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ON A PERTURBATION THEORY AND ON STRONG CONVERGENCE RATES FOR STOCHASTIC ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS WITH NONGLOBALLY MONOTONE COEFFICIENTS

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We develop a perturbation theory for stochastic differential equations (SDEs) by which we mean both stochastic ordinary differential equations (SODEs) and stochastic partial differential equations (SPDEs). In particular, we estimate the L^p-distance between the solution process of an SDE and an arbitrary Itô process, which we view as a perturbation of the solution process of the SDE, by the L^q-distances of the differences of the local characteristics for suitable $p, q > 0$. As one application of the developed perturbation theory, we establish strong convergence rates for numerical approximations of a class of SODEs with nonglobally monotone coefficients. As another application of the developed perturbation theory, we prove strong convergence rates for spatial spectral Galerkin approximations of solutions of semilinear SPDEs with nonglobally monotone nonlinearities including Cahn–Hilliard–Cook-type equations and stochastic Burgers equations. Further applications of the developed perturbation theory include regularity analyses of solutions of SDEs with respect to their initial values as well as small-noise analyses for ordinary and partial differential equations.

1. Introduction. In this article we develop a *perturbation theory* for stochastic differential equations (SDEs) by which we mean both stochastic ordinary differential equations (SODEs) and stochastic partial differential equations (SPDEs). To illustrate this perturbation theory, we use the following setting in this introductory section. Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $D \subseteq H$ be a Borel measurable set, let $\mu\colon D \to H$ and $\sigma\colon D \to \mathrm{HS}(U, H)$ be Borel measurable functions, let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t \in [0,T]})$ be a filtered probability space which fulfills the usual conditions, let $(W_t)_{t \in [0,T]}$ be an Id_U-cylindrical $(\mathbb{F}_t)_{t \in [0,T]}$-Wiener process, let $X, Y\colon [0, T] \times \Omega \to D$ be adapted stochastic processes with continuous sample paths (c.s.p.), and let $a\colon [0, T] \times \Omega \to H$ and $b\colon [0, T] \times \Omega \to \mathrm{HS}(U, H)$ be predictable stochastic processes which satisfy that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\, ds < \infty$ and

$$X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s, \tag{1}$$

$$Y_t = Y_0 + \int_0^t a_s\, ds + \int_0^t b_s\, dW_s. \tag{2}$$

The process X is thus a solution process of the SDE (1) and the process Y is a general Itô process with drift process a, diffusion process b, and Wiener process W. We view the stochastic process Y as a *perturbation* of the solution process of the SDE (1) and we are interested in

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estimates for the strong *perturbation error* $\|X_t - Y_t\|_{L^p(\Omega;H)} = (\mathbb{E}[\|X_t - Y_t\|_H^p])^{1/p}$ at some fixed (or random) time $t \in [0, T]$ for $p \in (0, \infty)$.

Informally speaking, we *estimate the global perturbation error by the local perturbation error*. More formally, for every $p \in (0, \infty)$ we estimate the global perturbation error $\|X_T - Y_T\|_{L^p(\Omega;H)}$ by the L^q-norms of the difference $X_0 - Y_0$ at time 0 and of the differences $a - \mu(Y) = (a_t - \mu(Y_t))_{t\in[0,T]}$ and $b - \sigma(Y) = (b_t - \sigma(Y_t))_{t\in[0,T]}$ of the local characteristics where $q \in (0, \infty)$ is appropriate; see Theorem 1.2 below for details. This perturbation result can then be applied to any stochastic process that is an Itô process with respect to the Wiener process W. Possible applications include:

(i). *Local Lipschitz continuity of solutions of SDEs with respect to their initial values* (choose $a_t = \mu(Y_t)$ and $b_t = \sigma(Y_t)$ for $t \in [0, T]$; cf. Corollary 2.8 in Section 2.3 below and Cox, Hutzenthaler and Jentzen [13] for details),

(ii). Strong convergence rates for *time-discrete numerical approximations of SODEs* (e.g., the Euler–Maruyama approximation with $N \in \mathbb{N} = \{1, 2, 3, \dots\}$ discretization time steps is given by $a_t = \mu(Y_{kT/NN})$ and $b_t = \sigma(Y_{kT/N})$ for $t \in [nT/N, (n + 1)T/N)$, $n \in \{0, 1, \dots, N - 1\}$; cf. Section 3.1 below),

(iii). Strong convergence rates for *spatial Galerkin approximations of SPDEs* (choose $a_t = P(\mu(Y_t))$ and $b_t u = P(\sigma(Y_t)u)$ for $u \in U$, $t \in [0, T]$ and some suitable projection operator $P \in L(H)$; cf. Section 3.2 below) and

(iv). Strong convergence rates for *small noise perturbations* of solutions of deterministic differential equations (choose $\sigma = 0$, $a_t = \mu(Y_t)$, $b_t = \varepsilon\tilde{\sigma}(Y_t)$ for $t \in [0, T]$ where $\tilde{\sigma}: D \to \mathrm{HS}(U, H)$ is a suitable Borel measurable function and where $\varepsilon \in (0, \infty)$ is a sufficiently small parameter; cf. Section 3.3 below).

In the scientific literature, a frequently used method to estimate strong perturbation errors is to employ Gronwall's lemma together with the popular *global monotonicity* assumption (cf., e.g., Minty [54, 55] for deterministic equations and Pardoux [60] condition (4.19), for SODEs) that there exists a real number $c \in \mathbb{R}$ such that for all $x, y \in D$ it holds that

$$\langle x - y, \mu(x) - \mu(y)\rangle_H + \tfrac{1}{2}\|\sigma(x) - \sigma(y)\|^2_{\mathrm{HS}(U,H)} \le c\,\|x - y\|_H^2. \tag{3}$$

Under the global monotonicity assumption (3), there are a multitute of mathematical results in the scientific literature and, at least partially, the above problems (i)–(iv) have been solved under this assumption (cf., e.g., Prévôt and Röckner [61], Proposition 4.2.10, Cerrai [8] for problem (i), cf., e.g., Hu [33], Higham, Mao and Stuart [32], Hutzenthaler, Jentzen and Kloeden [36], Sabanis [64] for problem (ii), and cf., for example, Liu [51], Sauer and Stannat [65] for problem (iii)). Unfortunately, the global monotonicity assumption (3) is too restrictive in the sense that the nonlinearities in the coefficient functions of the majority of nonlinear (stochastic) differential equations from applications do not satisfy the global monotonicity assumption (3) (see, e.g., Section 3.1 and Section 3.2 below for a few example SDEs which fail to satisfy (3)).

Beyond the global monotonicity assumption (3), we are not aware of a general technique for estimating global perturbation errors by local perturbation errors. In the scientific literature, there exist the following results for SDEs with nonglobally monotone nonlinearities for the problems (i)–(iv). Problem (i)—which is, in a certain sense, the simplest of problems (i)–(iv), as there is only a perturbation of the initial value but no perturbation of the dynamics of (1)—is already solved for a large class of SDEs with nonglobally monotone nonlinearities (cf., e.g., Li [50], Hairer and Mattingly [28], Zhang [70], and Cox, Hutzenthaler and Jentzen [13]). Problem (ii) has been solved for a large class of one-dimensional square-root diffusion processes with inaccessible boundaries (cf., e.g., Gyöngy and Rasonyi [26], Dereich, Neuenkirch and Szpruch [18], Alfonsi [3], Neuenkirch and Szpruch [59]). We are not aware of any result in the scientific literature that solves problem (ii) in the case of a multidimensional SODE which fails to satisfy (3). Regarding problem (iii), we are aware of exactly one result in the scientific literature on SPDEs with nonglobally monotone nonlinearities, that is, the work of Dörsek [19]. More precisely, [19], Corollary 3.2, establishes the strong convergence rate 1 for spatial spectral Galerkin approximations of the vorticity formulation of the two-dimensional stochastic Navier–Stokes equations with degenerate additive noise. For problem (iv), we have not found results in the scientific literature on SDEs with nonglobally monotone nonlinearities.

An important observation of this article is that there exist exponential integrating factors $\exp(\int_0^t \chi_s\, ds)$, $t \in [0, T]$, such that, informally speaking, the rescaled squared distances $\|X_t - Y_t\|_H^2 \exp(-\int_0^t \chi_s\, ds)$, $t \in [0, T]$, are sums and integrals over local perturbation errors where $(\chi_t)_{t\in[0,T]}$ is a suitable stochastic process. The following proposition, Proposition 1.1 below, formalizes this idea and establishes a *pathwise perturbation formula*. In Proposition 1.1 the squared Hilbert-space distance $\|v - w\|_H^2$, $v, w \in H$, is replaced by a more general function $V(v, w)$, $v, w \in H$, to measure distances. It proved very beneficial in the case of some SDEs such as Cox–Ingersoll–Ross processes or the Cahn–Hilliard–Cook equation with space-time white noise to measure the distance between the solution X and its perturbation Y with a general function $V \in C^2(H^2, \mathbb{R})$ rather than with the squared Hilbert space distance (cf., e.g., Cox et al. [13], Section 4.10 for details). Next we note that in the perturbation formula (4) below, there appears an operator $\overline{\mathcal{G}}_{\mu,\sigma} : C^2(H^2, \mathbb{R}) \to C(H^2, \mathbb{R})$ defined in (15) below which is the formal generator of the bivariate process consisting of two solution processes of the SDE (1); cf. also Ichikawa [39], Maslowski [53], and, for example, Leha and Ritter [48, 49] for references in the scientific literature where this operator has been used.

PROPOSITION 1.1 (Perturbation formula). *Assume the above setting, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $V = (V(x, y))_{(x,y)\in H^2} \in C^2(H^2, \mathbb{R})$, and let $\chi : [0, T] \times \Omega \to \mathbb{R}$ be a predictable stochastic process with $\mathbb{P}(\int_0^T |\chi_s|\, ds < \infty) = 1$. Then for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that*

$$
\begin{aligned}
\frac{V(X_t, Y_t)}{\exp(\int_0^t \chi_r\, dr)} &= V(X_0, Y_0) + \int_0^t \frac{(\partial_x V)(X_s, Y_s)\,\sigma(X_s) + (\partial_y V)(X_s, Y_s)\, b_s}{\exp(\int_0^s \chi_r\, dr)}\, dW_s \\
&\quad + \int_0^t \frac{(\overline{\mathcal{G}}_{\mu,\sigma} V)(X_s, Y_s) - \chi_s V(X_s, Y_s) + \sum_{u\in\mathbb{U}} (\partial_x \partial_y V)(X_s, Y_s)(\sigma(X_s)u, [b_s - \sigma(Y_s)]u)}{\exp(\int_0^s \chi_r\, dr)}\, ds \\
&\quad + \int_0^t \frac{(\partial_y V)(X_s, Y_s)[a_s - \mu(Y_s)] + \frac{1}{2}\operatorname{trace}([b_s + \sigma(Y_s)]^* (\operatorname{Hess}_y V)(X_s, Y_s)[b_s - \sigma(Y_s)])}{\exp(\int_0^s \chi_r\, dr)}\, ds.
\end{aligned}
\tag{4}
$$

Proposition 1.1 follows immediately from Itô's formula together with the addition and the subtraction of a suitable term; see Proposition 2.5 below for details. Proposition 1.1 turned out to be rather useful to develop a perturbation theory for the SDE (1) and, thereby, to partially solve problems (i)–(iv) without assuming global monotonicity. In the formulation of Proposition 1.1, the exponential integrating factor $\exp(\int_0^t \chi_s\, ds)$, $t \in [0, T]$, can be quite arbitrary. However, it is essential to observe that if the stochastic process $\chi : [0, T] \times \Omega \to \mathbb{R}$ can be chosen such that $\forall s \in [0, T]$: $\mathbb{P}((\overline{\mathcal{G}}_{\mu,\sigma} V)(X_s, Y_s) - \chi_s V(X_s, Y_s) \leq 0) = 1$, then the expectation of the right-hand side of (4) is, informally speaking, dominated by sums and integrals over the local perturbation errors $a - \mu(Y)$ and $b - \sigma(Y)$ times random factors. The exponential integrating factors $\exp(\int_0^t \chi_s\, ds)$, $t \in [0, T]$, on the left-hand side of (4) and the random factors on the right-hand side of (4) can then, roughly speaking, be estimated by using Hölder's inequality and Young's inequality. In the case where there exists $p \in [2, \infty)$ such that for all $x, y \in H$ it holds that $V(x, y) = \|x - y\|_H^p$, this leads to the *perturbation estimate* in (5) below. We also refer to Section 2.3 below for more general perturbation estimates including a general "distance-type" function V.

