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ANOMALOUS DIFFUSION FOR MULTI-DIMENSIONAL CRITICAL KINETIC FOKKER–PLANCK EQUATIONS

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We consider a particle moving in $d \geq 2$ dimensions, its velocity being a reversible diffusion process, with identity diffusion coefficient, of which the invariant measure behaves, roughly, like $(1 + |v|)^{-\beta}$ as $|v| \to \infty$, for some constant $\beta > 0$. We prove that for large times, after a suitable rescaling, the position process resembles a Brownian motion if $\beta \geq 4+d$, a stable process if $\beta \in [d, 4+d)$ and an integrated multi-dimensional generalization of a Bessel process if $\beta \in (d-2, d)$. The critical cases $\beta = d$, $\beta = 1+d$ and $\beta = 4+d$ require special rescalings.

1. Introduction and results.

1.1. *Motivation and references.* Describing the motion of a particle with complex dynamics, after space-time rescaling, by a simple diffusion, is a natural and classical subject. See, for example, Langevin [24], Larsen–Keller [25], Bensoussans–Lions–Papanicolaou [5] and Bodineau–Gallagher–St-Raymond [7]. Particles undergoing anomalous diffusion are often observed in physics, and many mathematical works show how to modify some Boltzmann-like linear equations to asymptotically get some fractional diffusion limit (i.e., a radially symmetric Lévy stable jumping position process). See Mischler–Mouhot–Mellet [29], Jara–Komorowski–Olla [20], Mellet [28], Ben Abdallah–Mellet–Puel [3, 4], etc.

The kinetic Fokker–Planck equation is also of constant use in physics, because it is rather simpler than the Boltzmann equation: assume that the density $f_t(x,v)$ of particles with position $x \in \mathbb{R}^d$ and velocity $v \in \mathbb{R}^d$ at time $t \geq 0$ solves

$$\partial_t f_t(x,v) + v \cdot \nabla_x f_t(x,v) = \frac{1}{2}\big(\Delta_v f_t(x,v) + \beta \,\mathrm{div}_v\big[F(v) f_t(x,v)\big]\big) \tag{1}$$

for some force field $F : \mathbb{R}^d \to \mathbb{R}^d$ and some constant $\beta > 0$ that will be useful later. We then try to understand the behavior of the *density* $\rho_t(x) = \int_{\mathbb{R}^d} f_t(x,v)\,\mathrm{d}v$ for large times.

The *trajectory* corresponding to (1) is the following stochastic kinetic model:

$$V_t = v_0 + B_t - \frac{\beta}{2}\int_0^t F(V_s)\,\mathrm{d}s \quad \text{and} \quad X_t = x_0 + \int_0^t V_s\,\mathrm{d}s. \tag{2}$$

Here $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion. For $(V_t, X_t)_{t\geq 0}$ (with values in $\mathbb{R}^d \times \mathbb{R}^d$) solving (2), the family of time-marginals $f_t = \mathrm{Law}(X_t, V_t)$ solves (1) in the sense of distributions.

It is well known that if F is sufficiently confining, then the velocity process $(V_t)_{t\geq 0}$ is close to equilibrium, its invariant distribution has a fast decay, and after rescaling, the position process $(X_t)_{t\geq 0}$ resembles a Brownian motion in large time. In other words, $(\rho_t)_{t\geq 0}$ is close to the solution to the heat equation.

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If on the contrary F is not sufficiently confining, for example, if $F \equiv 0$, then $(X_t)_{t\geq 0}$ cannot be reduced to an autonomous Markov process in large times. In other words, $(\rho_t)_{t\geq 0}$ does not solve an autonomous time-homogeneous PDE.

The only way to hope for some anomalous diffusion limit, for a Fokker–Planck toy model like (1), is to choose the force in such a way that the invariant measure of the velocity process has a fat tail. One realizes that one has to choose F behaving like $F(v) \sim 1/|v|$ as $|v| \to \infty$, and the most natural choice is $F(v) = v/(1+|v|^2)$. Now the asymptotic behavior of the model may depend on the value of $\beta > 0$, since the invariant distribution of the velocity process is given by $(1+|v|^2)^{-\beta/2}$, up to some normalization constant.

The Fokker–Planck model (1), with the force $F(v) = v/(1+|v|^2)$, is the object of the papers by Nasreddine–Puel [31] ($d \geq 1$ and $\beta > 4+d$, diffusive regime), Cattiaux–Nasreddine–Puel [11] ($d \geq 1$ and $\beta = 4+d$, critical diffusive regime) and Lebeau–Puel [26] ($d = 1$ and $\beta \in (1,5) \setminus \{2,3,4\}$). In this last paper, the authors show that after time/space rescaling, the density $(\rho_t)_{t\geq 0}$ is close to the solution to the fractional heat equation with index $\alpha/2$, where $\alpha = (\beta+1)/3$. In other words, $(X_t)_{t\geq 0}$ resembles a symmetric α-stable process. This work relies on a spectral approach and involves many explicit computations.

Using an alternative probabilistic approach, we studied the one-dimensional case in [15], treating all the cases $\beta \in (0,\infty)$ in a rather concise way. We allowed for a more general (symmetric) force field F.

Physicists observed that atoms subjected to Sisyphus cooling anomalously diffuse; see Castin–Dalibard–Cohen–Tannoudji [9], Sagi–Brook–Almog–Davidson [34] and Marksteiner–Ellinger–Zoller [27]. A theoretical study has been proposed by Barkai–Aghion–Kessler [2]. They precisely model the motion of atoms by (1) with $F(v) = v/(1+v^2)$ induced by the laser field, simplifying very slightly the model derived in [9]. They predict, in dimension $d = 1$ and with a quite high level of rigor, the results of [15], Theorem 1, excluding the critical cases, with the following terminology: normal diffusion when $\beta > 5$, Lévy diffusion when $\beta \in (1,5)$ and *Obukhov–Richardson phase* when $\beta \in (0,1)$. This last case is treated in a rather confused way in [2], mainly because no tractable explicit computation can be handled, since the limit process is an integrated symmetric Bessel process.

In [22], Kessler–Barkai mention other fields of applications of this model, such as single particle models for long-range interacting systems (Bouchet–Dauxois [8]), condensation describing a charged particle in the vicinity of a charged polymer (Manning [28]), and motion of nanoparticles in an appropriately constructed force field (Cohen [12]). We refer to [11, 26, 31] and especially [2, 22] for many other references and motivations.

The goal of the present paper is to study what happens in higher dimension. We also allow for some nonradially symmetric force, to understand more deeply what happens, in particular in the stable regime. To our knowledge, the results are completely new. The proofs are technically much more involved than in dimension 1.

1.2. *Main results*. In the whole paper, we assume that the initial condition $(v_0, x_0) \in \mathbb{R}^d \times \mathbb{R}^d$ is deterministic and, for simplicity, that $v_0 \neq 0$. We also assume that the force is of the following form.

ASSUMPTION 1. There is a potential $U : \mathbb{R}^d \setminus \{0\} \to (0,\infty)$ of the form $U(v) = \Gamma(|v|)\gamma(v/|v|)$, for some $\gamma : \mathbb{S}_{d-1} \to (0,\infty)$ of class C^∞ and some $\Gamma : \mathbb{R}_+ \to (0,\infty)$ of class C^∞ satisfying $\Gamma(r) \sim r$ as $r \to \infty$, such that for any $v \in \mathbb{R}^d \setminus \{0\}$, $F(v) = \nabla[\log U(v)] = [U(v)]^{-1}\nabla U(v)$.

Observe that F is of class C^∞ on $\mathbb{R}^d \setminus \{0\}$. We will check the following well-posedness result.

PROPOSITION 2. *Under Assumption 1, (2) has a pathwise unique solution* $(V_t, X_t)_{t\geq 0}$, *which is furthermore* $(\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d$*-valued.*

REMARK 3. Assume that $\beta > d$. As we will see, $(V_t)_{t\geq 0}$ has a unique invariant probability measure given by $\mu_\beta(\mathrm{d}v) = c_\beta[U(v)]^{-\beta}\,\mathrm{d}v$, for $c_\beta = [\int_{\mathbb{R}^d}[U(v)]^{-\beta}\,\mathrm{d}v]^{-1}$.

As already mentioned, the main example we have in mind is $\Gamma(r) = \sqrt{1+r^2}$ and $\gamma \equiv 1$, whence $U(v) = \sqrt{1+|v|^2}$ and $F(v) = v/(1+|v|^2)$. We also allow for some non radially symmetric potentials to understand more deeply what may happen.

In the whole paper, we denote by $\mathcal{S}_d^+$ the set of symmetric positive-definite $d \times d$ matrices. We also denote by $\varsigma(\mathrm{d}\theta)$ the uniform probability measure on $\mathbb{S}_{d-1}$.

For $((Z_t^\epsilon)_{t\geq 0})_{\epsilon\geq 0}$ a family of $\mathbb{R}^d$-valued processes, we write $(Z_t^\epsilon)_{t\geq 0} \xrightarrow{f.d.} (Z_t^0)_{t\geq 0}$ if for any finite subset $S \subset [0,\infty)$ the vector $(Z_t^\epsilon)_{t\in S}$ goes in law to $(Z_t^0)_{t\in S}$ as $\epsilon \to 0$; and we write $(Z_t^\epsilon)_{t\geq 0} \xrightarrow{d} (Z_t^0)_{t\geq 0}$ if the convergence in law holds in the usual sense of continuous processes. Here is our main result.

THEOREM 4. *Fix* $\beta > 0$, *suppose Assumption 1 and consider the solution* $(V_t, X_t)_{t\geq 0}$ *to (2). We set* $a_\beta = [\int_{\mathbb{S}_{d-1}}[\gamma(\theta)]^{-\beta}\varsigma(\mathrm{d}\theta)]^{-1} > 0$, *as well as* $M_\beta = a_\beta \int_{\mathbb{S}_{d-1}} \theta[\gamma(\theta)]^{-\beta}\varsigma(\mathrm{d}\theta) \in \mathbb{R}^d$ *and, if* $\beta > 1+d$, $m_\beta = \int_{\mathbb{R}^d} v\mu_\beta(\mathrm{d}v) \in \mathbb{R}^d$.

(a) *If* $\beta > 4+d$, *there is* $\Sigma \in \mathcal{S}_d^+$ *such that*

$$\big(\epsilon^{1/2}[X_{t/\epsilon} - m_\beta t/\epsilon]\big)_{t\geq 0} \xrightarrow{f.d.} (\Sigma B_t)_{t\geq 0},$$

where $(B_t)_{t\geq 0}$ *is a* d*-dimensional Brownian motion.*

(b) *If* $\beta = 4+d$ *and if* $\int_1^\infty r^{-1}|r\Gamma'(r)/\Gamma(r) - 1|^2\,\mathrm{d}r < \infty$, *then*

$$\big(\epsilon^{1/2}|\log\epsilon|^{-1/2}[X_{t/\epsilon} - m_\beta t/\epsilon]\big)_{t\geq 0} \xrightarrow{f.d.} (\Sigma B_t)_{t\geq 0}$$

for some $\Sigma \in \mathcal{S}_d^+$, *where* $(B_t)_{t\geq 0}$ *is a* d*-dimensional Brownian motion.*

(c) *If* $\beta \in (1+d, 4+d)$, *set* $\alpha = (\beta+2-d)/3$. *Then*

$$\big(\epsilon^{1/\alpha}[X_{t/\epsilon} - m_\beta t/\epsilon]\big)_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0},$$

where $(S_t)_{t\geq 0}$ *is a nontrivial* α*-stable Lévy process.*

(d) *If* $\beta = 1+d$ *and if* $\int_1^\infty r^{-1}|r/\Gamma(r) - 1|\,\mathrm{d}r < \infty$ *there is* $c > 0$ *such that*

$$\big(\epsilon\big[X_{t/\epsilon} - cM_\beta|\log\epsilon|t/\epsilon\big]\big)_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0},$$

where $(S_t)_{t\geq 0}$ *is a nontrivial* 1*-stable Lévy process.*

(e) *If* $\beta \in (d, 1+d)$, *set* $\alpha = (\beta+2-d)/3$. *Then*

$$\big(\epsilon^{1/\alpha}X_{t/\epsilon}\big)_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0},$$

where $(S_t)_{t\geq 0}$ *is a nontrivial* α*-stable Lévy process.*

(f) *If* $\beta = d$, *then*

$$\big(|\epsilon\log\epsilon|^{3/2}X_{t/\epsilon}\big)_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0},$$

where $(S_t)_{t\geq 0}$ *is a nontrivial* 2/3*-stable Lévy process.*

(g) *If* $\beta \in (d-2, d)$,

$$(\epsilon^{3/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{d} \left(\int_0^t \mathcal{V}_s \, \mathrm{d}s\right)_{t\geq 0},$$

where $(\mathcal{V}_t)_{t\geq 0}$ *is a* $\mathbb{R}^d$*-valued continuous process* (*see Definition* 25) *of which the norm* $(|\mathcal{V}_t|)_{t\geq 0}$ *is a Bessel process with dimension* $d-\beta$ *issued from* 0.

The strong regularity of U is only used to apply as simply as possible some classical PDE results.

REMARK 5. (i) In the diffusive regimes (a) and (b), the matrix Σ depends only on U and β; see Remarks 31(i) and 36(i). The additional condition when $\beta = 4+d$ more or less imposes that $\Gamma'(r) \to 1$ as $r \to \infty$ and that this convergence does not occur too slowly. This is slightly restrictive, but found no way to get rid of this assumption.

(ii) In cases (c), (d), (e) and (f), the Lévy measure of the α-stable process $(S_t)_{t\geq 0}$ only depends on U and β: a complicated formula involving Itô's excursion measure can be found in Proposition 23(i). The additional condition when $\beta = 1+d$ requires that $r^{-1}\Gamma(r)$ does not converge too slowly to 1 as $r \to \infty$ and is very weak. The constant $c > 0$ in point (d) is explicit; see Remark 24.

(iii) In point (g), the law of $(\mathcal{V}_t)_{t\geq 0}$ depends only on γ and on β.

(iv) Actually, point (g) should extend to any value of $\beta \in (-\infty, d)$, with a rather simple proof, the definition of the limit process $(\mathcal{V}_t)_{t\geq 0}$ being less involved: see Definition 25 and observe that for $\beta \leq d-2$, the set of zeros of a Bessel process with dimension $d-\beta$ issued from 0 is trivial. We did not include this uninteresting case because the paper is already technical enough.

For the main model we have in mind, Theorem 4 applies and its statement simplifies. See Remarks 31(ii) and 36(ii) and Proposition 23(ii).

REMARK 6. Assume $\Gamma(r) = \sqrt{1+r^2}$ and $\gamma \equiv 1$, that is, $F(v) = v/(1+|v|^2)$.

(a) If $\beta > 4+d$, then $(\epsilon^{1/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (qB_t)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion, for some explicit $q > 0$.

(b) If $\beta = 4+d$, then $(\epsilon^{1/2}|\log\epsilon|^{-1/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (qB_t)_{t\geq 0}$, where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion, for some explicit $q > 0$.

(c)–(d)–(e) If $\beta \in (d, 4+d)$, then $(\epsilon^{1/\alpha} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0}$, where $(S_t)_{t\geq 0}$ is a radially symmetric α-stable process, where $\alpha = (\beta+2-d)/3$ and with nonexplicit multiplicative constant.

(f) If $\beta = d$, then $(|\epsilon\log\epsilon|^{3/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0}$, where $(S_t)_{t\geq 0}$ is a radially symmetric 2/3-stable process with nonexplicit multiplicative constant.

(g) If $\beta \in (d-2, d)$, $(\epsilon^{3/2} X_{t/\epsilon})_{t\geq 0} \xrightarrow{d} (\int_0^t \mathcal{V}_s \, \mathrm{d}s)_{t\geq 0}$, with $(\mathcal{V}_t)_{t\geq 0}$ introduced in Definition 25.

1.3. *Comments*. Pardoux–Veretennikov [32] studied in great generality the diffusive case, allowing for some much more general SDEs with nonconstant diffusion coefficient and general drift coefficient. Their results are sufficiently sharp to include the diffusive case $\beta > 4+d$ when $F(v) = v/(1+|v|^2)$. Hence, the diffusive case (a) is rather classical.

We studied the one-dimensional case $d = 1$ with an even potential U in [15]. Many technical difficulties appear in higher dimension. In the diffusive and critical diffusive regime, the main difficulty is that we cannot solve explicitly the Poisson equation $\mathcal{L}\phi(v) = v$ (with $\mathcal{L}$ the generator of $(V_t)_{t\geq 0}$), while this is feasible in dimension 1. Observe that such a problem would disappear if dealing only with the force $F(v) = v/(1+|v|^2)$.

We use a spherical decomposition $V_t = R_t\Theta_t$ of the velocity process. This is of course very natural in this context, and we do not see how to proceed in another way. However, since in some sense, after rescaling, the radius process $(R_t)_{t\geq 0}$ resembles a Bessel process with dimension $d-\beta \in (-\infty, 2)$, which hits 0, spherical coordinates are rather difficult to deal with, the process Θ_t moving very fast each time R_t touches 0.

In dimension 1, the most interesting *stable* regime is derived as follows. We write $(V_t)_{t\geq 0}$ as a function of a time-changed Brownian motion $(W_t)_{t\geq 0}$, using the classical *speed measures* and *scale functions* of one-dimensional SDEs and express $\epsilon^{1/\alpha} X_{t/\epsilon}$ accordingly. Passing to the limit as $\epsilon \to 0$, we find the expression of the (symmetric) stable process in terms of the Brownian motion $(W_t)_{t\geq 0}$ and of its inverse local time at 0 discovered by Biane–Yor [6]; see also Itô–McKean [18], page 226, and Jeulin–Yor [21]. In higher dimension, the situation is much more complicated, and we found no simpler way than writing our limiting stable processes using some *excursion Poisson point processes*.

Let us emphasize that our proofs are qualitative. On the contrary, even in dimension 1, the informal proofs of Barkai–Aghion–Kessler [2] rely on very explicit computations and explicit solutions to O.D.E.s in terms of modified Bessel functions, and Lebeau–Puel [26] also use rather explicit computations.

1.4. *Plan of the paper.* To start with, we explain informally in Section 2 our proof of Theorem 4 in the most interesting case, that is when $F(v) = v/(1+|v|^2)$ and when $\beta \in (d, 4+d)$.

In Section 3, we introduce some notation of constant use in the paper.

In Section 4, we write the velocity process $(V_t)_{t\geq 0}$ as $(R_t\Theta_t)_{t\geq 0}$, the radius process $(R_t)_{t\geq 0}$ solving an autonomous SDE, and the process $(\Theta_t)_{t\geq 0}$ being $\mathbb{S}_{d-1}$-valued. We also write down a representation of the radius as a function of a time-changed Brownian motion, using the classical theory of speed measures and scale functions of one-dimensional SDEs.

We designed the other sections to be as independent as possible.

Sections 5, 6, 7 and 8 treat respectively, the stable regime (cases (c)–(d)–(e)–(f)), integrated Bessel regime (case (g)), diffusive regime (case (a)) and critical diffusive regime (case (b)).

Finally, an Appendix at the end of the paper contains some more or less classical results about ergodicity of diffusion processes, about Itô's excursion measure, about Bessel processes, about convergence of inverse functions and, finally, a few technical estimates.

2. Informal proof in the stable regime with a symmetric force. We assume in this section that $F(v) = v/(1+|v|^2)$ and that $\beta \in (d, 4+d)$ and explain informally how to prove Theorem 4(c)–(d)–(e). We also assume, for example, that $x_0 = 0$ and that $v_0 = \theta_0 \in \mathbb{S}_{d-1}$.

Step 1. Writing the velocity process in spherical coordinates, we find that $V_t = R_t\hat{\Theta}_{H_t}$, where

$$R_t = 1 + \tilde{B}_t + \int_0^t \left(\frac{d-1}{2R_s} - \frac{\beta R_s}{1+R_s^2}\right) \mathrm{d}s \tag{3}$$

for some $1D$-Brownian motion $(\tilde{B}_t)_{t\geq 0}$, independent of a spherical $\mathbb{S}_{d-1}$-valued Brownian motion $(\hat{\Theta}_t)_{t\geq 0}$ starting from θ_0, and where $H_t = \int_0^t R_s^{-2}\,\mathrm{d}s$.

Step 2. Using the classical *speed measure* and *scale function*, we may write the radius process $(R_t)_{t\geq 0}$ as a space and time changed Brownian motion. For that we introduce $h(r) = (\beta+2-d)\int_1^r u^{1-d}[1+u^2]^{\beta/2}\,\mathrm{d}u$, which is an increasing bijection from $(0,\infty)$ into $\mathbb{R}$. We denote by $h^{-1}: \mathbb{R} \to (0,\infty)$ its inverse function and by $\sigma(w) = h'(h^{-1}(w))$ from $\mathbb{R}$ to

$(0, \infty)$. For $(W_t)_{t\geq 0}$ a one-dimensional Brownian motion, consider the continuous increasing process $A_t = \int_0^t [\sigma(W_s)]^{-2}\,\mathrm{d}s$ and its inverse $(\rho_t)_{t\geq 0}$. One can classically check that $R_t = h^{-1}(W_{\rho_t})$ is a (weak) solution to (3), so that we can write the position process as

$$X_t = \int_0^t h^{-1}(W_{\rho_s})\hat{\Theta}_{H_s}\,\mathrm{d}s = \int_0^{\rho_t} \frac{h^{-1}(W_u)}{[\sigma(W_u)]^2}\hat{\Theta}_{H_{A_u}}\,\mathrm{d}u.$$

We used the substitution $\rho_s = u$, that is, $s = A_u$, whence $\mathrm{d}s = [\sigma(W_u)]^{-2}\,\mathrm{d}u$. We next observe that $T_t = H_{A_t} = \int_0^{A_t}[h^{-1}(W_{\rho_s})]^{-2}\,\mathrm{d}s = \int_0^t [\psi(W_u)]^{-2}\,\mathrm{d}u$, where we have set $\psi(w) = h^{-1}(w)\sigma(w)$. Finally,

$$X_{t/\epsilon} = \int_0^{\rho_{t/\epsilon}} \frac{h^{-1}(W_u)}{[\sigma(W_u)]^2}\hat{\Theta}_{T_u}\,\mathrm{d}u.$$

Step 3. To study the large time behavior of the position process, it is more convenient to start from a fixed Brownian motion $(W_t)_{t\geq 0}$ and to use Step 2 with the Brownian motion $(W_t^\epsilon = (c\epsilon)^{-1}W_{(c\epsilon)^2 t})_{t\geq 0}$, for some constant $c > 0$ to be chosen later. After a few computations, we find that

$$X_{t/\epsilon} = \int_0^{\rho_t^\epsilon} \frac{h^{-1}(W_s/(c\epsilon))\hat{\Theta}_{T_s^\epsilon}}{(c\epsilon)^2[\sigma(W_s/(c\epsilon))]^2}\,\mathrm{d}s \quad \text{where}$$

$$T_t^\epsilon = \int_0^t \frac{\mathrm{d}u}{[c\epsilon\psi(W_u/c\epsilon)]^2} \quad \text{and} \quad A_t^\epsilon = \int_0^t \frac{\mathrm{d}u}{c^2\epsilon[\sigma(W_u/(c\epsilon))]^2},$$

and where $(\rho_t^\epsilon)_{t\geq 0}$ is the inverse of $(A_t^\epsilon)_{t\geq 0}$.

Step 4. If choosing $c = \int_{\mathbb{R}}[\sigma(x)]^{-2}\,\mathrm{d}x$, it holds that $\lim_{\epsilon\to 0} A_t^\epsilon = L_t^0$ a.s. for all $t \geq 0$, where $(L_t^0)_{t\geq 0}$ is the local time of $(W_t)_{t\geq 0}$: by the occupation times formula (see Revuz–Yor [33], Corollary 1.6, p. 224)

$$A_t^\epsilon = \int_{\mathbb{R}} \frac{L_t^x\,\mathrm{d}x}{c^2\epsilon[\sigma(x/(c\epsilon))]^2} = \int_{\mathbb{R}} \frac{L_t^{c\epsilon y}\,\mathrm{d}y}{c[\sigma(y)]^2} \longrightarrow \int_{\mathbb{R}} \frac{\mathrm{d}y}{c[\sigma(y)]^2}L_t^0 = L_t^0.$$

As a consequence, ρ_t^ϵ tends to τ_t, the inverse of L_t^0.

Step 5. Studying the function h near 0 and ∞, and then h^{-1}, σ and ψ near $-\infty$ and ∞, we find that, with $\alpha = (\beta + 1 - d)/3$ (see Lemma 42(ix) and (v)):

- $\lim_{\epsilon\to 0}\epsilon^{1/\alpha}(c\epsilon)^{-2}h^{-1}(w/(c\epsilon))[\sigma(w/(c\epsilon))]^{-2} = c'w^{1/\alpha-2}\mathbf{1}_{\{w>0\}}$,
- $\lim_{\epsilon\to 0}[c\epsilon\psi(w/c\epsilon)]^{-2} = c''w^{-2}\mathbf{1}_{\{w>0\}} + \varphi(w)\mathbf{1}_{\{w\leq 0\}}$,

for some constants $c', c'' > 0$ and some unimportant function $\varphi \geq 0$. Here appears the scaling $\epsilon^{1/\alpha}$.

Passing to the limit informally in the expression of Step 3, we find that

$$\epsilon^{1/\alpha}X_{t/\epsilon} \longrightarrow S_t = c'\int_0^{\tau_t} W_s^{1/\alpha-2}\mathbf{1}_{\{W_s>0\}}\hat{\Theta}_{U_s}\,\mathrm{d}s \quad \text{where}$$

$$U_t = c''\int_0^t W_u^{-2}\mathbf{1}_{\{W_u>0\}}\,\mathrm{d}u + \int_0^t \varphi(W_u)\mathbf{1}_{\{W_u\leq 0\}}\,\mathrm{d}u.$$

Unfortunately, this expression does not make sense, because $U_t = \infty$ for all $t > 0$, since the Brownian motion is (almost) 1/2-Hölder continuous and since it hits 0. But in some sense, $U_t - U_s$ is well-defined if $W_u > 0$ for all $u \in (s, t)$. And in some sense, the processes $(\hat{\Theta}_{U_s})_{s\in[a,b]}$ and $(\hat{\Theta}_{U_s})_{s\in[a',b']}$ are independent if $W_u > 0$ on $[a, b] \cup [a', b']$ and if there exists $t \in (b, a')$ such that $W_t = 0$, since then $U_{a'} - U_b = \infty$, so that the spherical Brownian motion $\hat{\Theta}$, at time $U_{a'}$, has completely forgotten the values it has taken during $[U_a, U_b]$.

Since $(\tau_t)_{t\geq 0}$ is the inverse local time of $(W_t)_{t\geq 0}$, it holds that τ_t is a stopping-time and that $W_{\tau_{t-}} = W_{\tau_t} = 0$ for each $t \geq 0$. Hence, by the strong Markov property, for any reasonable function $f : \mathbb{R} \to \mathbb{R}^d$, the process $Z_t = \int_0^{\tau_t} f(W_s)\,\mathrm{d}s$ is Lévy, and its jumps are given by $\Delta Z_t = \int_{\tau_{t-}}^{\tau_t} f(W_s)\,\mathrm{d}s$, for $t \in J = \{s \geq 0 : \Delta\tau_s > 0\}$.

The presence of $\hat{\Theta}_{U_s}$ in the expression of $(S_t)_{t\geq 0}$ does not affect its Lévy character, because $(\hat{\Theta}_t)_{t\geq 0}$ is independent of $(W_t)_{t\geq 0}$ and because in some sense, the family $\{(\hat{\Theta}_{U_u})_{u\in[\tau_{s-},\tau_s]} : s \in J\}$ is independent. Hence, $(S_t)_{t\geq 0}$ is Lévy and its jumps are given by

$$\Delta S_t = c' \int_{\tau_{t-}}^{\tau_t} W_s^{1/\alpha-2} \mathbf{1}_{\{W_s>0\}} \hat{\Theta}^t_{[c'' \int_{(\tau_t+\tau_{t-})/2}^s W_u^{-2}\,\mathrm{d}u]}\,\mathrm{d}s, \quad t \in J$$

for some i.i.d. family $\{(\hat{\Theta}^t_u)_{u\in\mathbb{R}} : t \in J\}$ of eternal spherical Brownian motions. Informally, for each $t \in J$, we have set $\hat{\Theta}^t_u = \hat{\Theta}_{U_{(\tau_t+\tau_{t-})/2+u}}$ for all $u \in \mathbb{R}$. The choice of $(\tau_t + \tau_{t-})/2$ for the time origin of the eternal spherical Brownian motion $\hat{\Theta}^t$ is arbitrary, any time in (τ_{t-}, τ_t) would be suitable. Observe that the clock $c'' \int_{(\tau_t+\tau_{t-})/2}^s W_u^{-2}\,\mathrm{d}u$ is well-defined for all $s \in (\tau_{t-}, \tau_t)$ because W_u is continuous and does not vanish on $u \in (\tau_{t-}, \tau_t)$. This clock tends to ∞ as $u \to \tau_t$, and to $-\infty$ as $u \to \tau_{t-}$.

It only remains to verify that the Lévy measure q of $(S_t)_{t\geq 0}$ is radially symmetric, which is more or less obvious by symmetry of the law of the eternal spherical Brownian motion; and enjoys the scaling property that $q(A_a) = a^\alpha q(A)$ for all $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ and all $a > 0$, where $A_a = \{x \in \mathbb{R}^d : ax \in A\}$. This property is inherited from the scaling property of the Brownian motion (this uses that the clock in the spherical Brownian motion is precisely proportional to $c'' \int_{(\tau_t+\tau_{t-})/2}^s W_u^{-2}\,\mathrm{d}u$).

To write all this properly, we have to use Itô's excursion theory.

Let us mention one last difficulty: when $\alpha \geq 1$, $\int_0^t W_s^{1/\alpha-2}\mathbf{1}_{\{W_s>0\}}\,\mathrm{d}s$ is a.s. infinite for all $t > 0$. Hence to study S_t, one really has to use the symmetries of the spherical Brownian motion and that the clock driving it explodes each time W hits 0.

3. Notation. In the whole paper, we suppose Assumption 1. We summarize here some notation of constant use.

Recall that $\mathcal{S}_d^+$ is the set of symmetric positive-definite $d \times d$ matrices.

We write the initial velocity as $v_0 = r_0\theta_0$, with $r_0 > 0$ and $\theta_0 \in \mathbb{S}_{d-1}$.

For $u \in \mathbb{R}^d \setminus \{0\}$, let $\pi_{u^\perp} = (I_d - \frac{uu^*}{|u|^2})$ be the $d \times d$-matrix of the orthogonal projection on $u^\perp$.

For $\Psi : \mathbb{R}^d \to \mathbb{R}^d$, let $\nabla^*\Psi = (\nabla\Psi_1 \cdots \nabla\Psi_d)^*$.

Recall that $a_\beta = [\int_{\mathbb{S}_{d-1}} [\gamma(\theta)]^{-\beta} \varsigma(\mathrm{d}\theta)]^{-1} > 0$, where ς is the uniform probability measure on $\mathbb{S}_{d-1}$. We introduce the probability measure $\nu_\beta(\mathrm{d}\theta) = a_\beta[\gamma(\theta)]^{-\beta}\varsigma(\mathrm{d}\theta)$ on $\mathbb{S}_{d-1}$. It holds that $M_\beta = \int_{\mathbb{S}_{d-1}} \theta\nu_\beta(\mathrm{d}\theta) \in \mathbb{R}^d$.

