The Annals of Probability
2020, Vol. 48, No. 2, 1002–1013
https://doi.org/10.1214/19-AOP1380

ON THE NATURE OF THE SWISS CHEESE IN DIMENSION 3

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We study scenarii linked with the *Swiss cheese picture* in dimension 3 obtained when two random walks are forced to meet often, or when one random walk is forced to squeeze its range. In the case of two random walks, we show that they most likely meet in a region of *optimal density*. In the case of one random walk, we show that a small range is reached by a strategy uniform in time. Both results rely on an original inequality estimating the cost of visiting sparse sites, and in the case of one random walk on the precise large deviation principle of van den Berg, Bolthausen and den Hollander (*Ann. of Math. (2)* **153** (2001) 355–406), including their sharp estimates of the rate functions in the neighborhood of the origin.

1. Introduction. In this note, we are concerned with describing the geometry of the range of a random walk on $\mathbb{Z}^3$, when forced to having a small volume, deviating from its mean by a small fraction of it, or to intersecting often the range of another independent random walk.

These issues were raised in two landmark papers of van den Berg, Bolthausen and den Hollander (referred to as BBH in the sequel) written two decades ago [10, 11]. Both papers dealt with the continuous counterpart of the range of a random walk, the Wiener sausage. They showed a large deviation principle, in two related contexts: (i) in [10] for the downward deviation of the volume of the sausage, (ii) in [11] for the upward deviation of the volume of the intersection of two independent sausages. They also expressed the rate function with a variational formula.

Their sharp asymptotics are followed with a heuristic description of the optimal scenario dubbed the *Swiss cheese* picture where, in case (i), the *Wiener sausage covers only part of the space leaving random holes whose sizes are of order 1 and whose density varies on space scale* $t^{1/d}$, and in case (ii) both Wiener sausages form apparently independent Swiss cheeses. However, they acknowledge that to show that conditioned on the deviation, the sausages *actually follow the Swiss cheese strategy requires substantial extra work*.

Remarkably, the Swiss cheese heuristic also highlights a crucial difference between dimension 3 and dimensions 5 and higher. Indeed, in dimension 3 the typical scenario is time homogeneous, in the sense that the Wiener sausage considered up to time t, would spend all its time localized in a region of typical scale $(t/\varepsilon)^{1/3}$, filling a fraction of order ε of every volume element, when the deviation of the volume occurs by a fraction ε of the mean. On the other hand in dimension 5 and higher, the typical scenario would be time inhomogeneous: the Wiener sausage would localize in a smaller region of space with scale of order $(\varepsilon t)^{1/d}$, only during a fraction ε of its time and, therefore, would produce a localization region where the density is of order one, no matter how small ε is.

Recently, Sznitman [9] suggested that the Swiss cheese could be described in terms of so-called tilted random interlacements. In the same time, in [1] we obtained a first result in the discrete setting, which expressed the folding of the range of a random walk in terms of

Received July 2018; revised January 2019.
MSC2010 subject classifications. 60F10, 60G50.
Key words and phrases. Random walk, large deviations, range.

its capacity. More precisely, we showed that a positive fraction of the path, considered up to some time n, was spent in a subset having a volume of order n and a capacity of order $n^{1-2/d}$, that is to say of the same order as the capacity of a ball with volume n.

In this note, we present a simple and powerful estimate on the probability a random walk visits sites which are *far from each others*; see Proposition 1.3 below. We then deduce two applications in dimension 3 that we state vaguely as follows:

- One random walk, when forced to have a small range, *folds in a time-homogeneous way*.
- Two random walks, when forced to meet often, do so in a region of *optimal density*.

To state our results more precisely, we introduce some notation. We denote by $\{S_n\}_{n\geq 0}$ the simple random walk on $\mathbb{Z}^d$ (in most of the paper $d=3$), and by $\mathcal{R}(I) := \{S_n\}_{n\in I}$, its range in a time window $I \subset \mathbb{N}$, which we simply write as $\mathcal{R}_n$, when I is the interval $I_n := \{0, \dots, n\}$. We let $Q(x,r) := (x + [-r, r[^d) \cap \mathbb{Z}^d$, be the cube of side length $2r$, and then define regions of *high density* as follows. First, we define the (random) set of centers, depending on a density ρ, a scale r and a time window I as

$$\mathcal{C}(\rho, r, I) := \{x \in 2r\mathbb{Z}^d : |\mathcal{R}(I) \cap Q(x,r)| \geq \rho \cdot |Q(x,r)|\},$$

and then the corresponding region

$$\mathcal{V}(\rho, r, I) := \bigcup_{x\in\mathcal{C}(\rho,r,I)} Q(x,r),$$

which we simply write $\mathcal{V}_n(\rho, r)$, when $I = I_n$. Thus, $\mathcal{R}(I) \cap \mathcal{V}(\rho, r, I)$ is the set of visited sites around which the range has density, on a scale r, above ρ in the time window I.

Our first result concerns the problem of forcing a single random walk having a small range. For $\varepsilon \in (0,1)$, we denote by $\mathbb{Q}_n^\varepsilon$ the law of the random walk conditionally on the event $\{|\mathcal{R}_n| - \mathbb{E}[|\mathcal{R}_n|] \leq -\varepsilon n\}$, and for $I \subset I_n$, we set $I^c := I_n \setminus I$.

THEOREM 1.1. *Assume $d = 3$. There exist positive constants β, K_0 and ε_0, such that for any $\varepsilon \in (0, \varepsilon_0)$, and any $1 \leq r \leq \varepsilon^{5/6} n^{1/3}$,*

$$\lim_{n\to\infty} \mathbb{Q}_n^\varepsilon \left[\begin{array}{c} |\mathcal{R}(I) \cap \mathcal{R}(I^c) \cap \mathcal{V}_n(\beta\varepsilon, r)| \geq \frac{\varepsilon}{8}|I|, \\ \textit{for all intervals } I \subseteq I_n, \textit{ with } |I| = \lfloor K_0 \varepsilon n \rfloor \end{array} \right] = 1.$$