THEOREM 1.2. *Assume the above setting, let* $\varepsilon \in [0,\infty]$, $p \in [2,\infty)$, *let* $\tau : \Omega \to [0,T]$ *be a stopping time and assume that* $\mathbb{P}\big(\int_0^\tau [\langle X_s - Y_s, \mu(X_s) - \mu(Y_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|\sigma(X_s) - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}]^+ / \|X_s - Y_s\|^2_H \, ds < \infty\big) = 1$. *Then for all* $\alpha, \beta \in (0,\infty)$, $r, q \in (0,\infty]$ *with* $\frac{1}{p} 1/p + \frac{1}{q} = \frac{1}{r}$ *it holds that*

(5)

$$\begin{aligned} \|X_\tau - Y_\tau\|_{L^r(\Omega;H)} &\le \Big[\|X_0 - Y_0\|_{L^p(\Omega;H)} \\ &+ \alpha^{(1-\frac{1}{p})} \|a - \mu(Y)\|_{L^p([\![0,\tau]\!];H)} + \beta^{(\frac{1}{2}-\frac{1}{p})} \sqrt{\tfrac{(p-1)(1+\varepsilon)}{\varepsilon}} \|b - \sigma(Y)\|_{L^p([\![0,\tau]\!];\mathrm{HS}(U,H))}\Big] \\ &\cdot \left\| \exp\left(\int_0^\tau \left[\frac{\langle X_s - Y_s, \mu(X_s) - \mu(Y_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|\sigma(X_s) - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}}{\|X_s - Y_s\|^2_H} + \frac{1-\frac{1}{p}}{\alpha} + \frac{\frac{1}{2}-\frac{1}{p}}{\beta} \right]^+ ds \right) \right\|_{L^q(\Omega;\mathbb{R})}. \end{aligned}$$

In the formulation of Theorem 1.2 the expression $[\![0,\tau]\!] := \{(t,\omega) \in [0,T] \times \Omega : t \le \tau(\omega)\}$ denotes the stochastic interval from 0 to τ (cf., e.g., Kühn [47]) and in the formulation of Theorem 1.2 the convention $\frac{0}{0} := 0$ is used. Theorem 1.2 follows immediately from Corollary 2.12 below which, in turn, follows from Theorem 2.10 below. Theorem 1.2 can be applied to prove local Lipschitz continuity in the strong sense with respect to the initial value by choosing $\tau = T$, $\varepsilon = 0$, $a = \mu(Y)$, $b = \sigma(Y)$. Thereby one obtains a quite similar inequality as in Cox, Hutzenthaler and Jentzen [13], Corollary 2.19 (see also Corollary 2.8 below). Local Lipschitz continuity with respect to the initial value follows then from finiteness of the *exponential moment* on the right-hand side of (5) which, in turn, is implied by conditions similar to (6) and (7) below in the case $a = \mu(Y)$ and $b = \sigma(Y)$ (cf., e.g., Cox, Hutzenthaler and Jentzen [13], Lemma 2.22 for details and cf., e.g., also [5, 20, 21, 28, 31] for some instructive results on exponential moments). Note that the counterexamples in Hairer, Hutzenthaler and Jentzen [27] show that some condition on μ and σ beyond smoothness and global boundedness is necessary to ensure that the exponential moment on the right-hand side of (5) is finite and, thereby, that solutions of (1) are locally Lipschitz continuous with respect to the initial values.

In order to demonstrate the flexibility of Theorem 1.2 (and Theorem 2.10 below), we partially solve two well-known approximation problems by means of Theorem 1.2 and Theorem 2.10, respectively. In our first application of Theorem 1.2, we establish in Theorem 1.3 below the strong convergence rate 1/2 for suitable numerical approximations for a large class of *finite-dimensional SODEs with nonglobally monotone coefficients*. We point out that strong convergence rates for numerical approximations are particularly important in order to construct efficient multilevel Monte Carlo approximation methods (cf. Giles [23, 24], Heinrich [29, 30], and Kebaier [43]). In the scientific literature, strong convergence rates for time-discrete approximation processes for multidimensional SODEs are only known under the global monotonicity assumption (3) (cf., e.g., [32, 33, 36, 44, 52, 63, 64, 69] and the references mentioned therein). In addition, strong convergence without rates has been established for time-discrete approximation processes for multidimensional SDEs with nonglobally monotone coefficients in [6, 34, 45, 64, 68]. To the best of our knowledge, Theorem 1.3 is the first result in the scientific literature which proves a strong convergence rate of time-discrete approximation processes for a multidimensional SODE with nonglobally monotone coefficients. In particular, to the best of our knowledge, Theorem 1.3 is the first result in the scientific literature which implies a strong convergence rate for the stochastic Lorenz equation with bounded noise (see Section 3.1.2), for the stochastic van der Pol oscillator (see Section 3.1.3), for the stochastic Duffing–van der Pol oscillator (see Section 3.1.4), for a model from experimental psychology (see Section 3.1.5), for the overdamped Langevin dynamics under suitable assumptions (see Section 3.1.6), or for the stochastic Duffing oscillator with additive noise (see Section 3.1.7). In inequality (7) below, there appears an operator $\mathcal{G}_{\mu,\sigma}\colon C^2(H,\mathbb{R}) \to C(H,\mathbb{R})$ defined in (13) below which is the generator associated with the SDE (1). Theorem 1.3 follows immediately from Proposition 3.3 below.

THEOREM 1.3 (Strong convergence rates for numerical approximations). *Assume the above setting, let* $d, m \in \mathbb{N}$, $c, r \in (0,\infty)$, $q_0, q_1 \in (0,\infty]$, $\alpha \in [0,\infty)$, $p, q \in [2,\infty)$ *with* $\frac{1}{p} + \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{r}$, *assume* $H = D = \mathbb{R}^d$, $U = \mathbb{R}^m$, *let* $U_1 \in C^1(\mathbb{R}^d, [0,\infty))$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in C^1(\mathbb{R}^d, \mathbb{R}^{d\times m})$ *have at most polynomially growing derivatives, let* $U_0 \in C^3(\mathbb{R}^d, [1,\infty))$ *satisfy for all* $x, y \in \mathbb{R}^d$ *with* $x \neq y$ *that* $\sum_{i=1}^{3} \|(U_0^{(i)})(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \leq c\,|U_0(x)|^{(1-1/q)}$, $\|x\|_{\mathbb{R}^d}^{1/c} \leq c(1 + U_0(x))$, $\mathbb{E}[e^{U_0(X_0)}] < \infty$ *and*

$$\frac{\langle x-y, \mu(x)-\mu(y)\rangle_{\mathbb{R}^d} + \frac{(p-1)(1+1/c)}{2}\|\sigma(x)-\sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|x-y\|^2_{\mathbb{R}^d}} \leq c + \frac{U_0(x)+U_0(y)}{2q_0 T e^{\alpha T}} + \frac{U_1(x)+U_1(y)}{2q_1 e^{\alpha T}}, \tag{6}$$

$$(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U_0)(x)\|^2_{\mathbb{R}^m} + U_1(x) \leq \alpha\, U_0(x) + c, \tag{7}$$

and let $Z^N\colon \{0,1,\ldots,N\} \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, *satisfy for all* $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$ *that* $Z_0^N = X_0$ *and*

$$Z_{n+1}^N = Z_n^N + \mathbb{1}_{\{\|Z_n^N\|_{\mathbb{R}^d} < \exp(|\ln(T/N)|^{1/2})\}} \left[\frac{\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})}{1 + \|\mu(Z_n^N)\frac{T}{N} + \sigma(Z_n^N)(W_{(n+1)T/N} - W_{nT/N})\|^2_{\mathbb{R}^d}}\right]. \tag{8}$$

Then there exists a real number $C \in [0,\infty)$ *such that for all* $N \in \mathbb{N}$ *it holds that*

$$\sup\nolimits_{n\in\{0,1,\ldots,N\}} \|X_{\frac{nT}{N}} - Z_n^N\|_{L^r(\Omega;\mathbb{R}^d)} \leq C N^{-1/2}. \tag{9}$$

The numerical scheme (8) has been proposed in [38]. Note that we cannot replace scheme (8) by the well-known Euler–Maruyama scheme since Euler–Maruyama approximations diverge in the strong sense in the case of superlinearly growing coefficient functions (see Theorem 2.1 in [35] and Theorem 2.1 in [37]). As sketched above, *exponential integrability properties* play an important role in the perturbation theory developed in this article. The advantage of the numerical approximations (8) is to preserve *exponential integrability* properties of the exact solution under minor additional assumptions (see [38] for more details). Condition (7) ensures that both the exact solution and the numerical approximations admit suitable exponential integrability properties and assumption (6) ensures that the exponential term on the right-hand side of (5) can be estimated in an appropriate way. Observe that if we choose $q_0 = q_1 = \infty$ in Theorem 1.3, then condition (6) essentially reduces to the global monotonicity assumption (3).

Our second application of Theorem 1.2 and of the more general Theorem 2.10 below concerns **the approximation and the analysis of SPDEs**. In the literature, there are a number of results which prove pathwise convergence rates or convergence rates for convergence in probability for spatially discrete approximation processes of SPDEs with nonglobally monotone nonlinearities (see, e.g., [1, 4, 7, 42, 45, 62]) or which prove strong convergence without convergence rates for spatially discrete approximation processes of SPDEs with nonglobally monotone nonlinearities (see, e.g., [6, 25, 45, 46]). We are aware of only one result which establishes a strong convergence rate for spatially discrete approximation processes of SPDEs with nonglobally monotone nonlinearities namely the above mentioned Corollary 3.2 in Dörsek [19]. Now our perturbation estimate (5) in Theorem 1.2 and its more general version (30) in Theorem 2.10 below, respectively, result in Theorem 1.4 below which can be applied to semilinear SPDEs with nonglobally monotone nonlinearities to establish strong convergence rates for Galerkin approximations. In particular, we apply Theorem 1.4 below to obtain for the first time a strong convergence rate for spectral Galerkin approximations for Cahn–Hilliard–Cook-type SPDEs (see inequality (77) in Section 3.2.2 below for details) and for stochastic Burgers equations with bounded diffusion coefficients (see inequality (99) in Section 3.2.3 below for details). Theorem 1.4 follows immediately from Proposition 3.7 below.

THEOREM 1.4 (Strong convergence rates for Galerkin approximations). *Assume the above setting, let $\varphi\colon D \to \mathbb{R}$ be a Borel measurable mapping, let $\varepsilon \in [0,\infty]$, $r \in (0,\infty)$, $q_0, q_1, \hat{q}_0, \hat{q}_1 \in (0,\infty]$, $c, \alpha, \beta, \hat{\alpha}, \hat{\beta} \in [0,\infty)$, $p \in [2,\infty)$, $U_0, \hat{U}_0 \in C^2(H,[0,\infty))$, $U_1, \hat{U}_1 \in C(D,[0,\infty))$, $P \in L(H)$ satisfy for all $x \in D$, $y \in P(H)$ that $P(H) \subseteq D$, $\frac{1}{p} + \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{\hat{q}_0} + \frac{1}{\hat{q}_1} = \frac{1}{r}$, $\mathbb{E}[e^{U_0(X_0)} + e^{\hat{U}_0(Y_0)}] < \infty$ and*

$$(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U_0)(x)\|_U^2 + U_1(x) \le \alpha\, U_0(x) + \beta,$$

$$(\mathcal{G}_{P\mu,P\sigma} \hat{U}_0)(y) + \tfrac{1}{2}\,\|\sigma(y)^* P^*(\nabla \hat{U}_0)(y)\|_U^2 + \hat{U}_1(y) \le \hat{\alpha}\, \hat{U}_0(y) + \hat{\beta},$$

$$\begin{aligned} &\langle Px - y, P\mu(Px) - P\mu(y)\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\,\|P\sigma(Px) - P\sigma(y)\|^2_{\mathrm{HS}(U,H)} \\ &\quad + \langle y - Px, P\mu(Px) - P\mu(x)\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\,\|P\sigma(Px) - P\sigma(x)\|^2_{\mathrm{HS}(U,H)} \\ &\quad \le \tfrac{|\varphi(x)|^2}{2} + \left[c + \tfrac{U_0(x)}{q_0 T e^{\alpha T}} + \tfrac{\hat{U}_0(y)}{\hat{q}_0 T e^{\hat{\alpha} T}} + \tfrac{U_1(x)}{q_1 e^{\alpha T}} + \tfrac{\hat{U}_1(y)}{\hat{q}_1 e^{\hat{\alpha} T}}\right] \|Px - y\|_H^2, \end{aligned} \tag{10}$$

and assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^T \|\mu(PX_s)\|_H + \|\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)}\,ds$ and $Y_t = PX_0 + \int_0^t P\mu(Y_s)\,ds + \int_0^t P\sigma(Y_s)\,dW_s$. Then

$$\begin{aligned} \sup_{t\in[0,T]} \|X_t - Y_t\|_{L^r(\Omega;H)} &\le T^{(\frac{1}{2}-\frac{1}{p})} \exp\left(\tfrac{1}{2} - \tfrac{1}{p} + \int_0^T c + \sum_{i=0}^{1}\left[\tfrac{\beta}{q_i e^{\alpha s}} + \tfrac{\hat{\beta}}{\hat{q}_i e^{\hat{\alpha} s}}\right] ds\right) \\ &\quad \cdot \|\varphi(X)\|_{L^p([0,T]\times\Omega;\mathbb{R})} \left|\mathbb{E}\left[e^{U_0(X_0)}\right]\right|^{\left[\frac{1}{q_0}+\frac{1}{q_1}\right]} \left|\mathbb{E}\left[e^{\hat{U}_0(Y_0)}\right]\right|^{\left[\frac{1}{\hat{q}_0}+\frac{1}{\hat{q}_1}\right]} \\ &\quad + \sup_{t\in[0,T]} \|(I-P)X_t\|_{L^r(\Omega;H)}. \end{aligned} \tag{11}$$

As a third application of Theorem 1.2, we study *SDEs with small noise* (cf., e.g., Theorem 1.2 in Freidlin and Wentzell [22] for the case of globally Lipschitz continuous coefficients). In particular, Corollary 3.12 below can be applied to a number of nonlinear ordinary and partial differential equations perturbed by a small noise term such as the examples in Sections 3.1.2–3.1.7 as well as the examples in Sections 3.2.2–3.2.3. We refer the reader to Section 3.3 for more details.