If $\beta > d$, we set $b_\beta = [\int_0^\infty [\Gamma(r)]^{-\beta} r^{d-1}\,\mathrm{d}r]^{-1}$ and introduce the probability measure $\nu'_\beta(\mathrm{d}r) = b_\beta[\Gamma(r)]^{-\beta} r^{d-1}\,\mathrm{d}r$ on $(0, \infty)$. It has a finite mean $m'_\beta = \int_0^\infty r\nu'_\beta(\mathrm{d}r) > 0$ if $\beta > 1 + d$.

Still in the case where $\beta > d$, we recall that $c_\beta = [\int_{\mathbb{R}^d} [U(v)]^{-\beta}\,\mathrm{d}v]^{-1}$ and that $\mu_\beta(\mathrm{d}v) = c_\beta \times [U(v)]^{-\beta}\,\mathrm{d}v$ on $\mathbb{R}^d$. It holds that $c_\beta = a_\beta b_\beta$ and

$$\int_{\mathbb{R}^d} \varphi(v)\mu_\beta(\mathrm{d}v) = \int_0^\infty \int_{\mathbb{S}_{d-1}} \varphi(r\theta)\nu_\beta(\mathrm{d}\theta)\nu'_\beta(\mathrm{d}r)$$

for any measurable $\varphi : \mathbb{R}^d \to \mathbb{R}_+$. In particular, $m_\beta = M_\beta m'_\beta$ if $\beta > 1 + d$.

In the whole paper, we implicitly extend all the functions on $\mathbb{S}_{d-1}$ to $\mathbb{R}^d \setminus \{0\}$ as follows: for $\psi : \mathbb{S}_{d-1} \to \mathbb{R}$ and $v \in \mathbb{R}^d \setminus \{0\}$, we set $\psi(v) = \psi(v/|v|)$.

We endow $\mathbb{S}_{d-1}$ with its natural Riemannian metric, denote by $T\mathbb{S}_{d-1}$ its tangent bundle and by ∇_S, div_S and Δ_S the associated gradient, divergence and Laplace operators. With the above convention, for a function $\psi : \mathbb{S}_{d-1} \to \mathbb{R}$ and a vector field $\Psi : \mathbb{S}_{d-1} \to T\mathbb{S}_{d-1}$, it holds that, for $\theta \in \mathbb{S}_{d-1} \subset \mathbb{R}^d \setminus \{0\}$,

$$\nabla_S \psi(\theta) = \nabla\psi(\theta), \qquad \mathrm{div}_S\, \Psi(\theta) = \mathrm{div}\, \Psi(\theta) \quad \text{and} \quad \Delta_S \psi(\theta) = \Delta\psi(\theta).$$

4. Representation of the solution. Here we show that (2) is well-posed and explain how to build a solution (in law) from some independent radial and spherical processes, in a way that will allow us to study the large time behavior of the position process by coupling.

LEMMA 7. *Consider a d-dimensional Brownian motion $(\hat{B}_t)_{t\geq 0}$. The following equation, of which the unknown $(\hat{\Theta}_t)_{t\geq 0}$ is $\mathbb{R}^d \setminus \{0\}$-valued,*

$$\hat{\Theta}_t = \theta_0 + \int_0^t \pi_{\hat{\Theta}_s^\perp}\, \mathrm{d}\hat{B}_s - \frac{d-1}{2}\int_0^t \frac{\hat{\Theta}_s}{|\hat{\Theta}_s|^2}\, \mathrm{d}s - \frac{\beta}{2}\int_0^t \pi_{\hat{\Theta}_s^\perp} \frac{\nabla\gamma(\hat{\Theta}_s)}{\gamma(\hat{\Theta}_s)}\, \mathrm{d}s, \tag{4}$$

has a unique strong solution, which is furthermore $\mathbb{S}_{d-1}$-valued.

Recall that we have extended γ to $\mathbb{R}^d \setminus \{0\}$ by setting $\gamma(v) = \gamma(v/|v|)$.

PROOF. The coefficients of this equation being of class C^1 on $\mathbb{R}^d \setminus \{0\}$, there classically exists a unique maximal strong solution (defined until it reaches 0 or explodes to infinity), and we only have to check that this solution a.s. remains in $\mathbb{S}_{d-1}$ for all times. By a classical computation using the Itô formula, $|\hat{\Theta}_t|^2 = |\theta_0|^2 = 1$ for all $t \geq 0$ a.s. This uses that for $\phi(\theta) = |\theta|^2$ defined on $\mathbb{R}^d$, we have $\nabla\phi(\theta) = 2\theta$, so that $(\nabla\phi(\theta))^*\pi_{\theta^\perp} = 0$ and $\partial_{ij}\phi(\theta) = 2\delta_{ij}$, from which $\frac{1}{2}\sum_{i,j=1}^d \partial_{ij}\phi(\theta)(\pi_{\theta^\perp})_{ij} - \frac{d-1}{2}\nabla\phi(\theta)\cdot|\theta|^{-2}\theta = 0$. □

The SDE (5) below has a unique strong solution: it has a unique local strong solution (until it reaches 0 or ∞) because its coefficients are C^1 on $(0,\infty)$ and we will see in Lemma 10 that one can build a $(0,\infty)$-valued global weak solution, so that the unique strong solution is global.

LEMMA 8. *For two independent Brownian motions $(\tilde{B}_t)_{t\geq 0}$ (in dimension 1) and $(\hat{B}_t)_{t\geq 0}$ (in dimension d), consider the $\mathbb{S}_{d-1}$-valued process $(\hat{\Theta}_t)_{t\geq 0}$ solution to (4) and the $(0,\infty)$-valued process $(R_t)_{t\geq 0}$ solution to*

$$R_t = r_0 + \tilde{B}_t + \frac{d-1}{2}\int_0^t \frac{\mathrm{d}s}{R_s} - \frac{\beta}{2}\int_0^t \frac{\Gamma'(R_s)}{\Gamma(R_s)}\, \mathrm{d}s. \tag{5}$$

Setting $H_t = \int_0^t R_s^{-2}\, \mathrm{d}s$, $V_t = R_t \hat{\Theta}_{H_t}$ and $X_t = x_0 + \int_0^t V_s\, \mathrm{d}s$, the $(\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d$-valued process $(V_t, X_t)_{t\geq 0}$ is a weak solution to (2).

PROOF. For each $t \geq 0$, $\nu_t = \inf\{s > 0 : H_s > t\}$ is a $(\tilde{\mathcal{F}}_s)_{s\geq 0}$-stopping time, where $\tilde{\mathcal{F}}_s = \sigma(\tilde{B}_u : u \leq s)$, so that we can set $\mathcal{H}_t = \tilde{\mathcal{F}}_{\nu_t} \vee \sigma(\hat{B}_s : s \leq t)$. Now for each $t \geq 0$, $H_t = \inf\{s > 0 : \nu_s > t\}$ is a $(\mathcal{H}_s)_{s\geq 0}$-stopping time and we can define the filtration $\mathcal{G}_t = \mathcal{H}_{H_t}$. One classically checks that:

(a) $(\tilde{B}_t)_{t\geq 0}$ is a $(\mathcal{G}_t)_{t\geq 0}$-Brownian motion, because $(\tilde{B}_{\nu_t})_{t\geq 0}$ is a $(\mathcal{H}_t)_{t\geq 0}$-martingale, so that $(\tilde{B}_t = \tilde{B}_{\nu_{H_t}})_{t\geq 0}$ is a $(\mathcal{H}_{H_t} = \mathcal{G}_t)_{t\geq 0}$-martingale, and we have $\langle \tilde{B}\rangle_t = t$ because $(\tilde{B}_t)_{t\geq 0}$ is a Brownian motion;

(b) $\bar{B}_t = \int_0^{H_t} R_{\nu_s}\,\mathrm{d}\hat{B}_s$ is a $(\mathcal{G}_t)_{t\geq 0}$-Brownian motion with dimension d, since $(\bar{B}_{\nu_t})_{t\geq 0}$ is a $(\mathcal{H}_t)_{t\geq 0}$-martingale, so that $(\bar{B}_t)_{t\geq 0}$ is a $(\mathcal{G}_t)_{t\geq 0}$-martingale, and because $\langle \bar{B}\rangle_t = I_d\int_0^{H_t} R^2_{\nu_s}\,\mathrm{d}s = I_d t$;

(c) these two Brownian motions are independent because for all $i = 1,\dots,d$, $\langle \tilde{B}, \bar{B}^i\rangle \equiv 0$;

(d) for any continuous $(\mathcal{H}_t)_{t\geq 0}$-adapted $(S_t)_{t\geq 0}$, we have $\int_0^{H_t} S_s\,\mathrm{d}\hat{B}_s = \int_0^t R_s^{-1} S_{H_s}\,\mathrm{d}\bar{B}_s$. Indeed, it suffices to verify that for any $(\mathcal{G}_t)_{t\geq 0}$-martingale $(M_t)_{t\geq 0}$, $\langle \int_0^{H.} S_s\,\mathrm{d}\hat{B}_s, M\rangle_t = \int_0^t R_s^{-1} S_{H_s}\,\mathrm{d}\langle \bar{B}, M\rangle_s$. But $(N_t = M_{\nu_t})_{t\geq 0}$ is a $(\mathcal{H}_t)_{t\geq 0}$-martingale, and we have

$$\begin{aligned}\left\langle \int_0^{H.} S_s\,\mathrm{d}\hat{B}_s, M\right\rangle_t &= \left\langle \int_0^{H.} S_s\,\mathrm{d}\hat{B}_s, \int_0^{H.}\mathrm{d}N_s\right\rangle_t = \int_0^{H_t} S_s\,\mathrm{d}\langle \hat{B}, N\rangle_s \\ &= \int_0^t S_{H_u}\,\mathrm{d}\big(\langle \hat{B}, N\rangle_{H_u}\big) = \int_0^t S_{H_u} R_u^{-1}\,\mathrm{d}\langle \bar{B}, M\rangle_u,\end{aligned}$$

because $R_u\,\mathrm{d}(\langle \hat{B}, N\rangle_{H_u}) = \mathrm{d}\langle \bar{B}, M\rangle_u$. Indeed, we have $\langle \bar{B}, M\rangle_t = \langle \int_0^{H.} R_{\nu_s}\,\mathrm{d}\hat{B}_s, \int_0^{H.}\mathrm{d}N_s\rangle_t = \int_0^{H_t} R_{\nu_s}\,\mathrm{d}\langle \hat{B}, N\rangle_s = \int_0^t R_u\,\mathrm{d}(\langle \hat{B}, N\rangle_{H_u})$.

Next, since $\Theta_t = \hat{\Theta}_{H_t}$ is $(\mathcal{G}_t)_{t\geq 0}$-adapted, recalling (4) and that $|\hat{\Theta}_t| = 1$,

$$\Theta_t = \theta_0 + \int_0^t R_s^{-1}\pi_{\Theta_s^\perp}\,\mathrm{d}\bar{B}_s - \frac{d-1}{2}\int_0^t R_s^{-2}\Theta_s\,\mathrm{d}s - \frac{\beta}{2}\int_0^t R_s^{-2}\pi_{\Theta_s^\perp}\frac{\nabla\gamma(\Theta_s)}{\gamma(\Theta_s)}\,\mathrm{d}s. \tag{6}$$

Applying the Itô formula, we find, setting $V_t = R_t\Theta_t$ as in the statement,

$$\begin{aligned} V_t &= v_0 + \int_0^t \Theta_s\,\mathrm{d}\tilde{B}_s + \int_0^t \pi_{\Theta_s^\perp}\,\mathrm{d}\bar{B}_s + \int_0^t\left(\frac{d-1}{2R_s} - \frac{\beta}{2}\frac{\Gamma'(R_s)}{\Gamma(R_s)}\right)\Theta_s\,\mathrm{d}s \\ &\quad - \int_0^t\left(\frac{d-1}{2R_s}\Theta_s + \frac{\beta}{2}\pi_{\Theta_s^\perp}\frac{\nabla\gamma(\Theta_s)}{R_s\gamma(\Theta_s)}\right)\mathrm{d}s \\ &= v_0 + B_t - \frac{\beta}{2}\int_0^t\left(\frac{\Gamma'(R_s)}{\Gamma(R_s)}\Theta_s + \pi_{\Theta_s^\perp}\frac{\nabla\gamma(\Theta_s)}{R_s\gamma(\Theta_s)}\right)\mathrm{d}s,\end{aligned}$$

where we have set $B_t = \int_0^t \Theta_s\,\mathrm{d}\tilde{B}_s + \int_0^t \pi_{\Theta_s^\perp}\,\mathrm{d}\bar{B}_s$. This is a $\mathbb{R}^d$-valued $(\mathcal{G}_t)_{t\geq 0}$-martingale with quadratic variation matrix $\int_0^t[\Theta_s\Theta_s^* + \pi_{\Theta_s^\perp}]\,\mathrm{d}s = I_d t$ and thus a Brownian motion. It only remains to verify that, for $v = r\theta$ with $r > 0$ and $\theta \in \mathbb{S}_{d-1}$, one has

$$F(v) = \big[\Gamma(r)\big]^{-1}\Gamma'(r)\theta + \big[r\gamma(\theta)\big]^{-1}\pi_{\theta^\perp}\nabla\gamma(\theta), \tag{7}$$

which follows from $F = \nabla[\log U]$ with $U(v) = \Gamma(|v|)\gamma(v/|v|)$. $\square$

We next build the radial process using classical tools, namely speed measures and scale functions; see Revuz–Yor [33], Chapter VII, Paragraph 3.

NOTATION 9. Fix $\beta > d-2$. Let $h(r) = (\beta + 2 - d)\int_{r_0}^r u^{1-d}[\Gamma(u)]^\beta\,\mathrm{d}u$, which is an increasing bijection from $(0,\infty)$ into $\mathbb{R}$. We denote by $h^{-1}:\mathbb{R}\to(0,\infty)$ its inverse function, for which $h^{-1}(0) = r_0$. We also introduce $\sigma(w) = h'(h^{-1}(w))$ and $\psi(w) = [\sigma(w)h^{-1}(w)]^2$, both from $\mathbb{R}$ to $(0,\infty)$

In the following statement, we introduce a parameter $\epsilon \in (0,1)$, which may seem artificial at this stage, but this will be crucial to work by coupling.

LEMMA 10. *Fix $\beta > d-2$ and consider a Brownian motion $(W_t)_{t\geq 0}$. For $\epsilon \in (0,1)$ and $a_\epsilon > 0$, introduce $A_t^\epsilon = \epsilon a_\epsilon^{-2}\int_0^t[\sigma(W_s/a_\epsilon)]^{-2}\,\mathrm{d}s$ and its inverse ρ_t^ϵ. Set $R_t^\epsilon = \sqrt{\epsilon}h^{-1}(W_{\rho_t^\epsilon}/a_\epsilon)$. For each $\epsilon \in (0,1)$, the process $(S_t^\epsilon = \epsilon^{-1/2}R_{\epsilon t}^\epsilon)_{t\geq 0}$ is $(0,\infty)$-valued and is a weak solution to* (5).

This can be rephrased as follows: $(R_t^\epsilon)_{t\geq 0}$ has the same law as $(\sqrt{\epsilon}R_{t/\epsilon})_{t\geq 0}$, with $(R_t)_{t\geq 0}$ solving (5). Of course, $(\sqrt{\epsilon}R_{t/\epsilon})_{t\geq 0}$ is a natural object when studying the large time behavior of $(R_t)_{t\geq 0}$.

PROOF OF LEMMA 10. First, $(S_t^\epsilon)_{t\geq 0}$ is $(0,\infty)$-valued by definition. Next, there classically exists a Brownian motion $(\bar{B}_t)_{t\geq 0}$ (see, e.g., Revuz–Yor [33], Proposition 1.13, p. 373) such that $Y_t^\epsilon = W_{\rho_t^\epsilon}$ solves $Y_t^\epsilon = \epsilon^{-1/2}a_\epsilon \int_0^t \sigma(Y_s^\epsilon/a_\epsilon)\,\mathrm{d}\bar{B}_s$, whence $Z_t^\epsilon = a_\epsilon^{-1}Y_t^\epsilon = \epsilon^{-1/2}\int_0^t \sigma(Z_s^\epsilon)\,\mathrm{d}\bar{B}_s$. Thus,

$$
\begin{aligned}
R_t^\epsilon &= \sqrt{\epsilon}h^{-1}(Z_t^\epsilon)\\
&= \sqrt{\epsilon}h^{-1}(0) + \int_0^t (h^{-1})'(Z_s^\epsilon)\sigma(Z_s^\epsilon)\,\mathrm{d}\bar{B}_s \frac{1}{2\sqrt{\epsilon}}\int_0^t (h^{-1})''(Z_s^\epsilon)\sigma^2(Z_s^\epsilon)\,\mathrm{d}s.
\end{aligned}
$$

But $h^{-1}(0) = r_0$, $(h^{-1})'(z)\sigma(z) = 1$ and

$$
(h^{-1})''(z)\sigma^2(z) = -\sigma'(z) = -h''(h^{-1}(z))/h'(h^{-1}(z))\frac{d-1}{h^{-1}(z)} - \beta\frac{\Gamma'(h^{-1}(z))}{\Gamma(h^{-1}(z))}
$$

because $h''(u)/h'(u) = [\log(u^{1-d}\Gamma^\beta(u))]' = (1-d)/u + \beta\Gamma'(u)/\Gamma(u)$. Hence,

$$
\begin{aligned}
R_t^\epsilon &= \sqrt{\epsilon}r_0 + \bar{B}_t + \frac{d-1}{2\sqrt{\epsilon}}\int_0^t \frac{1}{h^{-1}(Z_s^\epsilon)}\,\mathrm{d}s - \frac{\beta}{2\sqrt{\epsilon}}\int_0^t \frac{\Gamma'(h^{-1}(Z_s^\epsilon))}{\Gamma(h^{-1}(Z_s^\epsilon))}\,\mathrm{d}s\\
&= \sqrt{\epsilon}r_0 + \bar{B}_t + \frac{d-1}{2}\int_0^t \frac{\mathrm{d}s}{R_s^\epsilon} - \frac{\beta}{2}\int_0^t \frac{\Gamma'(R_s^\epsilon/\sqrt{\epsilon})}{\sqrt{\epsilon}\Gamma(R_s^\epsilon/\sqrt{\epsilon})}\,\mathrm{d}s.
\end{aligned}
$$

Hence, $S_t^\epsilon = \epsilon^{-1/2}R_{\epsilon t}^\epsilon$ solves (5) with the Brownian motion $\tilde{B}_t = \epsilon^{-1/2}\bar{B}_{\epsilon t}$. □

Finally, we can give the proof of Proposition 2.

PROOF OF PROPOSITION 2. The global weak existence of a $\mathbb{R}^d \setminus \{0\}$-valued solution proved in Lemma 8, together with the local strong existence and pathwise uniqueness (until the velocity process reaches 0 or explodes to infinity), which follows from the fact that the drift F is of class C^1 on $\mathbb{R}^d \setminus \{0\}$, imply the global existence and pathwise uniqueness for (2). □

5. The stable regime. Here we prove Theorem 4(c)–(d)–(e)–(f). We fix $\beta \in [d, 4+d)$ and set $\alpha = (\beta + 2 - d)/3$. We introduce some notation that will be used during the whole section. We recall Notation 9. We fix $\epsilon \in (0,1)$ and introduce

$$
a_\epsilon = \kappa\epsilon \quad \text{if } \beta \in (d, 4+d) \quad \text{and} \quad a_\epsilon = \frac{\epsilon|\log\epsilon|}{4} \quad \text{if } \epsilon = d,
$$

where $\kappa = \int_{\mathbb{R}}[\sigma(w)]^{-2}\,\mathrm{d}w < \infty$ when $\beta > d$; see Lemma 42(i). We consider a one-dimensional Brownian motion $(W_t)_{t\geq 0}$, set $A_t^\epsilon = \epsilon a_\epsilon^{-2}\int_0^t[\sigma(W_s/a_\epsilon)]^{-2}\,\mathrm{d}s$, introduce its inverse ρ_t^ϵ and put $R_t^\epsilon = \sqrt{\epsilon}h^{-1}(W_{\rho_t^\epsilon}/a_\epsilon)$. We know from Lemma 10 that $S_t^\epsilon = \epsilon^{-1/2}R_{\epsilon t}^\epsilon = h^{-1}(W_{\rho_{\epsilon t}^\epsilon}/a_\epsilon)$ solves (5). We also consider the solution $(\hat{\Theta}_t)_{t\geq 0}$ of (4), independent of $(W_t)_{t\geq 0}$.

LEMMA 11. *For each* $\epsilon \in (0,1)$, $(X_{t/\epsilon} - x_0)_{t\geq 0} \overset{d}{=} (\tilde{X}_t^\epsilon)_{t\geq 0}$, *where*

$$
\tilde{X}_t^\epsilon = \frac{1}{a_\epsilon^2}\int_0^{\rho_t^\epsilon} \frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon}}{[\sigma(W_u/a_\epsilon)]^2}\,\mathrm{d}u \quad \textit{where } T_t^\epsilon = \frac{1}{a_\epsilon^2}\int_0^t \frac{\mathrm{d}s}{\psi(W_s/a_\epsilon)}. \tag{8}
$$

Furthermore, for any $m \in \mathbb{R}^d$, *any* $t \geq 0$, *it holds that*

$$\tilde{X}_{t/\epsilon} - mt/\epsilon = \frac{1}{a_\epsilon^2} \int_0^{\rho_t^\epsilon} \frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon} - m}{[\sigma(W_u/a_\epsilon)]^2} \, \mathrm{d}u. \tag{9}$$

PROOF. We know from Lemma 8 that, setting $H_t^\epsilon = \int_0^t [S_s^\epsilon]^{-2} \, \mathrm{d}s$, it holds that $(S_t^\epsilon \hat{\Theta}_{H_t^\epsilon})_{t\geq 0} \stackrel{d}{=} (V_t)_{t\geq 0}$. Since $X_t - x_0 = \int_0^t V_s \, \mathrm{d}s$, we conclude that $(X_{t/\epsilon} - x_0)_{t\geq 0} \stackrel{d}{=} (\tilde{X}_t^\epsilon)_{t\geq 0}$, where $\tilde{X}_t^\epsilon = \int_0^{t/\epsilon} S_s^\epsilon \hat{\Theta}_{H_s^\epsilon} \, \mathrm{d}s = \int_0^{t/\epsilon} h^{-1}(W_{\rho_{\epsilon s}^\epsilon}/a_\epsilon)\hat{\Theta}_{H_s^\epsilon} \, \mathrm{d}s$. Performing the change of variables $u = \rho_{\epsilon s}^\epsilon$, that is, $s = \epsilon^{-1} A_u^\epsilon$, so $\mathrm{d}s = a_\epsilon^{-2}[\sigma(W_u/a_\epsilon)]^{-2} \, \mathrm{d}u$, we find

$$\tilde{X}_t^\epsilon = \frac{1}{a_\epsilon^2} \int_0^{\rho_t^\epsilon} \frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{H^\epsilon_{\epsilon^{-1}A_u^\epsilon}}}{[\sigma(W_u/a_\epsilon)]^2} \, \mathrm{d}u.$$

Using the same change of variables, one verifies that

$$\begin{aligned} H^\epsilon_{\epsilon^{-1}A_t^\epsilon} &= \int_0^{\epsilon^{-1}A_t^\epsilon} \frac{\mathrm{d}s}{[h^{-1}(W_{\rho_{\epsilon s}^\epsilon}/a_\epsilon)]^2} \\ &= \frac{1}{a_\epsilon^2} \int_0^t \frac{\mathrm{d}u}{[\sigma(W_u/a_\epsilon)]^2 [h^{-1}(W_u/a_\epsilon)]^2} = \frac{1}{a_\epsilon^2} \int_0^t \frac{\mathrm{d}u}{\psi(W_u/a_\epsilon)}. \end{aligned}$$

The last claim follows from $a_\epsilon^{-2} \int_0^{\rho_t^\epsilon} [\sigma(W_u/a_\epsilon)]^{-2} \, \mathrm{d}u = \epsilon^{-1} A^\epsilon_{\rho_t^\epsilon} = \epsilon^{-1} t$. □

We first study the convergence of the time-change.

LEMMA 12. (i) *For all* $T > 0$, *a.s.*, $\sup_{[0,T]} |A_t^\epsilon - L_t^0| \to 0$ *as* $\epsilon \to 0$, *where* $(L_t^0)_{t\geq 0}$ *is the local time at* 0 *of* $(W_t)_{t\geq 0}$.

(ii) *For all* $t \geq 0$, *a.s.*, $\rho_t^\epsilon \to \tau_t = \inf\{u \geq 0 : L_u^0 > t\}$, *the generalized inverse of* $(L_s^0)_{s\geq 0}$.

PROOF. Point (ii) follows from point (i) by Lemma 41 and since $\mathbb{P}(\tau_t \neq \tau_{t-}) = 0$. Concerning point (i), we first assume that $\beta > d$. Since $a_\epsilon = \kappa\epsilon$, by the occupation times formula (see Revuz–Yor [33], Corollary 1.6, p. 224)

$$A_t^\epsilon = \frac{\epsilon}{a_\epsilon^2} \int_0^t \frac{\mathrm{d}s}{[\sigma(W_s/a_\epsilon)]^2} = \frac{1}{\kappa^2\epsilon} \int_{\mathbb{R}} \frac{L_t^x \, \mathrm{d}x}{\sigma^2(x/(\kappa\epsilon))} = \int_{\mathbb{R}} \frac{L_t^{\kappa\epsilon y} \, \mathrm{d}y}{\kappa\sigma^2(y)},$$

where $(L_t^x)_{t\geq 0}$ is the local time of $(W_t)_{t\geq 0}$ at x. Since $\kappa = \int_{\mathbb{R}} [\sigma(w)]^{-2} \, \mathrm{d}w$, which is finite by Lemma 42(i), we write

$$|A_t^\epsilon - L_t^0| \leq \int_{\mathbb{R}} \frac{|L_t^{\kappa\epsilon y} - L_t^0| \, \mathrm{d}y}{\kappa\sigma^2(y)}.$$

This a.s. tends uniformly (on $[0, T]$) to 0 as $\epsilon \to 0$ by dominated convergence, since $\sup_{[0,T]} |L_t^{\kappa\epsilon y} - L_t^0|$ a.s. tends to 0 for each fixed y by [33], Corollary 1.8, page 226, and since $\sup_{[0,T]\times\mathbb{R}} L_t^x < \infty$ a.s.

We next treat the case where $\beta = d$, which is more complicated. We recall that $a_\epsilon = \epsilon|\log\epsilon|/4$. By Lemma 42(vi)–(vii), we know that $[\sigma(w)]^{-2} \leq C(1 + |w|)^{-1}$ and that

$$\int_{-x}^x \frac{\mathrm{d}w}{[\sigma(w)]^2} \stackrel{x\to\infty}{\sim} \frac{\log x}{4}. \tag{10}$$

We fix $\delta > 0$ and write $A_t^\epsilon = J_t^{\epsilon,\delta} + Q_t^{\epsilon,\delta}$, where

$$J_t^{\epsilon,\delta} = \frac{\epsilon}{a_\epsilon^2} \int_0^t \frac{\mathbf{1}_{\{|W_s|>\delta\}} \, \mathrm{d}s}{[\sigma(W_s/a_\epsilon)]^2} \quad \text{and} \quad Q_t^{\epsilon,\delta} = \frac{\epsilon}{a_\epsilon^2} \int_0^t \frac{\mathbf{1}_{\{|W_s|\leq\delta\}} \, \mathrm{d}s}{[\sigma(W_s/a_\epsilon)]^2}.$$

One checks that $\sup_{[0,T]} J_t^{\epsilon,\delta} \le CT\epsilon/[a_\epsilon^2(1+\delta/a_\epsilon)] \le CT\epsilon/(\delta a_\epsilon)$, which tends to 0 as $\epsilon \to 0$. We next use the occupation times formula (see Revuz–Yor [33], Corollary 1.6, p. 224) to write

$$Q_t^{\epsilon,\delta} = \frac{\epsilon}{a_\epsilon^2}\int_{-\delta}^{\delta} \frac{L_t^x \,\mathrm{d}x}{[\sigma(x/a_\epsilon)]^2} = \frac{\epsilon}{a_\epsilon^2}\int_{-\delta}^{\delta} \frac{\mathrm{d}x}{[\sigma(x/a_\epsilon)]^2} L_t^0 + \frac{\epsilon}{a_\epsilon^2}\int_{-\delta}^{\delta} \frac{(L_t^x - L_t^0)\,\mathrm{d}x}{[\sigma(x/a_\epsilon)]^2}$$
$$= r_{\epsilon,\delta} L_t^0 + R_t^{\epsilon,\delta},$$

the last identity standing for a definition. By a substitution and (10),

$$r_{\epsilon,\delta} = \frac{\epsilon}{a_\epsilon}\int_{-\delta/a_\epsilon}^{\delta/a_\epsilon} \frac{\mathrm{d}y}{[\sigma(y)]^2} \overset{\epsilon\to 0}{\sim} \frac{\epsilon \log(\delta/a_\epsilon)}{4a_\epsilon} \longrightarrow 1 \quad \text{as } \epsilon \to 0.$$

All this proves that a.s., for all $\delta > 0$,

$$\limsup_{\epsilon\to 0} \sup_{[0,T]} |A_t^\epsilon - L_t^0| \le \limsup_{\epsilon\to 0} \sup_{[0,T]} |R_t^{\epsilon,\delta}|.$$

But we have $|R_t^{\epsilon,\delta}| \le r_{\epsilon,\delta} \times \sup_{[-\delta,\delta]} |L_t^x - L_t^0|$, so that $\limsup_{\epsilon\to 0}\sup_{[0,T]} |A_t^\epsilon - L_t^0| \le \sup_{[0,T]\times[-\delta,\delta]} |L_t^x - L_t^0|$ a.s., and it suffices to let $\delta \to 0$, using Revuz–Yor [33], Corollary 1.8, page 226, to complete the proof. □

We next proceed to three first approximations: in the formula (9), we show that one may replace ρ_t^ϵ by its limiting value τ_t, that the negative values of W have a negligible influence, and that we may introduce a cutoff that will allow us to neglect the small jumps of the limiting stable process. All this is rather tedious in the infinite variation case $\alpha \in [1,2)$. We recall that $m'_\beta > 0$, $M_\beta \in \mathbb{R}^d$ and $m_\beta = m'_\beta M_\beta$ were defined in Section 3.