We note that the proof of the theorem gives in fact a stretched exponential speed of convergence to 1. This result expresses the fact that under $\mathbb{Q}_n^\varepsilon$, in any time interval of length of order εn, the random walk intersects the other part of its range a fraction ε of its time, which is in agreement with the intuitive idea that, if during some time interval, the walk moves in a region with density of visited sites of order ε, it should intersect it a fraction ε of its time. Note that it brings complementary information to the results obtained in [1], where it was shown that for some positive constants β, C and $\varepsilon_0 \in (0,1)$, for any $\varepsilon \in (0, \varepsilon_0)$, and any $C\varepsilon^{-5/9} n^{2/9} \log n \leq r \leq \frac{1}{C}\sqrt{\varepsilon} n^{1/3}$,

$$\lim_{n\to\infty} \mathbb{Q}_n^\varepsilon \left[|\mathcal{V}_n(\beta\varepsilon, r)| \geq \frac{n}{C\varepsilon} \text{ and } \mathrm{cap}(\mathcal{V}_n(\beta\varepsilon, r)) \leq C\left(\frac{n}{\varepsilon}\right)^{\frac{1}{3}} \right] = 1, \tag{1.1}$$

with $\mathrm{cap}(\Lambda)$ being the capacity of a finite set $\Lambda \subset \mathbb{Z}^d$. We refer to [7] for a definition, but let us just recall that for any Λ, one has $c|\Lambda|^{1-\frac{2}{d}} \leq \mathrm{cap}(\Lambda) \leq |\Lambda|$, with $c > 0$ some universal constant, and that the lower bound is achieved (at least up to a constant) when Λ is a discrete ball. Now by definition of $\mathcal{V}_n(\beta\varepsilon, r)$, one can observe that (1.1) also shows that for some constant $\alpha > 0$ (independent of ε), one has $|\mathcal{R}_n \cap \mathcal{V}_n(\beta\varepsilon, r)| \geq \alpha n$, with $\mathbb{Q}_n^\varepsilon$-probability going to one. This is in a sense stronger than the result of Theorem 1.1, which only gives (by summation over disjoint intervals) that with high $\mathbb{Q}_n^\varepsilon$-probability, $|\mathcal{R}_n \cap \mathcal{V}_n(\beta\varepsilon, r)| \geq \frac{1}{8}\varepsilon n$. On the other hand, (1.1) says nothing on the distribution of the times when the random walk visits the sets $\mathcal{V}_n(\beta\varepsilon, r)$, while Theorem 1.1 shows that they are in a sense uniformly distributed. Both results are proved using different techniques: while (1.1) was obtained by using only elementary arguments, Theorem 1.1 relies on the sharp and intricate results of [10] (which have been obtained in the discrete setting by Phetpradap in his thesis [8]).

Our second result concerns the problem of intersection of two independent ranges. For $n \geq 1$, and $\rho \in (0,1)$, we denote by $\widetilde{\mathbb{Q}}_n^\rho$ the law of two independent walks S and $\widetilde{S}$, both starting from the origin, conditionally on the event $\{|\mathcal{R}_n \cap \widetilde{\mathcal{R}}_n| > \rho n\}$, where $\mathcal{R}_n$ and $\widetilde{\mathcal{R}}_n$ denote the respective ranges of these two walks up to time n. We also let $\widetilde{\mathcal{V}}_n(\rho, r)$ be the corresponding high-density region for the walk $\widetilde{S}$.

THEOREM 1.2. *Assume $d = 3$. There exist positive constants c and κ, such that for any $n \geq 1$, $\rho \in (0,1)$, $\delta \in (0,1)$ and $r \leq c\delta^{2/3}(\rho n)^{1/3}$,*

$$\lim_{n\to\infty} \widetilde{\mathbb{Q}}_n^\rho\big[\big|\mathcal{R}_n \cap \widetilde{\mathcal{R}}_n \cap \mathcal{V}_n(\delta\rho, r) \cap \widetilde{\mathcal{V}}_n(\delta\rho, r)\big| > (1-\kappa\delta)\rho n\big] = 1. \tag{1.2}$$

Recall that the heuristic picture is that as ρ goes to zero, under the law $\widetilde{\mathbb{Q}}_n^\rho$, both random walks should localize in a region of typical diameter $(n/\rho)^{1/3}$, during their whole time-period. Thus, the occupation density in the localization region is expected to be of order ρ, and (1.2) provides a precise statement of this picture. Let us stress that unlike Theorem 1.1, the proof of this second result does not rely on BBH's fine large deviation principle, but only on relatively soft arguments.

Our main technical tool for proving both Theorems 1.1 and 1.2 is the following proposition, which allows us to estimate visits in a region of *low density* at a given space scale r. For $x \in \mathbb{Z}^d$, we let $\mathbb{P}_x$ be the law of the simple random walk starting from x, that we simply denote by $\mathbb{P}$ when x is the origin.

PROPOSITION 1.3. *Assume that Λ is a subset of $\mathbb{Z}^d$ with the following property. For some $\rho \in (0,1)$, and $r \geq 1$,*

$$\big|\Lambda \cap Q(z,r)\big| \leq \rho \cdot \big|Q(z,r)\big|, \quad \textit{for all } z \in 2r\mathbb{Z}^d. \tag{1.3}$$

There is a constant $\kappa > 1$ independent of r, ρ and Λ, such that for any $n \geq \rho^{\frac{2}{d}-1}r^2$, any $t \geq \kappa\rho n$,

$$\mathbb{P}\big[|\mathcal{R}_n \cap \Lambda| \geq t\big] \leq \exp\left(-\rho^{1-\frac{2}{d}} \cdot \frac{t}{2r^2}\right). \tag{1.4}$$

Note that in (1.4) the smaller is the scale the smaller is the probability. Note also that this result holds in any dimension $d \geq 3$.

Now some remarks on the limitation of our results are in order. Let us concentrate on Theorem 1.1 which is our main result. First, the size of the time-window is constrained by the degree of precision in BBH's asymptotics. The fact that one can only consider windows of size order εn, and not say order $\varepsilon^K n$, for some $K > 1$, is related to the asymptotic of the rate function in the neighborhood of the origin; see (2.3) below. From [10], one knows the first-order term in dimension 3. However, pushing further the precision of this asymptotic would allow to consider higher exponents K. On the other hand, it would be even more interesting to allow time windows of smaller size, say of polynomial order n^κ, with $\kappa \in (2/3, 1)$. Going below the exponent 2/3 does not seem reasonable, as the natural belief is that strands of the path of length $n^{2/3}$ should typically move freely, and might visit from time to time regions with very low occupation density. Thus, we believe that a result in the same vein as Theorem 1.1 should hold for exponents $\kappa \in (2/3, 1)$. One would need however a much better understanding of the speed of convergence in the large deviation principle; see (2.2) below.