1.1. *Notation*. Throughout this article, the following notation is used. For all sets A and B, let $\mathcal{M}(A,B)$ be the set of all mappings from A to B. For all measurable spaces $(A,\mathcal{A})$ and $(B,\mathcal{B})$ let $\mathcal{L}^0(A;B)$ be the set of all $\mathcal{A}/\mathcal{B}$-measurable functions. For every $d \in \mathbb{N}$ let $\mathcal{C}^3_{\mathcal{D}}(\mathbb{R}^d,\mathbb{R})$ be the set given by

$$\mathcal{C}^3_{\mathcal{D}}(\mathbb{R}^d,\mathbb{R}) = \bigcup_{p,c\in[3,\infty)} \left\{ f \in C^2(\mathbb{R}^d,\mathbb{R}) \colon \begin{array}{c} f'' \text{ is locally Lipschitz continuous and for} \\ \text{all } i \in \{1,2,3\} \text{ and } \lambda_{\mathbb{R}^d}\text{-almost all } x \in \mathbb{R}^d \text{ it} \\ \text{holds that } \|f^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \le c\,|f(x)|^{[1-i/p]} \end{array} \right\}.$$

For every $d \in \mathbb{N}$ and every metric space (E,δ), let $\mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d,E)$ be the set given by

$$\begin{aligned} \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d,E) = \{ f \in C(\mathbb{R}^d,E) : (\exists c \in [0,\infty) : \forall x,y \in \mathbb{R}^d : \\ \delta(f(x),f(y)) \le c(1 + \|x\|^c_{\mathbb{R}^d} + \|y\|^c_{\mathbb{R}^d})\|x-y\|_{\mathbb{R}^d})\}. \end{aligned} \tag{12}$$

For all separable $\mathbb{R}$-Hilbert spaces $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, every orthonormal basis $\mathbb{U}$ of U, every open set $O \subseteq H$, every nonempty set $\mathcal{O} \subseteq O$ and all $\mu \in \mathcal{M}(\mathcal{O}, H)$, $\sigma \in \mathcal{M}(\mathcal{O}, \mathrm{HS}(U, H))$, let $\mathcal{G}_{\mu,\sigma}\colon C^2(O, \mathbb{R}) \to \mathcal{M}(\mathcal{O}, \mathbb{R})$, $G_\sigma\colon C^1(O, \mathbb{R}) \to \mathcal{M}(\mathcal{O}, U^*)$, $\overline{\mathcal{G}}_{\mu,\sigma}\colon\ C^2(O^2, \mathbb{R}) \to \mathcal{M}(\mathcal{O}^2, \mathbb{R})$, and $\overline{G}_\sigma\colon C^1(O^2, \mathbb{R}) \to \mathcal{M}(\mathcal{O}^2, U^*)$ be the functions which satisfy for all $\psi \in C^1(O, \mathbb{R})$, $\bar{\psi} \in C^1(O^2, \mathbb{R})$, $\phi \in C^2(O, \mathbb{R})$, $\bar{\phi} \in C^2(O^2, \mathbb{R})$, $x, y \in \mathcal{O}$ that

$$(\mathcal{G}_{\mu,\sigma}\phi)(x) = \phi'(x)\mu(x) + \frac{1}{2}\operatorname{trace}\bigl(\sigma(x)\sigma(x)^*(\operatorname{Hess}\phi)(x)\bigr), \tag{13}$$

$$(G_\sigma\psi)(x) = \psi'(x)\sigma(x), \tag{14}$$

$$\begin{aligned}(\overline{\mathcal{G}}_{\mu,\sigma}\bar{\phi})(x, y) &= \Bigl(\frac{\partial}{\partial x}\bar{\phi}\Bigr)(x, y)\mu(x) + \Bigl(\frac{\partial}{\partial y}\bar{\phi}\Bigr)(x, y)\mu(y)\\ &\quad + \frac{1}{2}\sum_{u\in\mathbb{U}}\Bigl(\frac{\partial^2}{\partial x^2}\bar{\phi}\Bigr)(x, y)\bigl((\sigma(x))(u), (\sigma(x))(u)\bigr)\\ &\quad + \sum_{u\in\mathbb{U}}\Bigl(\frac{\partial}{\partial y}\frac{\partial}{\partial x}\bar{\phi}\Bigr)(x, y)\bigl((\sigma(x))(u), (\sigma(y))(u)\bigr)\\ &\quad + \frac{1}{2}\sum_{u\in\mathbb{U}}\Bigl(\frac{\partial^2}{\partial y^2}\bar{\phi}\Bigr)(x, y)\bigl((\sigma(y))(u), (\sigma(y))(u)\bigr),\end{aligned} \tag{15}$$

$$(\overline{G}_\sigma\bar{\psi})(x, y) = \Bigl(\frac{\partial}{\partial x}\bar{\psi}\Bigr)(x, y)\sigma(x) + \Bigl(\frac{\partial}{\partial y}\bar{\psi}\Bigr)(x, y)\sigma(y). \tag{16}$$

We call the linear operator $\mathcal{G}_{\mu,\sigma}$ in (13) *generator*, we call the linear operator G_σ in (14) *noise operator*, we call the linear operator $\overline{\mathcal{G}}_{\mu,\sigma}$ in (15) *extended generator* (cf. Ichikawa [39] and Maslowski [53]), and we call the linear operator $\overline{G}_\sigma$ in (16) *extended noise operator* (cf., e.g., Cox, Hutzenthaler and Jentzen [13]). For every $T \in (0, \infty)$ let $\mathcal{P}_T$ be the set given by $\mathcal{P}_T = \cup_{n\in\mathbb{N}}\{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}\colon 0 = t_0 < t_1 < \cdots < t_n = T\}$. For every $T \in (0, \infty)$, every filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ which fulfills the usual conditions, and all adapted and product measurable stochastic processes $\chi\colon [0, T] \times \Omega \to \mathbb{R}$ and $\zeta\colon [0, T] \times \Omega \to U^* = L(U, \mathbb{R})$ with $\mathbb{P}(\int_0^T |\chi_s| + \|\zeta_s\|_{U^*}^2\, ds < \infty) = 1$ let $\Psi[\chi, \zeta]$ be the equivalence class (with respect to indistinguishability) of adapted $\mathbb{R}$-valued stochastic processes on $[0, T]$ with c.s.p. which satisfies that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that

$$\Psi[\chi, \zeta]_t = \exp\Biggl(\int_0^t \chi_s - \tfrac{1}{2}\|\zeta_s\|_{U^*}^2\, ds + \int_0^t \zeta_s\, dW_s\Biggr). \tag{17}$$

For every $a \in \mathbb{R}$ let a^+ be the real number given by $a^+ = \max(a, 0)$. For every $T \in (0, \infty)$, every filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ which fulfills the usual conditions, and every stopping time $\tau\colon \Omega \to [0, T]$ let $[\![0, \tau]\!]$ be the set given by $[\![0, \tau]\!] = \{(t, \omega) \in [0, T] \times \Omega\colon t \le \tau(\omega)\}$ (see, e.g., Kühn [47], Definition 3.1). Throughout this article we also often calculate and formulate expressions in the extended real numbers $[-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$. In particular, we frequently use the convention $\frac{0}{0} = 0 \cdot \infty = 0$.

1.2. *Setting*. Throughout this article, the following setting is frequently used.

SETTING 1.5. Consider the notation in Section 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $O \subseteq H$ be an open set, let $\mathcal{O} \in \mathcal{B}(O)$, $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space which fulfills the usual conditions, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process and let $\mathbb{U} \subseteq U$ be an orthonormal basis of U.

2. A perturbation theory for stochastic differential equations (SDEs).

2.1. *Itô's formula and an exponential integrating factor.*

LEMMA 2.1. *Assume Setting 1.5, let $V \in C^2(O, \mathbb{R})$ and let $X: [0,T] \times \Omega \to \mathcal{O}$, $a: [0,T] \times \Omega \to H$, $b: [0,T] \times \Omega \to \mathrm{HS}(U,H)$, $\chi: [0,T] \times \Omega \to \mathbb{R}$, $\zeta: [0,T] \times \Omega \to U^*$ be predictable stochastic processes which satisfy that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + |\chi_s| + \|\zeta_s\|^2_{U^*}\, ds < \infty$ and $X_t = X_0 + \int_0^t a_s\, ds + \int_0^t b_s\, dW_s$. Then for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that*

$$
\begin{aligned}
\frac{V(X_t)}{\Psi[\chi,\zeta]_t} &= V(X_0) + \int_0^t \tfrac{V'(X_s)\,b_s - V(X_s)\zeta_s}{\Psi[\chi,\zeta]_s}\, dW_s \\
&\quad + \int_0^t \tfrac{V'(X_s)\,a_s + \frac{1}{2}\operatorname{trace}(b_s^*\,(\operatorname{Hess} V)(X_s)\,b_s) + \operatorname{trace}(\zeta_s^*[V(X_s)\zeta_s - V'(X_s)\,b_s]) - V(X_s)\chi_s}{\Psi[\chi,\zeta]_s}\, ds.
\end{aligned} \tag{18}
$$

PROOF. Applying Itô's formula to the process $\frac{V(X_t)}{\Psi[\chi,\zeta]_t}$, $t \in [0,T]$, shows that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
\frac{V(X_t)}{\Psi[\chi,\zeta]_t} &= V(X_0) + \int_0^t \frac{V'(X_s)b_s - V(X_s)\zeta_s}{\Psi[\chi,\zeta]_s}\, dW_s \\
&\quad + \int_0^t \frac{V'(X_s)a_s + \frac{1}{2}\operatorname{trace}((b_s)^*(\operatorname{Hess} V)(X_s)b_s) - V(X_s)[\chi_s - \frac{1}{2}\|\zeta_s\|^2_{U^*}]}{\Psi[\chi,\zeta]_s}\, ds \\
&\quad + \int_0^t \frac{\frac{1}{2}V(X_s)\|\zeta_s\|^2_{U^*} - \operatorname{trace}(\zeta_s^* V'(X_s)b_s)}{\Psi[\chi,\zeta]_s}\, ds.
\end{aligned} \tag{19}
$$

Combining this with the elementary fact that for all $s \in [0,T]$ it holds that

$$
\begin{aligned}
V(X_s)\,\|\zeta_s\|^2_{U^*} - \operatorname{trace}(\zeta_s^*\, V'(X_s)\,b_s) &= V(X_s)\,\|\zeta_s\|^2_{\mathrm{HS}(U,\mathbb{R})} - \operatorname{trace}(\zeta_s^*\, V'(X_s)\,b_s) \\
&= \operatorname{trace}(\zeta_s^* V(X_s)\zeta_s) - \operatorname{trace}(\zeta_s^*\, V'(X_s)\,b_s) = \operatorname{trace}(\zeta_s^*\,[V(X_s)\zeta_s - V'(X_s)\,b_s])
\end{aligned} \tag{20}
$$

completes the proof of Lemma 2.1. □

In Lemma 2.2, we present a slightly different formulation of Lemma 2.1, that is, we add and substract in (18) the generator in (13) and the noise operator in (14). Lemma 2.2 is an immediate consequence of Lemma 2.1.

LEMMA 2.2. *Assume Setting 1.5, let $V \in C^2(O, \mathbb{R})$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U,H))$, and let $X: [0,T] \times \Omega \to \mathcal{O}$, $a: [0,T] \times \Omega \to H$, $b: [0,T] \times \Omega \to \mathrm{HS}(U,H)$, $\chi: [0,T] \times \Omega \to \mathbb{R}$, $\zeta: [0,T] \times \Omega \to U^*$ be predictable stochastic processes which satisfy that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + |\chi_s| + \|\zeta_s\|^2_{U^*} + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)}\, ds < \infty$ and $X_t = X_0 + \int_0^t a_s\, ds + \int_0^t b_s\, dW_s$. Then for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that*

$$
\begin{aligned}
\frac{V(X_t)}{\Psi[\chi,\zeta]_t} &= V(X_0) + \int_0^t \tfrac{(\mathcal{G}_{\mu,\sigma}V)(X_s) - \chi_s V(X_s) + \operatorname{trace}(\zeta_s^*[V(X_s)\zeta_s - V'(X_s)\,b_s])}{\Psi[\chi,\zeta]_s}\, ds \\
&\quad + \int_0^t \tfrac{V'(X_s)[a_s - \mu(X_s)] + \frac{1}{2}\operatorname{trace}([b_s + \sigma(X_s)]^*(\operatorname{Hess} V)(X_s)[b_s - \sigma(X_s)])}{\Psi[\chi,\zeta]_s}\, ds \\
&\quad + \int_0^t \tfrac{V'(X_s)[b_s - \sigma(X_s)] + (G_\sigma V)(X_s) - V(X_s)\zeta_s}{\Psi[\chi,\zeta]_s}\, dW_s.
\end{aligned} \tag{21}
$$

2.2. *A perturbation formula.* In the next result, Proposition 2.3, we formulate the special case of Lemma 2.2 where the stochastic process $(X_t)_{\in[0,T]}$ in Lemma 2.2 is the pairing of two stochastic processes $X = (X^1, X^2)$.