NOTATION 13. (i) If $\beta \in [d, 1+d)$, we set, for $\delta \in (0,1]$ and $\epsilon \in (0,1)$,

$$Z_t^{\epsilon,\delta} = a_\epsilon^{1/\alpha-2}\int_0^{\tau_t} \frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon}}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u>\delta\}}\,\mathrm{d}u \quad \text{and} \quad U_t^{\epsilon,\delta} = a_\epsilon^{1/\alpha}\tilde{X}_t^\epsilon - Z_t^{\epsilon,\delta}.$$

(ii) If $\beta = 1+d$, we put

$$\zeta_\epsilon = \frac{\int_{-\infty}^1 h^{-1}(w/a_\epsilon)[\sigma(w/a_\epsilon)]^{-2}\,\mathrm{d}w}{\int_{-\infty}^1 [\sigma(w/a_\epsilon)]^{-2}\,\mathrm{d}w} = \frac{\int_{-\infty}^{1/a_\epsilon} h^{-1}(w)[\sigma(w)]^{-2}\,\mathrm{d}w}{\int_{-\infty}^{1/a_\epsilon} [\sigma(w)]^{-2}\,\mathrm{d}w}$$

(so that $\kappa_{\epsilon,1}$ defined below vanishes) and we set, for $\delta \in (0,1]$ and $\epsilon \in (0,1)$,

$$Z_t^{\epsilon,\delta} = \frac{1}{a_\epsilon}\int_0^{\tau_t} \frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon} - \zeta_\epsilon M_\beta}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u>\delta\}}\,\mathrm{d}u,$$
$$\kappa_{\epsilon,\delta} = \frac{1}{a_\epsilon}\int_{-\infty}^{\delta} \frac{h^{-1}(w/a_\epsilon) - \zeta_\epsilon}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w,$$
$$U_t^{\epsilon,\delta} = a_\epsilon[\tilde{X}_t^\epsilon - \zeta_\epsilon M_\beta t/\epsilon] - Z_t^{\epsilon,\delta} - \kappa_{\epsilon,\delta} M_\beta t.$$

(iii) If $\beta \in (1+d, 4+d)$ we introduce, for $\delta \in (0,1]$ and $\epsilon \in (0,1)$,

$$Z_t^{\epsilon,\delta} = a_\epsilon^{1/\alpha-2}\int_0^{\tau_t} \frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon} - m_\beta}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u>\delta\}}\,\mathrm{d}u$$
$$\kappa_{\epsilon,\delta} = a_\epsilon^{1/\alpha-2}\int_{-\infty}^{\delta} \frac{h^{-1}(w/a_\epsilon) - m'_\beta}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w,$$
$$U_t^{\epsilon,\delta} = a_\epsilon^{1/\alpha}[\tilde{X}_t^\epsilon - m_\beta t/\epsilon] - Z_t^{\epsilon,\delta} - \kappa_{\epsilon,\delta} M_\beta t.$$

Observe that ζ_ϵ and $\kappa_{\delta,\epsilon}$ are well defined by Lemma 42(i)–(viii).

LEMMA 14. *For all $\beta \in [d, 4+d)$, all $t \geq 0$, all $\eta > 0$,*

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\mathbb{P}\big[\big|U_t^{\epsilon,\delta}\big| > \eta\big] = 0.$$

PROOF. *Case (i)*: $\beta \in [d, 1+d)$, whence $\alpha \in [2/3, 1)$. Recalling (8),

$$U_t^{\epsilon,\delta} = a_\epsilon^{\frac{1}{\alpha}-2}\int_0^{\tau_t}\frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon}}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u\leq\delta\}}\,\mathrm{d}u + a_\epsilon^{\frac{1}{\alpha}-2}\int_{\tau_t}^{\rho_t^\epsilon}\frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon}}{[\sigma(W_u/a_\epsilon)]^2}\,\mathrm{d}u.$$

Since $h^{-1}(w)[\sigma(w)]^{-2} \leq C(1+w)^{1/\alpha-2}\mathbf{1}_{\{w\geq 0\}} + C(1+|w|)^{-2}\mathbf{1}_{\{w<0\}}$ by Lemma 42(viii),

$$\begin{aligned}&a_\epsilon^{1/\alpha-2}h^{-1}(w/a_\epsilon)\big[\sigma(w/a_\epsilon)\big]^{-2}\\&\quad\leq Cw^{1/\alpha-2}\mathbf{1}_{\{w\geq 0\}} + C|w|^{1/\alpha-2}\big(1+|w|/a_\epsilon\big)^{-1/\alpha}\mathbf{1}_{\{w<0\}} \leq C|w|^{1/\alpha-2},\end{aligned}$$

and thus

$$\begin{aligned}\big|U_t^{\epsilon,\delta}\big| &\leq C\int_0^{\tau_t} W_u^{1/\alpha-2}\mathbf{1}_{\{0\leq W_u\leq\delta\}}\,\mathrm{d}u\\&\quad + C\int_0^{\tau_t}|W_u|^{1/\alpha-2}\big(1+|W_u|/a_\epsilon\big)^{-1/\alpha}\mathbf{1}_{\{W_u<0\}}\,\mathrm{d}u + C\int_{\tau_t}^{\rho_t^\epsilon}|W_u|^{1/\alpha-2}\,\mathrm{d}u.\end{aligned}$$

But $1/\alpha - 2 > -1$, so that the integral $\int_0^T |W_u|^{1/\alpha-2}\,\mathrm{d}u$ is a.s. finite for all $T > 0$ (because its expectation is finite). One concludes by dominated convergence, using that $\rho_t^\epsilon \to \tau_t$ a.s. for each $t \geq 0$ fixed by Lemma 12(ii), that a.s.,

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\big|U_t^{\epsilon,\delta}\big| \leq C\lim_{\delta\to 0}\int_0^{\tau_t} W_u^{1/\alpha-2}\mathbf{1}_{\{0\leq W_u\leq\delta\}}\,\mathrm{d}u = 0.$$

Case (iii): $\beta \in (1+d, 4+d)$. This is much more complicated. By (9),

$$\begin{aligned}U_t^{\epsilon,\delta} &= a_\epsilon^{1/\alpha-2}\int_0^{\tau_t}\frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon} - m_\beta}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u\leq\delta\}}\,\mathrm{d}u - \kappa_{\epsilon,\delta}M_\beta t\\&\quad + a_\epsilon^{1/\alpha-2}\int_{\tau_t}^{\rho_t^\epsilon}\frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon} - m_\beta}{[\sigma(W_u/a_\epsilon)]^2}\,\mathrm{d}u\\&= K_{\tau_t}^{\epsilon,\delta} + M_\beta I_{\tau_t}^{\epsilon,\delta} + \big[K_{\rho_t^\epsilon}^{\epsilon,\infty} - K_{\tau_t}^{\epsilon,\infty}\big] + M_\beta\big[I_{\rho_t^\epsilon}^{\epsilon,\infty} - I_{\tau_t}^{\epsilon,\infty}\big],\end{aligned}$$

where we have set (extending the definition of $\kappa_{\epsilon,\delta}$ to all values of $\delta \in (0,\infty]$),

$$K_t^{\epsilon,\delta} = a_\epsilon^{1/\alpha-2}\int_0^t\frac{h^{-1}(W_u/a_\epsilon)[\hat{\Theta}_{T_u^\epsilon} - M_\beta]}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u\leq\delta\}}\,\mathrm{d}u,$$

$$I_t^{\epsilon,\delta} = a_\epsilon^{1/\alpha-2}\int_0^t\frac{h^{-1}(W_u/a_\epsilon) - m'_\beta}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u\leq\delta\}}\,\mathrm{d}u - \kappa_{\epsilon,\delta}L_t^0.$$

We used that $m_\beta = m'_\beta M_\beta$, that $L_{\tau_t}^0 = t$ and that by Lemma 42(ii),

$$\kappa_{\epsilon,\infty} = a_\epsilon^{\frac{1}{\alpha}-2}\int_{-\infty}^{\infty}\frac{h^{-1}(w/a_\epsilon) - m'_\beta}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w = a_\epsilon^{\frac{1}{\alpha}-1}\int_{-\infty}^{\infty}\frac{h^{-1}(y) - m'_\beta}{[\sigma(y)]^2}\,\mathrm{d}y = 0. \tag{11}$$

We first treat I. By the occupation times formula (see Revuz–Yor [33], Corollary 1.6, p. 224) and by definition of $\kappa_{\epsilon,\delta}$,

$$I_t^{\epsilon,\delta} = a_\epsilon^{1/\alpha-2}\int_{-\infty}^{\delta}\frac{h^{-1}(w/a_\epsilon) - m'_\beta}{[\sigma(w/a_\epsilon)]^2}\big(L_t^w - L_t^0\big)\,\mathrm{d}w.$$

For each $\delta \in (0, \infty]$, each $T \geq 0$, we a.s. have $\lim_{\epsilon\to 0} \sup_{[0,T]} |I_t^{\epsilon,\delta} - I_t^\delta| = 0$, where we have set $I_t^\delta = (\beta + 2 - d)^{-2} \int_0^\delta w^{1/\alpha-2}(L_t^w - L_t^0)\,\mathrm{d}w$. Indeed, this follows from dominated convergence, because:

- $a_\epsilon^{1/\alpha-2}|h^{-1}(w/a_\epsilon) - m'_\beta|[\sigma(w/a_\epsilon)]^{-2} \leq C|w|^{1/\alpha-2}$ by Lemma 42(viii),
- $\lim_{\epsilon\to 0} a_\epsilon^{1/\alpha-2}[h^{-1}(w/a_\epsilon) - m'_\beta][\sigma(w/a_\epsilon)]^{-2} = (\beta + 2 - d)^{-2} w^{1/\alpha-2}\mathbf{1}_{\{w\geq 0\}}$, see Lemma 42(ix),
- a.s., $\int_{\mathbb{R}} |w|^{1/\alpha-2} \sup_{[0,T]} |L_t^w - L_t^0|\,\mathrm{d}w < \infty$, since $1/\alpha - 2 \in (-3/2, -1)$ and since $\sup_{[0,T]} |L_t^w - L_t^0|$ is a.s. bounded and almost 1/2-Holdër continuous (as a function of w), see [33], Corollary 1.8, page 226.

We conclude that $\lim_{\delta\to 0} \limsup_{\epsilon\to 0} |I_{\tau_t}^{\epsilon,\delta}| = \lim_{\delta\to 0} |I_{\tau_t}^\delta| = 0$ a.s. and, using that $\rho_t^\epsilon \to \tau_t$ a.s. by Lemma 12(ii) (for each fixed $t \geq 0$) and that $t \to I_t^\infty$ is a.s. continuous on $[0, \infty)$, that $\lim_{\delta\to 0} \limsup_{\epsilon\to 0} |I_{\rho_t^\epsilon}^{\epsilon,\infty} - I_{\tau_t}^{\epsilon,\infty}| = 0$ a.s. All this proves that a.s.,

$$\lim_{\delta\to 0} \limsup_{\epsilon\to 0} \big[|I_{\tau_t}^{\epsilon,\delta}| + |I_{\rho_t^\epsilon}^{\epsilon,\infty} - I_{\tau_t}^{\epsilon,\infty}|\big] = 0.$$

We next treat K. We mention at once that all the computations below concerning K are also valid when $\beta = 1 + d$, that is, $\alpha = 1$. We introduce $\mathcal{W} = \sigma(W_t, t \geq 0)$. Assume for a moment that there is $C > 0$ such that for any $\delta \in (0, \infty]$, any $\epsilon \in (0, 1)$, any $0 \leq s \leq t$, a.s.,

$$\mathbb{E}\big[\big(K_t^{\epsilon,\delta} - K_s^{\epsilon,\delta}\big)^2|\mathcal{W}\big] \leq C \int_s^t |W_u|^{\frac{2}{\alpha}-2}\big[\mathbf{1}_{\{0\leq W_u\leq\delta\}} + \big(1 + |W_u|/\epsilon\big)^{-\frac{1}{\alpha}}\big]\,\mathrm{d}u. \tag{12}$$

Then, τ_t and ρ_t^ϵ being $\mathcal{W}$-measurable, we will deduce that

$$\begin{aligned}&\mathbb{E}\big[\big(K_{\tau_t}^{\epsilon,\infty} - K_{\rho_t^\epsilon}^{\epsilon,\infty}\big)^2 + \big(K_{\tau_t}^{\epsilon,\delta}\big)^2|\mathcal{W}\big]\\ &\quad\leq C\left|\int_{\rho_t^\epsilon}^{\tau_t} |W_u|^{2/\alpha-2}\,\mathrm{d}u\right| + C\int_0^{\tau_t} |W_u|^{2/\alpha-2}\big[\mathbf{1}_{\{0\leq W_u\leq\delta\}} + \big(1 + |W_u|/\epsilon\big)^{-1/\alpha}\big]\,\mathrm{d}u.\end{aligned}$$

Since $\int_0^T |W_u|^{2/\alpha-2}\,\mathrm{d}u < \infty$ a.s. for all $T > 0$ because $2/\alpha - 2 > -1$ and since $\rho_t^\epsilon \to \tau_t$ a.s. (for $t \geq 0$ fixed) by Lemma 12(ii), conclude, by dominated convergence that a.s.,

$$\lim_{\delta\to 0} \limsup_{\epsilon\to 0} \mathbb{E}\big[\big(K_{\tau_t}^{\epsilon,\delta}\big)^2 + \big(K_{\tau_t}^{\epsilon,\infty} - K_{\rho_t^\epsilon}^{\epsilon,\infty}\big)^2|\mathcal{W}\big] = 0,$$

from which the convergence $\lim_{\delta\to 0} \limsup_{\epsilon\to 0} |K_{\tau_t}^{\epsilon,\delta}| + |K_{\tau_t}^{\epsilon,\infty} - K_{\rho_t^\epsilon}^{\epsilon,\infty}| = 0$ in probability follows.

We now check (12), starting from

$$\begin{aligned}\big(K_t^{\epsilon,\delta} - K_s^{\epsilon,\delta}\big)^2 = a_\epsilon^{2/\alpha-4} \int_s^t \int_s^t \frac{h^{-1}(W_a/a_\epsilon)}{[\sigma(W_a/a_\epsilon)]^2} \frac{h^{-1}(W_b/a_\epsilon)}{[\sigma(W_b/a_\epsilon)]^2} \mathbf{1}_{\{W_a\leq\delta\}}\mathbf{1}_{\{W_b\leq\delta\}}\\ \times (\hat{\Theta}_{T_a^\epsilon} - M_\beta)(\hat{\Theta}_{T_b^\epsilon} - M_\beta)\,\mathrm{d}a\,\mathrm{d}b.\end{aligned}$$

Since $(T_t^\epsilon)_{t\geq 0}$ is $\mathcal{W}$-measurable, since $(\hat{\Theta}_t)_{t\geq 0}$ is independent of $\mathcal{W}$, and since $M_\beta = \int_{\mathbb{S}_{d-1}} \theta\nu_\beta(\mathrm{d}\theta)$, Lemma 38(ii) (and the Markov property) tells us that there are $C > 0$ and $\lambda > 0$ such that

$$\big|\mathbb{E}\big([\hat{\Theta}_{T_a^\epsilon} - M_\beta][\hat{\Theta}_{T_b^\epsilon} - M_\beta]|\mathcal{W}\big)\big| \leq C\exp\big(-\lambda\big|T_b^\epsilon - T_a^\epsilon\big|\big).$$

By Lemma 42(viii) and since $a_\epsilon = \kappa\epsilon$, we have

$$a_\epsilon^{1/\alpha-2}\frac{h^{-1}(w/a_\epsilon)}{[\sigma(w/a_\epsilon)]^2} \leq C\big(\epsilon + |w|\big)^{1/\alpha-2}\big[\mathbf{1}_{\{w\geq 0\}} + \big(1 + |w|/\epsilon\big)^{-1/\alpha}\mathbf{1}_{\{w<0\}}\big],$$

whence

$$\mathbb{E}\big[\big(K_t^{\epsilon,\delta}-K_s^{\epsilon,\delta}\big)^2|\mathcal{W}\big]\le C\int_s^t\int_s^t\big(\epsilon+|W_a|\big)^{1/\alpha-2}\big(\epsilon+|W_b|\big)^{1/\alpha-2}$$
$$\times\big[\mathbf{1}_{\{0<W_a\le\delta\}}+\big(1+|W_a|/\epsilon\big)^{-1/\alpha}\big]\big[\mathbf{1}_{\{0\le W_b\le\delta\}}+\big(1+|W_b|/\epsilon\big)^{-1/\alpha}\big]$$
$$\times\exp\big(-\lambda\big|T_a^\epsilon-T_b^\epsilon\big|\big)\,\mathrm{d}a\,\mathrm{d}b.$$

Next, we observe that, since $a_\epsilon^2\psi(w/a_\epsilon)\le C(\epsilon+|w|)^2$ by Lemma 42(iv),

$$\lambda\big|T_a^\epsilon-T_b^\epsilon\big|=\lambda\left|\frac{1}{a_\epsilon^2}\int_a^b\frac{\mathrm{d}s}{\psi(W_s/a_\epsilon)}\,\mathrm{d}s\right|\ge c\left|\int_a^b\big(\epsilon+|W_s|\big)^{-2}\,\mathrm{d}s\right|$$

for some $c>0$. Using that $(xy)^{1/\alpha}\le x^{2/\alpha}+y^{2/\alpha}$ and a symmetry argument, we conclude that

$$\mathbb{E}\big[\big(K_t^{\epsilon,\delta}-K_s^{\epsilon,\delta}\big)^2|\mathcal{W}\big]\le C\int_s^t\big(\epsilon+|W_b|\big)^{2/\alpha-2}\big[\mathbf{1}_{\{0\le W_b\le\delta\}}+\big(1+|W_b|/\epsilon\big)^{-1/\alpha}\big]$$
$$\times\int_s^t\big(\epsilon+|W_a|\big)^{-2}\exp\left(-c\left|\int_a^b\big(\epsilon+|W_s|\big)^{-2}\,\mathrm{d}s\right|\right)\mathrm{d}a\,\mathrm{d}b$$
$$\le C\int_s^t|W_b|^{2/\alpha-2}\big[\mathbf{1}_{\{0\le W_b\le\delta\}}+\big(1+|W_b|/\epsilon\big)^{-1/\alpha}\big]\,\mathrm{d}b$$

as desired. We finally used that for all $b\in[0,t]$, all continuous $\varphi:\mathbb{R}_+\to\mathbb{R}_+$,

$$\int_0^t\varphi(a)\exp\left(-\left|\int_a^b\varphi(s)\,\mathrm{d}s\right|\right)\mathrm{d}a\le 2. \tag{13}$$

Case (ii): $\beta=1+d$. Applying (9) with $m=\zeta_\epsilon M_\beta$, we see that

$$U_t^{\epsilon,\delta}=\frac{1}{a_\epsilon}\int_0^{\tau_t}\frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon}-\zeta_\epsilon M_\beta}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u\le\delta\}}\,\mathrm{d}u-\kappa_{\epsilon,\delta}M_\beta t$$
$$+\frac{1}{a_\epsilon}\int_{\tau_t}^{\rho_t^\epsilon}\frac{h^{-1}(W_u/a_\epsilon)\hat{\Theta}_{T_u^\epsilon}-\zeta_\epsilon M_\beta}{[\sigma(W_u/a_\epsilon)]^2}\,\mathrm{d}u$$
$$=K_{\tau_t}^{\epsilon,\delta}+\big[K_{\rho_t^\epsilon}^{\epsilon,\infty}-K_{\tau_t}^{\epsilon,\infty}\big]+M_\beta I_{\tau_t}^{\epsilon,\delta}+M_\beta\big[I_{\rho_t^\epsilon}^{\epsilon,\infty}-I_{\tau_t}^{\epsilon,\infty}\big],$$

where we have set, for $\delta\in(0,1)\cup\{\infty\}$, with the convention that $\kappa_{\epsilon,\infty}=0$,

$$K_t^{\epsilon,\delta}=\frac{1}{a_\epsilon}\int_0^t\frac{h^{-1}(W_u/a_\epsilon)[\hat{\Theta}_{T_u^\epsilon}-M_\beta]}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u\le\delta\}}\,\mathrm{d}u,$$
$$I_t^{\epsilon,\delta}=\frac{1}{a_\epsilon}\int_0^t\frac{h^{-1}(W_u/a_\epsilon)-\zeta_\epsilon}{[\sigma(W_u/a_\epsilon)]^2}\mathbf{1}_{\{W_u\le\delta\}}\,\mathrm{d}u-\kappa_{\epsilon,\delta}L_t^0.$$

As in Case (iii), $\lim_{\delta\to0}\limsup_{\epsilon\to0}[|K_{\tau_t}^{\epsilon,\delta}|+|K_{\tau_t}^{\epsilon,\infty}-K_{\rho_t^\epsilon}^{\epsilon,\infty}|]=0$ in probability.

We also have, for any $\delta\in(0,1)\cup\{\infty\}$, by definition of $\kappa_{\epsilon,\delta}$ (in particular since $\kappa_{\epsilon,1}=\kappa_{\epsilon,\infty}=0$),

$$I_t^{\epsilon,\delta}=\frac{1}{a_\epsilon}\int_{-\infty}^\delta\frac{h^{-1}(w/a_\epsilon)-\zeta_\epsilon}{[\sigma(w/a_\epsilon)]^2}\big(L_t^w-L_t^0\mathbf{1}_{\{w\le1\}}\big)\,\mathrm{d}w.$$

As in Case (iii), it is sufficient to verify that for each $\delta\in(0,1)\cup\{\infty\}$, each $T\ge0$, we a.s. have $\lim_{\epsilon\to0}\sup_{[0,T]}|I_t^{\epsilon,\delta}-I_t^\delta|=0$, where we have set $I_t^\delta=9^{-2}\int_0^\delta w^{-1}(L_t^w-L_t^0\mathbf{1}_{\{w\le1\}})\,\mathrm{d}w$. This, here again, follows from dominated convergence, because, recalling that $a_\epsilon=\kappa\epsilon$:

- $a_\epsilon^{-1}h^{-1}(w/a_\epsilon)[\sigma(w/a_\epsilon)]^{-2} \le Cw^{-1}\mathbf{1}_{\{w\ge 0\}} + C|w|^{-1}(1+|w|)^{-1}\mathbf{1}_{\{w<0\}}$ by Lemma 42(viii),
- $\lim_{\epsilon\to 0} a_\epsilon^{-1}h^{-1}(w/a_\epsilon)/[\sigma(w/a_\epsilon)]^2 = 9^{-1}w^{-1}\mathbf{1}_{\{w\ge 0\}}$, see Lemma 42(ix),
- $\zeta_\epsilon \le C\int_{-\infty}^{1/a_\epsilon} h^{-1}(w)[\sigma(w)]^{-2}\,\mathrm{d}w \le C(1+|\log\epsilon|)$ by Lemma 42(viii),
- $a_\epsilon^{-1}\zeta_\epsilon[\sigma(w/a_\epsilon)]^{-2} \le C\epsilon^{-1}(1+|\log\epsilon|)(1+|w|/\epsilon)^{-4/3}$, and by Lemma 42(vi), this is smaller than $C\epsilon^{1/3}(1+|\log\epsilon|)|w|^{-4/3}$.
- the integral

$$\int_{\mathbb{R}} \big[|w|^{-1}\mathbf{1}_{\{w>0\}} + |w|^{-1}\big(1+|w|\big)^{-1}\mathbf{1}_{\{w<0\}} + |w|^{-4/3}\big] \sup_{[0,T]} \big|L_t^w - L_t^0\mathbf{1}_{\{w\le 1\}}\big|\,\mathrm{d}w$$

is a.s. finite, since $\sup_{[0,T]}|L_t^w - L_t^0\mathbf{1}_{\{w\le 1\}}|$ is a.s. bounded, vanishes for w sufficiently large (namely, for $w > \sup_{[0,T]} W_s$) and is a.s. almost 1/2-Holdër continuous near 0; see [33], Corollary 1.8, page 226. □

We need the excursion theory for the Brownian motion; see Revuz–Yor [33], Chapter XII, Part 2. We introduce some notation and briefly summarize what we will use.

NOTATION 15. Recall that $(W_t)_{t\ge 0}$ is a Brownian motion, that $(L_t^0)_{t\ge 0}$ is its local time at 0, that $\tau_t = \inf\{u \ge 0 : L_u^0 > t\}$ is its inverse. We introduce $J = \{s > 0 : \tau_s > \tau_{s-}\}$ and, for $s \in J$,

$$e_s = (W_{\tau_{s-}+r}\mathbf{1}_{\{r\in[0,\tau_s-\tau_{s-}]\}})_{r\ge 0} \in \mathcal{E},$$

where $\mathcal{E}$ is the set of continuous functions e from $\mathbb{R}_+$ into $\mathbb{R}$ such that $e(0) = 0$, such that

$$\ell(e) = \sup\{r > 0 : e(r) \ne 0\} \in (0,\infty)$$

and such that $e(r)$ does not vanish on $(0,\ell(e))$. For $e \in \mathcal{E}$, we denote by $x(e) = \mathrm{sg}(e(\ell(e)/2)) \in \{-1,1\}$ and observe that $\mathrm{sg}(e(r)) = x(e)$ for all $r \in (0,\ell(e))$.

We introduce $\mathbf{M} = \sum_{s\in J}\delta_{(s,e_s)}$, which is a Poisson measure on $[0,\infty)\times\mathcal{E}$ with intensity measure $\mathrm{d}s\,\Xi(\mathrm{d}e)$, where Ξ is a σ-finite measure on $\mathcal{E}$ known as Itô's measure and that can be decomposed as follows: denoting by $\mathcal{E}_1 = \{e \in \mathcal{E} : \ell(e) = 1 \text{ and } x(e) = 1\}$ and by $\Xi_1 \in \mathcal{P}(\mathcal{E}_1)$ the law of the normalized Brownian excursion, for all measurable $A \subset \mathcal{E}$,

$$\Xi(A) = \int_0^\infty \frac{\mathrm{d}\ell}{\sqrt{2\pi\ell^3}} \int_{\{-1,1\}} \frac{1}{2}(\delta_{-1}+\delta_1)(\mathrm{d}x) \int_{\mathcal{E}_1} \Xi_1(\mathrm{d}e)\mathbf{1}_{\{(x\sqrt{\ell}e(r/\ell))_{r\ge 0}\in A\}}. \tag{14}$$

It holds that $\tau_t = \int_0^t\int_{\mathcal{E}} \ell(e)\mathbf{M}(\mathrm{d}s,\mathrm{d}e)$ and for all $t \in J$, all $s \in [\tau_{t-},\tau_t]$, we have $W_s = e_t(s-\tau_{t-})$. For any $\phi : \mathbb{R} \to \mathbb{R}_+$, any $t \ge 0$, we have

$$\int_0^{\tau_t} \phi(W_u)\,\mathrm{d}u = \sum_{s\in J\cap[0,t]} \int_{\tau_{s-}}^{\tau_s} \phi(W_u)\,\mathrm{d}u = \int_0^t \int_{\mathcal{E}} \bigg[\int_0^{\ell(e)} \phi\big(e(u)\big)\,\mathrm{d}u\bigg]\mathbf{M}(\mathrm{d}s,\mathrm{d}e). \tag{15}$$

We now rewrite the processes of Notation 13 in terms of the excursion Poisson measure. We recall that ψ, h, σ were defined in Notation 9.

NOTATION 16. Fix $\epsilon \in (0,1)$ and $0 \le \delta < A \le \infty$. For $e \in \mathcal{E}$, and $\theta = (\theta_r)_{r\in\mathbb{R}}$ in $\mathcal{H} = C(\mathbb{R},\mathbb{S}_{d-1})$, let

$$F_{\epsilon,\delta,A}(e,\theta) = a_\epsilon^{1/\alpha-2} \int_0^{\ell(e)} \frac{h^{-1}(e(u)/a_\epsilon)\theta_{r_{\epsilon,u}(e)} - m_{\beta,\epsilon}}{[\sigma(e(u)/a_\epsilon)]^2} \mathbf{1}_{\{\delta<e(u)<A\}}\,\mathrm{d}u,$$

where $m_{\beta,\epsilon} = 0$ if $\beta \in [d,1+d)$, $m_{\beta,\epsilon} = \zeta_\epsilon M_\beta$ if $\beta = 1+d$ and $m_{\beta,\epsilon} = m_\beta$ in the case $\beta \in (1+d,4+d)$ and where, for $u \in (0,\ell(e))$,

$$r_{\epsilon,u}(e) = \frac{1}{a_\epsilon^2}\int_{\ell(e)/2}^u \frac{\mathrm{d}v}{\psi(e(v)/a_\epsilon)}.$$

Observe that $F_{\epsilon,\delta,A}(e,\theta)=0$ if $x(e)=-1$. Also, we make start the clock $r_{\epsilon,u}(e)$ from the middle $\ell(e)/2$ of the excursion because at the limit, $a_\epsilon^2\psi(x/a_\epsilon)$ vanishes at $x=0$ sufficiently fast so that $a_\epsilon^{-2}\int_{0+}[\psi(e(v)/a_\epsilon)]^{-1}\,\mathrm{d}v$ and $a_\epsilon^{-2}\int^{\ell(e)-}[\psi(e(v)/a_\epsilon)]^{-1}\,\mathrm{d}v$ will tend to infinity as $\epsilon\to 0$.

REMARK 17. For all $\epsilon\in(0,1)$, all $\delta\in(0,1)$, all $t\geq 0$, we have

$$Z_t^{\epsilon,\delta}=\int_0^t\int_{\mathcal{E}}F_{\epsilon,\delta,\infty}\big(e,(\hat{\Theta}_{[P_{s-}^\epsilon+r-r_{\epsilon,0}(e)]\vee 0})_{r\in\mathbb{R}}\big)\mathbf{M}(\mathrm{d}s,\mathrm{d}e)\quad\text{where}\tag{16}$$

$$P_t^\epsilon=\int_0^t\int_{\mathcal{E}}\bigg[\frac{1}{a_\epsilon^2}\int_0^{\ell(e)}\frac{\mathrm{d}u}{\psi(e(u)/a_\epsilon)}\bigg]\mathbf{M}(\mathrm{d}s,\mathrm{d}e).$$

PROOF. For any reasonable $\phi_1:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\phi_2:\mathbb{R}\to\mathbb{R}$, if setting $\nu_t=\int_0^t\phi_2(W_s)\,\mathrm{d}s$, we have

$$\begin{aligned}\int_0^{\tau_t}\phi_1(W_s,\nu_s)\,\mathrm{d}s&=\sum_{s\in J\cap[0,t]}\int_{\tau_{s-}}^{\tau_s}\phi_1\bigg(W_u,\nu_{\tau_{s-}}+\int_{\tau_{s-}}^u\phi_2(W_v)\,\mathrm{d}v\bigg)\mathrm{d}u\\&=\int_0^t\int_{\mathcal{E}}\bigg[\int_0^{\ell(e)}\phi_1\bigg(e(u),\nu_{\tau_{s-}}+\int_0^u\phi_2\big(e(v)\big)\,\mathrm{d}v\bigg)\mathrm{d}u\bigg]\mathbf{M}(\mathrm{d}s,\mathrm{d}e).\end{aligned}$$

With $\phi_1(w,\nu)=a_\epsilon^{1/\alpha-2}[\sigma(w/a_\epsilon)]^{-2}[h^{-1}(w/a_\epsilon)\hat{\Theta}_\nu-m_{\beta,\epsilon}]\mathbf{1}_{\{w>\delta\}}$ and $\phi_2(w)=a_\epsilon^{-2}[\psi(w/a_\epsilon)]^{-1}$, so that $T_t^\epsilon=\int_0^t\phi_2(W_s)\,\mathrm{d}s$ and $P_t^\epsilon=T_{\tau_t}^\epsilon$ by (15), this gives

$$\begin{aligned}Z_t^{\epsilon,\delta}=\int_0^t\int_{\mathcal{E}}\bigg[a_\epsilon^{\frac{1}{\alpha}-2}\int_0^{\ell(e)}&\frac{h^{-1}(e(u)/a_\epsilon)\hat{\Theta}_{P_{s-}^\epsilon+a_\epsilon^{-2}\int_0^u[\psi(e(v))]^{-2}\,\mathrm{d}v}-m_{\beta,\epsilon}}{[\sigma(e(u)/a_\epsilon)]^2}\\&\times\mathbf{1}_{\{e(u)\geq\delta\}}\,\mathrm{d}u\bigg]\mathbf{M}(\mathrm{d}s,\mathrm{d}e),\end{aligned}$$

from which the result follows because by definition of $F_{\epsilon,\delta,\infty}$, we have

$$\begin{aligned}&F_{\epsilon,\delta,\infty}\big(e,(\hat{\Theta}_{[P_{s-}^\epsilon+r-r_{\epsilon,0}(e)]\vee 0})_{r\in\mathbb{R}}\big)\\&\quad=a_\epsilon^{1/\alpha-2}\int_0^{\ell(e)}\frac{h^{-1}(e(u)/a_\epsilon)\hat{\Theta}_{[P_{s-}^\epsilon+r_{\epsilon,u}(e)-r_{\epsilon,0}(e)]\vee 0}-m_{\beta,\epsilon}}{[\sigma(e(u)/a_\epsilon)]^2}\mathbf{1}_{\{e(u)\geq\delta\}}\,\mathrm{d}u\end{aligned}$$

and because P_{s-}^ϵ is positive, as well as $r_{\epsilon,u}(e)-r_{\epsilon,0}(e)$ which equals

$$\frac{1}{a_\epsilon^2}\int_{\ell(e)/2}^u\frac{\mathrm{d}v}{\psi(e(v)/a_\epsilon)}+\frac{1}{a_\epsilon^2}\int_0^{\ell(e)/2}\frac{\mathrm{d}v}{\psi(e(v)/a_\epsilon)}=\frac{1}{a_\epsilon^2}\int_0^u\frac{\mathrm{d}v}{\psi(e(v)/a_\epsilon)}$$

as desired. □

We now get rid of the correlation in the spherical process.