Similarly, one could ask whether our proof could show a kind of time inhomogeneity in dimension 5 and higher. However, a problem for this is the following. Given two small time windows (say of order εn), one would like to argue that the walk cannot visit regions with high occupation density in both, unless these two time windows were adjacent. However, even if they are not, the cost for the walk to come back at the origin at the beginning of each of them is only polynomially small, which is almost invisible when compared to the (stretched) exponentially small cost of the large deviations. Therefore, obtaining such result seems out of reach at the moment.

Let us sketch the proof of Theorem 1.1. The first step, which reveals also the link between [10] and [11] is to show that $\{|\mathcal{R}_n| - \mathbb{E}[|\mathcal{R}_n|] < -\varepsilon n\}$ implies large mutual intersections $\{|\mathcal{R}(I) \cap \mathcal{R}(I^c)| > \beta\varepsilon|I|\}$ for some constant β as soon as the interval $I \subset [0, n]$ is large enough. This requires to show a LDP on the same precision as [10] for $\mathcal{R}(I^c)$, where I^c typically consists of two subintervals: this step presents some subtleties, which we deal with by wrapping parts of the two trajectories, and use that the intersection essentially increases under such operation. Then one falls back on an estimate similar to Theorem 1.2.

The rest of the paper is organized as follows. In Section 2, we gather useful results on the range: BBH's results, as well as the upward large deviation principle of Hamana and Kesten [4], and estimates on probability that the walk covers a fixed region with low density for a long time. We then prove Proposition 1.3 and Theorem 1.2. In Section 3, we prove an extension of BBH's estimate when one considers two independent walks starting from different positions. The proof of Theorem 1.1 is concluded in Section 4.

2. Visiting sparse regions.

In this section, we prove our main tool, Proposition 1.3 and then Theorem 1.2, after we recall well-known results.

2.1. *Preliminaries.* Dvoretzky and Erdős [3] established that there exists a constant $\kappa_d > 0$, such that almost surely and in L^1,

$$\lim_{n\to\infty} \frac{|\mathcal{R}_n|}{n} = \kappa_d.$$

In addition, one has

$$\big|\mathbb{E}\big[|\mathcal{R}_n|\big] - \kappa_d n\big| = \begin{cases} \mathcal{O}(\sqrt{n}) & \text{when } d = 3, \\ \mathcal{O}(\log n) & \text{when } d = 4, \\ \mathcal{O}(1) & \text{when } d \geq 5. \end{cases} \tag{2.1}$$

Precise asymptotic of the variance and a central limit theorem are obtained by Jain and Orey [5] in dimensions $d \geq 5$, and by Jain and Pruitt [6] in dimensions $d \geq 3$.

The analogue of the LDP of [10] has been established in the discrete setting by Phetpradap in his thesis [8], and reads as follows: there exists a strictly positive function I_d, such that for any $\varepsilon \in (0, \kappa_d)$,

$$\lim_{n\to\infty} \frac{1}{n^{1-\frac{2}{d}}} \log \mathbb{P}\big[|\mathcal{R}_n| - \mathbb{E}\big[|\mathcal{R}_n|\big] \leq -\varepsilon n\big] = -I_d(\varepsilon). \tag{2.2}$$

Moreover, there exist positive constants C, μ_d and ν_d, such that for $\varepsilon \in (0, \nu_d)$,

$$\left.\begin{matrix} \mu_3 \varepsilon^{2/3} \\ \mu_4 \sqrt{\varepsilon} \end{matrix}\right\} \leq I_d(\varepsilon) \leq \begin{cases} \mu_3 \varepsilon^{2/3}(1 + C\varepsilon) & \text{when } d = 3, \\ \mu_4 \sqrt{\varepsilon}(1 + C\varepsilon^{1/3}) & \text{when } d = 4, \end{cases} \tag{2.3}$$

and if $d \geq 5$, $I_d(\varepsilon) = \mu_d \varepsilon^{1-\frac{2}{d}}$. These results were first obtained in [10] for the Wiener sausage: the lower bounds are given by their Theorem 4(ii) and Theorem 5(iii), respectively, for dimensions 3, 4 and for dimension 5 and higher (note that their constants μ_d differs from ours by a universal constant; see also [8] for details). The upper bound in dimension 5 and higher is also provided by their Theorem 5(iii). The upper bound in dimension 3 and 4 is obtained in the course of the proof of Theorem 4(ii); see their equations (5.73), (5.81) and (5.82). Note that, as they use Donsker–Varadhan's large deviation theory, their rate functions I_d are given by variational formulas.

Upward large deviations are obtained by Hamana and Kesten [4]. Their result implies that there exists a strictly positive function J_d, such that for $\varepsilon \in (0, 1 - \kappa_d)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}\big(|\mathcal{R}_n| - \mathbb{E}[|\mathcal{R}_n|] \geq \varepsilon n\big) = -J_d(\varepsilon). \tag{2.4}$$

Finally, an elementary fact we shall need is the following.

LEMMA 2.1. *There exists a constant $C > 0$, such that for any $\rho \in (0, 1)$, $r \geq 1$ and $\Lambda \subset \mathbb{Z}^d$, satisfying (1.3) one has for all $n \geq 1$, and all $x \in \mathbb{Z}^d$,*

$$\mathbb{E}_x\big[|\mathcal{R}_n \cap \Lambda|\big] \leq C\big(\rho^{2/d} r^2 + \rho n\big), \quad \textit{and} \quad \mathbb{E}_x\big[|\mathcal{R}_n \cap \Lambda|^2\big] \leq C\big(\rho^{2/d} r^2 + \rho n\big)^2.$$