PROPOSITION 2.3. *Assume Setting 1.5, let* $V = (V(x_1,x_2))_{(x_1,x_2)\in O^2} \in C^2(O^2,\mathbb{R})$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U,H))$, *let* $\chi\colon [0,T]\times\Omega \to \mathbb{R}$, $\zeta\colon [0,T]\times\Omega\to U^*$ *be predictable stochastic processes, let* $X^i\colon [0,T]\times\Omega\to\mathcal{O}$, $a^i\colon [0,T]\times\Omega\to H$, $b^i\colon [0,T]\times\Omega\to \mathrm{HS}(U,H)$, $i\in\{1,2\}$, *be predictable stochastic processes, and assume that for all* $t\in[0,T]$, $i\in\{1,2\}$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T \|a_s^i\|_H + \|b_s^i\|^2_{\mathrm{HS}(U,H)} + |\chi_s| + \|\zeta_s\|^2_{U^*} + \|\mu(X_s^i)\|_H + \|\sigma(X_s^i)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$ *and* $X_t^i = X_0^i + \int_0^t a_s^i\,ds + \int_0^t b_s^i\,dW_s$. *Then for all* $t\in[0,T]$ *it holds* $\mathbb{P}$*-a.s. that*

$$
\begin{aligned}
\frac{V(X_t^1,X_t^2)}{\Psi[\chi,\zeta]_t} &= V(X_0^1,X_0^2) + \int_0^t \frac{\sum_{i=1}^2 (\partial_{x_i}V)(X_s^1,X_s^2)\big[b_s^i-\sigma(X_s^i)\big] + (\overline{G}_\sigma V)(X_s^1,X_s^2) - V(X_s^1,X_s^2)\zeta_s}{\Psi[\chi,\zeta]_s}\,dW_s \\
&\quad + \int_0^t \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s^1,X_s^2) - \chi_s V(X_s^1,X_s^2) + \operatorname{trace}\Big(\zeta_s^*\Big[V(X_s^1,X_s^2)\zeta_s - \sum_{i=1}^2(\partial_{x_i}V)(X_s^1,X_s^2)\,b_s^i\Big]\Big)}{\Psi[\chi,\zeta]_s}\,ds \\
&\quad + \int_0^t \frac{\sum_{i=1}^2(\partial_{x_i}V)(X_s^1,X_s^2)\big[a_s^i-\mu(X_s^i)\big] + \frac12\sum_{i=1}^2 \operatorname{trace}\Big(\big[b_s^i+\sigma(X_s^i)\big]^*(\mathrm{Hess}_{x_i}V)(X_s^1,X_s^2)\big[b_s^i-\sigma(X_s^i)\big]\Big)}{\Psi[\chi,\zeta]_s}\,ds \\
&\quad + \sum_{u\in\mathbb{U}}\int_0^t \frac{\sum_{i=1}^2\big(\partial_{x_i}\partial_{x_{3-i}}V\big)(X_s^1,X_s^2)\Big(\big[b_s^i+\sigma(X_s^i)\big]u, \big[b_s^{3-i}-\sigma(X_s^{3-i})\big]u\Big)}{2\,\Psi[\chi,\zeta]_s}\,ds.
\end{aligned}
$$

Next we formulate the special case of Proposition 2.3 where the stochastic process $(X_t^1)_{t\in[0,T]}$ in Proposition 2.3 is a solution process of the SDE with drift coefficient μ and diffusion coefficient σ.

COROLLARY 2.4. *Assume Setting 1.5, let* $V = (V(x,y))_{(x,y)\in O^2} \in C^2(O^2,\mathbb{R})$, $\mu\in\mathcal{L}^0(\mathcal{O};H)$, $\sigma\in\mathcal{L}^0(\mathcal{O};\mathrm{HS}(U,H))$, *and let* $X, Y\colon [0,T]\times\Omega\to\mathcal{O}$, $a\colon[0,T]\times\Omega\to H$, $b\colon[0,T]\times\Omega\to\mathrm{HS}(U,H)$, $\chi\colon[0,T]\times\Omega\to\mathbb{R}$, $\zeta\colon[0,T]\times\Omega\to U^*$ *be predictable stochastic processes which satisfy that for all* $t\in[0,T]$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + |\chi_s| + \|\zeta_s\|^2_{U^*} + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, *and* $Y_t = Y_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s$. *Then for all* $t\in[0,T]$ *it holds* $\mathbb{P}$*-a.s. that*

$$
\begin{aligned}
\frac{V(X_t,Y_t)}{\Psi[\chi,\zeta]_t} &= V(X_0,Y_0) + \int_0^t \frac{(\partial_y V)(X_s,Y_s)[b_s-\sigma(Y_s)] + (\overline{G}_\sigma V)(X_s,Y_s) - V(X_s,Y_s)\zeta_s}{\Psi[\chi,\zeta]_s}\,dW_s \\
&\quad + \int_0^t \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s,Y_s) - \chi_s V(X_s,Y_s) + \operatorname{trace}\big(\zeta_s^*[V(X_s,Y_s)\zeta_s - (\partial_x V)(X_s,Y_s)\,\sigma(X_s) - (\partial_y V)(X_s,Y_s)\,b_s]\big)}{\Psi[\chi,\zeta]_s}\,ds \\
&\quad + \int_0^t \frac{(\partial_y V)(X_s,Y_s)[a_s-\mu(Y_s)] + \frac12\operatorname{trace}\big([b_s+\sigma(Y_s)]^*(\mathrm{Hess}_y V)(X_s,Y_s)[b_s-\sigma(Y_s)]\big)}{\Psi[\chi,\zeta]_s}\,ds \\
&\quad + \sum_{u\in\mathbb{U}}\int_0^t \frac{\big(\partial_x\partial_y V\big)(X_s,Y_s)(\sigma(X_s)u,[b_s-\sigma(Y_s)]u)}{\Psi[\chi,\zeta]_s}\,ds.
\end{aligned}
\tag{22}
$$

Note in the setting of Corollary 2.4 that if Y is also a solution of the SDE with drift coefficient μ and diffusion coefficient σ and if χ and ζ are appropriate (see Cox, Hutzenthaler and Jentzen [13], Proposition 2.12), then Corollary 2.4 essentially reduces to Cox, Hutzenthaler and Jentzen [13], Proposition 2.12, and can be used to study the regularity of solutions of SDEs in the initial value. The next result, Proposition 2.5, formulates the special case of Corollary 2.4 in which the process $\zeta\equiv 0$ vanishes.

PROPOSITION 2.5. *Assume Setting 1.5, let* $V = (V(x,y))_{(x,y)\in O^2} \in C^2(O^2,\mathbb{R})$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U,H))$, *let* $X, Y\colon [0,T]\times\Omega \to \mathcal{O}$, $a\colon [0,T]\times\Omega \to H$, $b\colon [0,T]\times\Omega\to \mathrm{HS}(U,H)$, $\chi\colon [0,T]\times\Omega\to\mathbb{R}$ *be predictable stochastic processes, and assume that for all* $t\in[0,T]$ *it holds* $\mathbb{P}$-*a.s. that* $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + |\chi_s| + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$ *and* $Y_t = Y_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s$. *Then for all* $t\in[0,T]$ *it holds* $\mathbb{P}$-*a.s. that*

$$\begin{aligned}\frac{V(X_t,Y_t)}{\exp(\int_0^t \chi_s\,ds)} &= V(X_0,Y_0) + \int_0^t \frac{(\partial_y V)(X_s,Y_s)[b_s-\sigma(Y_s)] + (\overline{G}_\sigma V)(X_s,Y_s)}{\exp(\int_0^s \chi_u\,du)}\,dW_s \\ &\quad + \int_0^t \frac{(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s,Y_s) - \chi_s V(X_s,Y_s) + \sum_{u\in\mathbb{U}} (\partial_x\partial_y V)(X_s,Y_s)(\sigma(X_s)u,[b_s-\sigma(Y_s)]u)}{\exp(\int_0^s \chi_u\,du)}\,ds \\ &\quad + \int_0^t \frac{(\partial_y V)(X_s,Y_s)[a_s-\mu(Y_s)] + \frac12 \operatorname{trace}\big([b_s+\sigma(Y_s)]^*(\operatorname{Hess}_y V)(X_s,Y_s)[b_s-\sigma(Y_s)]\big)}{\exp(\int_0^s \chi_u\,du)}\,ds.\end{aligned} \tag{23}$$

2.3. *Perturbation estimates.* In this subsection, our goal is to establish an estimate for the quantity $\sup_{t\in[0,T]} \|V(X_t,Y_t)\|_{L^r(\Omega;\mathbb{R})}$ for some $r\in(0,\infty)$ in (23) in Proposition 2.5. The following lemma follows from (23) by applying a localizing argument together with Hölder's inequality and Fatou's lemma.

LEMMA 2.6. *Assume Setting 1.5, let* $V = (V(x,y))_{(x,y)\in O^2} \in C^2(O^2,[0,\infty))$, $\mu\in\mathcal{L}^0(\mathcal{O};H)$, $\sigma\in\mathcal{L}^0(\mathcal{O};\mathrm{HS}(U,H))$, *let* $\tau\colon\Omega\to[0,T]$ *be a stopping time, let* $X, Y\colon[0,T]\times\Omega\to\mathcal{O}$ *be adapted stochastic processes with c.s.p., let* $a\colon[0,T]\times\Omega\to H$, $b\colon[0,T]\times\Omega\to\mathrm{HS}(U,H)$, $\chi\colon[0,T]\times\Omega\to\mathbb{R}$ *be predictable stochastic processes, and assume that for all* $t\in[0,T]$ *it holds* $\mathbb{P}$-*a.s. that* $\int_0^T |\chi_s| + \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$, $X_t = X_0 + \int_0^t\mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, *and* $Y_t = Y_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s$. *Then for all* $p\in(0,1]$ *it holds that*

$$\begin{aligned}&\|V(X_\tau,Y_\tau)\|_{L^p(\Omega;\mathbb{R})} \le \left\|\exp\left(\int_0^\tau \chi_s\,ds\right)\right\|_{L^{p/(1-p)}(\Omega;\mathbb{R})} \\ &\cdot \sup\Bigg\{\mathbb{E}\Bigg[V(X_0,Y_0) + \int_0^{\nu\wedge\tau}\Big[(\overline{\mathcal{G}}_{\mu,\sigma}V)(X_s,Y_s) - \chi_s V(X_s,Y_s) \\ &+ \textstyle\sum_{u\in\mathbb{U}}\big(\partial_x\partial_y V\big)(X_s,Y_s)\,(\sigma(X_s)u,[b_s-\sigma(Y_s)]u) + (\partial_y V)(X_s,Y_s)\,[a_s-\mu(Y_s)] \\ &+ \tfrac12\operatorname{trace}\big([b_s+\sigma(Y_s)]^*\,(\operatorname{Hess}_y V)(X_s,Y_s)\,[b_s-\sigma(Y_s)]\big)\Big]\exp\big(-\textstyle\int_0^s\chi_u\,du\big)\,ds\Bigg]\colon \begin{array}{c}\nu \text{ stopping}\\ \text{time such that}\end{array} \\ &\sum_{i=0}^2\int_0^\nu \left\|V^{(i)}(X_s,Y_s)\right\|^2_{L^{(i)}(H^2,\mathbb{R})}\Big[|\chi_s| + \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + \|\mu(X_s)\|_H + \|\mu(Y_s)\|_H \\ &+ \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\Big]\,ds \in L^\infty(\Omega;\mathbb{R})\Bigg\}.\end{aligned} \tag{24}$$

PROOF. Throughout this proof let $\tau_n\colon\Omega\to[0,T]$, $n\in\mathbb{N}$, be stopping times which satisfy for all $n\in\mathbb{N}$ that

$$\begin{aligned}\tau_n = \inf\Bigg(\{\tau\}\cup\Bigg\{t\in[0,T]\colon \sum_{i=0}^2\int_0^t \left\|V^{(i)}(X_s,Y_s)\right\|^2_{L^{(i)}(H^2,\mathbb{R})}\Big[|\chi_s| + \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} \\ + \|\mu(X_s)\|_H + \|\mu(Y_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\Big]\,ds \ge n\Bigg\}\Bigg).\end{aligned}$$

Hölder's inequality and Fatou's lemma prove that for all $p \in (0, 1]$ it holds that

$$\begin{aligned} \|V(X_\tau, Y_\tau)\|_{L^p(\Omega;\mathbb{R})} &= \left\| \frac{V(X_\tau, Y_\tau)}{\exp(\int_0^\tau \chi_s \, ds)} \exp\left(\int_0^\tau \chi_s \, ds \right) \right\|_{L^p(\Omega;\mathbb{R})} \\ &\leq \left\| \frac{V(X_\tau, Y_\tau)}{\exp(\int_0^\tau \chi_s \, ds)} \right\|_{L^1(\Omega;\mathbb{R})} \left\| \exp\left(\int_0^\tau \chi_s \, ds \right) \right\|_{L^{p/(1-p)}(\Omega;\mathbb{R})} \\ &\leq \sup_{n\in\mathbb{N}} \mathbb{E}\left[\frac{V(X_{\tau_n}, Y_{\tau_n})}{\exp(\int_0^{\tau_n} \chi_s \, ds)} \right] \left\| \exp\left(\int_0^\tau \chi_s \, ds \right) \right\|_{L^{p/(1-p)}(\Omega;\mathbb{R})}. \end{aligned} \tag{25}$$

Applying Proposition 2.5 to the right-hand side of (25) completes the proof of Lemma 2.6. □

If the right-hand side of (24) is further estimated in an appropriate way, then a more compact statement can be obtained. This is the subject of the next corollary.

COROLLARY 2.7. *Assume Setting 1.5, let* $V = (V(x, y))_{(x,y)\in O^2} \in C^2(O^2, [0, \infty))$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U, H))$, *let* $\tau : \Omega \to [0, T]$ *be a stopping time, let* $X, Y : [0, T] \times \Omega \to \mathcal{O}$ *be adapted stochastic processes with c.s.p., let* $a : [0, T] \times \Omega \to H$, $b : [0, T] \times \Omega \to \mathrm{HS}(U, H)$, $\chi : [0, T] \times \Omega \to [0, \infty)$ *be predictable stochastic processes, and assume that for all* $t \in [0, T]$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)} + \chi_s \, ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s$, *and* $Y_t = Y_0 + \int_0^t a_s \, ds + \int_0^t b_s \, dW_s$. *Then for all* $p \in (0, 1]$ *it holds that*

$$\begin{aligned} \|V(X_\tau, Y_\tau)\|_{L^p(\Omega;\mathbb{R})} &\leq \left\| \exp\left(\int_0^\tau \chi_s \, ds \right) \right\|_{L^{p/(1-p)}(\Omega;\mathbb{R})} \mathbb{E}\Big[V(X_0, Y_0) + \int_0^\tau \Big[(\overline{\mathcal{G}}_{\mu,\sigma} V)(X_s, Y_s) \\ &\quad - \chi_s V(X_s, Y_s) + \textstyle\sum_{u\in\mathbb{U}} (\partial_x \partial_y V)(X_s, Y_s) (\sigma(X_s)u, [b_s - \sigma(Y_s)]u) + (\partial_y V)(X_s, Y_s) [a_s - \mu(Y_s)] \\ &\quad + \tfrac{1}{2} \operatorname{trace}\big([b_s + \sigma(Y_s)]^* (\operatorname{Hess}_y V)(X_s, Y_s) [b_s - \sigma(Y_s)] \big) \Big]^+ ds \Big]. \end{aligned} \tag{26}$$

Lemma 2.6 can be used to study the regularity of solutions of SDEs with respect to the initial values. This is illustrated in the next result, Corollary 2.8, which follows immediately from Lemma 2.6.