LEMMA 18. *Let $\mathbf{N}$ be a Poisson measure on $[0,\infty)\times\mathcal{E}\times\mathcal{H}$ with intensity measure*

$$\pi(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta)=\mathrm{d}s\,\Xi(\mathrm{d}e)\Lambda(\mathrm{d}\theta)$$

for $\Xi\in\mathcal{P}(\mathcal{E})$ the law of the normalized Brownian excursion and $\Lambda\in\mathcal{P}(\mathcal{H})$ the law of the stationary eternal spherical process built in Lemma 38. For $\epsilon\in(0,1)$ and $\delta\in(0,1)$, we introduce the process

$$\bar{Z}_t^{\epsilon,\delta}=\int_0^t\int_{\mathcal{E}}\int_{\mathcal{H}}F_{\epsilon,\delta,\infty}(e,\theta)\mathbf{N}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta).$$

For all $T > 0$, all $\delta > 0$, there exists $q_{T,\delta} : (0,1) \to \mathbb{R}_+$ with $\lim_{\epsilon\to 0} q_{T,\delta}(\epsilon) = 1$ and such that for any $\epsilon \in (0,1)$, we can find a coupling between $(Z_t^{\epsilon,\delta})_{t\in[0,T]}$ and $(\bar{Z}_t^{\epsilon,\delta})_{t\in[0,T]}$ such that $\mathbb{P}[(Z_t^{\epsilon,\delta})_{t\in[0,T]} = (\bar{Z}_t^{\epsilon,\delta})_{t\in[0,T]}] \geq q_{T,\delta}(\epsilon)$.

Observe that the process $(\bar{Z}_t^{\epsilon,\delta})_{t\geq 0}$ is Lévy.

PROOF OF LEMMA 18. The proof is tedious, but simple in its principle: the main idea is that the clock of $\hat{\Theta}$ in (16) runs a very long way (asymptotically infinite when $\epsilon \to 0$) between two excursions, so that we can apply Lemma 38(iv).

Step 1. For all $\delta \in (0,1)$, all $e \in \mathcal{E}$, there is $s_\delta(e) > 0$ such that for all $\epsilon \in (0,1)$, all $\theta, \theta' \in \mathcal{H}$, we have $F_{\epsilon,\delta,\infty}(e,\theta) = F_{\epsilon,\delta,\infty}(e,\theta')$ as soon as $\theta_r = \theta'_r$ for all $r \in [-s_\delta(e), s_\delta(e)]$.

We recall that $F_{\epsilon,\delta,\infty}(e,\theta) = 0$ if $x(e) = -1$, so that it suffices to treat the case of positive excursions. We have $F_{\epsilon,\delta,\infty}(e,\theta) = F_{\epsilon,\delta,\infty}(e,\theta')$ if $\theta_u = \theta'_u$ for all $u \in [-s_{\delta,\epsilon}(e), s_{\delta,\epsilon}(e)]$, where

$$s_{\delta,\epsilon}(e) = \max\{-r_{\epsilon,\inf\{v>0:e(v)>\delta\}\wedge(\ell(e)/2)}(e), r_{\epsilon,\sup\{v>0:e(v)>\delta\}\vee(\ell(e)/2)}(e)\}$$

because then for all $u \in (0,\ell(e))$ such that $\theta_{r_{\epsilon,u}(e)} \neq \theta'_{r_{\epsilon,u}(e)}$, we have either $r_{\epsilon,u}(e) > r_{\epsilon,\sup\{v>0:e(v)>\delta\}\vee(\ell(e)/2)}(e)$ or $r_{\epsilon,u}(e) < r_{\epsilon,\inf\{v>0:e(v)>\delta\}\wedge(\ell(e)/2)}(e)$, whence in both cases $e(u) < \delta$, which makes vanish the indicator function $\mathbf{1}_{\{e(u)\geq\delta\}}$. Using now that $a_\epsilon^{-2}[\psi(w/a_\epsilon)]^{-1} \leq Cw^{-2}$ for all $w > 0$ by Lemma 42(iv), we realize that

$$s_{\delta,\epsilon}(e) \leq C\int_{\inf\{v>0:e(v)>\delta\}\wedge(\ell(e)/2)}^{\sup\{v>0:e(v)>\delta\}\vee(\ell(e)/2)} \frac{\mathrm{d}u}{[e(u)]^2}.$$

Denoting by $s_\delta(e)$ this last quantity, which is finite because e does not vanish during the interval $[\inf\{v > 0 : e(v) > \delta\} \wedge (\ell(e)/2), \sup\{v > 0 : e(v) > \delta\} \vee (\ell(e)/2)]$, completes the step.

Step 2. Since only a finite number of excursions exceed δ per unit of time we may rewrite (16) as

$$Z_t^{\epsilon,\delta} = \sum_{i=1}^{N_t^\delta} F_{\epsilon,\delta,\infty}(e_i^\delta, (\hat{\Theta}_{[T_i^{\epsilon,\delta}+r]\vee 0})_{r\geq 0}),$$

where $\mathcal{E}_\delta = \{e \in \mathcal{E} : \sup_{u\in[0,\ell(e)]} e(u) > \delta\}$, $N_t^\delta = \mathbf{M}([0,t] \times \mathcal{E}_\delta)$, of which we denote by $(s_i^\delta)_{i\geq 1}$ the chronologically ordered instants of jump. For each $i \geq 1$, we have introduced by $e_i^\delta \in \mathcal{E}_\delta$ the mark associated to s_i^δ, uniquely defined by the fact that $\mathbf{M}(\{(s_i^\delta, e_i^\delta)\}) = 1$. We also have set, for each $i \geq 1$,

$$T_i^{\epsilon,\delta} = P^\epsilon_{s_i^\delta -} - r_{\epsilon,0}(e_i^\delta).$$

Step 3. Here we show that, $\forall \delta \in (0,1), T > 0$, a.s., $\min_{i=1,\dots,N_T^\delta}(T_i^{\epsilon,\delta} - T_{i-1}^{\epsilon,\delta}) \to \infty$ as $\epsilon \to 0$. It suffices to observe that, since $\psi(u) \leq C(1 + |u|^2)$ by Lemma 42(iv) and since $P^\epsilon_{s_i^\delta -} \geq T_{i-1}^{\epsilon,\delta}$,

$$T_i^{\epsilon,\delta} - T_{i-1}^{\epsilon,\delta} \geq -r_{\epsilon,0}(e_i^\delta) = \frac{1}{a_\epsilon^2}\int_0^{\ell(e_i^\delta)/2} \frac{\mathrm{d}v}{\psi[e_i^\delta(v)/a_\epsilon]} \geq c\int_0^{\ell(e_i^\delta)/2} \frac{\mathrm{d}v}{a_\epsilon^2 + [e_i^\delta(v)]^2}.$$

By monotone convergence, we conclude that (see Lemma 39(i))

$$\liminf_{\epsilon\to 0}(T_i^{\epsilon,\delta} - T_{i-1}^{\epsilon,\delta}) \geq c\int_0^{\ell(e_i^\delta)/2} \frac{\mathrm{d}v}{[e_i^\delta(v)]^2} = \infty \quad \text{a.s.}$$

Step 4. We work conditionally on $\mathbf{M}$ and set $A_{T,\delta} = \sup_{i=1,\dots,N_T^\delta} s_\delta(e_i^\delta)$. By Lemma 38(iv), we can find, for each $\epsilon \in (0,1)$, an i.i.d. family of Λ-distributed eternal processes $(\hat\Theta_r^{\star,1,\epsilon})_{r\in\mathbb{R}}, \dots, (\hat\Theta_r^{\star,N_T^\delta,\epsilon})_{r\in\mathbb{R}}$ such that the probability that $(\hat\Theta_{[T_i^{\epsilon,\delta}+r]\vee 0})_{r\in[-A_{T,\delta},A_{T,\delta}]} = (\hat\Theta_r^{\star,i,\epsilon})_{r\in[-A_{T,\delta},A_{T,\delta}]}$ for all $i = 1,\dots,N_T^\delta$ (conditionally on $\mathbf{M}$) is greater than $p_{T,\delta,\epsilon} = p_{A_{T,\delta}}(T_1^{\epsilon,\delta}, T_2^{\epsilon,\delta} - T_1^{\epsilon,\delta}, \dots, T_{N_t^\delta}^{\epsilon,\delta} - T_{N_t^\delta-1}^{\epsilon,\delta})$, which a.s. tends to 1 as $\epsilon \to 0$ by Step 3.

Step 5. We set, for $t \in [0,T]$,

$$\bar Z_t^{\epsilon,\delta} = \sum_{i=1}^{N_t^\delta} F_{\epsilon,\delta,\infty}\big(e_i^\delta, (\hat\Theta_r^{\star,i,\epsilon})_{r\geq 0}\big).$$

This process has the same law as the process $(\bar Z_t^{\epsilon,\delta})_{t\in[0,T]}$ of the statement. Furthermore, we know from Step 1 that $Z_t^{\epsilon,\delta} = \bar Z_t^{\epsilon,\delta}$ for all $t\in[0,T]$ as soon as $(\hat\Theta_{[T_i^{\epsilon,\delta}+r]\vee 0})_{r\in[-A_{T,\delta},A_{T,\delta}]} = (\hat\Theta_t^{\star,i,\epsilon})_{r\in[-A_{T,\delta},A_{T,\delta}]}$ for all $i = 1,\dots,N_T^\delta$. This occurs with probability $q_{T,\delta}(\epsilon) = \mathbb{E}[p_{T,\delta,\epsilon}]$, which tends to 1 as $\epsilon \to 0$ by dominated convergence. □

We introduce the compensated Poisson measure $\tilde{\mathbf{N}} = \mathbf{N} - \pi$.

LEMMA 19. *We fix $\delta \in (0,1]$ and $\epsilon \in (0,1)$.*

(i) *If $\beta \in [d, 1+d)$, we simply set $\hat Z_t^{\epsilon,\delta} = \bar Z_t^{\epsilon,\delta}$.*

(ii) *If $\beta = 1+d$, we set $\hat Z_t^{\epsilon,\delta} = \bar Z_t^{\epsilon,\delta} + \kappa_{\epsilon,\delta} M_\beta t$ and we have*

$$\hat Z_t^{\epsilon,\delta} = \int_0^t \int_{\mathcal{E}}\int_{\mathcal{H}} F_{\epsilon,\delta,1}(e,\theta)\tilde{\mathbf{N}}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta) + \int_0^t \int_{\mathcal{E}}\int_{\mathcal{H}} F_{\epsilon,1,\infty}(e,\theta)\mathbf{N}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta).$$

(iii) *If $\beta \in (1+d, 4+d)$, we set $\hat Z_t^{\epsilon,\delta} = \bar Z_t^{\epsilon,\delta} + \kappa_{\epsilon,\delta} M_\beta t$ and we have*

$$\hat Z_t^{\epsilon,\delta} = \int_0^t \int_{\mathcal{E}}\int_{\mathcal{H}} F_{\epsilon,\delta,\infty}(e,\theta)\tilde{\mathbf{N}}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta).$$

PROOF. We recall that $\int_{\mathbb{R}} \phi(w)\,\mathrm{d}w = \int_{\mathcal{E}}[\int_0^{\ell(e)} \phi(e(u))\,\mathrm{d}u]\,\Xi(\mathrm{d}e)$ for all $\phi \in L^1(\mathbb{R})$; see Lemma 39(ii).

To verify (iii), we have to check that

$$I = \int_{\mathcal{E}}\int_{\mathcal{H}} F_{\epsilon,\delta,\infty}(e,\theta)\Lambda(\mathrm{d}\theta)\,\Xi(\mathrm{d}e) = -\kappa_{\epsilon,\delta} M_\beta.$$

Recalling the expression of $F_{\epsilon,\delta,\infty}$ and that Λ is the law of the eternal stationary spherical process (see Lemma 38) of which the invariant measure is ν_β, which satisfies $\int_{\mathbb{S}_{d-1}} \theta\,\nu_\beta(\mathrm{d}\theta) = M_\beta$, we find

$$\begin{aligned} I &= \int_{\mathcal{E}}\bigg[a_\epsilon^{1/\alpha-2}\int_0^{\ell(e)} \frac{h^{-1}(e(u)/a_\epsilon))M_\beta - m_\beta}{[\sigma(e(u)/a_\epsilon)]^2}\mathbf{1}_{\{e(u)>\delta\}}\,\mathrm{d}u\bigg]\,\Xi(\mathrm{d}e) \\ &= a_\epsilon^{1/\alpha-2}\int_\delta^\infty \frac{h^{-1}(w/a_\epsilon)M_\beta - m_\beta}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w. \end{aligned}$$

Recalling that $m_\beta = M_\beta m'_\beta$, the definition of $\kappa_{\epsilon,\delta}$ (see Notation 13(iii)) and that $\kappa_{\epsilon,\infty} = 0$ (see (11)),

$$I = M_\beta a_\epsilon^{\frac{1}{\alpha}-2}\int_\delta^\infty \frac{h^{-1}(w/a_\epsilon) - m'_\beta}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w = -M_\beta a_\epsilon^{\frac{1}{\alpha}-2}\int_{-\infty}^\delta \frac{h^{-1}(w/a_\epsilon) - m'_\beta}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w,$$

which equals $-M_\beta\kappa_{\epsilon,\delta}$ as desired.

Concerning (ii), since $F_{\epsilon,\delta,\infty} = F_{\epsilon,\delta,1} + F_{\epsilon,1,\infty}$, we have to verify that

$$J = \int_{\mathcal{E}} \int_{\mathcal{H}} F_{\epsilon,\delta,1}(e,\theta)\Lambda(\mathrm{d}\theta)\Xi(\mathrm{d}e) = -\kappa_{\epsilon,\delta} M_\beta.$$

Proceeding as above, we find

$$\begin{aligned} J &= M_\beta a_\epsilon^{-1} \int_\delta^1 \frac{h^{-1}(w/a_\epsilon) - \zeta_\epsilon}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w \\ &= -M_\beta a_\epsilon^{-1} \int_{-\infty}^\delta \frac{h^{-1}(w/a_\epsilon) - \zeta_\epsilon}{[\sigma(w/a_\epsilon)]^2}\,\mathrm{d}w \\ &= -M_\beta \kappa_{\epsilon,\delta} \end{aligned}$$

by definition of $\kappa_{\epsilon,\delta}$ and since $\kappa_{\epsilon,1} = 0$, recall Notation 13(ii). □

We introduce the limit (as $\epsilon \to 0$) of the function defined in Notation 16.

NOTATION 20. Fix $0 \le \delta < A \le \infty$. For $e \in \mathcal{E}$ and $\theta = (\theta_r)_{r\in\mathbb{R}}$ in $\mathcal{H} = C(\mathbb{R}, \mathbb{S}_{d-1})$, we set

$$F_{\delta,A}(e,\theta) = \frac{1}{(\beta+2-d)^2} \int_0^{\ell(e)} [e(u)]^{1/\alpha-2} \theta_{r_u(e)} \mathbf{1}_{\{\delta \le e(u) \le A\}}\,\mathrm{d}u,$$

where, for $u \in (0, \ell(e))$,

$$r_u(e) = \frac{1}{(\beta+2-d)^2} \int_{\ell(e)/2}^u \frac{\mathrm{d}v}{[e(v)]^2}.$$

Finally, we make tend ϵ and δ to 0.

LEMMA 21. *Let $(\hat{Z}_t^{\epsilon,\delta})_{t\ge0}$ be the processes introduced in Lemma 19, built with the same Poisson measure $\mathbf{N}$ for all values of $\epsilon, \delta \in (0,1)$. For all $T > 0$, $\sup_{[0,T]} |\hat{Z}_t^{\epsilon,\delta} - Z_t|$ goes to 0 in probability as $(\epsilon,\delta) \to (0,0)$, where*

(i) $Z_t = \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{0,\infty}(e,\theta)\mathbf{N}(\mathrm{d}s, \mathrm{d}e, \mathrm{d}\theta)$ *if* $\beta \in [d, 1+d)$,

(ii) $Z_t = \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{0,1}(e,\theta)\tilde{\mathbf{N}}(\mathrm{d}s, \mathrm{d}e, \mathrm{d}\theta) + \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{1,\infty}(e,\theta)\mathbf{N}(\mathrm{d}s, \mathrm{d}e, \mathrm{d}\theta)$ *if* $\beta = 1+d$,

(iii) $Z_t = \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{0,\infty}(e,\theta)\tilde{\mathbf{N}}(\mathrm{d}s, \mathrm{d}e, \mathrm{d}\theta)$ *if* $\beta \in (1+d, 4+d)$.

PROOF. We divide the proof in several steps.

Step 1. There is $C > 0$ such that for all $\epsilon \in (0,1]$, all $0 \le \delta \le A \le \infty$, all $e \in \mathcal{E}$, all $\theta \in \mathcal{H}$,

$$|F_{\epsilon,\delta,A}(e,\theta)| \le C \int_0^{\ell(e)} ([e(u)]^{1/\alpha-2} + \mathbf{1}_{\{\beta=1+d\}}[e(u)]^{-4/3})\mathbf{1}_{\{\delta\le e(u)\le A\}}\,\mathrm{d}u.$$

Indeed, by Lemma 42(viii), $[1 + h^{-1}(w)][\sigma(w)]^{-2} \le C(1+|w|)^{1/\alpha-2}$. This implies that $a_\epsilon^{1/\alpha-2}[1 + h^{-1}(w/a_\epsilon)][\sigma(w/a_\epsilon)]^{-2} \le C|w|^{1/\alpha-2}$, and it only remains to note that when $\beta = 1+d$ (so that $\alpha = 1$),

$$\left|\frac{m_{\beta,\epsilon}}{a_\epsilon[\sigma(w/a_\epsilon)]^2}\right| = \frac{|M_\beta|\zeta_\epsilon}{a_\epsilon[\sigma(w/a_\epsilon)]^2} \le C\frac{1+|\log\epsilon|}{\epsilon(1+|w|/\epsilon)^{\frac{4}{3}}} \le C\frac{\epsilon^{\frac{1}{3}}(1+|\log\epsilon|)}{|w|^{\frac{4}{3}}} \tag{17}$$

by Lemma 42(vi), since $a_\epsilon = \kappa\epsilon$ and since $\zeta_\epsilon \le C(1+|\log\epsilon|)$, see the end of the proof of Lemma 14.

Step 2. We fix $0 \le \delta_0 < A \le \infty$ and verify that for all $\theta \in \mathcal{H}$ and Ξ-almost every $e \in \mathcal{E}$, we have

$$\lim_{(\epsilon,\delta)\to(0,\delta_0)} F_{\epsilon,\delta,A}(e,\theta) = F_{\delta_0,A}(e,\theta).$$

Using precisely the same bounds as in Step 1, the result follows from dominated convergence, because

- $a_\epsilon^{1/\alpha-2} h^{-1}(w/a_\epsilon)[\sigma(w/a_\epsilon)]^{-2} \to (\beta+2-d)^{-2} w^{1/\alpha-2}$ for each fixed $w > 0$ by Lemma 42(ix),
- $\theta \in \mathcal{H}$ is continuous and $r_{\epsilon,u}(e) = a_\epsilon^{-2} \int_{\ell(e)/2}^{u} [\psi(e(v)/a_\epsilon)]^{-1}\,\mathrm{d}v \to r_u(e)$ for each $u \in (0,\ell(e))$ by Lemma 42(v) (and by dominated convergence),
- $a_\epsilon^{1/\alpha-2} m_{\beta,\epsilon}[\sigma(w/a_\epsilon)]^{-2} \to 0$ for each fixed $w > 0$, because

⋆ if $\beta \in [d, 1+d)$, $m_{\beta,\epsilon} = 0$,
⋆ if $\beta = 1+d$, see (17),
⋆ if $\beta \in (1+d, 4+d)$, then

$$a_\epsilon^{1/\alpha-2} |m_{\beta,\epsilon}| \big[\sigma(w/a_\epsilon)\big]^{-2} \le C\epsilon^{1/\alpha-2}(1+w/\epsilon)^{-2(\beta+1-d)/(\beta+2-d)} \to 0,$$

by Lemma 42(vi), since $m_{\beta,\epsilon} = m_\beta$ and since $2(\beta+1-d)/(\beta+2-d) > 2-1/\alpha$,

- $\int_0^{\ell(e)} ([e(u)]^{1/\alpha-2} + [e(u)]^{-4/3})\,\mathrm{d}u < \infty$ for Ξ-almost every $e \in \mathcal{E}$ by Lemma 39(iv).

Step 3. We write $\hat{Z}_t^{\epsilon,\delta} = Y_t^{\epsilon,1} - Y_t^{\epsilon,2} + Y_t^{\epsilon,\delta,3}$ and $Z_t = Y_t^1 - Y_t^2 + Y_t^3$, where

$$Y_t^1 = \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{1,\infty}(e,\theta) \mathbf{N}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta),$$

$$Y_t^{\epsilon,1} = \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{\epsilon,1,\infty}(e,\theta) \mathbf{N}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta),$$

$$Y_t^2 = \begin{cases} t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{1,\infty}(e,\theta)\Lambda(\mathrm{d}\theta)\Xi(\mathrm{d}e) & \text{if } \beta \in (1+d,4+d), \\ 0 & \text{if } \beta \in [d,1+d], \end{cases}$$

$$Y_t^{\epsilon,2} = \begin{cases} t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{\epsilon,1,\infty}(e,\theta)\Lambda(\mathrm{d}\theta)\Xi(\mathrm{d}e) & \text{if } \beta \in (1+d,4+d), \\ 0 & \text{if } \beta \in [d,1+d], \end{cases}$$

$$Y_t^3 = \begin{cases} \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{0,1}(e,\theta)\mathbf{N}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta) & \text{if } \beta \in [d,1+d), \\ \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{0,1}(e,\theta)\tilde{\mathbf{N}}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta) & \text{if } \beta \in [1+d,4+d), \end{cases}$$

$$Y_t^{\epsilon,\delta,3} = \begin{cases} \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{\epsilon,\delta,1}(e,\theta)\mathbf{N}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta) & \text{if } \beta \in [d,1+d), \\ \int_0^t \int_{\mathcal{E}} \int_{\mathcal{H}} F_{\epsilon,\delta,1}(e,\theta)\tilde{\mathbf{N}}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta) & \text{if } \beta \in [1+d,4+d). \end{cases}$$

Step 3.1. For any $\beta \in [d, 4+d)$, it holds that a.s.,

$$\lim_{\epsilon\to 0} \sup_{[0,T]} |Y_t^1 - Y_t^{\epsilon,1}| \le \lim_{\epsilon\to 0} \int_0^T \int_{\mathcal{E}} \int_{\mathcal{H}} \big|F_{1,\infty}(e,\theta) - F_{\epsilon,1,\infty}(e,\theta)\big| \mathbf{N}(\mathrm{d}s,\mathrm{d}e,\mathrm{d}\theta) = 0.$$

This uses only the facts that $F_{1,\infty}(e,\theta) = F_{\epsilon,1,\infty}(e,\theta) = 0$ as soon as $\sup_{r\in[0,\ell(e)]} e(r) < 1$, that $\mathbf{N}(\{(s,e,\theta) \in [0,T] \times \mathcal{E} \times \mathcal{H} : \sup_{[0,\ell(e)]} e \ge 1\})$ is a.s. finite, and that $\lim_{\epsilon\to 0} F_{\epsilon,1,\infty}(e,\theta) = F_{1,\infty}(e,\theta)$ for $\Xi \otimes \Lambda$-almost every $(e,\theta) \in \mathcal{E} \times \mathcal{H}$ by Step 2.

Step 3.2. If $\beta \in (1+d, 4+d)$, it holds that

$$\lim_{\epsilon\to 0}\sup_{[0,T]}\left|Y_t^2 - Y_t^{\epsilon,2}\right| \le T \lim_{\epsilon\to 0}\int_{\mathcal{E}}\int_{\mathcal{H}}\left|F_{1,\infty}(e,\theta) - F_{\epsilon,1,\infty}(e,\theta)\right|\Lambda(\mathrm{d}\theta)\,\Xi(\mathrm{d}e) = 0$$

by dominated convergence, thanks to Steps 1 and 2 and since

$$\int_{\mathcal{E}}\left[\int_0^{\ell(e)}\left(e(u)\right)^{1/\alpha-2}\mathbf{1}_{\{e(u)\ge 1\}}\,\mathrm{d}u\right]\Xi(\mathrm{d}e) = \int_1^\infty x^{1/\alpha-2}\,\mathrm{d}x < \infty$$

by Lemma 39(ii) and since $1/\alpha - 2 < -1$ because $\alpha = (\beta+2-d)/3 > 1$.

Step 3.3. If $\beta \in [d, 1+d)$,

$$\begin{aligned}
&\lim_{(\epsilon,\delta)\to(0,0)}\mathbb{E}\Big[\sup_{[0,T]}\left|Y_t^3 - Y_t^{\epsilon,\delta,3}\right|\Big]\\
&\quad\le T\lim_{(\epsilon,\delta\to(0,0)}\int_{\mathcal{E}}\int_{\mathcal{H}}\left|F_{0,1}(e,\theta) - F_{\epsilon,\delta,1}(e,\theta)\right|\Lambda(\mathrm{d}\theta)\,\Xi(\mathrm{d}e) = 0
\end{aligned}$$

by dominated convergence, using Steps 1 and 2 and that

$$\int_{\mathcal{E}}\left[\int_0^{\ell(e)}\left[e(u)\right]^{1/\alpha-2}\mathbf{1}_{\{0\le e(u)\le 1\}}\,\mathrm{d}u\right]\Xi(\mathrm{d}e) = \int_0^1 w^{1/\alpha-2}\,\mathrm{d}w < \infty$$

by Lemma 39(ii) and since $1/\alpha - 2 > -1$ because $\alpha = (\beta+2-d)/3 < 1$.

Step 3.4. If finally $\beta \in [1+d, 4+d)$, by Doob's inequality,

$$\begin{aligned}
&\lim_{(\epsilon,\delta)\to(0,0)}\mathbb{E}\Big[\sup_{[0,T]}\left|Y_t^3 - Y_t^{\epsilon,\delta,3}\right|^2\Big]\\
&\quad\le 4T\lim_{(\epsilon,\delta)\to(0,0)}\int_{\mathcal{E}}\int_{\mathcal{H}}\left|F_{0,1}(e,\theta) - F_{\epsilon,\delta,1}(e,\theta)\right|^2\Lambda(\mathrm{d}\theta)\,\Xi(\mathrm{d}e) = 0
\end{aligned}$$

by dominated convergence, using Steps 1 and 2 and since we know from Lemma 39(iii) that $\int_{\mathcal{E}}[\int_0^{\ell(e)}(|e(u)|^{1/\alpha-2} + |e(u)|^{-4/3})\mathbf{1}_{\{0\le e(u)\le 1\}}\,\mathrm{d}u]^2\,\Xi(\mathrm{d}e) \le 4[\int_0^1\sqrt{x}(x^{1/\alpha-2} + x^{-4/3})\,\mathrm{d}x]^2$ which is finite. □

Gathering all the previous lemmas, we deduce the following.

PROPOSITION 22. *Consider the process $(Z_t)_{t\ge 0}$ defined in Lemma 21 (its definition depending on β) and set $S_t = \kappa^{-1/\alpha}Z_t$ if $\beta \in (d, 4+d)$ and $S_t = 8Z_t$ if $\beta = d$.*

(i) *If $\beta \in (1+d, 4+d)$, then $(\epsilon^{1/\alpha}[X_{t/\epsilon} - m_\beta t/\epsilon])_{t\ge 0} \xrightarrow{f.d.} (S_t)_{t\ge 0}$.*
(ii) *If $\beta = 1+d$, then $(\epsilon[X_{t/\epsilon} - M_\beta\zeta_\epsilon t/\epsilon])_{t\ge 0} \xrightarrow{f.d.} (S_t)_{t\ge 0}$.*
(iii) *If $\beta \in (d, 1+d)$, then $(\epsilon^{1/\alpha}X_{t/\epsilon})_{t\ge 0} \xrightarrow{f.d.} (S_t)_{t\ge 0}$.*
(iv) *If $\beta = d$, then $([\epsilon|\log\epsilon|]^{3/2}X_{t/\epsilon})_{t\ge 0} \xrightarrow{f.d.} (S_t)_{t\ge 0}$.*

PROOF. Since $a_\epsilon = \kappa\epsilon$ when $\beta \in (d, 4+d)$ and $a_\epsilon = \epsilon|\log\epsilon|/4$ when $\beta = d$, it is sufficient to prove that, setting $m_{\beta,\epsilon} = m_\beta$ if $\beta \in (1+d, 4+d)$, $m_{\beta,\epsilon} = M_\beta\zeta_\epsilon$ if $\beta = 1+d$ and $m_{\beta,\epsilon} = 0$ if $\beta \in [d, 1+d)$, it holds that $(a_\epsilon^{1/\alpha}[X_{t/\epsilon} - m_{\beta,\epsilon}t/\epsilon])_{t\ge 0} \xrightarrow{f.d.} (Z_t)_{t\ge 0}$.

We know from Lemma 11 that $(X_{t/\epsilon})_{t\ge 0} \stackrel{d}{=} (x_0 + \tilde{X}_t^\epsilon)_{t\ge 0}$. Since $a_\epsilon^{1/\alpha}x_0 \to 0$, it thus suffices to verify that $(\check{Z}_t^\epsilon)_{t\ge 0} \xrightarrow{f.d.} (Z_t)_{t\ge 0}$, where we have set $\check{Z}_t^\epsilon = a_\epsilon^{1/\alpha}[\tilde{X}_{t/\epsilon} - m_{\beta,\epsilon}t/\epsilon]$.

We consider $\Phi : \mathbb{D}([0,\infty), \mathbb{R}^d) \to \mathbb{R}$ of the form $\Phi(x) = \phi(x_{t_1}, \dots, x_{t_n})$ for some continuous and bounded $\phi : \mathbb{R}^n \to \mathbb{R}$. Our goal is to check that $I_\epsilon = \mathbb{E}[\Phi((\check{Z}_t^\epsilon)_{t\ge 0})] \to \mathbb{E}[\Phi((Z_t)_{t\ge 0})] = I$ as $\epsilon \to 0$.

We know from Lemma 14 that for all $t \geq 0$, all $\eta > 0$,

$$\lim_{\delta\to 0}\limsup_{\epsilon\to 0}\mathbb{P}\big(\big|\check{Z}^\epsilon_t - \big[Z^{\epsilon,\delta}_t + \kappa_{\epsilon,\delta}M_\beta t\big]\big| \geq \eta\big) = 0,$$

with the convention that $\kappa_{\epsilon,\delta} = 0$ when $\beta \in [d, 1+d)$. Next we denote by $I_{\epsilon,\delta} = \mathbb{E}[\Phi(([Z^{\epsilon,\delta}_t + \kappa_{\epsilon,\delta}M_\beta t])_{t\geq 0})]$ and we deduce that $\lim_{\delta\to 0}\limsup_{\epsilon\to 0}|I_{\epsilon,\delta} - I_\epsilon| = 0$. We thus have to check that $\lim_{\delta\to 0}\limsup_{\epsilon\to 0}|I_{\epsilon,\delta} - I| = 0$.

