z) - E[B(z)|\mathbf{S}_n]) \qquad \text{a.s.-}P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}. \tag{A.15}$$

From the following two results (the first one can be easily proved by direct calculation, taking into account that the posterior distribution of $\lambda$ is a beta distribution; the second one is a consequence of Theorem 3 in Freedman (1963)]:

$$\sqrt{n}\,(\lambda - E[\lambda|\mathbf{T}_n]) \xrightarrow{d} N(0, \sigma_{\lambda_0}^2) \qquad \text{a.s.-}P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty},$$

$$\sqrt{n}\,(\mu - E[\mu|\mathbf{S}_n]) \xrightarrow{d} N(0, \sigma_{B_0}^2) \qquad \text{a.s.-}P_{\lambda_0}^{\infty} \times P_{b_0}^{\infty}$$

and from (8.21)–(8.24), Lemmas 3 and 4 and the independence between the posterior distributions of $\lambda$ and $B(\cdot)$, the theorem is proved. □

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