COROLLARY 2.8. *Assume Setting 1.5, let* $V \in C^2(O^2, [0, \infty))$, $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U, H))$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, *let* $\tau : \Omega \to [0, T]$ *be a stopping time, let* $X, Y : [0, T] \times \Omega \to \mathcal{O}$, $\chi : [0, T] \times \Omega \to \mathbb{R}$ *be predictable stochastic processes and assume that for all* $t \in [0, T]$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T |\chi_s| + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)} \, ds < \infty$, $\int_0^\tau [(\overline{\mathcal{G}}_{\mu,\sigma} V)(X_s, Y_s) - \chi_s V(X_s, Y_s)]^+ \, ds \leq 0$, $X_t = X_0 + \int_0^t \mu(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s$ *and* $Y_t = Y_0 + \int_0^t \mu(Y_s) \, ds + \int_0^t \sigma(Y_s) \, dW_s$. *Then for all* $p \in (0, 1]$ *it holds that*

$$\|V(X_\tau, Y_\tau)\|_{L^p(\Omega;\mathbb{R})} \leq \mathbb{E}\big[V(X_0, Y_0)\big] \, \|\exp(\textstyle\int_0^\tau \chi_s \, ds)\|_{L^{p/(1-p)}(\Omega;\mathbb{R})}. \tag{27}$$

Corollary 2.8 is a statement quite similar to Proposition 2.17 in Cox, Hutzenthaler and Jentzen [13] in the case $p = 1$ in the setting of the proposition. As Proposition 2.17 in Cox, Hutzenthaler and Jentzen [13], Corollary 2.8 can now be used to study the regularity with respect to the initial value for a number of nonlinear SDEs in the literature (such as the stochastic Duffing–van der Pol oscillator, the Cox–Ingersoll–Ross process or Cahn–Hilliard–Cook equations); see Cox, Hutzenthaler and Jentzen [13], Sections 4–5, for a list of examples.

2.4. *Perturbation estimates in the case of Hilbert space distances.* This subsection investigates the special case of Proposition 2.5 in which the distance-type function $V \in C^2(O^2, \mathbb{R})$ in Proposition 2.5 satisfies that there exists $p \in [2, \infty)$ such that for all $x, y \in O$ it holds that $V(x, y) = \|x - y\|_H^p$.

PROPOSITION 2.9. *Assume Setting 1.5, let $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U, H))$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, let $X, Y\colon [0, T] \times \Omega \to \mathcal{O}$, $a\colon [0, T] \times \Omega \to H$, $b\colon [0, T] \times \Omega \to \mathrm{HS}(U, H)$, $\chi\colon [0, T] \times \Omega \to \mathbb{R}$ be predictable stochastic processes, and assume that for all $t \in [0, T]$ it holds $\mathbb{P}$-a.s. that $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + |\chi_s| + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\, ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s$ and $Y_t = Y_0 + \int_0^t a_s\, ds + \int_0^t b_s\, dW_s$. Then for all $t \in [0, T]$, $\varepsilon \in [0, \infty]$, $p \in [2, \infty)$ it holds $\mathbb{P}$-a.s. that*

$$
\begin{aligned}
\frac{\|X_t - Y_t\|_H^p}{\exp(\int_0^t \chi_s\, ds)} &\le \|X_0 - Y_0\|_H^p + \int_0^t \Big\langle \frac{p\, \|X_s - Y_s\|_H^{(p-2)}\, [X_s - Y_s]}{\exp(\int_0^s \chi_u\, du)}, [\sigma(X_s) - b_s]\, dW_s \Big\rangle_H \\
&\quad + \int_0^t \frac{p\, \|X_s - Y_s\|_H^{(p-2)} \big[\langle X_s - Y_s, \mu(Y_s) - a_s\rangle_H + \frac{(p-1)(1+1/\varepsilon)}{2} \|b_s - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\big] - \chi_s \|X_s - Y_s\|_H^p}{\exp(\int_0^s \chi_u\, du)}\, ds \\
&\quad + \int_0^t \frac{p\, \|X_s - Y_s\|_H^{(p-2)} \big[\langle X_s - Y_s, \mu(X_s) - \mu(Y_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2} \|\sigma(X_s) - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\big]}{\exp(\int_0^s \chi_u\, du)}\, ds.
\end{aligned}
\tag{28}
$$

PROOF. Combining (23) in Proposition 2.5 together with Remark 2.14 in Cox, Hutzenthaler and Jentzen [13] and a straightforward generalization of Example 2.15 in Cox, Hutzenthaler and Jentzen [13] shows that for all $t \in [0, T]$, $p \in [2, \infty)$ it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
\frac{\|X_t - Y_t\|_H^p}{\exp(\int_0^t \chi_s\, ds)} &= \|X_0 - Y_0\|_H^p + \int_0^t \Big\langle \frac{p\, \|X_s - Y_s\|_H^{(p-2)}\, [X_s - Y_s]}{\exp(\int_0^s \chi_u\, du)}, [\sigma(X_s) - b_s]\, dW_s \Big\rangle_H \\
&\quad + \int_0^t \frac{\mathbb{1}_{\{X_s \neq Y_s\}}\, \frac{p\,(p-2)}{2}\, \|X_s - Y_s\|_H^{(p-4)} \big[\|[\sigma(X_s) - \sigma(Y_s)]^*(X_s - Y_s)\|_U^2 + \|[b_s - \sigma(Y_s)]^*[X_s - Y_s]\|_U^2\big]}{\exp(\int_0^s \chi_u\, du)}\, ds \\
&\quad + \int_0^t \frac{p\, \|X_s - Y_s\|_H^{(p-2)} \big[\langle X_s - Y_s, \mu(X_s) - \mu(Y_s)\rangle_H + \frac{1}{2} \|\sigma(X_s) - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\big] - \chi_s \|X_s - Y_s\|_H^p}{\exp(\int_0^s \chi_u\, du)}\, ds \\
&\quad + \int_0^t \frac{p\, \|X_s - Y_s\|_H^{(p-2)} \big[\langle X_s - Y_s, \mu(Y_s) - a_s\rangle_H + \frac{1}{2} \|b_s - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)} + \mathrm{trace}\big([\sigma(Y_s) - \sigma(X_s)]^*[b_s - \sigma(Y_s)]\big)\big]}{\exp(\int_0^s \chi_u\, du)}\, ds \\
&\quad + \int_0^t \frac{\mathbb{1}_{\{X_s \neq Y_s\}}\, p\,(p-2)\, \|X_s - Y_s\|_H^{(p-4)}\, \mathrm{trace}\big([\sigma(Y_s) - \sigma(X_s)]^*[X_s - Y_s][X_s - Y_s]^*[b_s - \sigma(Y_s)]\big)}{\exp(\int_0^s \chi_u\, du)}\, ds.
\end{aligned}
$$

The Cauchy–Schwarz inequality in the Hilbert space $\mathrm{HS}(U, H)$ (see, e.g., Prévôt and Röckner [61], Remark B.0.4 and Proposition B.0.8), the Hölder estimate for Schatten norms (see, e.g., Prévôt and Röckner [61], Remark B.0.6) and the fact that for all $a, b \in \mathbb{R}$, $\varepsilon \in [0, \infty]$ it holds that $ab \le \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2$ hence imply that for all $t \in [0, T]$, $\varepsilon \in [0, \infty]$, $p \in [2, \infty)$ it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
\frac{\|X_t - Y_t\|_H^p}{\exp(\int_0^t \chi_s\, ds)} &\le \|X_0 - Y_0\|_H^p + \int_0^t \Big\langle \frac{p\, \|X_s - Y_s\|_H^{(p-2)}\, [X_s - Y_s]}{\exp(\int_0^s \chi_u\, du)}, [\sigma(X_s) - b_s]\, dW_s \Big\rangle_H \\
&\quad + \int_0^t \frac{p\, \|X_s - Y_s\|_H^{(p-2)} \big[\langle X_s - Y_s, \mu(Y_s) - a_s\rangle_H + \frac{(p-1)(1+1/\varepsilon)}{2} \|b_s - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\big]}{\exp(\int_0^s \chi_u\, du)}\, ds \\
&\quad + \int_0^t \frac{p\, \|X_s - Y_s\|_H^{(p-2)} \big[\langle X_s - Y_s, \mu(X_s) - \mu(Y_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2} \|\sigma(X_s) - \sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\big] - \chi_s \|X_s - Y_s\|_H^p}{\exp(\int_0^s \chi_u\, du)}\, ds.
\end{aligned}
$$

This completes the proof of Proposition 2.9. □

The next result, Theorem 2.10, further develops our theory of perturbations for SDEs. In particular, we apply a localization argument to the right-hand side of (28), then take expectations on both sides and thereafter apply Hölder's inequality.

THEOREM 2.10. *Assume Setting 1.5, let* $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U,H))$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, $\varepsilon \in [0,\infty]$, $p \in [2,\infty)$, *let* $\tau\colon \Omega \to [0,T]$ *be a stopping time, let* $X, Y\colon [0,T]\times\Omega \to \mathcal{O}$ *be adapted stochastic processes with c.s.p., let* $a\colon [0,T]\times\Omega\to H$, $b\colon [0,T]\times\Omega \to \mathrm{HS}(U,H)$, $\chi\colon [0,T]\times\Omega\to\mathbb{R}$ *be predictable stochastic processes and assume that for all* $t\in[0,T]$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T \|a_s\|_H + \|b_s\|^2_{\mathrm{HS}(U,H)} + \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, $Y_t = Y_0 + \int_0^t a_s\,ds + \int_0^t b_s\,dW_s$ *and*

$$\int_0^\tau \left[\frac{\langle X_s - Y_s, \mu(X_s)-\mu(Y_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|\sigma(X_s)-\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}}{\|X_s-Y_s\|^2_H} + \chi_s\right]^+ ds < \infty. \tag{29}$$

Then for all $r, q \in (0,\infty]$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ *it holds that*

$$\begin{aligned}
&\|X_\tau - Y_\tau\|_{L^r(\Omega;H)} \\
&\le \left\| \exp\left(\int_0^\tau \left[\frac{\langle X_s - Y_s, \mu(X_s)-\mu(Y_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|\sigma(X_s)-\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}}{\|X_s-Y_s\|^2_H} + \chi_s\right]^+ ds\right)\right\|_{L^q(\Omega;\mathbb{R})} \\
&\quad \cdot \Bigg[\|X_0 - Y_0\|_{L^p(\Omega;H)} + \Big\| p\,\|X-Y\|_H^{(p-2)} \\
&\quad \cdot \big[\langle X-Y, \mu(Y)-a\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|b-\sigma(Y)\|^2_{\mathrm{HS}(U,H)} - \chi\|X-Y\|_H^2\big]^+\Big\|^{1/p}_{L^1([\![0,\tau]\!];\mathbb{R})}\Bigg].
\end{aligned}\tag{30}$$

PROOF. Throughout this proof let $\hat{\chi}\colon [0,T]\times\Omega\to[0,\infty)$ be a predictable stochastic process which satisfies for all $t\in[0,T]$ that

$$\hat{\chi}_t = p\,\mathbb{1}_{\{t\le\tau\}}\left[\frac{\langle X_t - Y_t, \mu(X_t)-\mu(Y_t)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|\sigma(X_t)-\sigma(Y_t)\|^2_{\mathrm{HS}(U,H)}}{\|X_t-Y_t\|^2_H} + \chi_t\right]^+. \tag{31}$$

Note that Proposition 2.9, the definition of $\hat{\chi}$, a localization of the involved stochastic integral, and Fatou's lemma prove that

$$\begin{aligned}
&\mathbb{E}\left[\frac{\|X_\tau - Y_\tau\|_H^p}{\exp(\int_0^\tau \hat{\chi}_s\,ds)}\right] \le \mathbb{E}\big[\|X_0-Y_0\|_H^p\big] \\
&\quad + \mathbb{E}\left[\int_0^\tau \frac{p\,\|X_s-Y_s\|_H^{(p-2)}\big[\langle X_s-Y_s, \mu(Y_s)-a_s\rangle_H + \frac{(p-1)(1+1/\varepsilon)}{2}\|b_s-\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)} - \chi_s\|X_s-Y_s\|_H^2\big]^+}{\exp(\int_0^s \hat{\chi}_u\,du)}\,ds\right].
\end{aligned}$$

This, the fact that $\hat{\chi}\ge 0$, and Hölder's inequality hence prove that for all $q\in(0,\infty]$, $r\in(0,p]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ it holds that

$$\begin{aligned}
\|X_\tau - Y_\tau\|^p_{L^r(\Omega;H)} &\le \left\|\exp\left(\frac{1}{p}\int_0^\tau \hat{\chi}_s\,ds\right)\right\|^p_{L^q(\Omega;\mathbb{R})} \left\|\frac{\|X_\tau - Y_\tau\|_H}{\exp(\frac{1}{p}\int_0^\tau \hat{\chi}_s\,ds)}\right\|^p_{L^p(\Omega;\mathbb{R})} \\
&\le \left\|\exp\left(\frac{1}{p}\int_0^\tau \hat{\chi}_s\,ds\right)\right\|^p_{L^q(\Omega;\mathbb{R})} \mathbb{E}\Bigg[\|X_0-Y_0\|_H^p + \int_0^\tau p\,\|X_s-Y_s\|_H^{(p-2)} \\
&\quad \cdot \Big[\langle X_s-Y_s, \mu(Y_s)-a_s\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|b_s-\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)} - \chi_s\|X_s-Y_s\|_H^2\Big]^+ ds\Bigg].
\end{aligned}$$

This implies (30). The proof of Theorem 2.10 is thus complete. □

Corollary 2.11 uses Theorem 2.10 to study the difference of solutions processes of two semilinear SPDEs with possibly different coefficient functions.