By Lemma 18, we know that for each $\delta > 0$, $\lim_{\epsilon\to 0}|I_{\epsilon,\delta} - J_{\epsilon,\delta}| = 0$ for each $\delta > 0$, where we have set $J_{\epsilon,\delta} = \mathbb{E}[\Phi((\bar{Z}^{\epsilon,\delta}_t + \kappa_{\epsilon,\delta}M_\beta t)_{t\geq 0})]$. It thus suffices to verify that $\lim_{\delta\to 0}\limsup_{\epsilon\to 0}|J_{\epsilon,\delta} - I| = 0$.

By Lemma 19, it holds that $J_{\epsilon,\delta} = \mathbb{E}[\Phi((\hat{Z}^{\epsilon,\delta}_t)_{t\geq 0})]$.

Finally, it follows from Lemma 21 that $\lim_{(\epsilon,\delta)\to(0,0)} J_{\epsilon,\delta} = I$, which completes the proof. $\square$

We still have to study a little our limiting processes.

PROPOSITION 23. *For any $\beta \in [d, 4+d)$, set $\alpha = (\beta+2-d)/3$ and consider the limit process $(S_t)_{t\geq 0}$ introduced in Proposition 22 (its definition depending on β).*

(i) *The process $(S_t)_{t\geq 0}$ is an α-stable Lévy process of which the Lévy measure q, depending only β and U, is given, for all $A \in \mathcal{B}(\mathbb{R}^d\setminus\{0\})$, by*

$$q(A) = \mathfrak{a}\int_0^\infty u^{-1-\alpha}\mathbb{P}(uY \in A)\,\mathrm{d}u,$$

where $\mathfrak{a} = \alpha/[\kappa\sqrt{2\pi}(\beta+2-d)^{2\alpha}]$ with $\kappa = (\beta+2-d)^{-1}\int_0^\infty u^{d-1}[\Gamma(u)]^{-\beta}\,\mathrm{d}u$ (see Lemma 42(i)) if $\beta \in (d, 4+d)$, where $\mathfrak{a} = 2^{7/6}/[3\sqrt{\pi}]$ if $\beta = d$ and where the $\mathbb{R}^d$-valued random variable Y is defined as follows. Consider a normalized Brownian excursion $\boldsymbol{e}$ (with unit length), independent of an eternal stationary spherical process $(\hat{\Theta}^\star_t)_{t\in\mathbb{R}}$ as in Lemma 38(iii) and set

$$Y = \int_0^1 \big[\boldsymbol{e}(u)\big]^{1/\alpha-2}\hat{\Theta}^\star_{[(\beta+2-d)^{-2}\int_{1/2}^u[\boldsymbol{e}(v)]^{-2}\,\mathrm{d}v]}\,\mathrm{d}u.$$

(ii) *Assume now that $\gamma \equiv 1$ (recall Assumption 1). Then $(\epsilon^{1/\alpha}X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0}$ if $\beta \in (d, 4+d)$ and $([\epsilon|\log\epsilon|]^{3/2}X_{t/\epsilon})_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0}$ if $\beta = d$. And in any case, $(S_t)_{t\geq 0}$ is a radially symmetric α-stable Lévy process, that is, there is a constant $\mathfrak{b} > 0$ depending on Γ, β and d such that $q(\mathrm{d}z) = \mathfrak{b}|z|^{-d-\alpha}\,\mathrm{d}z$ and thus $\mathbb{E}[\exp(i\xi\cdot S_t)] = \exp(-\mathfrak{b}'t|\xi|^\alpha)$ for all $\xi \in \mathbb{R}^d$, all $t \geq 0$, for some other constant $\mathfrak{b}' > 0$.*

Observe that in (i), the random variable Y is well-defined thanks to Lemma 39(iv).

PROOF. We start with point (i). It readily follows from its definition (see Proposition 22 and Lemma 21) that $(S_t)_{t\geq 0}$ is a Lévy process with Lévy measure given by

$$q(A) = \int_{\mathcal{E}}\Xi(\mathrm{d}e)\int_{\mathcal{H}}\Lambda(\mathrm{d}\theta)\mathbf{1}_{\{\mathfrak{c}F_{0,\infty}(e,\theta)\in A\}}, \quad A \in \mathcal{B}(\mathbb{R}^d\setminus\{0\}),$$

where $\mathfrak{c} = \kappa^{-1/\alpha}$ if $\beta \in (d, 4+d)$ and $\mathfrak{c} = 8$ if $\beta = d$. Using the decomposition (14) of Ξ and that $F_0(e,\theta) = 0$ if $x(e) = -1$, we have

$$\begin{aligned} q(A) &= \int_0^\infty \frac{\mathrm{d}\ell}{2\sqrt{2\pi\ell^3}}\int_{\mathcal{E}_1}\Xi_1(\mathrm{d}e)\int_{\mathcal{H}}\Lambda(\mathrm{d}\theta)\mathbf{1}_{\{\mathfrak{c}F_{0,\infty}(\sqrt{\ell}e(\cdot/\ell),\theta)\in A\}} \\ &= \int_0^\infty \frac{\mathrm{d}\ell}{2\sqrt{2\pi\ell^3}}\mathbb{P}\big(\mathfrak{c}F_{0,\infty}\big(\sqrt{\ell}\mathbf{e}(\cdot/\ell), \hat{\Theta}^\star\big) \in A\big) \end{aligned}$$

with the notation of the statement. But recalling Notation 20,

$$\begin{aligned}
&F_{0,\infty}(\sqrt{\ell}\mathbf{e}(\cdot/\ell), \hat{\Theta}^\star)\\
&\quad = \frac{1}{(\beta+2-d)^2}\int_0^\ell [\sqrt{\ell}\mathbf{e}(u/\ell)]^{1/\alpha-2}\hat{\Theta}^\star_{[(\beta+2-d)^{-2}\int_{\ell/2}^u[\sqrt{\ell}\mathbf{e}(v/\ell)]^{-2}\,\mathrm{d}v]}\,\mathrm{d}u\\
&\quad = \frac{\ell^{1/(2\alpha)}Y}{(\beta+2-d)^2},
\end{aligned}$$

whence

$$q(A) = \int_0^\infty \frac{\mathrm{d}\ell}{2\sqrt{2\pi\ell^3}}\mathbb{P}\Big(\frac{\mathfrak{c}\ell^{1/(2\alpha)}}{(\beta+2-d)^2}Y \in A\Big) = \int_0^\infty \frac{\mathfrak{a}\,\mathrm{d}u}{u^{1+\alpha}}\mathbb{P}(uY\in A).$$

Let us check that $(S_t)_{t\geq 0}$ is α-stable, that is, that its Lévy measure q satisfies $q(A_c) = c^\alpha q(A)$, for all $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, all $c > 0$, where we have set $A_c = \{z \in \mathbb{R}^d : cz \in A\}$. But

$$q(A_c) = \int_0^\infty \frac{\mathfrak{a}\,\mathrm{d}u}{u^{1+\alpha}}\mathbb{P}(cuY\in A) = c^\alpha \int_0^\infty \frac{\mathfrak{a}\,\mathrm{d}u}{u^{1+\alpha}}\mathbb{P}(uY\in A) = c^\alpha q(A).$$

We now turn to point (ii). If $\gamma \equiv 1$, then $M_\beta = m_\beta = 0$, so that the announced convergence to $(S_t)_{t\geq 0}$ follows from Proposition 22. Moreover, $(S_t)_{t\geq 0}$ is radially symmetric by definition, recalling Proposition 22, Lemma 21 and that $\mathbf{N}(\mathrm{d}s, \mathrm{d}e, \mathrm{d}\theta)$ is a Poisson measure with intensity $\mathrm{d}s\,\Xi(\mathrm{d}e)\Lambda(\mathrm{d}\theta)$ and observing that $\Lambda \in \mathcal{P}(\mathcal{H})$ is the law of $\hat{\Theta}^\star$, which is a stationary $\mathbb{S}_{d-1}$-valued Brownian motion (because $\gamma \equiv 1$; see Lemma 38). □

We can finally handle the following.

PROOF OF THEOREM 4(C)–(D)–(E)–(F). Points (c)–(e)–(f) immediately follow from Propositions 22 and 23. For point (d), which concerns the case where $\beta = 1 + d$, we know that $(\epsilon[X_{t/\epsilon} - M_\beta\zeta_\epsilon t/\epsilon])_{t\geq 0} \xrightarrow{f.d.} (S_t)_{t\geq 0}$, where $(S_t)_{t\geq 0}$ is a 1-stable Lévy process. We claim that under the additional condition $\int_1^\infty r^{-1}|[\Gamma(r)]^{-1}r - 1|\,\mathrm{d}r < \infty$, there is $b \in \mathbb{R}$ such that

$$\lim_{\epsilon\to 0}\Big(\zeta_\epsilon - \frac{1}{9\kappa}|\log\epsilon|\Big) = b, \tag{18}$$

whence $(\epsilon[X_{t/\epsilon} - M_\beta|\log\epsilon|t/(9\kappa\epsilon)])_{t\geq 0} \xrightarrow{f.d.} (S_t + bM_\beta t)_{t\geq 0}$. This completes the proof because the Lévy process $(S_t + bM_\beta t)_{t\geq 0}$ is also a 1-stable.

To check (18), we recall Notation 13 to write $\zeta_\epsilon = C_\epsilon/D_\epsilon$, where

$$C_\epsilon = \int_{-\infty}^{1/a_\epsilon} h^{-1}(w)[\sigma(w)]^{-2}\,\mathrm{d}w \quad \text{and} \quad D_\epsilon = \int_{-\infty}^{1/a_\epsilon}[\sigma(w)]^{-2}\,\mathrm{d}w.$$

By Lemma 42(i)–(vi), we have $|D_\epsilon - \kappa| \leq C\int_{1/a_\epsilon}^\infty (1+|w|)^{-4/3}\,\mathrm{d}w \leq Ca_\epsilon^{1/3} \leq C\epsilon^{1/3}$ since $a_\epsilon = \kappa\epsilon$.

We thus only have to verify that $\lim_{\epsilon\to 0}(C_\epsilon - |\log\epsilon|/9)$ exists. Recalling Notation 9 and using the substitution $r = h^{-1}(w)$, we find

$$C_\epsilon = \int_0^{h^{-1}(1/a_\epsilon)} r[h'(r)]^{-1}\,\mathrm{d}r = \frac{1}{3}\int_0^{A_\epsilon} r^d[\Gamma(r)]^{-1-d}\,\mathrm{d}r,$$

where we have set $A_\epsilon = h^{-1}(1/a_\epsilon)$. Since $h(r) = 3\int_{r_0}^r u^{1-d}[\Gamma(u)]^{1+d}\,\mathrm{d}u \sim r^3$ as $r \to \infty$ and since $a_\epsilon = \kappa\epsilon$, it comes that $A_\epsilon \sim_{\epsilon\to 0} [\kappa\epsilon]^{-1/3}$, and so $\lim_{\epsilon\to 0}(|\log\epsilon|/9 - (\log A_\epsilon)/3) =$

$(\log\kappa)/9$, and we are reduced to check that $\lim_{\epsilon\to 0}(C_\epsilon - (\log A_\epsilon)/3)$ exists. But

$$C_\epsilon - \frac{1}{3}\log A_\epsilon = \frac{1}{3}\int_0^{A_\epsilon}\left[\left(\frac{r}{\Gamma(r)}\right)^{1+d} - \mathbf{1}_{\{r\geq 1\}}\right]\frac{dr}{r}$$

$$\to \frac{1}{3}\int_0^{\infty}\left[\left(\frac{r}{\Gamma(r)}\right)^{1+d} - \mathbf{1}_{\{r\geq 1\}}\right]\frac{dr}{r}$$

as $\epsilon\to 0$. This last quantity is well-defined and finite, because $\Gamma:[0,\infty)\to(0,\infty)$ is bounded from below, because $\Gamma(r)\sim r$ as $r\to\infty$, and because $\int_1^\infty r^{-1}|(r/\Gamma(r))-1|\,dr<\infty$ by assumption. □

REMARK 24. In Theorem 4(d), that is, when $\beta = 1+d$, the constant c is given by $c = 1/(9\kappa) = (3\int_0^\infty u^{d-1}[\Gamma(u)]^{-1-d}\,du)^{-1}$ by Lemma 42(i).

6. The integrated Bessel regime. Here we give the proof of Theorem 4(g). We first define properly the limit process $(\mathcal{V}_t)_{t\geq 0}$.

DEFINITION 25. We fix $\beta\in(d-2,d)$ and consider a Bessel process $(\mathcal{R}_t)_{t\geq 0}$ starting from 0 with dimension $d-\beta\in(0,2)$, as well as an i.i.d. family $\{(\hat{\Theta}_t^{\star,i})_{t\in\mathbb{R}}, i\geq 1\}$ with common law Λ (see Lemma 38(iii)) independent of $(\mathcal{R}_t)_{t\geq 0}$. We set $\mathcal{Z}=\{t\geq 0:\mathcal{R}_t=0\}$ and we write $\mathcal{Z}^c=\bigcup_{i\geq 1}(\ell_i,r_i)$ as the (countable) union of its connected components: for all $i\geq 1$, we have $\mathcal{R}_{\ell_i}=\mathcal{R}_{r_i}=0$ and $\mathcal{R}_t>0$ for all $t\in(\ell_i,r_i)$. We then define

$$\mathcal{V}_t = \sum_{i\geq 1}\mathbf{1}_{\{t\in(\ell_i,r_i)\}}\mathcal{R}_t\hat{\Theta}^{\star,i}_{[\int_{(\ell_i+r_i)/2}^t \mathcal{R}_s^{-2}\,ds]}.$$

REMARK 26. In some sense to be precised, $(\mathcal{V}_t)_{t\geq 0}$ is the unique (in law) solution to

$$\mathcal{V}_t = B_t - \frac{\beta}{2}\int_0^t \mathcal{F}(\mathcal{V}_s)\,ds,$$

where $\mathcal{F}(v)=\mathcal{U}^{-1}(v)\nabla\mathcal{U}(v)$, with $\mathcal{U}(v)=|v|\gamma(v/|v|)$ (if $\gamma\equiv 1$, one finds $\mathcal{F}(v)=|v|^{-2}v$) and where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion. This equation is what one gets when informally searching for the limit of $\sqrt{\epsilon}V_{t/\epsilon}$ as $\epsilon\to 0$, $(V_t)_{t\geq 0}$ being the solution to (2). But it is not clearly well-defined because $\mathcal{F}$ is singular at 0. See [14], Section 6, for the detailed study of such an equation in dimension $d=2$ and when $\gamma\equiv 1$.

We now introduce some notation that will be used during the whole section. We fix $\beta\in(d-2,d)$, recall Notation 9 and set, for $\epsilon\in(0,1)$,

$$a_\epsilon = \epsilon^{(\beta+2-d)/2}.$$

For a $1D$-Brownian motion $(W_t)_{t\geq 0}$, we set $A_t^\epsilon = \epsilon a_\epsilon^{-2}\int_0^t[\sigma(W_s/a_\epsilon)]^{-2}\,ds$, introduce its inverse ρ_t^ϵ and put $R_t^\epsilon=\sqrt{\epsilon}h^{-1}(W_{\rho_t^\epsilon}/a_\epsilon)$ and $T_t^\epsilon=\int_0^t[R_s^\epsilon]^{-2}\,ds$. We also consider the solution $(\hat{\Theta}_t)_{t\geq 0}$ of (4), independent of $(W_t)_{t\geq 0}$.

LEMMA 27. *For all $\epsilon\in(0,1)$, $(\sqrt{\epsilon}V_{t/\epsilon})_{t\geq 0}\overset{d}{=}(R_t^\epsilon\hat{\Theta}_{T_t^\epsilon})_{t\geq 0}$, for $(V_t)_{t\geq 0}$ the velocity process of (2).*

PROOF. By Lemmas 8 and 10, setting $S_t^\epsilon=\epsilon^{-1/2}R_{\epsilon t}^\epsilon$ and $\bar{T}_t^\epsilon=\int_0^t[S_s^\epsilon]^{-2}\,ds$, it holds that $(S_t^\epsilon\hat{\Theta}_{\bar{T}_t^\epsilon})_{t\geq 0}\overset{d}{=}(V_t)_{t\geq 0}$, whence $(\sqrt{\epsilon}S_{t/\epsilon}^\epsilon\hat{\Theta}_{\bar{T}_{t/\epsilon}^\epsilon})_{t\geq 0}\overset{d}{=}(\sqrt{\epsilon}V_{t/\epsilon})_{t\geq 0}$. To conclude, observe that $\sqrt{\epsilon}S_{t/\epsilon}^\epsilon=R_t^\epsilon$ and $\bar{T}_{t/\epsilon}^\epsilon=\int_0^{t/\epsilon}[\epsilon^{-1/2}R_{\epsilon s}^\epsilon]^{-2}\,ds=\int_0^t[R_s^\epsilon]^{-2}\,ds=T_t^\epsilon$. □

We first study the convergence of the radius process.

LEMMA 28. *There is a Bessel process $(\mathcal{R}_t)_{t\geq 0}$ with dimension $d-\beta$ issued from 0 such that $(R_t^\epsilon)_{t\geq 0}$ a.s. converges to $(\mathcal{R}_t)_{t\geq 0}$, uniformly on compact time intervals.*

PROOF. Since $[\sigma(w)]^{-2} \leq C(1+|w|)^{-2(\beta+1-d)/(\beta+2-d)}$ by Lemma 42(vi) and since

$$\lim_{\epsilon\to 0} \epsilon\big[a_\epsilon \sigma(w/a_\epsilon)\big]^{-2} = (\beta+2-d)^{-2} w^{-2(\beta+1-d)/(\beta+2-d)} \mathbf{1}_{\{w>0\}}$$

by Lemma 42(xi), since $\int_0^T |W_s|^{-2(\beta+1-d)/(\beta+2-d)}\,\mathrm{d}s$ is finite a.s. for all $T>0$ because $2(\beta+1-d)/(\beta+2-d)<1$, we conclude, by dominated convergence, that a.s., for all $t\geq 0$, $(A_t^\epsilon)_{t\geq 0}$ converges to

$$A_t = (\beta+2-d)^{-2} \int_0^t W_s^{-2(\beta+1-d)/(\beta+2-d)} \mathbf{1}_{\{W_s>0\}}\,\mathrm{d}s.$$

Let $\rho_t = \inf\{s>0 : A_s > t\}$ be its generalized inverse and let $J=\{t>0 : \rho_t > \rho_{t-}\}$. We now verify that a.s., for all $T>0$,

$$\lim_{\epsilon\to 0} \sup_{u\in[0,T]} \big|(W_{\rho_u^\epsilon})_+ - (W_{\rho_u})_+\big| = 0. \tag{19}$$

(a) By Lemma 41, we know that a.s., for all $t\in[0,\infty)\setminus J$, $\rho_t^\epsilon \to \rho_t$.

(b) We a.s. have, for all $t\geq 0$, $A_{\rho_{t-}} = A_{\rho_t} = t$ (since A is continuous) and

$$\rho_{A_t} = \inf\{s>t : W_s>0\} = \begin{cases} t & \text{if } W_t\geq 0,\\ \inf\{s>t : W_s=0\} & \text{if } W_t<0.\end{cases}$$

Indeed, the second equality is clear and, setting $\nu_t = \inf\{s>t : W_s>0\}$, it holds that $\rho_{A_t} = \inf\{s>0 : A_s > A_t\} = \inf\{s>t : A_s > A_{\nu_t}\}$ (because $A_{\nu_t}=A_t$ by definition of A), whence clearly $\rho_{A_t} = \inf\{s>t : W_s>0\}$ (again by definition of A).

(c) Since A is continuous, we deduce from (a) that a.s., for a.e. $t\geq 0$, $A_{\rho_t^\epsilon}\to A_{\rho_t}$. Since moreover $t\to A_{\rho_t}$ is a.s. continuous (by (b)) and nondecreasing (as well as $t\to A_{\rho_t^\epsilon}$ for each $\epsilon>0$), we conclude from the Dini theorem that a.s., $\sup_{[0,T]}|A_{\rho_t^\epsilon} - A_{\rho_t}|\to 0$.

(d) By (b), we a.s. have $(W_u)_+ = W_{\rho_{A_u}}$ for all $u\geq 0$.

(e) Almost surely, $u\to W_{\rho_u}$ is nonnegative and continuous. First, by (b), we have $W_{\rho_u} = W_{\rho_{A_{\rho_u}}}$, which is nonnegative by (d). Next, it suffices to prove that $W_{\rho_{u-}} = W_{\rho_u}$ for all $u\geq 0$. Setting $t=\rho_{u-}$, we see that $W_t = W_{\rho_{A_t}}$ (by (b) and since $W_t\geq 0$). Hence $W_t = W_{\rho_{A_{\rho_{u-}}}} = W_{\rho_{A_{\rho_u}}}$ by (b), whence $W_t = W_{\rho_u}$ as desired.

(f) To complete the proof of (19), it suffices to note that $(W_{\rho_u^\epsilon})_+ - (W_{\rho_u})_+ = W_{\rho_{A_{\rho_u^\epsilon}}} - W_{\rho_u}$ by (d) and (e), that $u\to W_{\rho_u}$ is continuous by (e), and finally to use point (c).

By Lemma 42(x), $\sqrt{\epsilon}h^{-1}(w/a_\epsilon)\to w_+^{1/(\beta+2-d)}$, uniformly on compact subsets of $\mathbb{R}$. Together with (19), this implies that $(R_t^\epsilon = \sqrt{\epsilon}h^{-1}(W_{\rho_t^\epsilon}/a_\epsilon))_{t\geq 0}$ a.s. converges, uniformly on compact time intervals, to $((W_{\rho_t})_+^{1/(\beta+2-d)})_{t\geq 0}$, which is a Bessel process with dimension $d-\beta$ issued from 0 by Lemma 40. □

We can now give the following proof.

PROOF OF THEOREM 4(G). Our goal is to verify that $(R_t^\epsilon \hat{\Theta}_{T_t^\epsilon})_{t\geq 0}$ goes in law to $(\mathcal{V}_t)_{t\geq 0}$, for the usual convergence of continuous processes. This implies that $(\epsilon^{3/2}X_{t/\epsilon})_{t\geq 0}$ goes in law to $(\int_0^t \mathcal{V}_s\,\mathrm{d}s)_{t\geq 0}$, since by Lemma 27, $(\epsilon^{3/2}X_{t/\epsilon} = \epsilon^{3/2}x_0 + \int_0^t \sqrt{\epsilon}V_{s/\epsilon}\,\mathrm{d}s)_{t\geq 0}$ and $(\epsilon^{3/2}x_0 + \int_0^t R_s^\epsilon \hat{\Theta}_{T_s^\epsilon}\,\mathrm{d}s)_{t\geq 0}$ have the same law. We already know from Lemma 28 that a.s., $\sup_{[0,T]}|R_t^\epsilon - \mathcal{R}_t|\to 0$ for all $T>0$, where $(\mathcal{R}_t)_{t\geq 0}$ is a Bessel process as in Definition 25 and we introduce $\mathcal{Z}=\{t\geq 0 : \mathcal{R}_t = 0\}$ and write $\mathcal{Z}^c = \bigcup_{i\geq 1}(\ell_i, r_i)$ with, for all $i\geq 1$,

$\mathcal{R}_{\ell_i} = \mathcal{R}_{r_i} = 0$ and $\mathcal{R}_t > 0$ for all $t \in (\ell_i, r_i)$. Finally, we set $\mathcal{W} = \sigma(W_s, s \geq 0)$ and observe that $\mathcal{W} = \sigma(R_t^\epsilon, \mathcal{R}_t, t \geq 0, \epsilon \in (0,1))$ is independent of $(\hat{\Theta}_t)_{t\geq 0}$.

Step 1. For all $i > j \geq 1$, we have $\lim_{\epsilon\to 0}(\tau_i^\epsilon - \tau_j^\epsilon) = \infty$ a.s., where

$$\tau_i^\epsilon = T^\epsilon_{(\ell_i+r_i)/2} = \int_0^{(\ell_i+r_i)/2} \frac{\mathrm{d}s}{[R_s^\epsilon]^2}.$$

Indeed, by the Fatou lemma, we know that a.s.,

$$\liminf_{\epsilon\to 0}(\tau_i^\epsilon - \tau_j^\epsilon) \geq \int_{(\ell_j+r_j)/2}^{(\ell_i+r_i)/2} \frac{\mathrm{d}s}{[\mathcal{R}_s]^2} \geq \int_{\ell_i}^{(\ell_i+r_i)/2} \frac{\mathrm{d}s}{[\mathcal{R}_s]^2} = \infty$$

by Lemma 40(ii).

Step 2. For $T > 0$ and $\delta > 0$, we consider the (a.s. finite) set of indices

$$\mathcal{I}_{\delta,T} = \Big\{ i \geq 1 : \ell_i \leq T \text{ and } \sup_{s\in(\ell_i,r_i)} \mathcal{R}_s > \delta \Big\}$$

and for $i \in \mathcal{I}_{\delta,T}$, we introduce $\ell_i < \ell_i^\delta < r_i^\delta < r_i$ defined by

$$\ell_i^\delta = \inf\{s > \ell_i : \mathcal{R}_s > \delta\} \quad \text{and} \quad r_i^\delta = \sup\{s < r_i : \mathcal{R}_s > \delta\}.$$

We also set

$$A_{\delta,T} = 2 \max_{i\in\mathcal{I}_{\delta,T}} \bigg[\bigg| \int_{(\ell_i+r_i)/2}^{\ell_i^\delta} \frac{\mathrm{d}s}{[\mathcal{R}_s]^2} \bigg| + \bigg| \int_{(\ell_i+r_i)/2}^{r_i^\delta} \frac{\mathrm{d}s}{[\mathcal{R}_s]^2} \bigg| \bigg].$$

By Lemma 38(iv), knowing $\mathcal{W}$, there is an i.i.d. family $((\hat{\Theta}_t^{\star,i,\epsilon,\delta})_{t\in\mathbb{R}})_{i\in\mathcal{I}_{\delta,T}}$ of Λ-distributed processes such that, setting

$$\Omega_{\epsilon,\delta,T} = \big\{ \forall i \in \mathcal{I}_{\delta,T}, (\hat{\Theta}_t^{\star,i,\epsilon,\delta})_{t\in[-A_{\delta,T},A_{\delta,T}]} = (\hat{\Theta}_{(\tau_i^\epsilon+t)\vee 0})_{t\in[-A_{\delta,T},A_{\delta,T}]} \big\},$$

we have $\Pr(\Omega_{\epsilon,\delta,T}|\mathcal{W}) = \mathbf{p}_{\delta,T}(\epsilon)$, where $\mathbf{p}_{\delta,T}(\epsilon) = p_{A_{\delta,T}}(\tau_{i_1}^\epsilon, \tau_{i_2}^\epsilon - \tau_{i_1}^\epsilon, \ldots, \tau_{i_n}^\epsilon - \tau_{i_{n-1}}^\epsilon)$ and where we have written $\mathcal{I}_{\delta,T} = \{i_1, \ldots, i_n\}$. We know that $\mathbf{p}_{\delta,T}(\epsilon)$ a.s. tends to 1 as $\epsilon \to 0$, so that $r_{\delta,T}(\epsilon) = \mathbb{P}(\Omega_{\epsilon,\delta,T}) = \mathbb{E}[\mathbf{p}_{\delta,T}(\epsilon)]$ also tends to 1 as $\epsilon \to 0$.