PROOF. The first moment estimate is a consequence of Lemma 2.2 in [1]. Indeed, one has denoting by H_y the hitting time of a point $y \in \mathbb{Z}^d$,

$$\mathbb{E}_x\big[|\mathcal{R}_n \cap \Lambda|\big] = \sum_{y\in\Lambda} \mathbb{P}_x[H_y \leq n] \leq \sum_{y\in\Lambda} G_n(y-x) = \sum_{y\in(\Lambda-x)} G_n(y),$$

where $G_n(y) := \sum_{k=0}^n \mathbb{P}[S_n = y]$, is the restricted Green's function. Now, for any x, the set $\Lambda - x$ also satisfies (1.3), with a possibly larger constant than ρ: at least with the constant $2d\rho$ instead of it. Then the aforementioned lemma gives the desired estimate for the first moment. Concerning the second moment, using the Markov property, we get

$$\begin{aligned}
\mathbb{E}_x\big[|\mathcal{R}_n \cap \Lambda|^2\big] &= \sum_{y,y'\in\Lambda} \mathbb{P}_x[H_y \leq n, H_{y'} \leq n] \\
&\leq 2 \sum_{y,y'\in\Lambda} \mathbb{P}_x[H_y \leq H_{y'} \leq n] \\
&\leq 2 \sum_{y\in\Lambda} \mathbb{P}_x[H_y \leq n] \sum_{y'\in\Lambda} \mathbb{P}_y[H_{y'} \leq n] \\
&\leq 2\Big(\sup_{y\in\mathbb{Z}^d} \mathbb{E}_y\big[|\mathcal{R}_n \cap \Lambda|\big]\Big)^2,
\end{aligned}$$

and with the first moment estimate, we conclude the proof. □

2.2. *Proof of Proposition* 1.3. Let $T := \lfloor r^2/\rho^{1-\frac{2}{d}} \rfloor$, and $R_j := |\mathcal{R}[jT, jT + T] \cap \Lambda|$. Note first that

$$|\mathcal{R}_n \cap \Lambda| \leq \sum_{j=0}^{\lfloor n/T \rfloor + 1} R_j. \tag{2.5}$$

Now, consider the martingale $(M_\ell)_{\ell\geq 0}$, defined by

$$M_\ell := \sum_{j=0}^{\ell} \big(R_j - \mathbb{E}[R_j \mid \mathcal{F}_{jT}]\big),$$

where we denote by $(\mathcal{F}_n)_{n\geq 0}$ the natural filtration of the walk $(S_n)_{n\geq 0}$. By choosing $\kappa \geq 8C$, with C as in Lemma 2.1, we deduce from this lemma that for any $t \geq \kappa\rho n$ and $\rho n \geq \rho^{2/d} r^2$ that

$$\sum_{j=0}^{\lfloor n/T\rfloor+1} \mathbb{E}[R_j \mid \mathcal{F}_{jT}] \leq t/2.$$

Hence, using (2.5) we get

$$\mathbb{P}\big[|\mathcal{R}_n \cap \Lambda| \geq t\big] \leq \mathbb{P}[M_{\lfloor n/T\rfloor+1} \geq t/2].$$

Moreover, the increments of the martingale $(M_\ell)_{\ell\geq 0}$ are bounded by T, and by Lemma 2.1 their conditional variance is bounded by $C\rho^2T^2$. Thus, McDiarmid's concentration inequality (see Theorem 6.1 in [2]), gives

$$\mathbb{P}[M_{\lfloor n/T\rfloor+1} \geq t/2] \leq \exp\left(-\frac{t}{2T}\right),$$

by taking larger κ if necessary. This proves the desired result. □

2.3. *Proof of Theorem* 1.2. Fix $\rho \in (0, 1)$ and $\delta \in (0, 1)$. We proved in [1] a lower bound for visiting a set of density ρ in dimension 3: Proposition 4.1 of [1] indeed establishes that for some positive constants c and c',

$$\mathbb{P}\big[|\mathcal{R}_n \cap \Lambda| > \rho|\Lambda|\big] \geq \exp\big(-c'\rho^{2/3}n^{1/3}\big), \quad \text{for any } \Lambda \subset B\big(0, c(n/\rho)^{1/3}\big).$$

Moreover, it is well known that the probability for a random walk starting from the origin to stay in a ball of radius $R \geq 1$ (centered at the origin), for a time $n \geq 1$ is at least $\exp(-c''\frac{n}{R^2})$, for some positive constant c''. Therefore, by forcing one of the two walks to stay inside the desired ball, we deduce that for some positive constant c_0 we have the following rough bound on the intersection of two random walks:

$$\mathbb{P}\big[|\mathcal{R}_n \cap \widetilde{\mathcal{R}}_n| > \rho n\big] \geq \exp\big(-c_0\rho^{2/3}n^{1/3}\big). \tag{2.6}$$

Now, the (random) set $\mathcal{R}_n \cap \mathcal{V}_n^c(\delta\rho, r)$ satisfies the hypothesis (1.3) of Proposition 1.3 with density $\delta\rho$, thus giving with the constant κ of this proposition

$$\mathbb{P}\big[\big|\mathcal{R}_n \cap \widetilde{\mathcal{R}}_n \cap \mathcal{V}_n^c(\delta\rho, r)\big| > \kappa\delta\rho n\big] \leq \exp\left(-(\delta\rho)^{1/3}\frac{\kappa\delta\rho n}{2r^2}\right),$$

and similarly with $\widetilde{\mathcal{V}}_n^c(\delta\rho, r)$ in place of $\mathcal{V}_n^c(\delta\rho, r)$. The proof follows after we observe that this probability becomes negligible, compared to the one in (2.6), if we choose r, satisfying

$$r < \left(\frac{\kappa}{2c_0}\right)^{1/2}\delta^{2/3}(\rho n)^{1/3}.$$

3. Large deviation estimate for two random walks. In this section, we prove an extension of (2.2), when one considers two independent random walks starting from (possibly) different positions. We show that the upper bound in (2.2) still holds, up to a negligible factor, uniformly over all possible starting positions. While this result could presumably be also obtained by following carefully the proof of [10], we have preferred to follow here an alternative way and deduce it directly from (2.2), using no heavy machinery. We state the result for dimension 3 only, since this is the case of interest for us here, but note that a similar result could be proved in any dimension $d \geq 3$, using exactly the same proof.

For $x \in \mathbb{Z}^3$, we denote by $\mathbb{P}_{0,x}$ the law of two independent random walks S and $\widetilde{S}$ starting respectively from the origin and x. We write $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ for their respective ranges. Furthermore, for an integrable random variable X, we set $\overline{X} := X - \mathbb{E}[X]$, and for $x \in \mathbb{Z}^3$, we denote by $\|x\|$ its Euclidean norm.