COROLLARY 2.11. *Assume Setting 1.5, let* $A\colon D(A)\subseteq H\to H$ *be a densely defined linear operator with* $\mathcal{O}\subseteq D(A)$, *let* $F_1,F_2\in\mathcal{L}^0(\mathcal{O};H)$, $B_1,B_2\in\mathcal{L}^0(\mathcal{O};\mathrm{HS}(U,H))$, $\varepsilon\in[0,\infty]$, $p\in[2,\infty)$, *let* $X^1,X^2\colon[0,T]\times\Omega\to\mathcal{O}$, $\hat{X}\colon[0,T]\times\Omega\to\mathcal{O}$, $\chi\colon[0,T]\times\Omega\to\mathbb{R}$ *be predictable stochastic processes and assume that for all* $t\in[0,T]$, $(i,j)\in(\{1,2\}^2\backslash\{(1,2)\})$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T\|AX_s^j\|_H+\|A\hat{X}_s\|_H+\|F_i(X_s^j)\|_H+\|B_i(X_s^j)\|^2_{\mathrm{HS}(U,H)}+\|F_2(\hat{X}_s)\|_H+\|B_2(\hat{X}_s)\|^2_{\mathrm{HS}(U,H)}\,ds<\infty$, $X_t^i=X_0^i+\int_0^t AX_s^i+F_i(X_s^i)\,ds+\int_0^t B_i(X_s^i)\,dW_s$, $\hat{X}_t=X_0^2+\int_0^t A\hat{X}_s+F_2(X_s^1)\,ds+\int_0^t B_2(X_s^1)\,dW_s$ *and*

$$\int_0^T\left[\frac{\langle X_s^2-\hat{X}_s,A[X_s^2-\hat{X}_s]+F_2(X_s^2)-F_2(\hat{X}_s)\rangle_H+\frac{(p-1)(1+\varepsilon)}{2}\|B_2(X_s^2)-B_2(\hat{X}_s)\|^2_{\mathrm{HS}(U,H)}}{\|X_s^2-\hat{X}_s\|_H^2}+\chi_s\right]^+ds<\infty. \tag{32}$$

Then for all $t\in[0,T]$, $r,q\in(0,\infty]$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ *it holds that*

$$\begin{aligned}
&\|X_t^1-X_t^2\|_{L^r(\Omega;H)}\leq\|X_t^1-\hat{X}_t\|_{L^r(\Omega;H)}\\
&\quad+\Big\|p\,\|X^2-\hat{X}\|_H^{(p-2)}\big[\langle X^2-\hat{X},F_2(\hat{X})-F_2(X^1)\rangle_H\\
&\qquad+\tfrac{(p-1)(1+1/\varepsilon)}{2}\|B_2(X^1)-B_2(\hat{X})\|^2_{\mathrm{HS}(U,H)}-\chi\,\|X^2-\hat{X}\|_H^2\big]^+\Big\|^{1/p}_{L^1([0,t]\times\Omega;\mathbb{R})}\\
&\quad\cdot\left\|\exp\left(\int_0^t\left[\frac{\langle X_s^2-\hat{X}_s,A[X_s^2-\hat{X}_s]+F_2(X_s^2)-F_2(\hat{X}_s)\rangle_H}{\|X_s^2-\hat{X}_s\|_H^2}\right.\right.\right.\\
&\qquad\left.\left.\left.+\frac{\frac{(p-1)(1+\varepsilon)}{2}\|B_2(X_s^2)-B_2(\hat{X}_s)\|^2_{\mathrm{HS}(U,H)}}{\|X_s^2-\hat{X}_s\|_H^2}+\chi_s\right]^+ds\right)\right\|_{L^q(\Omega;\mathbb{R})}.
\end{aligned}\tag{33}$$

Corollary 2.11 follows immediately from the triangle inequality and an application of Theorem 2.10 to the stochastic process $(X_t^2)_{t\in[0,T]}$ with the perturbation process $(\hat{X}_t)_{t\in[0,T]}$. In a number of situations it is convenient to further estimate the right-hand side of (30) in Theorem 2.10 in an appropriate way. This is the subject of the next corollary of Theorem 2.10.

COROLLARY 2.12. *Assume Setting 1.5, let* $\sigma\in\mathcal{L}^0(\mathcal{O};\mathrm{HS}(U,H))$, $\mu\in\mathcal{L}^0(\mathcal{O};H)$, $\varepsilon\in[0,\infty]$, $p\in[2,\infty)$, *let* $\tau\colon\Omega\to[0,T]$ *be a stopping time, let* $X,Y\colon[0,T]\times\Omega\to\mathcal{O}$ *be adapted stochastic processes with c.s.p., let* $a\colon[0,T]\times\Omega\to H$, $b\colon[0,T]\times\Omega\to\mathrm{HS}(U,H)$ *be predictable stochastic processes, and assume that for all* $t\in[0,T]$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T\|a_s\|_H+\|b_s\|^2_{\mathrm{HS}(U,H)}+\|\mu(X_s)\|_H+\|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)}+\|\mu(Y_s)\|_H+\|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\,ds<\infty$, $X_t=X_0+\int_0^t\mu(X_s)\,ds+\int_0^t\sigma(X_s)\,dW_s$, $Y_t=Y_0+\int_0^t a_s\,ds+\int_0^t b_s\,dW_s$, *and*

$$\int_0^\tau\left[\frac{\langle X_s-Y_s,\mu(X_s)-\mu(Y_s)\rangle_H+\frac{(p-1)(1+\varepsilon)}{2}\|\sigma(X_s)-\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}}{\|X_s-Y_s\|_H^2}\right]^+ds<\infty. \tag{34}$$

Then for all $\delta,\rho,r\in(0,\infty)$, $q\in(0,\infty]$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ *it holds that*

$$\begin{aligned}
&\|X_\tau-Y_\tau\|_{L^r(\Omega;H)}\\
&\leq\left\|\exp\left(\int_0^\tau\left[\frac{\langle X_s-Y_s,\mu(X_s)-\mu(Y_s)\rangle_H+\frac{(p-1)(1+\varepsilon)}{2}\|\sigma(X_s)-\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}}{\|X_s-Y_s\|_H^2}+\frac{(1-\frac{1}{p})}{\delta}+\frac{(\frac{1}{2}-\frac{1}{p})}{\rho}\right]^+ds\right)\right\|_{L^q(\Omega;\mathbb{R})}
\end{aligned}$$

$$\cdot \Bigg[\|X_0 - Y_0\|_{L^p(\Omega;H)} + \delta^{(1-\frac{1}{p})} \|a - \mu(Y)\|_{L^p(⟦0,\tau⟧;H)} + \rho^{(\frac{1}{2}-\frac{1}{p})} \sqrt{(p-1)(1+1/\varepsilon)}\, \|b - \sigma(Y)\|_{L^p(⟦0,\tau⟧;\mathrm{HS}(U,H))} \Bigg].$$

Corollary 2.12 follows immediately from an application of Theorem 2.10 with $\chi_t = \frac{1}{\delta}(1 - \frac{1}{p}) + \frac{1}{\rho}(\frac{1}{2} - \frac{1}{p})$, $t \in [0, T]$ and an application of Young's inequality.

3. Applications of the perturbation theory for SDEs.

3.1. *Numerical approximations of SODEs.* This subsection uses Corollary 2.12 to establish strong convergence rates for the stopped-tamed Euler–Maruyama method in [38] (see (6) in [38]). To accomplish this, we employ the elementary result in Lemma 3.1 below. The proof of Lemma 3.1 is straightforward.

LEMMA 3.1. *Let $d \in \mathbb{N}$ and let $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ be the function which satisfies for all $v \in \mathbb{R}^d$ that $\psi(v) = \frac{v}{1+\|v\|_{\mathbb{R}^d}^2}$. Then for all $v \in \mathbb{R}^d$ it holds that $\|\psi'(v)\|_{L(\mathbb{R}^d)} \leq 3$, $\|\psi'(v) - I_{\mathbb{R}^d}\|_{L(\mathbb{R}^d)} \leq 3[1 \wedge \|v\|_{\mathbb{R}^d}]^2$ and $\sup_{u\in\mathbb{R}^d,\, \|u\|_{\mathbb{R}^d}\leq 1} \|\psi''(v)(u,u)\|_{\mathbb{R}^d} \leq 14[1 \wedge \|v\|_{\mathbb{R}^d}]$.*

We now use Lemma 3.1 together with Corollary 2.12 to prove a suitable strong convergence rate estimate (see (36) below) for the stopped-tamed Euler–Maruyama approximations in [38].

LEMMA 3.2. *Consider the notation in Section 1.1. Let $d, m, n \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_n = T < \infty$, $\mathcal{O} \in \mathcal{B}(\mathbb{R}^d)$, $\phi \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R})$, $\mu \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in \mathcal{L}^0(\mathbb{R}^d;\mathbb{R}^{d\times m})$ satisfy for all $x, y \in \mathbb{R}^d$ that $\max(\|\mu(x)-\mu(y)\|_{\mathbb{R}^d}, \|\sigma(x)-\sigma(y)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}) \leq (\phi(x)+\phi(y))\|x-y\|_{\mathbb{R}^d}$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space which fulfills the usual conditions, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motion, let $X, Y\colon [0,T]\times\Omega \to \mathbb{R}^d$ be adapted stochastic processes with c.s.p., assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^T \|\mu(X_s)\|_{\mathbb{R}^d} + \|\sigma(X_s)\|_{\mathbb{R}^{d\times m}}^2\, ds < \infty$ and $X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s$, assume for all $k \in \{0, 1, \ldots, n-1\}$, $t \in [t_k, t_{k+1}]$ that $Y_0 = X_0$ and*

$$Y_t = Y_{t_k} + \mathbb{1}_{\{Y_{t_k}\in\mathcal{O}\}} \left[\frac{\mu(Y_{t_k})(t-t_k)+\sigma(Y_{t_k})(W_t-W_{t_k})}{1+\|\mu(Y_{t_k})(t-t_k)+\sigma(Y_{t_k})(W_t-W_{t_k})\|_{\mathbb{R}^d}^2} \right] \tag{35}$$

and let $\tau\colon \Omega \to [0,T]$ be given by $\tau = \inf(\{T\} \cup \{t \in \{t_0, t_1, \ldots, t_n\}\colon Y_t \notin \mathcal{O}\})$. Then for all stopping times $\nu\colon \Omega \to [0,T]$ and all $\varepsilon, r \in (0,\infty)$, $p \in [2,\infty)$, $q, u, v \in (0,\infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $\frac{1}{u} + \frac{1}{v} = \frac{1}{p}$ it holds that

$$\begin{aligned} &\|X_{\nu\wedge\tau} - Y_{\nu\wedge\tau}\|_{L^r(\Omega;\mathbb{R}^d)} \leq 30\, p \left(1 + \tfrac{1}{\varepsilon}\right) e^T \left[\max_{0\leq k\leq n-1} |t_{k+1} - t_k| \right]^{\frac{1}{2}} \\ &\cdot \left\| \exp\left(\int_0^{\nu\wedge\tau} \left[\frac{\langle X_s - Y_s, \mu(X_s)-\mu(Y_s)\rangle_{\mathbb{R}^d} + \frac{(p-1)(1+\varepsilon)}{2} \|\sigma(X_s)-\sigma(Y_s)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}^2}{\|X_s - Y_s\|_{\mathbb{R}^d}^2} \right]^+ ds \right) \right\|_{L^q(\Omega;\mathbb{R})} \\ &\cdot \left[\sup_{s\in[0,T]} \big\| \|\mu(Y_s)\|_{\mathbb{R}^d} + [1 \vee \|\sigma(Y_s)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}]^2 + |\phi(Y_s)| \big\|_{L^u(\Omega;\mathbb{R})} \right] \\ &\cdot \sup_{s\in[0,T]} \max\Big(1, \sqrt{T}\, \|\mu(Y_s)\|_{L^v(\Omega;\mathbb{R}^d)} + v\, \|\sigma(Y_s)\|_{L^v(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))}\Big). \end{aligned} \tag{36}$$

PROOF. Throughout this proof let $\nu\colon \Omega \to [0,T]$ be a stopping time, let $e_1^{(m)} = (1,0,\dots,0),\dots,e^{(m)} = (0,\dots,0,1) \in \mathbb{R}^m$ be the Euclidean orthonormal basis of the $\mathbb{R}^m$, let $\psi\colon \mathbb{R}^d \to \mathbb{R}^d$ be the function which satisfies for all $v \in \mathbb{R}^d$ that $\psi(v) = v\,[1+\|v\|^2_{\mathbb{R}^d}]^{-1}$, and let $Z, a\colon [0,T]\times\Omega \to \mathbb{R}^d$, $b\colon [0,T]\times\Omega \to \mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)$ satisfy for all $k \in \{0,1,\dots,n-1\}$, $t \in [t_k, t_{k+1})$ that

$$Z_t = \mu(Y_{t_k})\,(t-t_k) + \sigma(Y_{t_k})\,(W_t - W_{t_k}), \tag{37}$$

$$a_t = \psi'(Z_t)\,\mu(Y_{t_k}) + \tfrac{1}{2}\sum_{j=1}^{m} \psi''(Z_t)\big(\sigma(Y_{t_k})\,e_j^{(m)}, \sigma(Y_{t_k})\,e_j^{(m)}\big), \tag{38}$$

and $b_t = \psi'(Z_t)\,\sigma(Y_{t_k})$. Itô's formula then proves that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $Y_t = Y_{t\wedge\tau} = X_0 + \int_0^t \mathbb{1}_{\{s<\tau\}}\,a_s\,ds + \int_0^t \mathbb{1}_{\{s<\tau\}}\,b_s\,dW_s$. This, Da Prato and Zabczyk [15], Lemma 7.7 and Lemma 3.1 imply that for all $p,u \in [2,\infty)$, $v \in (2,\infty]$ with $\frac{1}{u}+\frac{1}{v}=\frac{1}{p}$ it holds that

$$\begin{aligned}
&\|a-\mu(Y)\|_{L^p([\![0,\tau]\!];\mathbb{R}^d)}\\
&\le 14\,T^{\frac{1}{p}}\Big[\sup_{s\in[0,T]}\big\|\|\mu(Y_s)\|_{\mathbb{R}^d} + \|\sigma(Y_s)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + |\phi(Y_s)|\big\|_{L^v(\Omega;\mathbb{R})}\Big]\\
&\quad\cdot\sup_{s\in[0,T]}\Big[\sqrt{T}\,\|\mu(Y_s)\|_{L^u(\Omega;\mathbb{R}^d)} + \frac{\sqrt{u\,(u-1)}\,\|\sigma(Y_s)\|_{L^u(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))}}{\sqrt{2}}\Big]\Big[\max_{0\le k\le n-1}|t_{k+1}-t_k|\Big]^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\|b-\sigma(Y)\|_{L^p([\![0,\tau]\!];\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))}\\
&\le 6\,T^{\frac{1}{p}}\Big[\sup_{s\in[0,T]}\big\|\|\sigma(Y_s)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)} + |\phi(Y_s)|\big\|_{L^v(\Omega;\mathbb{R})}\Big]\\
&\quad\cdot\sup_{s\in[0,T]}\Big[\sqrt{T}\,\|\mu(Y_s)\|_{L^u(\Omega;\mathbb{R}^d)} + \frac{\sqrt{u\,(u-1)}\,\|\sigma(Y_s)\|_{L^u(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))}}{\sqrt{2}}\Big]\Big[\max_{0\le k\le n-1}|t_{k+1}-t_k|\Big]^{\frac{1}{2}}.
\end{aligned}$$