Step 3. Knowing $\mathcal{W}$, we consider an i.i.d. family $((\hat{\Theta}_t^{\star,i,\epsilon,\delta})_{t\in\mathbb{R}})_{i\in\mathbb{N}^*\setminus\mathcal{I}_{\delta,T}}$, independent of $((\hat{\Theta}_t^{\star,i,\epsilon,\delta})_{t\in\mathbb{R}})_{i\in\mathbb{N}^*\setminus\mathcal{I}_{\delta,T}}$, and we consider the process $(\mathcal{V}_t^{\epsilon,\delta})_{t\geq 0}$ built from $(\mathcal{R}_t)_{t\geq 0}$ and the i.i.d. family $((\hat{\Theta}_t^{\star,i,\epsilon,\delta})_{t\in\mathbb{R}})_{i\geq 1}$ as in Definition 25, that is,

$$\mathcal{V}_t^{\epsilon,\delta} = \sum_{i\geq 1} \mathbf{1}_{\{t\in(\ell_i,r_i)\}} \mathcal{R}_t \hat{\Theta}^{\star,i,\epsilon,\delta}_{[\int_{(\ell_i+r_i)/2}^t \mathcal{R}_s^{-2}\,\mathrm{d}s]}.$$

For all $\epsilon \in (0,1)$ and all $\delta \in (0,1)$, $(\mathcal{V}_t^{\epsilon,\delta})_{t\geq 0} \stackrel{d}{=} (\mathcal{V}_t)_{t\geq 0}$. We will show that for all $\eta > 0$,

$$\lim_{\delta\to 0} \limsup_{\epsilon\to 0} \mathbb{P}[\Delta_{\epsilon,\delta,T} > \eta] = 0 \quad \text{where } \Delta_{T,\delta,\epsilon} = \sup_{[0,T]} \big| R_t^\epsilon \hat{\Theta}_{T_t^\epsilon} - \mathcal{V}_t^{\epsilon,\delta} \big|$$

and this will conclude the proof. Recalling that $|\mathcal{V}_t^{\epsilon,\delta}| = \mathcal{R}_t$,

$$\begin{aligned}\Delta_{\epsilon,\delta,T} \leq {} & \sup_{[0,T]} \big| R_t^\epsilon - \mathcal{R}_t \big| + \sup_{[0,T]} \big| \mathcal{R}_t \hat{\Theta}_{T_t^\epsilon} - \mathcal{V}_t^{\epsilon,\delta} \big| \mathbf{1}_{\{\mathcal{R}_t \leq \delta\}} \\ & + \sup_{[0,T]} \big| \mathcal{R}_t \hat{\Theta}_{T_t^\epsilon} - \mathcal{V}_t^{\epsilon,\delta} \big| \mathbf{1}_{\{\mathcal{R}_t > \delta\}}.\end{aligned}$$

We already know that the first term a.s. tends to 0 as $\epsilon \to 0$, the second one is bounded by 2δ and the third one is bounded by $(\sup_{[0,T]} \mathcal{R}_t)\Delta'_{\epsilon,\delta,T}$, where we denote by $\Delta'_{\epsilon,\delta,T} = \sup_{[0,T]} |\hat{\Theta}_{T_t^\epsilon} - \mathcal{R}_t^{-1}\mathcal{V}_t^{\epsilon,\delta}| \mathbf{1}_{\{\mathcal{R}_t > \delta\}}$. All in all, we only have to check that

$$\lim_{\delta\to 0} \limsup_{\epsilon\to 0} \mathbb{P}\big[\Delta'_{\epsilon,\delta,T} > \eta\big] = 0.$$

Step 4. For all $t \in [0,T]$, $\mathcal{R}_t > \delta$ implies that $t \in \bigcup_{i\in\mathcal{I}_{\delta,T}}(\ell_i^\delta, r_i^\delta)$, whence

$$\mathcal{R}_t^{-1}\mathcal{V}_t^{\epsilon,\delta} - \hat{\Theta}_{T_t^\epsilon} = \sum_{i\in\mathcal{I}_{\delta,T}} \mathbf{1}_{\{t\in(\ell_i^\delta,r_i^\delta)\}}\big(\hat{\Theta}^{\star,i,\epsilon,\delta}_{[\int_{(\ell_i+r_i)/2}^t \mathcal{R}_s^{-2}\,\mathrm{d}s]} - \hat{\Theta}_{[\tau_i^\epsilon + \int_{(\ell_i+r_i)/2}^t [R_s^\epsilon]^{-2}\,\mathrm{d}s]}\big),$$

because $T_t^\epsilon = \tau_i^\epsilon + \int_{(\ell_i+r_i)/2}^t [R_s^\epsilon]^{-2}\,\mathrm{d}s$. Next, for $x \in (0,1)$, it holds that

$$\lim_{\epsilon\to 0}\mathbb{P}\big(\Omega'_{\epsilon,\delta,T}(x)\big) = 1,$$

where

$$\Omega'_{\epsilon,\delta,T}(x) = \left\{\forall i \in \mathcal{I}_{\delta,T}, \forall t \in (\ell_i^\delta, r_i^\delta), \left|\int_{(\ell_i+r_i)/2}^t \frac{\mathrm{d}s}{\mathcal{R}_s^2} - \int_{(\ell_i+r_i)/2}^t \frac{\mathrm{d}s}{[R_s^\epsilon]^2}\right| \le x\right\}.$$

Indeed, for each $i \in \mathcal{I}_{\delta,T}$, $\mathcal{R}_s$ is continuous and positive on $(\ell_i^\delta, r_i^\delta)$ and we have already seen that $\lim_{\epsilon\to 0}\sup_{[0,T]}|R_t^\epsilon - \mathcal{R}_t| = 0$. For the same reasons, it holds that $\lim_{\epsilon\to 0}\mathbb{P}(\Omega''_{\epsilon,\delta,T}) = 1$

$$\Omega''_{\epsilon,\delta,T} = \left\{\forall i \in \mathcal{I}_{\delta,T}, \forall t \in (\ell_i^\delta, r_i^\delta), \left|\int_{(\ell_i+r_i)/2}^t \frac{\mathrm{d}s}{\mathcal{R}_s^2}\right| \vee \left|\int_{(\ell_i+r_i)/2}^t \frac{\mathrm{d}s}{[R_s^\epsilon]^2}\right| \le A_{\delta,T}\right\}.$$

Now on $\bar{\Omega}_{\epsilon,\delta,T}(x) = \Omega_{\epsilon,\delta,T} \cap \Omega'_{\epsilon,\delta,T}(x) \cap \Omega''_{\epsilon,\delta,T}$, we have, for any $t \in [0,T]$,

$$\begin{aligned}&(\mathcal{R}_t^{-1}\mathcal{V}_t^{\epsilon,\delta} - \hat{\Theta}_{T_t^\epsilon})\mathbf{1}_{\{\mathcal{R}_t>\delta\}}\\ &\quad = \sum_{i\in\mathcal{I}_{\delta,T}} \mathbf{1}_{\{t\in(\ell_i^\delta,r_i^\delta)\}}\big(\hat{\Theta}^{\star,i,\epsilon,\delta}_{[\int_{(\ell_i+r_i)/2}^t \mathcal{R}_s^{-2}\,\mathrm{d}s]} - \hat{\Theta}^{\star,i,\epsilon,\delta}_{[\int_{(\ell_i+r_i)/2}^t [R_s^\epsilon]^{-2}\,\mathrm{d}s]}\big),\end{aligned}$$

whence

$$\begin{aligned}\Delta'_{\epsilon,\delta,T} \le \#(\mathcal{I}_{\delta,T})\sup\big\{&\big|\hat{\Theta}_a^{\star,i,\epsilon,\delta} - \hat{\Theta}_b^{\star,i,\epsilon,\delta}\big| :\\ &i \in \mathcal{I}_{\delta,T}, a,b \in [-A_{\delta,T}, A_{\delta,T}], |a-b| < x\big\}\end{aligned}$$

and we denote by $M_{\delta,T}^\epsilon(x)$ this last expression. But the law of $M_{\delta,T}^\epsilon(x)$ does not depend on $\epsilon \in (0,1)$ (because conditionally on $\mathcal{W}$, the family $((\hat{\Theta}_t^{\star,i,\epsilon,\delta})_{t\in\mathbb{R}})_{i\in\mathcal{I}_{\delta,T}}$ is i.i.d. and Λ-distributed. All in all, we have proved that for all $\delta > 0$, all $T > 0$, all $\eta > 0$, $x > 0$, with a small abuse of notation,

$$\begin{aligned}\limsup_{\epsilon\to 0}\mathbb{P}\big(\Delta'_{\epsilon,\delta,T} > \eta\big) &\le \mathbb{P}\big(M_{\delta,T}(x) > \eta\big) + \limsup_{\epsilon\to 0}\mathbb{P}\big(\big(\bar{\Omega}_{\epsilon,\delta,T}(x)\big)^c\big)\\ &= \mathbb{P}\big(M_{\delta,T}(x) > \eta\big).\end{aligned}$$

But $\lim_{x\to 0}\mathbb{P}(M_{\delta,T}(x) > \eta) = 0$, because the Λ-distributed processes are continuous. We thus have $\limsup_{\epsilon\to 0}\mathbb{P}(\Delta'_{\epsilon,\delta,T} > \eta) = 0$ for each $\delta > 0$, which completes the proof. □

7. The diffusive regime. The goal of this section is to prove Theorem 4(a). As already mentioned, this regime is almost treated in Pardoux–Veretennikov [32], which consider much more general problems. However, we can not strictly apply their result because F is not locally bounded (except if $\gamma \equiv 1$). Moreover, our proof is much simpler (because our model is much simpler). First, we adapt to our context a Poincaré inequality found in Cattiaux–Gozlan–Guillin–Roberto [10].

LEMMA 29. *For any $\beta > 2 + d$, there is a constant $C > 0$ such that for all $f \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, \mu_\beta)$ satisfying $\int_{\mathbb{R}^d} f(v)\mu_\beta(\mathrm{d}v) = 0$,*

$$\int_{\mathbb{R}^d}[f(v)]^2(1+|v|)^{-2}\mu_\beta(\mathrm{d}v) \le C\int_{\mathbb{R}^d}|\nabla f(v)|^2\mu_\beta(\mathrm{d}v).$$

PROOF. The constants below are allowed to depend only on U, β and d. By Assumption 1, there are $0 < C_1 < C_2$ such that $C_1(1+|v|)^{-\beta}\,\mathrm{d}v \le \mu_\beta(\mathrm{d}v) \le C_2(1+|v|)^{-\beta}\,\mathrm{d}v$.

We know from [10], Proposition 5.5, that for any $\alpha > d$, there is a constant C such that for $g \in H^1_{\mathrm{loc}}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d, (1+|v|)^{-\alpha}\,\mathrm{d}v)$ satisfying $\int_{\mathbb{R}^d} g(v)(1+|v|)^{-\alpha}\,\mathrm{d}v = 0$, we have the inequality $\int_{\mathbb{R}^d}[g(v)]^2(1+|v|)^{-\alpha}\,\mathrm{d}v \le C\int_{\mathbb{R}^d}|\nabla g(v)|^2(1+|v|)^{2-\alpha}\,\mathrm{d}v$.

For f as in the statement, we apply this inequality with $\alpha = \beta+2 > d$ and $g = f - a$, the constant $a \in \mathbb{R}$ being such that $\int_{\mathbb{R}^d} g(v)(1+|v|)^{-\beta-2}\,\mathrm{d}v = 0$. We finally obtain that $\int_{\mathbb{R}^d}[g(v)]^2(1+|v|)^{-2-\beta}\,\mathrm{d}v \le C_3\int_{\mathbb{R}^d}|\nabla g(v)|^2(1+|v|)^{-\beta}\,\mathrm{d}v$.

But $\int_{\mathbb{R}^d} f(v)\mu_\beta(\mathrm{d}v) = 0$, whence $a = -\int_{\mathbb{R}^d} g(v)\mu_\beta(\mathrm{d}v)$ and thus

$$\begin{aligned} a^2 &\le C_2^2\left[\int_{\mathbb{R}^d} g(v)\big(1+|v|\big)^{-\beta}\,\mathrm{d}v\right]^2 \\ &= C_2^2\left[\int_{\mathbb{R}^d}\big(1+|v|\big)^{-\beta/2-1}g(v)\big(1+|v|\big)^{1-\beta/2}\,\mathrm{d}v\right]^2, \end{aligned}$$

whence $a^2 \le C_2^2 C_4\int_{\mathbb{R}^d}[g(v)]^2(1+|v|)^{-\beta-2}\,\mathrm{d}v$ by the Cauchy–Schwarz inequality, where the constant $C_4 = \int_{\mathbb{R}^d}(1+|v|)^{2-\beta}\,\mathrm{d}v$ is finite because $\beta > 2+d$.

Using that $f^2 \le 2g^2 + 2a^2$ and setting $C_5 = \int_{\mathbb{R}^d}(1+|v|)^{-2-\beta}\,\mathrm{d}v$, we find that

$$\begin{aligned} &\int_{\mathbb{R}^d}\big[f(v)\big]^2\big(1+|v|\big)^{-2}\mu_\beta(\mathrm{d}v) \\ &\quad\le 2C_2\int_{\mathbb{R}^d}\big[g(v)\big]^2\big(1+|v|\big)^{-2-\beta}\,\mathrm{d}v + 2C_2a^2\int_{\mathbb{R}^d}\big(1+|v|\big)^{-2-\beta}\,\mathrm{d}v \\ &\quad\le 2C_2\big[1+C_2^2C_4C_5\big]\int_{\mathbb{R}^d}\big[g(v)\big]^2\big(1+|v|\big)^{-2-\beta}\,\mathrm{d}v \\ &\quad\le 2C_2C_3\big[1+C_2^2C_4C_5\big]\int_{\mathbb{R}^d}\big|\nabla g(v)\big|^2\big(1+|v|\big)^{-\beta}\,\mathrm{d}v \\ &\quad\le 2C_1^{-1}C_2C_3\big[1+C_2^2C_4C_5\big]\int_{\mathbb{R}^d}\big|\nabla f(v)\big|^2\mu_\beta(\mathrm{d}v). \end{aligned}$$

We finally used that $\nabla g = \nabla f$. □

We next state a lemma that will allow us to solve the Poisson equation $\mathcal{L}f(v) = v - m_\beta$, where $\mathcal{L}$ is the generator of the velocity process. We state a slightly more general version, that will be needed when treating the critical case $\beta = 4+d$

LEMMA 30. *Suppose that $\beta > 2+d$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be of class C^∞ and satisfy*

$$\int_{\mathbb{R}^d} g(v)\mu_\beta(\mathrm{d}v) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d}\big[g(v)\big]^2\big(1+|v|\big)^2\mu_\beta(\mathrm{d}v) < \infty. \tag{20}$$

There exists $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$, of class C^∞, such that $\int_{\mathbb{R}^d}|\nabla f(v)|^2\mu_\beta(\mathrm{d}v) < \infty$ and solving the equation $\frac{1}{2}[\Delta f - \beta F\cdot\nabla f] = g$ on $\mathbb{R}^d \setminus \{0\}$.

PROOF. We divide the proof in three steps.

Step 1. We introduce the weighted Sobolev space $H^1_\beta = \{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : |||\varphi|||_\beta < \infty$ and $\int_{\mathbb{R}^d}\varphi(v)\mu_\beta(\mathrm{d}v) = 0\}$, where we have set

$$|||\varphi|||_\beta^2 = \int_{\mathbb{R}^d}\big[\varphi(v)\big]^2\big(1+|v|\big)^{-2}\mu_\beta(\mathrm{d}v) + \int_{\mathbb{R}^d}\big|\nabla\varphi(v)\big|^2\mu_\beta(\mathrm{d}v).$$

By the Lax–Milgram theorem, there is a unique $f \in H^1_\beta$ such that for all $\varphi \in H^1_\beta$, $\int_{\mathbb{R}^d}\nabla f(v)\cdot\nabla\varphi(v)\mu_\beta(\mathrm{d}v) = -2\int_{\mathbb{R}^d}\varphi(v)g(v)\mu_\beta(\mathrm{d}v)$.

Indeed, the quadratic form $A(\varphi,\phi)=\int_{\mathbb{R}^d}\nabla\varphi(v)\cdot\nabla\phi(v)\mu_\beta(\mathrm{d}v)$ is continuous on the Hilbert space H^1_β and coercive (i.e., there is $c>0$ such that $A(\varphi,\varphi)\geq c|||\varphi|||_\beta$ for all $\varphi\in H^1_\beta$) by Lemma 29; and the linear form $L(\varphi)=2\int_{\mathbb{R}^d}\varphi(v)g(v)\mu_\beta(\mathrm{d}v)$ is continuous on H^1_β (here we use the moment condition on g).

Step 2. Since $\int_{\mathbb{R}^d}g(v)\mu_\beta(\mathrm{d}v)=0$, it comes by Step 1 that $\int_{\mathbb{R}^d}\nabla f(v)\cdot\nabla\varphi(v)\mu_\beta(\mathrm{d}v)=-2\int_{\mathbb{R}^d}\varphi(v)g(v)\mu_\beta(\mathrm{d}v)$ for all $\varphi\in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ with $|||\varphi|||_\beta<\infty$ (without the centering condition on φ).

Step 3. We can now apply Gilbarg–Trudinger [16], Corollary 8.11, page 186: F being of class C^∞ on $\mathbb{R}^d\setminus\{0\}$, as well as g, and f being a weak solution to $\frac{1}{2}[\Delta f-\beta F\cdot\nabla f]=g$, it is of class C^∞ on $\mathbb{R}^d\setminus\{0\}$. More precisely, we fix $v\in\mathbb{R}^d\setminus\{0\}$ and we apply the cited corollary on the open ball $B(v,|v|/2)$ to conclude that f is of class C^∞ on $B(v,|v|/2)$.

Step 4. We thus can proceed rigorously to some integrations by parts to deduce that for all $\varphi\in C^\infty_c(\mathbb{R}^d\setminus\{0\})$, recalling that $\mu_\beta(\mathrm{d}v)=c_\beta[U(v)]^{-\beta}\,\mathrm{d}v$, we have

$$\int_{\mathbb{R}^d}\mathrm{div}\big[(U(v))^{-\beta}\nabla f(v)\big]\varphi(v)\,\mathrm{d}v=2\int_{\mathbb{R}^d}\varphi(v)g(v)\big[U(v)\big]^{-\beta}\,\mathrm{d}v.$$

Hence $\mathrm{div}[U^{-\beta}\nabla f]=2gU^{-\beta}$ on $\mathbb{R}^d\setminus\{0\}$ by continuity, whence the conclusion, since $F(v)=[U(v)]^{-1}\nabla U(v)$. □

We can now give the following proof.

PROOF OF THEOREM 4(A). Fix $\beta>4+d$ and take, for each $1\leq i\leq d$, a C^∞ function $f_i:\mathbb{R}^d\setminus\{0\}\to\mathbb{R}$ such that $\int_{\mathbb{R}^d}|\nabla f_i(v)|^2\mu_\beta(\mathrm{d}v)<\infty$ and $\frac{1}{2}[\Delta f_i(v)-\beta F(v)\cdot\nabla f_i(v)]=v_i-m^i_\beta$, where $m^i_\beta=\int_{\mathbb{R}^d}v_i\mu_\beta(\mathrm{d}v)$ is the i-th coordinate of m_β. Such a function f_i exists by Lemma 30, because $g_i(v)=v_i-m^i_\beta$ is C^∞, μ_β-centered and $\int_{\mathbb{R}^d}[g_i(v)]^2(1+|v|)^2\mu_\beta(\mathrm{d}v)$ is finite because $\beta>4+d$.

We now set $f=(f_1f_2\cdots f_d)^*:\mathbb{R}^d\to\mathbb{R}^d$ and apply the Itô formula, which is licit because f is of class C^∞ on $\mathbb{R}^d\setminus\{0\}$ and because $(V_t)_{t\geq0}$ never visits 0: recalling (2) and that $\nabla^*f=(\nabla f_1\nabla f_2\cdots\nabla f_d)^*$,

$$\begin{aligned}f(V_t)&=f(v_0)+\int_0^t\nabla^*f(V_s)\,\mathrm{d}B_s+\int_0^t(V_s-m_\beta)\,\mathrm{d}s\\&=f(v_0)+\int_0^t\nabla^*f(V_s)\,\mathrm{d}B_s+X_t-m_\beta t-x_0.\end{aligned}$$

Hence $\sqrt{\epsilon}(X_{t/\epsilon}-m_\beta t/\epsilon)=M^\epsilon_t+Y^\epsilon_t$, where $M^\epsilon_t=-\sqrt{\epsilon}\int_0^{t/\epsilon}\nabla^*f(V_s)\,\mathrm{d}B_s$ and where $Y^\epsilon_t=\sqrt{\epsilon}[x_0+f(V_{t/\epsilon})-f(v_0)]$.

For each $t\geq0$, Y^ϵ_t goes to 0 in law (and thus in probability) as $\epsilon\to0$: this immediately follows from the fact that $f(V_{t/\epsilon})$ converges in law as $\epsilon\to0$; see Lemma 37(iii). It is not clear (and probably false) that $\sup_{[0,t]}|Y^\epsilon_s|\to0$, which explains why we deal with finite-dimensional distributions.

Next, $(M^\epsilon_t)_{t\geq0}$ converges in law, in the usual sense of continuous processes, to $(\Sigma B_t)_{t\geq0}$, where $\Sigma\in\mathcal{S}^+_d$ is the square root of $\int_{\mathbb{R}^d}\nabla^*f(v)\nabla f(v)\mu_\beta(\mathrm{d}v)\in\mathcal{S}^+_d$ (see below). Indeed, since $(M^\epsilon_t)_{t\geq0}$ is a continuous $\mathbb{R}^d$-valued martingale, it suffices, by Jacod–Shiryaev [19], Theorem VIII-3.11, page 473, to verify that for all $i,j\in\{1,\dots,d\}$, $\langle M^{\epsilon,i},M^{\epsilon,j}\rangle_t\to\Sigma^2_{ij}t$ in probability for each $t\geq0$. But this follows from the fact that the brackets $\langle M^{\epsilon,i},M^{\epsilon,j}\rangle_t=\epsilon\int_0^{t/\epsilon}\nabla f_i(V_s)\cdot\nabla f_j(V_s)\,\mathrm{d}s$, from Lemma 37(ii) and from the fact that $\int_{\mathbb{R}^d}|\nabla f(v)|^2\mu_\beta(\mathrm{d}v)<\infty$.

All this proves that indeed, $(\sqrt{\epsilon}(X_{t/\epsilon}-m_\beta t/\epsilon))_{t\geq0}$ converges, in the sense of finite-dimensional distributions, to $(\Sigma B_t)_{t\geq0}$, as $\epsilon\to0$.

Let us finally explain why Σ^2 is positive definite. For $\xi \in \mathbb{R}^d \setminus \{0\}$, we have, setting $f_\xi(v) = f(v) \cdot \xi$,

$$\xi^* \Sigma^2 \xi = \int_{\mathbb{R}^d} \left|\nabla f(v)\xi\right|^2 \mu_\beta(\mathrm{d}v) = \int_{\mathbb{R}^d} \left|\nabla f_\xi(v)\right|^2 \mu_\beta(\mathrm{d}v),$$

which is strictly positive because else we would have $\nabla f_\xi(v) = 0$ for a.e. $v \in \mathbb{R}^d$, so that f_ξ would be constant on $\mathbb{R}^d \setminus \{0\}$ (recall that f is smooth on $\mathbb{R}^d \setminus \{0\}$). This is impossible, because $\Delta f_\xi(v) - \beta F(v) \cdot \nabla f_\xi(v) = 2(v - m_\beta) \cdot \xi$ on $\mathbb{R}^d \setminus \{0\}$ and because constants do not solve this equation. □

REMARK 31. Consider some $\beta > 4 + d$.

(i) In Theorem 4(a), $\Sigma \in \mathcal{S}_d^+$ is the square root of $\int_{\mathbb{R}^d} \nabla^* f(v) \nabla f(v) \mu_\beta(\mathrm{d}v)$, with μ_β defined in Remark 3 and with $f = (f_1, \dots, f_d)$, where $f_i : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is the (unique) C^∞ solution to $\frac{1}{2}[\Delta f_i(v) - \beta F(v) \cdot \nabla f_i(v)] = v_i - m_\beta^i$ such that $\int_{\mathbb{R}^d} |\nabla f_i(v)|^2 \mu_\beta(\mathrm{d}v) < \infty$.

(ii) If $U(v) = (1 + |v|^2)^{1/2}$, then $\mu_\beta(\mathrm{d}v) = c_\beta (1 + |v|^2)^{-\beta/2} \mathrm{d}v$ and $m_\beta = 0$, so that $(\sqrt{\epsilon} X_{t/\epsilon})_{t \geq 0} \xrightarrow{f.d.} (\Sigma B_t)_{t \geq 0}$. Furthermore, it holds that $f_i(v) = -a(|v|^2 + 3)v_i$, with $a = 2/(3\beta - 4 - 2d)$, and a computation shows that $\Sigma = q I_d$, with

$$\begin{aligned} q^2 &= \int_{\mathbb{R}^d} \left|\nabla f_1(v)\right|^2 \mu_\beta(\mathrm{d}v) \\ &= -\int_{\mathbb{R}^d} f_1(v) \left[\Delta f_1(v) - \beta F(v) \cdot \nabla f_1(v)\right] \mu_\beta(\mathrm{d}v) \\ &= -2 \int_{\mathbb{R}^d} f_1(v) v_1 \mu_\beta(\mathrm{d}v) \\ &= 2ac_\beta \int_{\mathbb{R}^d} \left(|v|^2 + 3\right) v_1^2 \left(1 + |v|^2\right)^{-\beta/2} \mathrm{d}v \\ &= \frac{2ac_\beta}{d} \int_{\mathbb{R}^d} \left(|v|^2 + 3\right) |v|^2 \left(1 + |v|^2\right)^{-\beta/2} \mathrm{d}v. \end{aligned}$$

8. The critical diffusive regime. The goal of this section is to prove Theorem 4(b). We have not been able to solve the Poisson equation, so that we adopt a rather complicated strategy. This would not be necessary if considering only the case $U(v) = (1 + |v|^2)^{1/2}$ where the solution to the Poisson equation is explicit: we could omit Lemmas 32 and 34 below.

LEMMA 32. *Fix $\beta > 0$. There is $\Psi : \mathbb{S}_{d-1} \to \mathbb{R}^d$, of class C^∞, such that for all $\theta \in \mathbb{S}_{d-1}$, all $k = 1, \dots, d$,*

$$\frac{1}{2} \Delta_S \Psi_k(\theta) - \frac{\beta}{2} \frac{\nabla_S \gamma(\theta)}{\gamma(\theta)} \cdot \nabla_S \Psi_k(\theta) = \frac{9}{2} \Psi_k(\theta) + \theta_k.$$

PROOF. By Aubin [1], Theorem 4.18, page 114, for any $\lambda > 0$ and any smooth function $g : \mathbb{S}_{d-1} \to \mathbb{R}$, there is a unique smooth solution $f : \mathbb{S}_{d-1} \to \mathbb{R}$ to

$$\mathrm{div}_S\left(\gamma^{-\beta} \nabla_S f\right) = 2\gamma^{-\beta}(\lambda f + g).$$

This uses that $\gamma^{-\beta}$ is smooth and positive on $\mathbb{S}_{d-1}$. This equation can be also rewrite as $\frac{1}{2} \Delta_S f - \frac{\beta}{2} \gamma^{-1} \nabla_S \gamma \cdot \nabla_S f = \lambda f + g$. Applying this result, for each fixed $k = 1, \dots, d$, with $\lambda = 9/2$ and $g(\theta) = \theta_k$, completes the proof. □

We now introduce some notation for the rest of the section. We write $V_t = R_t \hat{\Theta}_{H_t}$ as in Lemma 8 and we set $\Theta_t = \hat{\Theta}_{H_t}$. We know that $(R_t)_{t \geq 0}$ solves (5) for some one-dimensional

Brownian motion $(\tilde{B}_t)_{t\geq 0}$, that $(\Theta_t)_{t\geq 0}$ solves (6) for some d-dimensional Brownian motion $(\bar{B}_t)_{t\geq 0}$, and that these two Brownian motions are independent.

LEMMA 33. *Assume that $\beta = 4 + d$ and consider the function Ψ introduced in Lemma 32. We have $R_t^3\Psi(\Theta_t) = r_0^3\Psi(\theta_0) - x_0 + (X_t - m_\beta t) + M_t + Y_t$, where*

$$M_t = \int_0^t R_s^2 \nabla_S^* \Psi(\Theta_s)\,\mathrm{d}\bar{B}_s + 3\int_0^t R_s^2 \Psi(\Theta_s)\,\mathrm{d}\tilde{B}_s,$$

$$Y_t = m_\beta t + \frac{3(4+d)}{2}\int_0^t \left(R_s - \frac{R_s^2\Gamma'(R_s)}{\Gamma(R_s)}\right)\Psi(\Theta_s)\,\mathrm{d}s.$$

PROOF. Applying Itô's formula with the function Ψ (extended to $\mathbb{R}^d \setminus \{0\}$ as in Section 3 so that we can use the usual derivatives of $\mathbb{R}^d$), we find

$$\Psi(\Theta_t) = \Psi(\theta_0) + \int_0^t R_s^{-1}\nabla^*\Psi(\Theta_s)\pi_{\Theta_s^\perp}\,\mathrm{d}\bar{B}_s - \frac{d-1}{2}\int_0^t R_s^{-2}\nabla^*\Psi(\Theta_s)\Theta_s\,\mathrm{d}s$$
$$- \frac{\beta}{2}\int_0^t R_s^{-2}\nabla^*\Psi(\Theta_s)\pi_{\Theta_s^\perp}\frac{\nabla\gamma(\Theta_s)}{\gamma(\Theta_s)}\,\mathrm{d}s + \frac{1}{2}\int_0^t R_s^{-2}\sum_{i,j=1}^d (\pi_{\Theta_s^\perp})_{ij}\partial_{ij}\Psi(\Theta_s)\,\mathrm{d}s.$$

But the way Ψ has been extended to $\mathbb{R}^d \setminus \{0\}$ implies that $\pi_{\theta^\perp}\nabla\Psi(\theta) = \nabla\Psi(\theta) = \nabla_S\Psi(\theta)$, that $\nabla^*\Psi(\theta)\theta = 0$ and that $\sum_{i,j=1}^d (\pi_{\theta^\perp})_{ij}\partial_{ij}\Psi(\theta) = \Delta\Psi(\theta) - \sum_{i,j=1}^d \theta_i\theta_j\partial_{ij}\Psi(\theta) = \Delta\Psi(\theta) = \Delta_S\Psi(\theta)$. Consequently,

$$\Psi(\Theta_t) = \Psi(\theta_0) + \int_0^t R_s^{-1}\nabla_S^*\Psi(\Theta_s)\,\mathrm{d}\bar{B}_s$$
$$- \frac{\beta}{2}\int_0^t R_s^{-2}\nabla_S^*\Psi(\Theta_s)\frac{\nabla_S\gamma(\Theta_s)}{\gamma(\Theta_s)}\,\mathrm{d}s + \frac{1}{2}\int_0^t R_s^{-2}\Delta_S\Psi(\Theta_s)\,\mathrm{d}s$$
$$= \Psi(\theta_0) + \int_0^t R_s^{-1}\nabla_S^*\Psi(\Theta_s)\,\mathrm{d}\bar{B}_s + \int_0^t R_s^{-2}\left[\frac{9}{2}\Psi(\Theta_s) + \Theta_s\right]\mathrm{d}s.$$

Recalling (5) and that $\beta = 4 + d$, Itô's formula tells us that

$$R_t^3 = r_0^3 + 3\int_0^t R_s^2\,\mathrm{d}\tilde{B}_s + \frac{3(d-1)}{2}\int_0^t R_s\,\mathrm{d}s - \frac{3\beta}{2}\int_0^t \frac{\Gamma'(R_s)R_s^2}{\Gamma(R_s)}\,\mathrm{d}s + 3\int_0^t R_s\,\mathrm{d}s$$
$$= r_0^3 + 3\int_0^t R_s^2\,\mathrm{d}\tilde{B}_s - \frac{9}{2}\int_0^t R_s\,\mathrm{d}s + \frac{3(4+d)}{2}\int_0^t \left(R_s - \frac{R_s^2\Gamma'(R_s)}{\Gamma(R_s)}\right)\mathrm{d}s.$$

We conclude that

$$R_t^3\Psi(\Theta_t) = r_0^3\Psi(\theta_0) + \int_0^t R_s^2\nabla_S^*\Psi(\Theta_s)\,\mathrm{d}\bar{B}_s + \int_0^t R_s\left[\frac{9}{2}\Psi(\Theta_s) + \Theta_s\right]\mathrm{d}s$$
$$+ 3\int_0^t R_s^2\Psi(\Theta_s)\,\mathrm{d}\tilde{B}_s - \frac{9}{2}\int_0^t R_s\Psi(\Theta_s)\,\mathrm{d}s$$
$$+ \frac{3(4+d)}{2}\int_0^t \left(R_s - \frac{R_s^2\Gamma'(R_s)}{\Gamma(R_s)}\right)\Psi(\Theta_s)\,\mathrm{d}s.$$

In other words, we have $R_t^3\Psi(\Theta_t) = r_0^3\Psi(\theta_0) - x_0 + (X_t - m_\beta t) + M_t + Y_t$ as desired. $\square$

We now treat the error term.

LEMMA 34. *Adopt the assumptions and notation of Lemma 33. Suppose the additional condition* $\int_1^\infty r^{-1}|r\Gamma'(r)/\Gamma(r)-1|^2 r^{-1}\,\mathrm{d}r<\infty$. *For each* $t\geq 0$, *in probability*,

$$\lim_{\epsilon\to 0}|\log\epsilon|^{-1/2}\epsilon^{1/2}\big[R^3_{t/\epsilon}\Psi(\Theta_{t/\epsilon})-r_0^3\Psi(\theta_0)+x_0-Y_{t/\epsilon}\big]=0.$$

PROOF. First, setting $\psi(v)=|v|^3\Psi(v/|v|)-r_0^3\Psi(\theta_0)+x_0$, we have

$$\lim_{\epsilon\to 0}|\log\epsilon|^{-1/2}\epsilon^{1/2}\big[R^3_{t/\epsilon}\Psi(\Theta_{t/\epsilon})-r_0^3\Psi(\theta_0)+x_0\big]=\lim_{\epsilon\to 0}|\log\epsilon|^{-1/2}\epsilon^{1/2}\psi(V_{t/\epsilon})=0$$

in probability, since by Lemma 37(ii), V_t converges in law as $t\to\infty$.