PROPOSITION 3.1. *Assume that $d = 3$. There exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists $n_0(\varepsilon)$, such that for all $n \geq n_0(\varepsilon)$ and $k \leq n$,*

$$\sup_{\|x\| \leq n^{2/3}} \mathbb{P}_{0,x}\big[\overline{|\mathcal{R}_k \cup \widetilde{\mathcal{R}}_{n-k}|} \leq -\varepsilon n\big] \leq \exp\biggl(-I_3\biggl(\frac{\varepsilon}{1+\varepsilon^2}\biggr) n^{1/3}\biggr).$$

PROOF. Set $m := \lfloor \varepsilon^2 n \rfloor$. First, using (2.2), we know that

$$\lim_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\big(\overline{|\mathcal{R}_{n+m}|} \leq -\varepsilon(1-\varepsilon^3)n\big) = -I_3\biggl(\frac{\varepsilon(1-\varepsilon^3)}{1+\varepsilon^2}\biggr)(1+\varepsilon^2)^{1/3}.$$

Now consider x, with $\|x\| \leq \varepsilon^{5/2} n^{2/3}$, and $k \leq n$. Note that by (2.1), we get for $\varepsilon > 0$,

$$\begin{aligned}\overline{|\mathcal{R}[0, n+m]|} &\leq \overline{|\mathcal{R}[0,k] \cup \mathcal{R}[k+m, n+m]|} \\ &\quad + \overline{|\mathcal{R}[k, k+m]} + \mathcal{O}(\sqrt{n}).\end{aligned}$$

Therefore, at least for n large enough,

$$\begin{aligned}&\mathbb{P}\big(\overline{|\mathcal{R}_{n+m}|} \leq -\varepsilon(1-\varepsilon^3)n\big) \\ &\quad \geq \mathbb{P}\big(\overline{|\mathcal{R}_k \cup \mathcal{R}[k+m, n+m]|} \leq -\varepsilon n, \\ &\qquad \overline{|\mathcal{R}[k, k+m]|} \leq \varepsilon^5 n, S_{k+m} - S_k = x\big) \\ &\quad \geq \mathbb{P}_{0,x}\big(\overline{|\mathcal{R}_k \cup \widetilde{\mathcal{R}}_{n-k}|} \leq -\varepsilon n\big) \cdot \mathbb{P}\big(\overline{|\mathcal{R}_m|} \leq \varepsilon^5 n, S_m = x\big),\end{aligned}$$

using the Markov property and reversibility of the random walk for the last inequality.

Now using Hamana and Kesten bound (2.4), the local central limit theorem (see Theorem 2.3.11 in [7]), and that $\|x\| \leq \varepsilon^{5/2} n^{2/3}$, we get that for some constant $c > 0$ (independent of x), and for all n large enough,

$$\begin{aligned}&\mathbb{P}\big(\overline{|\mathcal{R}_m|} \leq \varepsilon^5 n, S_m = x\big) \\ &\quad \geq \mathbb{P}(S_m = x) - \mathbb{P}\big(\overline{|\mathcal{R}_m|} \geq \varepsilon^5 n\big) \\ &\quad \geq \exp\big(-c\varepsilon^3 n^{1/3}\big) - \exp\big(-J_d(\varepsilon^3)(1-\varepsilon)\varepsilon^2 n\big) \\ &\quad \geq \frac{1}{2}\exp\big(-c\varepsilon^3 n^{1/3}\big).\end{aligned}$$

Moreover, it follows from (2.3), that for ε small enough,

$$I_3\biggl(\frac{\varepsilon(1-\varepsilon^3)}{1+\varepsilon^2}\biggr)(1+\varepsilon^2)^{1/3} - c\varepsilon^3 \geq I_3\biggl(\frac{\varepsilon}{1+\varepsilon^2}\biggr)\biggl(1+\frac{1}{4}\varepsilon^2\biggr).$$

Therefore, for ε small enough, and then for all n large enough,

$$\begin{aligned}&\sup_{\|x\| \leq \varepsilon^{5/2} n^{2/3}} \mathbb{P}_{0,x}\big(\overline{|\mathcal{R}_k \cup \widetilde{\mathcal{R}}_{n-k}|} \leq -\varepsilon n\big) \\ &\quad \leq \exp\biggl(-I_3\biggl(\frac{\varepsilon}{1+\varepsilon^2}\biggr)\biggl(1+\frac{1}{5}\varepsilon^2\biggr)n^{1/3}\biggr).\end{aligned} \tag{3.1}$$

It remains to consider x satisfying $\varepsilon^{5/2} n^{2/3} \leq \|x\| \leq n^{2/3}$. Our strategy is to show that there exists a constant $\rho \in (0,1)$, such that for any such x, the probability of the event

$$\mathcal{A} := \big\{\overline{|\mathcal{R}_k \cup \widetilde{\mathcal{R}}_{n-k}|} \leq -\varepsilon n\big\}, \tag{3.2}$$

under $\mathbb{P}_{0,x}$ is bounded by the probability of the same event under $\mathbb{P}_{0,x'}$, with x' satisfying $\|x'\| \le \rho\|x\|$, up to some negligible error term. Applying this estimate at most order $\log(1/\varepsilon)$ times, and using (3.1), will give the result.