Corollary 2.12 hence implies that for all $\varepsilon, r \in (0,\infty)$, $p \in [2,\infty)$, $q,u,v \in (0,\infty]$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ and $\frac{1}{u}+\frac{1}{v}=\frac{1}{p}$ it holds that

$$\begin{aligned}
&\|X_{\nu\wedge\tau} - Y_{\nu\wedge\tau}\|_{L^r(\Omega;\mathbb{R}^d)} \le 6\,T^{\frac{1}{p}}\Big[\frac{7\cdot 2^{(1-\frac{1}{p})}}{3} + \sqrt{(p-1)\,(1+1/\varepsilon)}\Big]\Big[\max_{0\le k\le n-1}|t_{k+1}-t_k|\Big]^{\frac{1}{2}}\\
&\cdot\Big\|\exp\Big(\int_0^{\nu\wedge\tau}\Big[\frac{\langle X_s-Y_s,\mu(X_s)-\mu(Y_s)\rangle_{\mathbb{R}^d} + \frac{(p-1)\,(1+\varepsilon)}{2}\|\sigma(X_s)-\sigma(Y_s)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|X_s-Y_s\|^2_{\mathbb{R}^d}} + 1 - \frac{3}{2p}\Big]^+ ds\Big)\Big\|_{L^q(\Omega;\mathbb{R})}\\
&\cdot\Big[\sup_{s\in[0,T]}\big\|\|\mu(Y_s)\|_{\mathbb{R}^d} + [1\vee\|\sigma(Y_s)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}]^2 + |\phi(Y_s)|\big\|_{L^v(\Omega;\mathbb{R})}\Big]\\
&\cdot\sup_{s\in[0,T]}\Big[\sqrt{T}\,\|\mu(Y_s)\|_{L^u(\Omega;\mathbb{R}^d)} + \frac{\sqrt{u\,(u-1)}\,\|\sigma(Y_s)\|_{L^u(\Omega;\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d))}}{\sqrt{2}}\Big].
\end{aligned}$$

This yields (36). The proof of Lemma 3.2 is thus complete. □

Lemma 3.2 is only of use if the right-hand side of (36) is finite. The next result (Proposition 3.3), in particular, provides sufficient conditions to ensure that the right-hande side of (36) is finite and thereby establishes strong convergence rates for the stopped-tamed Euler–Maruyama approximations in [38].

PROPOSITION 3.3. *Consider the notation in Section 1.1, let* $d, m \in \mathbb{N}$, $r, \varepsilon, c, T \in (0, \infty)$, $q_0, q_1 \in (0, \infty]$, $\alpha \in [0, \infty)$, $p \in [2, \infty)$, $U_0 \in \mathcal{C}^3_{\mathcal{D}}(\mathbb{R}^d, [0, \infty))$, $U_1 \in \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d, [0, \infty))$, $\mu \in \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^{d\times m})$ *satisfy for all* $x, y \in \mathbb{R}^d$ *that*

$$\begin{aligned}
&\|x\|_{\mathbb{R}^d}^{1/c} \leq c\,(1 + U_0(x)),\\
&(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U_0)(x)\|^2_{\mathbb{R}^m} + U_1(x) \leq \alpha\, U_0(x) + c,\\
&\langle x - y, \mu(x) - \mu(y)\rangle_{\mathbb{R}^d} + \tfrac{(p-1)\,(1+\varepsilon)}{2}\,\|\sigma(x) - \sigma(y)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\\
&\quad \leq \Big[c + \tfrac{U_0(x)+U_0(y)}{2q_0 T e^{\alpha T}} + \tfrac{U_1(x)+U_1(y)}{2q_1 e^{\alpha T}}\Big]\,\|x - y\|^2_{\mathbb{R}^d},
\end{aligned}$$

and $\frac{1}{p} + \frac{1}{q_0} + \frac{1}{q_1} = \frac{1}{r}$, *let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ *be a filtered probability space which fulfills the usual conditions, let* $W\colon [0, T] \times \Omega \to \mathbb{R}^m$ *be a standard* $(\mathbb{F}_t)_{t\in[0,T]}$*-Brownian motion, let* $X\colon [0, T] \times \Omega \to \mathbb{R}^d$ *and* $Y^\theta\colon [0, T] \times \Omega \to \mathbb{R}^d$, $\theta \in \mathcal{P}_T$, *be adapted stochastic processes with c.s.p., assume that* $\mathbb{E}\big[e^{U_0(X_0)}\big] < \infty$, *assume that for all* $t \in [0, T]$ *it holds* $\mathbb{P}$*-a.s. that* $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, *and assume for all* $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$, $k \in \{0, 1, \dots, n-1\}$, $t \in [t_k, t_{k+1}]$ *that* $Y^\theta_0 = X_0$ *and*

$$Y^\theta_t = Y^\theta_{t_k} + \mathbb{1}_{\left\{\|Y^\theta_{t_k}\|_{\mathbb{R}^d} < \exp(|\ln(\max_{0\leq i\leq n-1} t_{i+1}-t_i)|^{1/2})\right\}} \left[\frac{\mu(Y^\theta_{t_k})(t-t_k)+\sigma(Y^\theta_{t_k})(W_t-W_{t_k})}{1+\|\mu(Y^\theta_{t_k})(t-t_k)+\sigma(Y^\theta_{t_k})(W_t-W_{t_k})\|^2_{\mathbb{R}^d}}\right]. \tag{39}$$

Then there exists $C \in [0, \infty)$ *such that for all* $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ *it holds that*

$$\sup\nolimits_{t\in[0,T]} \|X_t - Y^\theta_t\|_{L^r(\Omega;\mathbb{R}^d)} \leq C\,\big[\max\nolimits_{k\in\{1,2,\dots,n\}} |t_k - t_{k-1}|\big]^{1/2}. \tag{40}$$

PROOF. Throughout this proof let $q \in (0, \infty]$ satisfy $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{q_1}$ and let $\tau_\theta\colon \Omega \to [0, T]$, $\theta \in \mathcal{P}_T$, be the functions which satisfy for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ that

$$\tau_\theta = \inf\!\left(\{T\} \cup \left\{t \in \{t_0, t_1, \dots, t_n\}\colon \|Y^\theta_t\|_{\mathbb{R}^d} \geq \exp\!\left(\left|\ln\!\left(\max_{i\in\{0,1,\dots,n-1\}} [t_{i+1} - t_i]\right)\right|^{1/2}\right)\right\}\right).$$

Note that the assumption that $\mu \in \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$ and the assumption that $\sigma \in \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^{d\times m})$ ensure that there exists $\hat{c} \in \big[1 + \|\mu(0)\|_{\mathbb{R}^d} + \|\sigma(0)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}, \infty\big)$ such that for all $x, y \in \mathbb{R}^d$ it holds that

$$\max\{\|\mu(x)-\mu(y)\|_{\mathbb{R}^d}, \|\sigma(x)-\sigma(y)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\} \leq \hat{c}\,\big(1+\|x\|^{\hat{c}}_{\mathbb{R}^d}+\|y\|^{\hat{c}}_{\mathbb{R}^d}\big)\,\|x-y\|_{\mathbb{R}^d}. \tag{41}$$

Next let $\phi\colon \mathbb{R}^d \to \mathbb{R}$ be the function which satisfies for all $x \in \mathbb{R}^d$ that $\phi(x) = 4\,|\hat{c}|^2\,[1 + \|x\|_{\mathbb{R}^d}]^{(2\hat{c}+2)}$. Note that for all $x, y \in \mathbb{R}^d$ it holds that $\max\{1, \|\mu(x)\|_{\mathbb{R}^d}, \|\sigma(x)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\} \leq \phi(x)$ and $\max\{\|\mu(x) - \mu(y)\|_{\mathbb{R}^d}, \|\sigma(x) - \sigma(y)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}\} \leq (\phi(x) + \phi(y))\,\|x - y\|_{\mathbb{R}^d}$. Corollary 2.4 in [13], Corollary 2.9 in [38], and the fact that for all $x \in \mathbb{R}^d$ it holds that $\frac{1}{c}\|x\|^{1/c}_{\mathbb{R}^d} \leq 1 + U_0(x)$ imply that there exist $C_1, C_2 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that

$$\begin{aligned}
&\left[\sup_{s\in[0,T]} \mathbb{E}\!\left[\exp\!\left(\frac{U_0(X_s)}{e^{\alpha s}} + \int_0^s \frac{U_1(X_u)}{e^{\alpha u}}\,du\right)\right]\right]\\
&\quad\cdot\left[\sup_{s\in[0,T]} \mathbb{E}\!\left[\exp\!\left(\frac{U_0(Y^\theta_s)}{e^{\alpha s}} + \int_0^{s\wedge\tau_\theta} \frac{U_1(Y^\theta_u)}{e^{\alpha u}}\,du\right)\right]\right] \leq C_1,
\end{aligned} \tag{42}$$

$$\sup_{t\in[0,T]} \|\phi(Y^\theta_t)\|_{L^{2p}(\Omega;\mathbb{R})} + \sup_{t\in[0,T]} \Big[\|X_t\|_{L^{2r}(\Omega;\mathbb{R}^d)} + \|Y^\theta_t\|_{L^{2r}(\Omega;\mathbb{R}^d)}\Big] \leq C_2. \tag{43}$$

Lemma 3.2 hence shows that for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_{t\wedge\tau_\theta} - Y^\theta_{t\wedge\tau_\theta}\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\le 360\, p^2 \left[1 + \tfrac{1}{\varepsilon}\right] e^{2T} (C_2)^2 \left[\max_{0\le k\le n-1} |t_{k+1} - t_k|\right]^{1/2} \\
&\quad\cdot \left\| \exp\left(\int_0^{\tau_\theta} \left[\frac{\langle X_s - Y^\theta_s, \mu(X_s) - \mu(Y^\theta_s)\rangle_{\mathbb{R}^d} + \frac{(p-1)(1+\varepsilon)}{2} \|\sigma(X_s) - \sigma(Y^\theta_s)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|X_s - Y^\theta_s\|^2_{\mathbb{R}^d}} \right]^+ ds \right) \right\|_{L^q(\Omega;\mathbb{R})}.
\end{aligned}
\tag{44}
$$

Moreover, note that the assumptions of Proposition 3.3, Hölder's inequality, Jensen's inequality, and nonnegativity of U_0 and U_1 imply that for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that

$$
\begin{aligned}
&\left\| \exp\left(\int_0^{\tau_\theta} \left[\frac{\langle X_s - Y^\theta_s, \mu(X_s) - \mu(Y^\theta_s)\rangle_{\mathbb{R}^d} + \frac{(p-1)(1+\varepsilon)}{2} \|\sigma(X_s) - \sigma(Y^\theta_s)\|^2_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^d)}}{\|X_s - Y^\theta_s\|^2_{\mathbb{R}^d}} \right]^+ ds \right) \right\|_{L^q(\Omega;\mathbb{R})} e^{-cT} \\
&\le \left\| \exp\left(\int_0^{\tau_\theta} \frac{U_0(X_s) + U_0(Y^\theta_s)}{2 q_0 T e^{\alpha T}} + \frac{U_1(X_s) + U_1(Y^\theta_s)}{2 q_1 e^{\alpha T}}\, ds \right) \right\|_{L^q(\Omega;\mathbb{R})} \\
&\le \sup_{s\in[0,T]} \left| \mathbb{E}\left[\exp\left(\frac{U_0(X_s)}{e^{\alpha s}} + \int_0^s \frac{U_1(X_u)}{e^{\alpha u}}\, du \right) \right] \right|^{\frac{1}{2q}} \sup_{s\in[0,T]} \left| \mathbb{E}\left[\exp\left(\frac{U_0(Y^\theta_s)}{e^{\alpha s}} + \int_0^{s\wedge\tau_\theta} \frac{U_1(Y^\theta_u)}{e^{\alpha u}}\, du \right) \right] \right|^{\frac{1}{2q}}.
\end{aligned}
$$

Combining this with (44) and (42) implies that there exists $C_3 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that

$$
\sup_{t\in[0,T]} \|X_{t\wedge\tau_\theta} - Y^\theta_{t\wedge\tau_\theta}\|_{L^r(\Omega;\mathbb{R}^d)} \le C_3 \left[\max_{0\le k\le n-1} |t_{k+1} - t_k|\right]^{1/2}. \tag{45}
$$