Next, we have $Y_t=\int_0^t g(V_s)\,\mathrm{d}s$, where we have set

$$g(v)=m_\beta+\frac{3(4+d)}{2}\bigg(r-\frac{r^2\Gamma'(r)}{\Gamma(r)}\bigg)\Psi(\theta),$$

where $r=|v|$ and $\theta=v/|v|$. This function is of class C^∞ on $\mathbb{R}^d\setminus\{0\}$ and, as we will see below,

$$\text{(a)}\quad\int_{\mathbb{R}^d}|g(v)|^2(1+|v|)^2\mu_\beta(\mathrm{d}v)<\infty\quad\text{and}\quad\text{(b)}\quad\int_{\mathbb{R}^d}g(v)\mu_\beta(\mathrm{d}v)=0.$$

Applying Lemma 30 (coordinate by coordinate), there exists $f:\mathbb{R}^d\setminus\{0\}\to\mathbb{R}^d$ of class C^∞, satisfying $\int_{\mathbb{R}^d}|\nabla f(v)|^2\mu_\beta(\mathrm{d}v)<\infty$ and, for each $k=1,\dots,d$, $\frac{1}{2}[\Delta f_k-\beta F\cdot\nabla f_k]=g_k$. By Itô's formula, starting from (2),

$$f(V_t)=f(v_0)+N_t+Y_t\quad\text{where } N_t=\int_0^t\nabla^* f(V_s)\,\mathrm{d}B_s.$$

To conclude that $|\log\epsilon|^{-1/2}\epsilon^{1/2}Y_{t/\epsilon}$ converges to 0 in probability, as $\epsilon\to 0$, we observe that $|\log\epsilon|^{-1/2}\epsilon^{1/2}[f(V_{t/\epsilon})-f(v_0)]$ tends to 0 in probability, which follows from the fact that V_t converges in law as $t\to\infty$, and that $|\log\epsilon|^{-1/2}\epsilon^{1/2}N_{t/\epsilon}\to 0$ in probability, which follows from the fact that $(\epsilon^{1/2}N_{t/\epsilon})_{t\geq 0}$ converges in law by Jacod–Shiryaev [19], Theorem VIII-3.11, page 473. Indeed, $(\epsilon^{1/2}N_{t/\epsilon})_{t\geq 0}$ is a continuous local martingale of which the bracket $\epsilon\int_0^{t/\epsilon}\nabla^* f(V_s)\nabla^* f(V_s)\,\mathrm{d}s$ a.s. converges to $[\int_{\mathbb{R}^d}\nabla^* f(v)\nabla f(v)\mu_\beta(\mathrm{d}v)]t$ as $\epsilon\to 0$ by Lemma 37(ii).

We now check (a). Since $|g(v)|\leq C(1+|v|)|1-|v|\Gamma'(|v|)/\Gamma(|v|)|$ and since $\beta=4+d$,

$$\begin{aligned}\int_{\mathbb{R}^d}|g(v)|^2(1+|v|)^2\mu_\beta(\mathrm{d}v)&\leq C\int_{\mathbb{R}^d}|g(v)|^2(1+|v|)^{-2-d}\,\mathrm{d}v\\&\leq C\int_0^\infty\bigg|1-\frac{r\Gamma'(r)}{\Gamma(r)}\bigg|^2\frac{r^{d-1}\,\mathrm{d}r}{(1+r)^d},\end{aligned}$$

which converges since, by assumption, $\int_1^\infty r^{-1}|r\Gamma'(r)/\Gamma(r)-1|^2 r^{-1}\,\mathrm{d}r$.

We finally check (b), recalling the notation introduced in Section 3:

$$\begin{aligned}J&=\int_{\mathbb{R}^d}g(v)\mu_\beta(\mathrm{d}v)\\&=m_\beta+\frac{3(4+d)}{2}\int_0^\infty\bigg(r-\frac{r^2\Gamma'(r)}{\Gamma(r)}\bigg)\nu'_\beta(\mathrm{d}r)\int_{\mathbb{S}_{d-1}}\Psi(\theta)\nu_\beta(\mathrm{d}\theta)\\&=m_\beta+\frac{3(4+d)}{2}J_1J_2,\end{aligned}$$

the first and last equalities standing for definitions. First,

$$J_1 = b_\beta \int_0^\infty \left(r - \frac{r^2\Gamma'(r)}{\Gamma(r)}\right) r^{d-1}[\Gamma(r)]^{-\beta}\,\mathrm{d}r$$
$$= b_\beta \int_0^\infty r^d[\Gamma(r)]^{-\beta}\,\mathrm{d}r + \frac{b_\beta}{\beta}\int_0^\infty r^{1+d}([\Gamma(r)]^{-\beta})'\,\mathrm{d}r,$$

whence $J_1 = b_\beta[1-(1+d)/\beta]\int_0^\infty r^d[\Gamma(r)]^{-\beta}$ and thus $J_1 = [1-(1+d)/\beta]m'_\beta$.

Next, recall that $\frac{1}{2}\Delta_S\Psi(\theta) - \frac{\beta}{2}[\gamma(\theta)]^{-1}\nabla_S\gamma(\theta)\cdot\nabla_S\Psi(\theta) = \frac{9}{2}\Psi(\theta)+\theta$ by Lemma 32 and observe that for any smooth $\psi : \mathbb{S}_{d-1}\to\mathbb{R}$, we have

$$\int_{\mathbb{S}_{d-1}}\left[\Delta_S\psi(\theta) - \beta\frac{\nabla_S\gamma(\theta)}{\gamma(\theta)}\cdot\nabla_S\psi(\theta)\right]\nu_\beta(\mathrm{d}\theta)$$
$$= a_\beta\int_{\mathbb{S}_{d-1}}\mathrm{div}_S([\gamma(\theta)]^{-\beta}\nabla_S\psi(\theta))\varsigma(\mathrm{d}\theta) = 0.$$

Hence, $J_2 = \int_{\mathbb{S}_{d-1}}\Psi(\theta)\nu_\beta(\mathrm{d}\theta) = -(2/9)\int_{\mathbb{S}_{d-1}}\theta\nu_\beta(\mathrm{d}\theta) = -(2/9)M_\beta$, so that

$$J = m_\beta - \frac{3(4+d)}{2}\left(1-\frac{1+d}{\beta}\right)\frac{2}{9}m'_\beta M_\beta = 0$$

because $\beta = 4+d$ and $m_\beta = m'_\beta M_\beta$. □

We finally treat the main martingale term.

LEMMA 35. *With the assumptions and notation of Lemma 33, as* $\epsilon\to 0$,

$$(|\log\epsilon|^{-1/2}\epsilon^{1/2}M_{t/\epsilon})_{t\ge 0}\xrightarrow{d}(\Sigma B_t)_{t\ge 0}$$

for some $\Sigma\in\mathcal{S}_d^+$, *where* $(B_t)_{t\ge0}$ *is a* d*-dimensional Brownian motion.*

PROOF. Using one more time Jacod–Shiryaev [19], Theorem VIII-3.11, page 473, it suffices to check that there is $\Sigma^2\in\mathcal{S}_d^+$ such that $\lim_{\epsilon\to0}Z_t^\epsilon = \Sigma^2 t$ in probability for each $t\ge0$, where Z_t^ϵ is the matrix of brackets of the martingale $|\log\epsilon|^{-1/2}\epsilon^{1/2}M_{t/\epsilon}$, namely

$$Z_t^\epsilon = \frac{\epsilon}{|\log\epsilon|}\int_0^{t/\epsilon}R_s^4 D(\Theta_s)\,\mathrm{d}s,$$

where $D(\theta) = \nabla_S^*\Psi(\theta)\nabla_S\Psi(\theta) + 9\Psi(\theta)\Psi^*(\theta)$. We proceed by coupling.

Step 1. We recall Notation 9 and use Lemma 10 with $a_\epsilon = \kappa\epsilon$, where $\kappa = \int_{\mathbb{R}}[\sigma(w)]^{-2}\,\mathrm{d}w < \infty$; see Lemma 42(i). We consider a one-dimensional Brownian motion $(W_t)_{t\ge0}$, introduce $A_t^\epsilon = \epsilon a_\epsilon^{-2}\int_0^t[\sigma(W_s/a_\epsilon)]^{-2}\,\mathrm{d}s$ and its inverse ρ_t^ϵ and put $R_t^\epsilon = \sqrt{\epsilon}h^{-1}(W_{\rho_t^\epsilon}/a_\epsilon)$. We know from Lemma 10 that $S_t^\epsilon = \epsilon^{-1/2}R_{\epsilon t}^\epsilon$ solves (5). We also consider the solution $(\hat\Theta_t)_{t\ge0}$ of (4), independent of $(W_t)_{t\ge0}$. We then know from Lemma 8 that, setting $H_t^\epsilon = \int_0^t[S_s^\epsilon]^{-2}\,\mathrm{d}s$, $(S_t^\epsilon\hat\Theta_{H_t^\epsilon})_{t\ge0}\stackrel{d}{=}(V_t)_{t\ge0}$. In particular, for each $t\ge0$, $Z_t^\epsilon\stackrel{d}{=}\tilde Z_t^\epsilon$, where

$$\tilde Z_t^\epsilon = \frac{\epsilon}{|\log\epsilon|}\int_0^{t/\epsilon}(S_s^\epsilon)^4 D(\hat\Theta_{H_s^\epsilon})\,\mathrm{d}s.$$

Step 2. Here we verify that $\tilde Z_t^\epsilon = K_{\rho_t^\epsilon}^\epsilon$, where, recalling Notation 9,

$$K_t^\epsilon = \frac{\epsilon}{|\log\epsilon|a_\epsilon^2}\int_0^t\frac{[h^{-1}(W_s/a_\epsilon)]^4 D(\hat\Theta_{T_s^\epsilon})}{[\sigma(W_s/a_\epsilon)]^2}\,\mathrm{d}s \quad\text{and}\quad T_t^\epsilon = \frac{1}{a_\epsilon^2}\int_0^t\frac{\mathrm{d}u}{\psi(W_u/a_\epsilon)}.$$

Recalling that $S_s^\epsilon = \epsilon^{-1/2} R_{\epsilon t}^\epsilon = h^{-1}(W_{\rho_{\epsilon s}^\epsilon}/a_\epsilon)$ and using the change of variables $u = \rho_{\epsilon s}^\epsilon$, that is, $s = \epsilon^{-1} A_u^\epsilon$, whence $\mathrm{d}s = a_\epsilon^{-2}[\sigma(W_u/a_\epsilon)]^{-2}\,\mathrm{d}u$, we find

$$\tilde{Z}_t^\epsilon = \frac{\epsilon}{|\log\epsilon| a_\epsilon^2} \int_0^{\rho_t^\epsilon} \frac{[h^{-1}(W_u/a_\epsilon)]^4 D(\hat{\Theta}_{H^\epsilon_{\epsilon^{-1}A_u^\epsilon}})}{[\sigma(W_u/a_\epsilon)]^2}\,\mathrm{d}u,$$

and it only remains to check that $H^\epsilon_{\epsilon^{-1}A_t^\epsilon} = T_t^\epsilon$. With the same substitution,

$$\begin{aligned} H^\epsilon_{\epsilon^{-1}A_t^\epsilon} &= \int_0^{\epsilon^{-1}A_t^\epsilon} \frac{\mathrm{d}s}{[h^{-1}(W_{\rho_{\epsilon s}^\epsilon}/a_\epsilon)]^2} \\ &= \frac{1}{a_\epsilon^2}\int_0^t \frac{\mathrm{d}u}{[\sigma(W_u/a_\epsilon)]^2[h^{-1}(W_u/a_\epsilon)]^2} = \frac{1}{a_\epsilon^2}\int_0^t \frac{\mathrm{d}u}{\psi(W_u/a_\epsilon)}. \end{aligned}$$

Step 3. We now prove that there is $C>0$ such that $\mathbb{E}[|K_t^\epsilon - G_D I_t^\epsilon|^2|\mathcal{W}] \le Ct/|\log\epsilon|^2$ for all $t\ge 0$, all $\epsilon\in(0,1)$, where $\mathcal{W} = \sigma(W_t, t\ge 0)$, where $G_D = \int_{\mathbb{S}_{d-1}} D(\theta)\nu_\beta(\mathrm{d}\theta)$ and where

$$I_t^\epsilon = \frac{\epsilon}{|\log\epsilon| a_\epsilon^2}\int_0^t \frac{[h^{-1}(W_s/a_\epsilon)]^4}{[\sigma(W_s/a_\epsilon)]^2}\,\mathrm{d}s.$$

We set $\Delta_t^\epsilon = \mathbb{E}[|K_t^\epsilon - G_D I_t^\epsilon|^2|\mathcal{W}]$ and write

$$\begin{aligned} \Delta_t^\epsilon = {} & \frac{\epsilon^2}{|\log\epsilon|^2 a_\epsilon^4}\int_0^t\int_0^t \frac{[h^{-1}(W_a/a_\epsilon)]^4}{[\sigma(W_a/a_\epsilon)]^2}\frac{[h^{-1}(W_b/a_\epsilon)]^4}{[\sigma(W_b/a_\epsilon)]^2} \\ & \times \mathbb{E}\big(\big[D(\hat{\Theta}_{T_a^\epsilon}) - G_D\big]\big[D(\hat{\Theta}_{T_b^\epsilon}) - G_D\big]|\mathcal{W}\big)\,\mathrm{d}a\,\mathrm{d}b. \end{aligned}$$

Using that $(T_t^\epsilon)_{t\ge 0}$ is $\mathcal{W}$-measurable, that $(\hat{\Theta}_t)_{t\ge 0}$ is independent of $\mathcal{W}$, that D is bounded and that $G_D = \int_{\mathbb{S}_{d-1}} D\,\mathrm{d}\nu_\beta$, we deduce from Lemma 38(ii) and the Markov property that there are $C>0$ and $\lambda>0$ such that a.s.,

$$\big|\mathbb{E}\big(\big[D(\hat{\Theta}_{T_a^\epsilon}) - G_D\big]\big[D(\hat{\Theta}_{T_b^\epsilon}) - G_D\big]|\mathcal{W}\big)\big| \le C\exp\big(-\lambda\big|T_b^\epsilon - T_a^\epsilon\big|\big).$$

By Lemma 42(iii) with $a_\epsilon = \kappa\epsilon$, we have $\epsilon a_\epsilon^{-2}[h^{-1}(w/a_\epsilon)]^4[\sigma(w/a_\epsilon)]^{-2} \le C(\epsilon+|w|)^{-1}$, whence

$$\Delta_t^\epsilon \le \frac{C}{|\log\epsilon|^2}\int_0^t\int_0^t \big(\epsilon+|W_a|\big)^{-1}\big(\epsilon+|W_b|\big)^{-1}\exp\big(-\lambda\big|T_a^\epsilon - T_b^\epsilon\big|\big)\,\mathrm{d}a\,\mathrm{d}b.$$

Next, since $a_\epsilon^2\psi(w/a_\epsilon) \le C(\epsilon+|w|)^2$ by Lemma 42(iv),

$$\lambda\big|T_a^\epsilon - T_b^\epsilon\big| = \lambda\left|\frac{1}{a_\epsilon^2}\int_a^b \frac{\mathrm{d}s}{\psi(W_s/a_\epsilon)}\,\mathrm{d}s\right| \ge c\left|\int_a^b (\epsilon+|W_s|)^{-2}\,\mathrm{d}s\right|$$

for some $c>0$. Using furthermore that $xy\le x^2+y^2$ and a symmetry argument, we conclude that

$$\Delta_t^\epsilon \le \frac{C}{|\log\epsilon|^2}\int_0^t\int_0^t \big(\epsilon+|W_a|\big)^{-2}\exp\left(-c\left|\int_a^b(\epsilon+|W_s|)^{-2}\,\mathrm{d}s\right|\right)\mathrm{d}a\,\mathrm{d}b.$$

The conclusion follows; see (13).

Step 4. One can check precisely as in Lemma 12 that for all $T\ge 0$, $\sup_{[0,T]}|A_t^\epsilon - L_t^0| \to 0$ a.s. as $\epsilon\to 0$, where $(L_t^0)_{t\ge 0}$ is the local time at 0 of $(W_t)_{t\ge 0}$. Actually, the proof of Lemma 12 works (without any modification) for any $\beta > d$.

Step 5. We verify that for each $T \geq 0$, a.s., $\lim_{\epsilon\to 0}\sup_{[0,T]}|I_t^\epsilon - (\kappa/36)L_t^0| = 0$. This resembles the proof of Lemma 12. By Lemma 42(iii), we know that $[h^{-1}(w)]^4/[\sigma(w)]^2 \leq C(1+|w|)^{-1}$ and that

$$\int_{-x}^{x} \frac{[h^{-1}(w)]^4\,\mathrm{d}w}{[\sigma(w)]^2} \overset{x\to\infty}{\sim} \frac{\log x}{36}. \tag{21}$$

We fix $\delta > 0$ and write $I_t^\epsilon = J_t^{\epsilon,\delta} + Q_t^{\epsilon,\delta}$, where

$$J_t^{\epsilon,\delta} = \frac{\epsilon}{|\log\epsilon| a_\epsilon^2}\int_0^t \frac{[h^{-1}(W_s/a_\epsilon)]^4}{[\sigma(W_s/a_\epsilon)]^2}\mathbf{1}_{\{|W_s|>\delta\}}\,\mathrm{d}s$$

and

$$Q_t^{\epsilon,\delta} = \frac{\epsilon}{|\log\epsilon| a_\epsilon^2}\int_0^t \frac{[h^{-1}(W_s/a_\epsilon)]^4}{[\sigma(W_s/a_\epsilon)]^2}\mathbf{1}_{\{|W_s|\leq\delta\}}\,\mathrm{d}s.$$

Since $a_\epsilon = \kappa\epsilon$ and since $|w| > \delta$ implies that $[h^{-1}(w/a_\epsilon)]^4/[\sigma(w/a_\epsilon)]^2 \leq C(1+|\delta/\epsilon|)^{-1}$, we find $\sup_{[0,T]} J_t^{\epsilon,\delta} \leq CT/[\delta|\log\epsilon|]$, which tends to 0 as $\epsilon \to 0$. We next use the occupation times formula to write

$$\begin{aligned} Q_t^{\epsilon,\delta} &= \frac{\epsilon}{|\log\epsilon| a_\epsilon^2}\int_{-\delta}^{\delta} \frac{[h^{-1}(x/a_\epsilon)]^4 L_t^x\,\mathrm{d}x}{[\sigma(x/a_\epsilon)]^2} \\ &= \frac{\epsilon}{|\log\epsilon| a_\epsilon^2}\int_{-\delta}^{\delta} \frac{[h^{-1}(x/a_\epsilon)]^4\,\mathrm{d}x}{[\sigma(x/a_\epsilon)]^2} L_t^0 + \frac{\epsilon}{|\log\epsilon| a_\epsilon^2}\int_{-\delta}^{\delta} \frac{[h^{-1}(x/a_\epsilon)]^4 (L_t^x - L_t^0)\,\mathrm{d}x}{[\sigma(x/a_\epsilon)]^2} \\ &= r_{\epsilon,\delta} L_t^0 + R_t^{\epsilon,\delta}, \end{aligned}$$

the last identity standing for a definition. By a substitution and (21),

$$r_{\epsilon,\delta} = \frac{\epsilon}{|\log\epsilon| a_\epsilon}\int_{-\delta/a_\epsilon}^{\delta/a_\epsilon} \frac{[h^{-1}(y)]^4\,\mathrm{d}y}{[\sigma(y)]^2} \overset{\epsilon\to 0}{\sim} \frac{\epsilon\log(\delta/a_\epsilon)}{36|\log\epsilon| a_\epsilon} \longrightarrow \frac{1}{36\kappa}$$

as $\epsilon \to 0$ because $a_\epsilon = \kappa\epsilon$. Recalling that $I_t^\epsilon = r_{\epsilon,\delta}L_t^0 + R_t^{\epsilon,\delta} + J_t^{\epsilon,\delta}$, we have proved that a.s.,

$$\text{for all } \delta > 0, \quad \limsup_{\epsilon\to 0}\sup_{[0,T]}\left|I_t^\epsilon - L_t^0/(36\kappa)\right| \leq \limsup_{\epsilon\to 0}\sup_{[0,T]}\left|R_t^{\epsilon,\delta}\right|.$$

But $|R_t^{\epsilon,\delta}| \leq r_{\epsilon,\delta} \times \sup_{[-\delta,\delta]}|L_t^x - L_t^0|$, whence

$$\limsup_{\epsilon\to 0}\sup_{[0,T]}\left|I_t^\epsilon - L_t^0/(36\kappa)\right| \leq \sup_{[0,T]\times[-\delta,\delta]}\left|L_t^x - L_t^0\right|/(36\kappa)$$

a.s. Letting $\delta \to 0$, using Revuz–Yor [33], Corollary 1.8, page 226, completes the step.

Step 6. We finally conclude. We fix $t \geq 0$ and recall from Steps 1 and 2 that $Z_t^\epsilon \overset{d}{=} \tilde{Z}_t^\epsilon = K_{\rho_t^\epsilon}^\epsilon$. By Step 4, we know that $A_s^\epsilon \to L_s^0$ a.s. for each $s \geq 0$, so that Lemma 41 tells us that ρ_t^ϵ a.s. converges to $\tau_t = \inf\{u \geq 0 : L_u^0 > t\}$, because t is a.s. not a jump time of $(\tau_s)_{s\geq 0}$. Using that ρ_t^ϵ is $\mathcal{W}$-measurable, we deduce from Step 3 that for any $A > 0$,

$$\mathbb{E}\left[\left|K_{\rho_t^\epsilon}^\epsilon - G_D I_{\rho_t^\epsilon}^\epsilon\right|\mathbf{1}_{\{\rho_t^\epsilon \leq A\}}\right] \leq \frac{CA}{|\log\epsilon|^2} \to 0.$$

Since ρ_t^ϵ a.s. tends to τ_t, one deduces that $\tilde{Z}_t^\epsilon - G_D I_{\rho_t^\epsilon}^\epsilon$ converges in probability to 0. We then infer from Step 5, using again that ρ_t^ϵ a.s. tends to τ_t, that $|I_{\rho_t^\epsilon}^\epsilon - L_{\rho_t^\epsilon}^0/(36\kappa)|$ a.s. tends to 0.

But $(L^0_s)_{s\ge 0}$ being continuous, we see that $L^0_{\rho^\epsilon_t}$ a.s. tends to $L^0_{\tau_t}=t$. All this proves that $\tilde Z^\epsilon_t$, and thus also Z^ϵ_t, converges in probability, as $\epsilon\to 0$, to $\Sigma^2 t$, where

$$\Sigma^2=\frac{G_D}{36\kappa}.$$

This matrix is positive definite: for $\xi\in\mathbb{R}^d\setminus\{0\}$ and $\Psi_\xi(\theta)=\Psi(\theta)\cdot\xi$,

$$\xi^* G_D\xi=\int_{\mathbb{R}^d}\big[\big|\nabla_S\Psi_\xi(\theta)\big|^2+9\big|\Psi_\xi(\theta)\big|^2\big]\nu_\beta(\mathrm{d}\theta)\ge 9\int_{\mathbb{R}^d}\big|\Psi_\xi(\theta)\big|^2\nu_\beta(\mathrm{d}\theta)$$

which cannot vanish, because else we would have $\Psi_\xi(\theta)=0$ for all $\theta\in\mathbb{S}_{d-1}$, which is impossible because Ψ_ξ solves $\frac12\Delta_S\Psi_\xi(\theta)-\frac{\beta}{2}[\gamma(\theta)]^{-1}\nabla_S\gamma(\theta)\cdot\nabla_S\Psi_\xi(\theta)=\frac92\Psi_\xi(\theta)+\xi\cdot\theta$. □

We now have all the tools to give the following proof.

PROOF OF THEOREM 4(B). Recall first that we know from Lemma 33 that $X_t-m_\beta t=[R^3_t\Psi(\Theta_t)-r^3_0\Psi(\theta_0)-Y_t]-M_t$, from Lemma 34 that for each $t\ge 0$,

$$\lim_{\epsilon\to 0}|\log\epsilon|^{-1/2}\epsilon^{1/2}\big[R^3_{t/\epsilon}\Psi(\Theta_{t/\epsilon})-r^3_0\Psi(\theta_0)+x_0-Y_{t/\epsilon}\big]=0$$

in probability, and from Lemma 35 that $(|\log\epsilon|^{-1/2}\epsilon^{1/2}M_{t/\epsilon})_{t\ge 0}\xrightarrow{d}(\Sigma B_t)_{t\ge 0}$ as $\epsilon\to 0$. We conclude that, as desired, $(|\log\epsilon|^{-1/2}\epsilon^{1/2}(X_{t/\epsilon}-m_\beta t/\epsilon)_{t\ge 0}\xrightarrow{f.d.}(\Sigma B_t)_{t\ge 0}$ as $\epsilon\to 0$. □

By Lemma 42(i), κ can be computed slightly more explicitly.

REMARK 36. Assume that $\beta=4+d$.

(i) In Theorem 4(b), $\Sigma\in\mathcal{S}^+_d$ is the square root of

$$\frac{1}{36\kappa}\int_{\mathbb{S}_{d-1}}\big[\nabla^*_S\Psi(\theta)\nabla_S\Psi(\theta)+9\Psi(\theta)\Psi^*(\theta)\big]\nu_\beta(\mathrm{d}\theta),$$

with ν_β defined in Section 3, Ψ introduced in Lemma 32 and with

$$\kappa=\frac16\int_0^\infty r^{d-1}\big[\Gamma(r)\big]^{-4-d}\,\mathrm{d}r.$$

(ii) If $U(v)=(1+|v|^2)^{1/2}$, then $\mu_\beta(\mathrm{d}v)=c_\beta(1+|v|^2)^{-\beta/2}\,\mathrm{d}v$ and $m_\beta=0$, so that we have $(\epsilon^{1/2}|\log\epsilon|^{-1/2}X_{t/\epsilon})_{t\ge 0}\xrightarrow{f.d.}(\Sigma B_t)_{t\ge 0}$. Moreover, $\gamma\equiv 1$, whence $\nu_\beta(\mathrm{d}\theta)=\varsigma(\mathrm{d}\theta)$ and $\Psi(\theta)=-a\theta$, where $a=2/(8+d)$ (a computation shows that $\Delta_S\Psi(\theta)=a(d-1)\theta$, whence $\frac12\Delta_S\Psi(\theta)=\frac92\Psi(\theta)+\theta$). Since now $\nabla_S\Psi(\theta)=-a\pi_{\theta^\perp}$, whence $\nabla^*_S\Psi(\theta)\nabla_S\Psi(\theta)=a^2\pi_{\theta^\perp}$, we find

$$\begin{aligned}\Sigma^2&=\frac{a^2}{36\kappa}\int_{\mathbb{S}_{d-1}}[\pi_{\theta^\perp}+9\theta\theta^*]\varsigma(\mathrm{d}\theta)\\&=\frac{a^2}{36\kappa}\int_{\mathbb{S}_{d-1}}[I_d+8\theta\theta^*]\varsigma(\mathrm{d}\theta)=\frac{a^2}{36\kappa}\bigg[\int_{\mathbb{S}_{d-1}}(1+8\theta_1^2)\varsigma(\mathrm{d}\theta)\bigg]I_d.\end{aligned}$$

Observing that $\int_{\mathbb{S}_{d-1}}\theta_1^2\varsigma(\mathrm{d}\theta)=1/d$, we conclude that $\Sigma=qI_d$, with $q=[9\kappa d(8+d)]^{-1/2}$.

APPENDIX

We still work in dimension $d \geq 2$ in the whole section.

A.1. Ergodicity and convergence in law. We first recall some classical properties of the velocity process.

LEMMA 37. *Assume that $\beta > d$ and consider the $\mathbb{R}^d \setminus \{0\}$-valued velocity process $(V_t)_{t\geq 0}$; see* (2).

(i) *The measure with density μ_β defined in Remark 3 is its unique invariant probability measure.*
(ii) *For any $\phi \in L^1(\mathbb{R}^d, \mu_\beta)$, $\lim_{T\to\infty} T^{-1} \int_0^T \phi(V_s)\,\mathrm{d}s = \int_{\mathbb{R}^d} \phi\,\mathrm{d}\mu_\beta$ a.s.*
(iii) *It holds that V_t goes in law to μ_β as $t \to \infty$.*

PROOF. We denote by $\mathcal{L}$ the generator of the velocity process, we have $\mathcal{L}\varphi(v) = \frac{1}{2}[\Delta\varphi(v) - \beta F(v) \cdot \nabla\varphi(v)]$ for all $\varphi \in C^2(\mathbb{R}^d \setminus \{0\})$, all $v \in \mathbb{R}^d \setminus \{0\}$. We also denote by $P_t(v, \mathrm{d}w)$ its semi-group: for $t \geq 0$ and $v \in \mathbb{R}^d \setminus \{0\}$, $P_t(v, \mathrm{d}w)$ is the law of V_t when $V_0 = v$.

Recalling that $\mu_\beta(\mathrm{d}v) = c_\beta[U(v)]^{-\beta}\,\mathrm{d}v$ and that $\mathcal{L}\varphi(v) = \frac{1}{2}[U(v)]^\beta \operatorname{div}([U(v)]^{-\beta}\nabla\varphi(v))$, we see that $\int_{\mathbb{R}^d} \mathcal{L}\varphi(v)\mu_\beta(\mathrm{d}v) = 0$ for all $\varphi \in C^2(\mathbb{R}^d \setminus \{0\})$, and μ_β is an invariant probability measure. The uniqueness of this invariant probability measure follows from point (iii). In a few lines below, we will verify the two following points.

(a) There is $\Phi : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ of class C^2 such that

$$\lim_{|v|\to 0+} \Phi(v) = \lim_{|v|\to\infty} \Phi(v) = \infty$$

and, for some $b, c > 0$ and some compact set $C \subset \mathbb{R}^d \setminus \{0\}$, for all $v \in \mathbb{R}^d \setminus \{0\}$, $\mathcal{L}\Phi(v) \leq -b + c\mathbf{1}_{\{v\in C\}}$.

(b) There is $t_0 > 0$ such that for any compact set $C \subset \mathbb{R}^d \setminus \{0\}$, there is $\alpha_C > 0$ and a probability measure ζ_C on $\mathbb{R}^d \setminus \{0\}$ such that for all $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $\inf_{x\in C} P_{t_0}(x, A) \geq \alpha_C \zeta_C(A)$.

These two conditions allow us to apply Theorems 4.4 and 5.1 of Meyn–Tweedie [30], which tell us that $(V_t)_{t\geq 0}$ is Harris recurrent, whence point (ii) (by Revuz–Yor [33], Theorem 3.12, page 427, any Harris recurrent process with an invariant probability measure satisfies the ergodic theorem) and $\mathrm{Law}(V_t) \to \mu_\beta$, whence point (iii). Indeed, in the terminology of [30], (a) implies condition (CD2) and (b) implies that all compact sets are *petite*.

Point (a). For some $q > 0$ to be chosen later, set, for $r \in (0, \infty)$, $g(r) = -q + \mathbf{1}_{\{r\in[1,3]\}}$ and $\varphi(r) = \int_2^r y^{1-d}[\Gamma(y)]^\beta\,\mathrm{d}y \int_2^y g(x)x^{d-1}[\Gamma(x)]^{-\beta}\,\mathrm{d}x$. For $v \in \mathbb{R}^d \setminus \{0\}$, set $\Phi(v) = \varphi(|v|) + m$, for some constant m to be chosen later.

But it holds that $\varphi'(r) = r^{1-d}[\Gamma(r)]^\beta \int_2^r g(x)x^{d-1}[\Gamma(x)]^{-\beta}\,\mathrm{d}x$, $\nabla\Phi(v) = \frac{\varphi'(|v|)}{|v|}v$, $\varphi''(r) = g(r) - [\frac{d-1}{r} - \beta\frac{\Gamma'(r)}{\Gamma(r)}]\varphi'(r)$, and $\Delta\Phi(v) = \varphi''(|v|) + \frac{d-1}{|v|}\varphi'(|v|)$, whence $F(v) \cdot \nabla\Phi(v) = \frac{\Gamma'(|v|)}{\Gamma(|v|)}\varphi'(|v|)$; see (7). We find that $\mathcal{L}\Phi(v) = g(|v|)/2$.