Fix x such that $\varepsilon^{5/2}n^{2/3} \le \|x\| \le n^{2/3}$, and assume, without loss of generality, that its largest coordinate in absolute value is the first one, say x_1, and that it is positive. In this case, $\|x\| \ge x_1 \ge \|x\|/\sqrt{3}$. Recall next that we consider two random walks S and $\widetilde{S}$, one starting from the origin and running up to time k, and the other one starting from x and running up to time $n-k$. For $x_1/3 \le y \le 2x_1/3$, consider the hyperplane $\mathcal{H}_y$ of vertices having first coordinate equal to y. Call excursion away from $\mathcal{H}_y$ (for any of the two walks), any part of their trajectory (of length at least two) whose starting and ending points belong to $\mathcal{H}_y$ and whose other points are outside this hyperplane. By extension, let us also call initial excursion the part of the trajectory from time 0 up to the hitting time of $\mathcal{H}_y$. Denote by N_y the number of excursions away from $\mathcal{H}_y$, made by any of the two walks S or $\widetilde{S}$, which hit the hyperplane $\mathcal{H}_{y+[\varepsilon^{-10}]}$. Note that one can order them by time of arrival, considering first those made by S and then those made by $\widetilde{S}$. Let N'_y be the number of excursions among the first $N_y \wedge (2\varepsilon^3 n^{1/3})$ previous ones, which spend a time at least ε^{-17} in the region between $\mathcal{H}_y$ and $\mathcal{H}_{y+[\varepsilon^{-10}]}$. Note that for a single excursion, the probability to hit $\mathcal{H}_{y+[\varepsilon^{-10}]}$ in less than ε^{-17} steps, is of order $\exp(-c\varepsilon^{-3})$, for some constant $c > 0$. By independence between the first $2\varepsilon^3 n^{1/3}$ excursions, we deduce that

$$\mathbb{P}_{0,x}\big(N'_y \le \varepsilon^3 n^{1/3}, N_y \ge 2\varepsilon^3 n^{1/3}\big) \le \exp\big(-cn^{1/3}\big), \tag{3.3}$$

for some possibly smaller constant $c > 0$. On the other hand, let T_y be the cumulated total time spent by S and $\widetilde{S}$ in the region between $\mathcal{H}_y$ and $\mathcal{H}_{y+[\varepsilon^{-10}]}$. Observe that the number of levels y between $x_1/3$ and $2x_1/3$ which are integer multiples of $[\varepsilon^{-10}]$ is of order $\varepsilon^{10}x_1/3$, and that the latter is (at least for ε small enough) larger than $\varepsilon^{13}n^{2/3}$. Thus, for at least one such y, one must have both $N'_y \le \varepsilon^3 n^{1/3}$ and $T_y \le \varepsilon^{-13}n^{1/3}$ (otherwise the total time spent in the region between the hyperplanes $\mathcal{H}_{x_1/3}$ and $\mathcal{H}_{2x_1/3}$ would exceed n). Using (3.3), we deduce that

$$\begin{aligned}&\mathbb{P}_{0,x}\big(N_y \ge 2\varepsilon^3 n^{1/3} \text{ or } T_y \ge \varepsilon^{-13}n^{1/3}, \text{ for all } y \in \{x_1/3, \dots, 2x_1/3\}\big)\\ &\quad \le x_1 \exp\big(-cn^{1/3}\big),\end{aligned}$$

for some constant $c > 0$. Then as a consequence of the pigeonhole principle, there exists (a deterministic) $y_0 \in \{x_1/3, \dots, 2x_1/3\}$, such that

$$\mathbb{P}_{0,x}\big(N_{y_0} \le 2\varepsilon^3 n^{1/3}, T_{y_0} \le \varepsilon^{-13}n^{1/3}, \mathcal{A}\big) \ge \frac{1}{x_1}\mathbb{P}_{0,x}(\mathcal{A}) - \exp\big(-cn^{1/3}\big),$$

with the event $\mathcal{A}$ as defined in (3.2).

Denote now by x' the symmetric of x with respect to $\mathcal{H}_{y_0}$. First, observe that since $x_1 \ge \|x\|/\sqrt{3}$, and $x_1/3 \le y_0 \le 2x_1/3$, there exists $\rho \in (0,1)$ (independent of x), such that $\|x'\| \le \rho\|x\|$. Next, recall that any excursion away from $\mathcal{H}_{y_0}$ and its symmetric with respect to $\mathcal{H}_{y_0}$ has the same probability to happen under the law $\mathbb{P}_{0,x'}$ (and note that this is even true for the initial excursion made by $\widetilde{S}$, from time 0 up to the hitting time of $\mathcal{H}_{y_0}$). Moreover, by reflecting (with respect to $\mathcal{H}_{y_0}$) all the excursions away from $\mathcal{H}_{y_0}$ (including the initial one) which hit $\mathcal{H}_{y_0+[\varepsilon^{-10}]}$, one can only increase the size of the range by at most T_{y_0}. Therefore,

$$\begin{aligned}&\mathbb{P}_{0,x'}\big(N_{y_0} = 0, \overline{|\mathcal{R}_k \cup \widetilde{\mathcal{R}}_{n-k}|} \le -\varepsilon n + \varepsilon^{-13}n^{1/3}\big)\\ &\quad \ge 2^{-2\varepsilon^3 n^{1/3}} \cdot \mathbb{P}_{0,x}\big(N_{y_0} \le 2\varepsilon^3 n^{1/3}, T_{y_0} \le \varepsilon^{-13}n^{1/3}, \mathcal{A}\big).\end{aligned}$$

Combining the last two displays, we conclude that for n large enough,

$$\mathbb{P}_{0,x}(\mathcal{A}) \le 2^{4\varepsilon^3 n^{1/3}} \mathbb{P}_{0,x'}(\overline{|\mathcal{R}_k \cup \widetilde{\mathcal{R}}_{n-k}|} \le -\varepsilon n + \varepsilon^{-13} n^{1/3}) + \exp(-cn^{1/3}),$$

for some (possibly smaller) constant $c > 0$. Repeating the same argument $(5/2)\log\varepsilon/(\log\rho)$ times, and using (3.1) and (2.3), we obtain the desired result. □

4. Proof of Theorem 1.1. Let $k \le \ell \le n$ be given, satisfying $\ell - k = \lfloor K_0 \varepsilon n \rfloor$, with K_0 a constant to be fixed later. Write $I = \{k, \dots, \ell\}$. Using that

$$\overline{|\mathcal{R}_n|} = \overline{|\mathcal{R}(I)|} + \overline{|\mathcal{R}(I^c)|} - \overline{|\mathcal{R}(I) \cap \mathcal{R}(I^c)|},$$

we have, with $\nu := 1 - \frac{K_0\varepsilon}{3}$,

$$\mathbb{P}(\overline{|\mathcal{R}_n|} \le -\varepsilon n) \le \mathbb{P}(\overline{|\mathcal{R}(I)|} + \overline{|\mathcal{R}(I^c)|} \le -\nu\varepsilon n) + \mathbb{P}\Big(|\mathcal{R}(I) \cap \mathcal{R}(I^c)| \ge \frac{K_0}{3}\varepsilon^2 n\Big). \tag{4.1}$$

We start by showing that the first probability on the right-hand side is negligible (when compared to the probability on the left-hand side). For this let $N = \lfloor \varepsilon^{-2} \rfloor$, and for $i = 0, \dots, N$, let $\alpha_i := i\varepsilon^2$. Then note that for n large enough,

$$\begin{aligned}
&\mathbb{P}(\overline{|\mathcal{R}(I)|} + \overline{|\mathcal{R}(I^c)|} \le -\nu\varepsilon n) \\
&\quad \le \sum_{i=0}^{N} \mathbb{P}(\overline{|\mathcal{R}(I)|} \le -\nu\alpha_i\varepsilon n, \overline{|\mathcal{R}(I^c)|} \le -\nu(1-\alpha_{i+1})\varepsilon n) \\
&\qquad + \mathbb{P}(\overline{|\mathcal{R}(I^c)|} \le -\nu\varepsilon n) \\
&\quad \le \sum_{i=0}^{N} \sum_{\|x\| \le n^{2/3}} \mathbb{P}(\overline{|\mathcal{R}(I)|} \le -\nu\alpha_i\varepsilon n, \\
&\qquad \overline{|\mathcal{R}(I^c)|} \le -\nu(1-\alpha_{i+1})\varepsilon n, S_\ell - S_k = x) \\
&\qquad + \mathbb{P}(\overline{|\mathcal{R}(I^c)|} \le -\nu\varepsilon n) + \mathcal{O}(\exp(-cn^{1/3})),
\end{aligned} \tag{4.2}$$

using Chernoff's bound for the last inequality (see, for instance, Theorem 3.1 in [2]). Now applying the Markov property and using Proposition 3.1 and (2.2), we get that for any $i \le N$, there exists $n_i \ge 1$, such that for all $n \ge n_i$,

$$\begin{aligned}
&\sum_{\|x\| \le n^{2/3}} \mathbb{P}(\overline{|\mathcal{R}(I)|} \le -\nu\alpha_i\varepsilon n, \overline{|\mathcal{R}(I^c)|} \le -\nu(1-\alpha_{i+1})\varepsilon n, S_\ell - S_k = x) \\
&\quad \le \sum_{\|x\| \le n^{2/3}} \mathbb{P}(\overline{|\mathcal{R}(I)|} \le -\nu\alpha_i\varepsilon n, S_\ell - S_k = x) \\
&\qquad \cdot \exp\Big(-I_3\Big(\frac{\varepsilon_i}{1+\varepsilon_i^2}\Big)(1-K_0\varepsilon)^{1/3} n^{1/3}\Big) \\
&\quad \le \exp\Big(-I_3(\widetilde{\varepsilon}_i)(1-\varepsilon^2)(K_0\varepsilon n)^{1/3} - I_3\Big(\frac{\varepsilon_i}{1+\varepsilon_i^2}\Big)(1-K_0\varepsilon)^{1/3} n^{1/3}\Big),
\end{aligned} \tag{4.3}$$

with

$$\varepsilon_i := \nu \frac{(1-\alpha_{i+1})\varepsilon}{1-K_0\varepsilon}, \quad \text{and} \quad \widetilde{\varepsilon}_i := \frac{\nu\alpha_i}{K_0}.$$

Note that by choosing larger n_i if necessary, one can also assume that (4.2) holds for $n \geq n_i$. Note furthermore that by (2.3), the first term in the exponential in (4.3) is already larger than $2I_3(\varepsilon)n^{1/3}$, when $\alpha_i \geq K_0\sqrt{\varepsilon}$, at least provided K_0 is large enough and ε small enough. Thus in the following, one can assume that $\alpha_i \leq K_0\sqrt{\varepsilon}$. We will also assume that ε is small enough, so that $K_0\sqrt{\varepsilon} < 1/2$. Then, using in particular that $\alpha_{i+1} = \alpha_i + \varepsilon^2$, we get (recall that $\nu = 1 - \frac{K_0\varepsilon}{3}$)

$$I_3(\widetilde{\varepsilon}_i)(1-\varepsilon^2)(K_0\varepsilon)^{1/3} + I_3\Big(\frac{\varepsilon_i}{1+\varepsilon_i^2}\Big)(1-K_0\varepsilon)^{1/3}$$
$$\geq \mu_3\Big(1-\frac{K_0\varepsilon}{3}\Big)^{2/3}\Big(\frac{\alpha_i^{2/3}}{K_0^{1/3}\varepsilon^{1/3}} + \frac{(1-\alpha_i)^{2/3}}{(1-K_0\varepsilon)^{1/3}}\Big)\varepsilon^{2/3} - \mathcal{O}(\varepsilon^2).$$

Now we claim that the bound in the parentheses above reaches its infimum when $i = 0$, or equivalently when $\alpha_i = 0$. To see this, it suffices to consider the variations of the function f defined for $u \in (0,1)$ by $f(u) = c_1^{1/3}u^{2/3} + c_2^{1/3}(1-u)^{2/3}$, with $c_1 = (K_0\varepsilon)^{-1}$, and $c_2 = (1-K_0\varepsilon)^{-1}$. A straightforward computation shows that $f'(u) > 0$ on $(0, u_0)$, with $u_0 = 1 - K_0\varepsilon$. Since we assumed that $u_0 > 1/2 > K_0\sqrt{\varepsilon}$, this proves our claim. By taking $K_0 = 100C$, with C the constant appearing in the upper bound of I_3 in (2.3), one deduces that for ε small enough,

$$I_3(\widetilde{\varepsilon}_i)(1-\varepsilon^2)(K_0\varepsilon)^{1/3} + I_3\Big(\frac{\varepsilon_i}{1+\varepsilon_i^2}\Big)(1-K_0\varepsilon)^{1/3}$$
$$\geq \mu_3\Big(1+\frac{K_0\varepsilon}{10}\Big)\varepsilon^{2/3} - \mathcal{O}(\varepsilon^2)$$
$$\geq I_3(\varepsilon)\Big(1+\frac{K_0\varepsilon}{20}\Big).$$