The integrals $\int_0^2 g(x)x^{d-1}[\Gamma(x)]^{-\beta}\,\mathrm{d}x$ and $\int_2^\infty g(x)x^{d-1}[\Gamma(x)]^{-\beta}\,\mathrm{d}x$ converge and are positive if $q > 0$ is small enough, so that

$$\lim_{r\to 0} \varphi(r) = \int_0^2 y^{1-d}[\Gamma(y)]^\beta\,\mathrm{d}y \int_y^2 g(x)x^{d-1}[\Gamma(x)]^{-\beta}\,\mathrm{d}x = \infty \quad \text{and}$$

$$\lim_{r\to\infty} \varphi(r) = \int_2^\infty y^{1-d}[\Gamma(y)]^\beta\,\mathrm{d}y \int_2^y g(x)x^{d-1}[\Gamma(x)]^{-\beta}\,\mathrm{d}x = \infty.$$

Hence it holds that $\lim_{|v|\to 0+}\Phi(v)=\lim_{|v|\to\infty}\Phi(v)=\infty$. With the choice $m=-\min_{r>0}\varphi(r)\in\mathbb{R}$, the function Φ is nonnegative and thus suitable.

Point (b). We will prove, and this is sufficient, that for any compact set $C\subset\mathbb{R}^d\setminus\{0\}$, there exists a constant $\kappa_C>0$ such that for all $v\in C$ all measurable $A\subset C$, $P_1(v,A)\ge\kappa_C|A|$, $|A|$ being the Lebesgue measure of A.

Consider $a'>a>0$ such that the annulus $D=\{x\in\mathbb{R}^d, a<|x|<a'\}$ contains C. Recall (2) and that the force F is bounded on D; see Assumption 1. By the Girsanov theorem, for any $A\in\mathcal{B}(\mathbb{R}^d)$,

$$P_1(v,A)=\mathbb{P}_v(V_1\in A)\ge\mathbb{E}_v[\mathbf{1}_{\{\forall s\in[0,1],V_s\in D\}}\mathbf{1}_{\{V_1\in A\}}]$$
$$\ge c\mathbb{E}[\mathbf{1}_{\{\forall s\in[0,1],v+B_s\in D\}}\mathbf{1}_{\{v+B_1\in A\}}]$$

for some constant $c>0$, where $(B_t)_{t\in[0,1]}$ is a d-dimensional Brownian motion issued from 0. But the density $g(v,w)$ of $v+B_1$ restricted to the event that $(v+B_s)_{s\in[0,1]}$ does not get out of D is bounded below, as a function of (v,w), on $C\times C$, whence the conclusion. $\square$

We recall some facts about the total variation distance: for two probability measures P, Q on some measurable set E,

$$\|P-Q\|_{\mathrm{TV}}=\frac{1}{2}\sup_{\|\phi\|_\infty\le 1}\left|\int_E\phi(x)(P-Q)(\mathrm{d}x)\right| \tag{22}$$
$$=\inf\{\mathbb{P}(X\neq Y):\mathrm{Law}(X)=P,\mathrm{Law}(Y)=Q\}.$$

Furthermore, if P and Q have some densities f and g with respect to some measure R on E, then

$$\|P-Q\|_{\mathrm{TV}}=\frac{1}{2}\int_E|f(x)-g(x)|R(\mathrm{d}x). \tag{23}$$

LEMMA 38. *We consider the $\mathbb{S}_{d-1}$-valued process $(\hat\Theta_t)_{t\ge0}$, solution to* (4).

(i) *The measure $\nu_\beta(\mathrm{d}\theta)=a_\beta[\gamma(\theta)]^{-\beta}\varsigma(\mathrm{d}\theta)$ on $\mathbb{S}_{d-1}$ is its unique invariant probability measure.*

(ii) *There is $C>0$ and $\lambda>0$ such that for all $t\ge0$, all measurable and bounded function $\phi:\mathbb{S}_{d-1}\to\mathbb{R}$,*

$$\sup_{\theta_0\in\mathbb{S}_{d-1}}\left|\mathbb{E}_{\theta_0}[\phi(\hat\Theta_t)]-\int_{\mathbb{S}_{d-1}}\phi\,\mathrm{d}\nu_\beta\right|\le C\|\phi\|_\infty e^{-\lambda t}.$$

(iii) *There exists a (unique in law) stationary eternal version $(\hat\Theta^\star_t)_{t\in\mathbb{R}}$ of this $\mathbb{S}_{d-1}$-valued process process and it holds that $\mathrm{Law}(\hat\Theta^\star_t)=\nu_\beta$ for all $t\in\mathbb{R}$. We denote by $\Lambda\in\mathcal{P}(\mathcal{H})$, where $\mathcal{H}=C(\mathbb{R},\mathbb{S}_{d-1})$, the law of this stationary process.*

(iv) *Consider the process $(\hat\Theta_t)_{t\ge0}$ starting from some given $\theta_0\in\mathbb{S}_{d-1}$. Fix $k\ge1$ and consider some positive sequences $(t^1_n)_{n\ge1},\dots,(t^k_n)_{n\ge1}$, all tending to infinity as $n\to\infty$. We can find, for each $A\ge1$ and each $n\ge1$, an i.i.d. family of Λ-distributed eternal processes $(\hat\Theta^{\star,1,n,A}_t)_{t\in\mathbb{R}},\dots,(\hat\Theta^{\star,k,n,A}_t)_{t\in\mathbb{R}}$ such that, defining $p_A(t^n_1,\dots,t^n_k)$ by the formula*

$$\mathbb{P}[(\hat\Theta_{(t^n_1+t)\vee0},\dots,\hat\Theta_{(t^n_1+\dots+t^n_k+t)\vee0})_{t\in[-A,A]}=(\hat\Theta^{\star,1,n,A}_t,\dots,\hat\Theta^{\star,k,n,A}_t)_{t\in[-A,A]}],$$

it holds that $\lim_{n\to\infty}p_A(t^n_1,\dots,t^n_k)=1$.

PROOF. We recall that the generator $\hat{\mathcal{L}}$ of the $(\hat\Theta_t)_{t\ge0}$ is given, for $\varphi\in C^2(\mathbb{S}_{d-1})$ and $\theta\in\mathbb{S}_{d-1}$, by $\hat{\mathcal{L}}\varphi(\theta)=\frac{1}{2}[\Delta_S\varphi(\theta)-\beta\frac{\nabla_S\gamma(\theta)}{\gamma(\theta)}\cdot\nabla_S\varphi(\theta)]=\frac{1}{2}[\gamma(\theta)]^\beta\,\mathrm{div}_S([\gamma(\theta)]^{-\beta}\nabla_S\varphi(\theta))$, so that $\nu_\beta(\mathrm{d}\theta) = a_\beta[\gamma(\theta)]^{-\beta}\varsigma(\mathrm{d}\theta)$ is an invariant probability measure. The uniqueness of this invariant probability measure follows from point (ii). We denote by $Q_t(x, \mathrm{d}y)$ the semi-group, defined as the law of $\hat{\Theta}_t$ when $\hat{\Theta}_0 = x \in \mathbb{S}_{d-1}$. Grigor'yan [17], Theorem 3.3, page 103, tells us that $Q_t(x, \mathrm{d}y)$ has a density $q_t(x, y)$ with respect to the uniform measure ς on $\mathbb{S}_{d-1}$, which is positive and smooth as a function of $(t, x, y) \in (0, \infty) \times \mathbb{S}_{d-1} \times \mathbb{S}_{d-1}$.

For (ii), it suffices that $b = \sup_{x,x' \in \mathbb{S}_{d-1}} \|Q_1(x, \cdot) - Q_1(x', \cdot)\|_{\mathrm{TV}} < 1$, because then the semi-group property implies that $\|Q_t(x, \cdot) - \nu_\beta\|_{\mathrm{TV}} \le b^{\lfloor t \rfloor}$, whence the result by (22). But, setting $a = \min\{q_1(x, y) : x, y \in \mathbb{S}_{d-1}\} > 0$ and recalling (23), we have

$$\begin{aligned}\|Q_1(x, \cdot) - Q_1(x', \cdot)\|_{\mathrm{TV}} &= \frac{1}{2}\int_{\mathbb{S}_{d-1}} |q_1(x, y) - q_1(x', y)|\varsigma(\mathrm{d}y)\\ &= \frac{1}{2}\int_{\mathbb{S}_{d-1}} \big|(q_1(x, y) - a) - (q_1(x', y) - a)\big|\varsigma(\mathrm{d}y),\end{aligned}$$

which is bounded by $\frac{1}{2}\int_{\mathbb{S}_{d-1}}[(q_1(x, y) - a) + (q_1(x', y) - a)]\varsigma(\mathrm{d}y) = 1 - a < 1$.

Point (iii) follows from the Kolmogorov extension theorem. Indeed, consider, for each $n \ge 0$, the solution $(\hat{\Theta}^n_t)_{t \ge -n}$ starting at time $-n$ with initial law ν_β and observe that for all $m > n$, $\mathrm{Law}((\hat{\Theta}^n_t)_{t\ge -n}) = \mathrm{Law}((\hat{\Theta}^m_t)_{t\ge -n})$ because $\mathrm{Law}(\hat{\Theta}^m_{-n}) = \nu_\beta$.

Next, we consider n large enough so that $\min\{t^n_1, \dots, t^n_k\} \ge 2A$. We will check by induction that for all $\ell = 1, \dots, k$, $\|\Gamma^{n,\ell}_A - \Lambda^{\otimes \ell}_A\|_{\mathrm{TV}} \le p_{A,\ell,n}$ where $\Lambda_A = \mathrm{Law}((\hat{\Theta}^\star_t)_{t\in[0,2A]})$, where $\Gamma^{n,\ell}_A \in \mathcal{P}(C([0, 2A], \mathbb{S}_{d-1})^\ell)$ is the law of $((\hat{\Theta}_{t^n_1 - A + t})_{t\in[0,2A]}, \dots, (\hat{\Theta}_{t^n_1 + \dots + t^\ell_k - A + t})_{t\in[0,2A]})$, and where

$$p_{A,\ell,n} = C\sum_{i=1}^{\ell} \exp\big(-\lambda\big(t^n_i - 2A\big)\big),$$

with $C > 0$ and $\lambda > 0$ introduced in (ii). By (22), this will prove point (iv). We recall that, by (ii), $\sup_{\theta_0 \in \mathbb{S}_{d-1}} \|Q_t(\theta_0, \cdot) - \nu_\beta\|_{\mathrm{TV}} \le C\exp(-\lambda t)$, and we introduce $\Lambda_{A,x} \in \mathcal{P}(C([0, 2A], \mathbb{S}_{d-1}))$ the law of $(\hat{\Theta}_t)_{t\in[0,2A]}$ when starting from $\hat{\Theta}_0 = x \in \mathbb{S}_{d-1}$.

Writing $\Gamma^{n,1}_A = \int_{\mathbb{S}_{d-1}} Q_{t^n_1 - A}(\theta_0, \mathrm{d}x)\Lambda_{A,x}(\cdot)$ and $\Lambda_A = \int_{\mathbb{S}_{d-1}} \nu_\beta(\mathrm{d}x)\Lambda_{A,x}(\cdot)$, we find that indeed,

$$\|\Gamma^{n,1}_A - \Lambda_A\|_{\mathrm{TV}} \le \|Q_{t^n_1 - A}(\theta_0, \cdot) - \nu_\beta\|_{\mathrm{TV}} \le C\exp\big(-\lambda\big(t^n_1 - A\big)\big) \le p_{A,1,n}.$$

Assuming next that $\|\Gamma^{n,\ell-1}_A - \Lambda^{\otimes(\ell-1)}_A\|_{\mathrm{TV}} \le p_{A,\ell-1,n}$ for some $\ell \in \{2, \dots, k\}$, we write

$$\begin{aligned}\Gamma^{n,\ell}_A\big(\mathrm{d}\theta^{(1)}, \dots, \mathrm{d}\theta^{(\ell)}\big) &= \int_{x\in\mathbb{S}_{d-1}} \Gamma^{n,\ell-1}_A\big(\mathrm{d}\theta^{(1)}, \dots, \mathrm{d}\theta^{(\ell-1)}\big)\\ &\qquad \times Q_{t^n_\ell - 2A}\big(\theta^{(\ell-1)}_{2A}, \mathrm{d}x\big)\Lambda_{A,x}\big(\mathrm{d}\theta^{(\ell)}\big),\\ \Lambda^{\otimes\ell}\big(\mathrm{d}\theta^{(1)}, \dots, \mathrm{d}\theta^{(\ell)}\big) &= \int_{x\in\mathbb{S}_{d-1}} \Lambda^{\otimes(\ell-1)}_A\big(\mathrm{d}\theta^{(1)}, \dots, \mathrm{d}\theta^{(\ell-1)}\big)\nu_\beta(\mathrm{d}x)\Lambda_{A,x}\big(\mathrm{d}\theta^{(\ell)}\big).\end{aligned}$$

We conclude that

$$\begin{aligned}\|\Gamma^{n,\ell}_A - \Lambda^{\otimes\ell}_A\|_{\mathrm{TV}} &\le \sup_{y\in\mathbb{S}_{d-1}} \|Q_{t^n_\ell - 2A}(y, \cdot) - \nu_\beta\|_{\mathrm{TV}} + \|\Gamma^{n,\ell-1}_A - \Lambda^{\otimes(\ell-1)}_A\|_{\mathrm{TV}}\\ &\le Ce^{-\lambda(t^n_\ell - 2A)} + p_{A,\ell-1,n},\end{aligned}$$

which equals $p_{A,\ell,n}$ as desired. □

A.2. On Itô's measure. We recall that Itô's measure $\Xi \in \mathcal{P}(\mathcal{E})$ was introduced in Notation 15.

LEMMA 39. (i) *For Ξ-almost every $e \in \mathcal{E}$, $\int_0^{\ell(e)/2} |e(u)|^{-2}\,du = \infty$.*

(ii) *For all $\phi \in L^1(\mathbb{R})$, $\int_{\mathcal{E}}[\int_0^{\ell(e)} \phi(e(u))\,du]\Xi(de) = \int_{\mathbb{R}} \phi(x)\,dx$.*

(iii) *For all measurable $\phi : \mathbb{R} \to \mathbb{R}_+$,*

$$\int_{\mathcal{E}}\left[\int_0^{\ell(e)} \phi(e(u))\,du\right]^2 \Xi(de) \le 4\left[\int_{\mathbb{R}} \sqrt{|x|}\phi(x)\,dx\right]^2.$$

(iv) *For $q < 3/2$, for Ξ-almost every $e \in \mathcal{E}$, we have $\int_0^{\ell(e)} |e(u)|^{-q}\,du < \infty$.*

PROOF. For (i), it suffices to use that $\int_{0+}(r|\log r|)^{-1}\,dr = \infty$ and Lévy's modulus of continuity (see Revuz–Yor [33], Theorem 2.7, p. 30), which implies that for Ξ-a.e. $e \in \mathcal{E}$, $\limsup_{t\searrow 0} \sup_{r\in[0,t]}(2r|\log r|)^{-1}|e(r)|^2 = 1$.

Next (iv) follows from (iii), since the integral $\int_0^{\ell(e)} |e(u)|^{-q}\,du$ is finite if and only if $\int_0^{\ell(e)} |e(u)|^{-q}\mathbf{1}_{\{|e(u)|\le 1\}}\,du < \infty$ (for any $e \in \mathcal{E}$) and since

$$\int_{\mathcal{E}}\left[\int_0^{\ell(e)} |e(u)|^{-q}\mathbf{1}_{\{|e(u)|\le 1\}}\,du\right]^2 \Xi(de) \le 4\left[\int_{-1}^{1} |x|^{1/2-q}\,dx\right]^2 < \infty.$$

We now check points (ii) and (iii). We recall that for $(W_t)_{t\ge 0}$ a Brownian motion, for $(L_t^x)_{t\ge 0, x\in\mathbb{R}}$ its family of local times, for $(\tau_t)_{t\ge 0}$ the inverse of $(L_t^0)_{t\ge 0}$, the second Ray–Knight theorem (see Revuz–Yor [33], Theorem 2.3, p. 456) tells us that $(L_{\tau_1}^w)_{w\ge 0}$ is a squared Bessel process with dimension 0 issued from 1. Hence, for some Brownian motion $(B_w)_{w\ge 0}$, we have $L_{\tau_1}^w = 1 + 2\int_0^w \sqrt{L_{\tau_1}^w}\,dB_v$, so that $\mathbb{E}[L_{\tau_1}^w] = 1$ and $\mathbb{E}[(L_{\tau_1}^w - 1)^2] = 4\mathbb{E}[(\int_0^w \sqrt{L_{\tau_1}^w}\,dB_v)^2] = 4\int_0^w \mathbb{E}[L_{\tau_1}^v]\,dv = 4w$. By symmetry, for any $w \in \mathbb{R}$, we have $\mathbb{E}[L_{\tau_1}^w] = 1$ and $\mathbb{E}[(L_{\tau_1}^w - 1)^2] = 4|w|$. Applying (15) with $t = 1$, we see that

$$\begin{aligned}\int_{\mathcal{E}}\left[\int_0^{\ell(e)} \phi(e(u))\,du\right]\Xi(de) &= \mathbb{E}\left[\int_0^1 \int_{\mathcal{E}}\left[\int_0^{\ell(e)} \phi(e(u))\,du\right]\mathbf{M}(ds, de)\right] \\ &= \mathbb{E}\left[\int_0^{\tau_1} \phi(W_s)\,ds\right].\end{aligned}$$

But finally, by the occupation times formula and the Fubini theorem,

$$\mathbb{E}\left[\int_0^{\tau_1} \phi(W_s)\,ds\right] = \mathbb{E}\left[\int_{\mathbb{R}} \phi(w)L_{\tau_1}^w\,dw\right] = \int_{\mathbb{R}} \phi(w)\,dw$$

which proves (ii). Similarly,

$$\begin{aligned}\int_{\mathcal{E}}\left[\int_0^{\ell(e)} \phi(e(u))\,du\right]^2 \Xi(de) &= \mathbb{E}\left[\left(\int_0^1 \int_{\mathcal{E}}\left[\int_0^{\ell(e)} \phi(e(u))\,du\right]\tilde{\mathbf{M}}(ds, de)\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_0^{\tau_1} \phi(W_s)\,ds - \int_{\mathbb{R}} \phi(w)\,dw\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_{\mathbb{R}} \phi(w)(L_{\tau_1}^w - 1)\,dw\right)^2\right] \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \phi(w)\phi(u)\mathbb{E}\left[(L_{\tau_1}^w - 1)(L_{\tau_1}^u - 1)\right]dw\,du.\end{aligned}$$

We complete the proof of (iii) using that $\mathbb{E}[(L_{\tau_1}^w - 1)^2] = 4|x|$ and the Cauchy–Schwarz inequality. □

A.3. On Bessel processes.

LEMMA 40. (i) *Fix* $\delta \in (0,2)$, *consider a Brownian motion* $(W_t)_{t\geq 0}$, *introduce the inverse* ρ_t *of* $A_t = (2-\delta)^{-2}\int_0^t W_s^{-2(1-\delta)/(2-\delta)}\mathbf{1}_{\{W_s>0\}}\,\mathrm{d}s$ *and set* $\mathcal{R}_t = (W_{\rho_t})_+^{1/(2-\delta)}$. *Then* $(\mathcal{R}_t)_{t\geq 0}$ *is a Bessel process with dimension* $2-\delta$ *issued from* 0.

(ii) *For* $(\mathcal{R}_t)_{t\geq 0}$ *a Bessel process with dimension* $\delta > 0$, *a.s., for all* $t \geq 0$ *such that* $\mathcal{R}_t = 0$ *and all* $h > 0$, *we have* $\int_t^{t+h}\mathcal{R}_s^{-2}\,\mathrm{d}s = \infty$.

PROOF. Point (i) is more or less included in Donati–Roynette–Vallois–Yor [13], Corollary 2.2, who state that for $(R_t)_{t\geq 0}$ a Bessel process with dimension $\delta \in (0,2)$ issued from 0, for $C_t = (2-\delta)^2\int_0^t R_s^{2(1-\delta)}\,\mathrm{d}s$ and for D_t the inverse of C_t, $(R_{D_t})^{2-\delta}$ is a reflected Brownian motion. Moreover, this is clearly an *if and only if* condition.

But for $\mathcal{C}_t = (2-\delta)^2\int_0^t \mathcal{R}_s^{2(1-\delta)}\,\mathrm{d}s = (2-\delta)^2\int_0^t (W_{\rho_s})_+^{2(1-\delta)/(2-\delta)}\,\mathrm{d}s = \int_0^{\rho_t}\mathbf{1}_{\{W_u>0\}}\,\mathrm{d}u$ and for $\mathcal{D}_t$ its inverse, we have $\mathcal{D}_t = A_{\mathcal{E}_t}$, where $\mathcal{E}_t$ is the inverse of $\int_0^t \mathbf{1}_{\{W_s>0\}}\,\mathrm{d}s$. It is then clear that $\mathcal{R}_{\mathcal{D}_t}^{2-\delta} = (W_{\rho_{\mathcal{D}_t}})_+ = (W_{\mathcal{E}_t})_+$ is a reflected Brownian motion.

Point (ii) follows from Khoshnevisan [23], (2.1a), page 1299, that asserts that a.s., for all $T > 0$, $\limsup_{h\searrow 0}\sup_{t\in[0,T]}[h(1\vee\log(1/h))]^{1/2}|\mathcal{R}_{t+h}-\mathcal{R}_t| = \sqrt{2}$. Indeed we have that $\int_{0+}[h(1\vee\log(1/h))]^{-1}\,\mathrm{d}h = \infty$. □

A.4. Inverting time changes. We recall a classical result about the convergence of inverse functions.

LEMMA 41. *Consider, for each* $n \geq 1$, *a continuous increasing bijective function* $(a_t^n)_{t\geq 0}$ *from* $[0,\infty)$ *into itself, as well as its inverse* $(r_t^n)_{t\geq 0}$. *Assume that* $(a_t^n)_{t\geq 0}$ *converges pointwise to some function* $(a_t)_{t\geq 0}$ *such that* $\lim_\infty a_t = \infty$, *denote by* $r_t = \inf\{u \geq 0 : a_u > t\}$ *its right-continuous generalized inverse and set* $J = \{t \in [0,\infty) : r_{t-} < r_t\}$. *For all* $t \in [0,\infty)\setminus J$, *we have* $\lim_{t\to\infty} r_t^n = r_t$.

A.5. Technical estimates. Finally, we study the functions h, ψ, σ introduced in Notation 9. We recall that $h(r) = (\beta+2-d)\int_{r_0}^r u^{1-d}[\Gamma(u)]^\beta\,\mathrm{d}u$ is an increasing bijection from $(0,\infty)$ into $\mathbb{R}$, that $h^{-1} : \mathbb{R} \to (0,\infty)$ is its inverse function. We have set $\sigma(w) = h'(h^{-1}(w))$ and $\psi(w) = [\sigma(w)h^{-1}(w)]^2$, both being functions from $\mathbb{R}$ to $(0,\infty)$.

LEMMA 42. *Fix* $\beta > d-2$ *and set* $\alpha = (\beta+2-d)/3$. *There are some constants* $0 < c < C$ *such that the results below are valid for all* $w \in \mathbb{R}$ (*except in point* (v)).

(i) *If* $\beta > d$, $\kappa = \int_{\mathbb{R}}[\sigma(z)]^{-2}\,\mathrm{d}z = (\beta+2-d)^{-1}\int_0^\infty r^{d-1}[\Gamma(r)]^{-\beta}\,\mathrm{d}r < \infty$.

(ii) *If* $\beta > 1+d$, $m'_\beta = (\int_{\mathbb{R}} h^{-1}(z)[\sigma(z)]^{-2}\,\mathrm{d}z)/(\int_{\mathbb{R}}[\sigma(z)]^{-2}\,\mathrm{d}z)$.

(iii) *If* $\beta = 4+d$, $\frac{[h^{-1}(w)]^4}{[\sigma(w)]^2} \leq C(1+|w|)^{-1}$ *and* $\int_{-x}^x \frac{[h^{-1}(z)]^4\,\mathrm{d}z}{[\sigma(z)]^2} \overset{x\to\infty}{\sim} \frac{\log x}{36}$.

(iv) *If* $\beta \in [d,4+d]$, $c(1+w)^2\mathbf{1}_{\{w>0\}} \leq \psi(w) \leq C(1+|w|)^2$.

(v) *If* $\beta \in [d,4+d)$, $\lim_{\eta\to 0}\eta^2\psi(w/\eta) = (\beta+2-d)^2 w^2$ *for any* $w > 0$.

(vi) *If* $\beta > d-2$, $[\sigma(w)]^{-2} \leq C(1+|w|)^{-2(\beta+1-d)/(\beta+2-d)}$.

(vii) *If* $\beta = d$, $\int_{-x}^x [\sigma(z)]^{-2}\,\mathrm{d}z \overset{x\to\infty}{\sim} \frac{\log x}{4}$.

(viii) *If* $\beta \in [d,4+d)$, $\frac{1+h^{-1}(w)}{[\sigma(w)]^2} \leq C(1+w)^{1/\alpha-2}\mathbf{1}_{\{w\geq 0\}} + C(1+|w|)^{-2}\mathbf{1}_{\{w<0\}}$.

(ix) *If* $\beta \in [d,4+d)$, *for all* $m \in \mathbb{R}$, *we have*

$$\lim_{\eta\to 0}\eta^{1/\alpha-2}\frac{h^{-1}(w/\eta)-m}{[\sigma(w/\eta)]^2} = (\beta+2-d)^{-2}w^{1/\alpha-2}\mathbf{1}_{\{w\geq 0\}}.$$

(x) *If* $\beta \in (d-2, d)$ *and* $a_\epsilon = \epsilon^{(\beta+2-d)/2}$, $\sqrt{\epsilon}h^{-1}(w/a_\epsilon) \to w_+^{1/(\beta+2-d)}$ *uniformly on compact sets*.

(xi) *If* $\beta \in (d-2, d)$ *and* $a_\epsilon = \epsilon^{(\beta+2-d)/2}$,

$$\lim_{\epsilon\to 0} \epsilon\big[a_\epsilon \sigma(w/a_\epsilon)\big]^{-2} = (\beta+2-d)^{-2} w^{-2(\beta+1-d)/(\beta+2-d)} \mathbf{1}_{\{w>0\}}.$$

PROOF. The three following points will be of constant use.

(a) We have $h^{-1}(w) \sim w^{1/(\beta+2-d)}$, $\sigma(w) \sim (\beta+2-d)w^{(\beta+1-d)/(\beta+2-d)}$ and $\psi(w) \sim (\beta+2-d)^2 w^2$ as $w \to \infty$.

(b) If $d \geq 3$, there are $c, c', c'' > 0$ such that, as $w \to -\infty$, $h^{-1}(w) \sim c|w|^{-1/(d-2)}$, $\sigma(w) \sim c'|w|^{(d-1)/(d-2)}$ and $\psi(w) \sim c''|w|^2$.

(c) If $d = 2$, there are a function ϵ satisfying $\lim_{w\to-\infty} \epsilon(w) = 0$ and $c, c', c'' > 0$ such that $h^{-1}(w) = \exp[-c|w|(1+\epsilon(w))]$, such that $\sigma(w) \sim c' \exp[c|w|(1+\epsilon(w))]$ as $w \to -\infty$ and $\lim_{w\to-\infty} \psi(w) = c''$.

To check (a), it suffices to note that by Assumption 1, $h(r) \sim r^{\beta+2-d}$ as $r \to \infty$. Next, (b) follows from the fact that $h(r) \sim -\mathfrak{c}r^{2-d}$ as $r \to 0$ (with $\mathfrak{c} = [\Gamma(0)]^\beta(\beta+2-d)/(d-2) > 0$), while (c) uses that $h(r) \sim -\mathfrak{c}\log(1/r)$ (with $\mathfrak{c} = \beta[\Gamma(0)]^\beta$, the result then holds with $\epsilon(w) = \mathfrak{c}[\log h^{-1}(w)]/w - 1$, $c = 1/\mathfrak{c}$, $c' = \mathfrak{c}$ and $c'' = \mathfrak{c}^2$).

We now prove (i). Using the substitution $r = h^{-1}(z)$,

$$\kappa = \int_{\mathbb{R}} \frac{\mathrm{d}z}{[h'(h^{-1}(z))]^2} = \int_0^\infty \frac{\mathrm{d}r}{h'(r)} = \frac{1}{\beta+2-d}\int_0^\infty \frac{r^{d-1}}{[\Gamma(r)]^\beta}\,\mathrm{d}r,$$

which is finite if and only if $d-1-\beta < -1$, that is, $\beta > d$. Recall that $\Gamma : [0,\infty) \to (0,\infty)$ is supposed to be continuous and that $\Gamma(r) \sim r$ as $r \to \infty$.

We proceed similarly for (ii). With m'_β defined in Section 3,

$$\frac{\int_{\mathbb{R}} h^{-1}(z)[\sigma(z)]^{-2}\,\mathrm{d}z}{\int_{\mathbb{R}}[\sigma(z)]^{-2}\,\mathrm{d}z} = \frac{\int_0^\infty r[h'(r)]^{-1}\,\mathrm{d}r}{\int_0^\infty [h'(r)]^{-1}\,\mathrm{d}r} = \frac{\int_0^\infty r^d[\Gamma(r)]^{-\beta}\,\mathrm{d}r}{\int_0^\infty r^{d-1}[\Gamma(r)]^{-\beta}\,\mathrm{d}r} = m'_\beta.$$

For (iii), we see that when $\beta = 4+d$, (a) implies that $[h^{-1}(w)]^4/[\sigma(w)]^2 \sim 36^{-1}w^{-1}$ as $w \to \infty$, whence the bound $[h^{-1}(w)]^4/[\sigma(w)]^2 \leq C(1+|w|)^{-1}$ on $\mathbb{R}_+$ and the estimate $\int_0^x \frac{[h^{-1}(w)]^4\,\mathrm{d}w}{[\sigma(w)]^2} \overset{x\to\infty}{\sim} \frac{\log x}{36}$. If $d \geq 3$, (b) tells us that $[h^{-1}(w)]^4/[\sigma(w)]^2 \sim c|w|^{-2(d+1)/(d-2)}$ as $w \to -\infty$ (for some constant $c > 0$), and we conclude using that $2(d+1)/(d-2) > 1$. If $d = 2$, (c) gives us $[h^{-1}(w)]^4/[\sigma(w)]^2 \sim [c']^{-2}\exp(-6c|w|(1+\epsilon(w)))$ as $w \to -\infty$, from which the estimates follow.

Point (iv) immediately follows from (a) (concerning the lowerbound and the upperbound on $\mathbb{R}_+$) and (b) or (c) (concerning the upperbound on $\mathbb{R}_-$).

Point (v) is a consequence of (a).

Point (vi) follows from (a) (concerning the bound on $\mathbb{R}_+$) and from (b) (and the fact that $(d-1)/(d-2) > (\beta+1-d)/(\beta+2-d)$) or (c).

With the same arguments as in (vi), we see that $\int_{-x}^x [\sigma(w)]^{-2}\,\mathrm{d}w \overset{x\to\infty}{\sim} \int_0^x [\sigma(w)]^{-2}\,\mathrm{d}w$, which is equivalent to $[\log x]/4$ as $x \to \infty$ by (a), whence (vii).

Points (viii) and (ix) follow from (b) or (c) (when $w < 0$) or (a) and using the fact that $1/(\beta+2-d) - 2(\beta+1-d)/(\beta+2-d) = 1/\alpha - 2$ (when $w \geq 0$).

Points (x) and (xi) follow from (a) (when $w \geq 0$) and (b) or (c) (when $w < 0$). Observe that in (x), the convergence is uniform on compact sets for free by the Dini theorem, since for each $\epsilon > 0$, $w \to \sqrt{\epsilon}h^{-1}(w/a_\epsilon)$ is nondecreasing and since the limit function $w \to w_+^{1/(\beta+2-d)}$ is continuous and nondecreasing. □

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