As a consequence, letting $N(\varepsilon) := \max_i n_i$, we get that for ε small enough, for all $n \geq N(\varepsilon)$,

$$\sum_{i=0}^{N}\mathbb{P}\big(\overline{|\mathcal{R}(I)|} \leq -\nu\alpha_i\varepsilon n, \overline{|\mathcal{R}(I^c)|} \leq -\nu(1-\alpha_{i+1})\varepsilon n\big)$$
$$\leq \exp\Big(-I_3(\varepsilon)\Big(1+\frac{K_0\varepsilon}{20}\Big)n^{1/3}\Big).$$

One can obtain similarly the same bound for the term $\mathbb{P}(\overline{|\mathcal{R}(I^c)|} \leq -\nu\varepsilon n)$, also for all $n \geq N(\varepsilon)$, possibly by taking a larger constant $N(\varepsilon)$ if necessary. Finally, using again (2.2), we get that for all ε small enough,

$$\mathbb{P}\big(\overline{|\mathcal{R}(I)|} + \overline{|\mathcal{R}(I^c)|} \leq -\nu\varepsilon n\big) = o\Big(\frac{1}{n}\mathbb{P}\big(\overline{|\mathcal{R}_n|} \leq -\varepsilon n\big)\Big). \tag{4.4}$$

It remains to estimate the second term in the right-hand side of (4.1). This is similar to the proof of Theorem 1.2. Assume given $1 \leq r \leq \varepsilon^{5/6}n^{1/3}$, and $\beta > 0$, whose value will be made more precise in a moment. To simplify notation, write $\mathcal{R}_1 = \mathcal{R}[0,k]$ and $\mathcal{R}_2 := \mathcal{R}[\ell,n]$. Next, set

$$\Lambda_1 = \bigcup_{x\in\mathcal{C}_1} Q(x,r), \quad \text{and} \quad \Lambda_2 = \bigcup_{x\in\mathcal{C}_2} Q(x,r),$$

with

$$\mathcal{C}_1 := \{x \in 2r\mathbb{Z}^d : |Q(x,r)\cap\mathcal{R}_1| \geq \beta\varepsilon r^d\},$$

and

$$\mathcal{C}_2 := \{x \in 2r\mathbb{Z}^d : |Q(x,r) \cap \mathcal{R}_2| \geq \beta\varepsilon r^d\}.$$

Since $\mathcal{R}(I^c) = \mathcal{R}_1 \cup \mathcal{R}_2$, one has

$$|\mathcal{R}(I) \cap \mathcal{R}(I^c)| \leq |\mathcal{R}(I) \cap \mathcal{R}_1| + |\mathcal{R}(I) \cap \mathcal{R}_2|$$

and, therefore,

$$\begin{aligned}&\mathbb{P}\Big(|\mathcal{R}(I) \cap \mathcal{R}(I^c)| \geq \frac{K_0}{3}\varepsilon^2 n\Big)\\ &\quad\leq \mathbb{P}\Big(|\mathcal{R}(I) \cap \mathcal{R}_1| \geq \frac{K_0}{6}\varepsilon^2 n\Big) + \mathbb{P}\Big(|\mathcal{R}(I) \cap \mathcal{R}_2| \geq \frac{K_0}{6}\varepsilon^2 n\Big).\end{aligned}$$

Both terms on the right-hand side are treated similarly. We first fix $\beta < 1/(24\kappa)$, with κ the constant appearing in statement of Lemma 1.3. Then applying Lemma 1.3 with $\rho = \beta\varepsilon$, $n = \lfloor K_0\varepsilon n \rfloor$, and $t = \frac{K_0}{24}\varepsilon^2 n$, we get (using also the Markov property at time k),

$$\begin{aligned}\mathbb{P}\Big(|\mathcal{R}(I) \cap \mathcal{R}_1 \cap \Lambda_1^c| \geq \frac{K_0}{24}\varepsilon^2 n\Big) &\leq \exp\Big(-(\beta\varepsilon)^{1/3}\frac{K_0\varepsilon^2 n}{48r^2}\Big)\\ &\leq \exp\Big(-\frac{\beta^{1/3}K_0}{48}\varepsilon^{2/3}n^{1/3}\Big),\end{aligned}$$

using that $r \leq \varepsilon^{5/6}n^{1/3}$, for the last inequality. By taking larger K_0 if necessary, one can ensure that this bound is $o((1/n)\cdot\mathbb{P}(\overline{|\mathcal{R}_n|} \leq -\varepsilon n))$. This way we obtain

$$\begin{aligned}&\mathbb{P}\Big(|\mathcal{R}(I) \cap \mathcal{R}_1 \cap \Lambda_1^c| \vee |\mathcal{R}(I) \cap \mathcal{R}_2 \cap \Lambda_2^c| \geq \frac{K_0}{24}\varepsilon^2 n\Big)\\ &\quad= o\Big(\frac{1}{n}\mathbb{P}(\overline{|\mathcal{R}_n|} \leq -\varepsilon n)\Big).\end{aligned} \tag{4.5}$$

Coming back to (4.1), dividing both sides of the inequality by the term on the left-hand side, and using (4.4) and (4.5), we get that for all ε small enough,

$$\begin{aligned}&\mathbb{Q}_n^\varepsilon\Big(|\mathcal{R}(I) \cap \Lambda_1 \cap \mathcal{R}_1| \geq \frac{K_0}{8}\varepsilon^2 n, \text{ or } |\mathcal{R}(I) \cap \Lambda_2 \cap \mathcal{R}_2| \geq \frac{K_0}{8}\varepsilon^2 n\Big)\\ &\quad\geq 1 - o\Big(\frac{1}{n}\Big).\end{aligned}$$

Since both

$$\Lambda_1 \cap \mathcal{R}_1 \subseteq \mathcal{V}_n(\beta\varepsilon, r) \cap \mathcal{R}(I^c), \quad \text{and} \quad \Lambda_2 \cap \mathcal{R}_2 \subseteq \mathcal{V}_n(\beta\varepsilon, r) \cap \mathcal{R}(I^c),$$

we get

$$\mathbb{Q}_n^\varepsilon\Big(|\mathcal{R}(I) \cap \mathcal{R}(I^c) \cap \mathcal{V}_n(\beta\varepsilon, r)| \geq \frac{K_0}{8}\varepsilon^2 n\Big) \geq 1 - o\Big(\frac{1}{n}\Big).$$

The proof of Theorem 1.1 follows by a union bound, since there are at most n intervals I of fixed length in I_n. $\square$

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