Hölder's inequality and (43) hence prove that for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - Y^\theta_t\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\le \|\mathbb{1}_{\{\tau_\theta < T\}}\|_{L^{2r}(\Omega;\mathbb{R})} \left[\sup_{t\in[0,T]} \left\| X_t - Y^\theta_t \right\|_{L^{2r}(\Omega;\mathbb{R}^d)} \right] + \sup_{t\in[0,T]} \|X_{t\wedge\tau_\theta} - Y^\theta_{t\wedge\tau_\theta}\|_{L^r(\Omega;\mathbb{R}^d)} \\
&\le C_2\, |\mathbb{P}[\tau_\theta < T]|^{\frac{1}{2r}} + C_3 \left[\max_{0\le k\le n-1} |t_{k+1} - t_k|\right]^{1/2}.
\end{aligned}
\tag{46}
$$

Next observe that Markov's inequality, the fact that for all $x \in \mathbb{R}^d$ it holds that $\frac{1}{c}\|x\|^{1/c}_{\mathbb{R}^d} \le 1 + U_0(x)$, nonnegativity of U_1, (42) and the fact that for all $x \in [0,\infty)$ it holds that $\frac{1}{4!}x^4 \le e^x$ show that for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that

$$
\begin{aligned}
\mathbb{P}[\tau_\theta < T] &\le \mathbb{P}\Big[\|Y^\theta_T\|_{\mathbb{R}^d} \ge \exp\Big(\big|\ln\big(\max_{i\in\{0,1,\dots,n-1\}} t_{i+1} - t_i\big)\big|^{1/2} \Big) \Big] \\
&\le \mathbb{P}\left[\frac{1 + U_0(Y^\theta_T)}{e^{\alpha T}} \ge \frac{1}{c\, e^{\alpha T}} \exp\left(\frac{|\ln(\max_{i\in\{0,1,\dots,n-1\}} t_{i+1} - t_i)|^{1/2}}{c} \right) \right] \\
&\le \mathbb{E}\left[\exp\left(\frac{1 + U_0(Y^\theta_T)}{e^{\alpha T}} \right) \right] \exp\left(\frac{-1}{c\, e^{\alpha T}} \exp\left(\frac{|\ln(\max_{i\in\{0,1,\dots,n-1\}} t_{i+1} - t_i)|^{1/2}}{c} \right) \right) \\
&\le C_1 \exp\left(\frac{1}{e^{\alpha T}} - \frac{|\ln(\max_{i\in\{0,1,\dots,n-1\}} t_{i+1} - t_i)|^2}{24\, c^5\, e^{\alpha T}} \right).
\end{aligned}
\tag{47}
$$

Therefore, we obtain that there exists $C_4 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that

$$|\mathbb{P}[\tau_\theta < T]|^{\frac{1}{2r}} \leq C_4 \left[\max_{k \in \{0,1,\dots,n-1\}} |t_{k+1} - t_k| \right]^{1/2}. \tag{48}$$

Combining this with (46) completes the proof of Proposition 3.3. □

Proposition 3.3 establishes under suitable assumptions strong convergence rates for the stopped-tamed Euler–Maruyama approximations in [38] in the case of SDEs with possibly nonglobally Lipschitz continuous drift and possibly nonglobally Lipschitz continuous diffusion coefficient functions. A number of SDEs from the literature have a globally Lipschitz continuous diffusion coefficient. This special case of Proposition 3.3 is the subject of the statement of Corollary 3.4 below. Corollary 3.4 follows immediately from Proposition 3.3.

COROLLARY 3.4. *Let $d, m \in \mathbb{N}$, let $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be globally Lipschitz continuous, let $c, T \in (0,\infty)$, $U_0 \in \mathcal{C}^3_{\mathcal{D}}(\mathbb{R}^d, [0,\infty))$, $U_1 \in \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d, [0,\infty))$, $\mu \in \mathcal{C}^1_{\mathcal{P}}(\mathbb{R}^d, \mathbb{R}^d)$ satisfy for all $\varepsilon \in (0,\infty)$ that*

$$\lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^d} \left[(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2} \|\sigma(x)^*(\nabla U_0)(x)\|^2_{\mathbb{R}^m} + U_1(x) - \eta\, U_0(x) \right] < \infty, \tag{49}$$

$$\sup_{x,y\in\mathbb{R}^d,\, x\neq y} \left[\frac{\langle x-y, \mu(x)-\mu(y)\rangle_{\mathbb{R}^d}}{\|x-y\|^2_{\mathbb{R}^d}} - \varepsilon \big(U_0(x) + U_0(y) + U_1(x) + U_1(y)\big) \right] < \infty \tag{50}$$

and $\sup_{x\in\mathbb{R}^d} \big[\|x\|^{1/c}_{\mathbb{R}^d} - c\, U_0(x) \big] < \infty$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space which fulfills the usual conditions, let $W : [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motion, let $X : [0,T]\times\Omega \to \mathbb{R}^d$ and $Y^\theta : [0,T]\times\Omega \to \mathbb{R}^d$, $\theta \in \mathcal{P}_T$, be adapted stochastic processes with c.s.p., assume that $\mathbb{E}\big[e^{U_0(X_0)}\big] < \infty$, assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, and assume for all $n \in \mathbb{N}$, $\theta = (t_0, \dots, t_n) \in \mathcal{P}_T$, $k \in \{0, 1, \dots, n-1\}$, $t \in [t_k, t_{k+1}]$ that $Y^\theta_0 = X_0$ and

$$Y^\theta_t = Y^\theta_{t_k} + \mathbb{1}_{\left\{ \|Y^\theta_{t_k}\|_{\mathbb{R}^d} < \exp(|\ln(\max_{0\leq i\leq n-1} t_{i+1}-t_i)|^{1/2}) \right\}} \left[\frac{\mu(Y^\theta_{t_k})(t-t_k) + \sigma(Y^\theta_{t_k})(W_t - W_{t_k})}{1 + \|\mu(Y^\theta_{t_k})(t-t_k) + \sigma(Y^\theta_{t_k})(W_t - W_{t_k})\|^2_{\mathbb{R}^d}} \right]. \tag{51}$$

Then there exist $C_r \in \mathbb{R}$, $r \in (0,\infty)$, such that for all $r \in (0,\infty)$, $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that $\sup_{t\in[0,T]} \|X_t - Y^\theta_t\|_{L^r(\Omega;\mathbb{R}^d)} \leq C_r \big[\max_{k\in\{1,2,\dots,n\}} |t_k - t_{k-1}|\big]^{1/2}$.

We now apply Corollary 3.4 and Proposition 3.3, respectively, to a selection of example SODEs with nonglobally monotone coefficients. In each of these example SODEs, the particular choice of the functions of U_0 and U_1 in Corollary 3.4 and the estimates associated with them are particularly inspired from the article Cox, utzenthaler and Jentzen [13] in which regularity with respect to the initial value for these example SODEs has been analyzed. The following common setting is used in our investigations of the example SODEs.

3.1.1. *Setting.* Throughout Section 3.1 the following setting is frequently used.

SETTING 3.5. Let $d, m \in \mathbb{N}$, $T \in (0,\infty)$, $\mu \in C(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in C(\mathbb{R}^d, \mathbb{R}^{d\times m})$, $x_0 \in \mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space which fulfills the usual conditions, let $W : [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motion, let $X : [0,T]\times\Omega \to \mathbb{R}^d$ and $Y^\theta : [0,T]\times\Omega \to \mathbb{R}^d$, $\theta \in \mathcal{P}_T$, be adapted stochastic processes with c.s.p., assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $X_t = x_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, and assume for all $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$, $k \in \{0, 1, \dots, n-1\}$, $t \in [t_k, t_{k+1}]$ that $Y^\theta_0 = X_0$ and

$$Y^\theta_t = Y^\theta_{t_k} + \mathbb{1}_{\left\{ \|Y^\theta_{t_k}\|_{\mathbb{R}^d} < \exp(|\ln(\max_{1\leq i\leq n} t_i - t_{i-1})|^{1/2}) \right\}} \left[\frac{\mu(Y^\theta_{t_k})(t-t_k) + \sigma(Y^\theta_{t_k})(W_t - W_{t_k})}{1 + \|\mu(Y^\theta_{t_k})(t-t_k) + \sigma(Y^\theta_{t_k})(W_t - W_{t_k})\|^2_{\mathbb{R}^d}} \right]. \tag{52}$$

3.1.2. *Stochastic Lorenz equation with bounded noise*. In this subsection, assume Setting 3.5, let $\alpha_1, \alpha_2, \alpha_3 \in [0, \infty)$, assume that $d = m = 3$, assume that σ is globally bounded and globally Lipschitz continuous[1], assume for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ that $\mu(x_1, x_2, x_3) = \big(\alpha_1 (x_2 - x_1), \alpha_2 x_1 - x_2 - x_1 x_3, x_1 x_2 - \alpha_3 x_3\big)$, and let $U_0 \in C(\mathbb{R}^3, [0, \infty))$ satisfy for all $x \in \mathbb{R}^3$ that $U_0(x) = \|x\|_{\mathbb{R}^3}^2$. Note that

$$
\begin{aligned}
&\lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^3} \Big[(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2} \|\sigma(x)^* (\nabla U_0)(x)\|_{\mathbb{R}^3}^2 - \eta\, U_0(x)\Big] \\
&\leq \lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^3} \Big[2 \langle x, \mu(x)\rangle_{\mathbb{R}^3} + \big[2 \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^3)}^2 - \eta\big] U_0(x) + \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^3)}^2\Big] < \infty.
\end{aligned} \tag{53}
$$

This proves that (49) is fulfilled. Moreover, note that for all $\varepsilon \in (0, \infty)$ it holds that

$$
\begin{aligned}
&\sup_{x,y\in\mathbb{R}^3,\, x\neq y} \left[\frac{\langle x-y, \mu(x)-\mu(y)\rangle_{\mathbb{R}^3}}{\|x-y\|_{\mathbb{R}^3}^2} - \varepsilon \Big(\|x\|_{\mathbb{R}^3}^2 + \|y\|_{\mathbb{R}^3}^2\Big)\right] \\
&\leq \sup_{x,y\in\mathbb{R}^3,\, x\neq y} \left[\frac{\|\mu(x)-\mu(y)\|_{\mathbb{R}^3}}{\|x-y\|_{\mathbb{R}^3}} - \varepsilon \Big(\|x\|_{\mathbb{R}^3}^2 + \|y\|_{\mathbb{R}^3}^2\Big)\right] < \infty.
\end{aligned} \tag{54}
$$

This shows that (50) is satisfied. We can thus apply Corollary 3.4 to obtain that there exist $C_r \in \mathbb{R}$, $r \in (0, \infty)$, such that for all $r \in (0, \infty)$, $n \in \mathbb{N}$, $\theta = (t_0, t_1, \ldots, t_n) \in \mathcal{P}_T$ it holds that $\sup_{t\in[0,T]} \|X_t - Y_t^\theta\|_{L^r(\Omega;\mathbb{R}^d)} \leq C_r \big[\max_{k\in\{1,2,\ldots,n\}} |t_k - t_{k-1}|\big]^{1/2}$.

3.1.3. *Stochastic van der Pol oscillator.* In this subsection assume Setting 3.5, let $c, \alpha \in (0, \infty)$, $\gamma, \delta \in [0, \infty)$, let $g\colon \mathbb{R} \to \mathbb{R}^{1\times m}$ be a globally Lipschitz continuous function, assume for all for all $x \in \mathbb{R}$ that $\|g(x)^*\|_{\mathbb{R}^m}^2 \leq c\Big(1 + x^2\Big)$, assume that $d = 2$, assume for all $x = (x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}^m$ that $\mu(x) = (x_2, (\gamma - \alpha(x_1)^2) x_2 - \delta x_1)$ and $\sigma(x)u = (0, g(x_1)u)$, and let $\vartheta \in (0, \frac{\alpha}{2c})$, $U_0, U_1 \in C(\mathbb{R}^2, [0, \infty))$ satisfy for all $x = (x_1, x_2) \in \mathbb{R}^2$ that $U_0(x) = \frac{\vartheta}{2} \|x\|_{\mathbb{R}^2}^2$ and $U_1(x) = \vartheta\, [\alpha - 2c\vartheta]\, (x_1 x_2)^2$. Note that

$$
\begin{aligned}
&\lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^2} \Big[(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2} \|\sigma(x)^* (\nabla U_0)(x)\|_{\mathbb{R}^m}^2 + U_1(x) - \eta\, U_0(x)\Big] \\
&= \vartheta \lim_{\eta\to\infty} \sup_{\substack{x=\\(x_1,x_2)\\ \in\mathbb{R}^2}} \Bigg[(1-\delta)\, x_1 x_2 + \gamma\, (x_2)^2 - \alpha\, (x_1 x_2)^2 + \frac{\|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^m,\mathbb{R}^2)}^2}{2} \\
&\qquad + \tfrac{\vartheta}{2} \|\sigma(x)^* x\|_{\mathbb{R}^m}^2 + \tfrac{U_1(x)}{\vartheta} - \tfrac{\eta \|x\|_{\mathbb{R}^2}^2}{2} \Bigg] \\
&\leq \vartheta \lim_{\eta\to\infty} \sup_{\substack{x=\\(x_1,x_2)\\ \in\mathbb{R}^2}} \Bigg[\|g(x_1)^*\|_{\mathbb{R}^m}^2 + \Big(1 + \gamma + \delta - \tfrac{\eta}{2}\Big) \|x\|_{\mathbb{R}^2}^2 \\
&\qquad + 2\vartheta\, |x_2|^2\, \|g(x_1)^*\|_{\mathbb{R}^m}^2 + \tfrac{U_1(x)}{\vartheta} - \alpha\, (x_1 x_2)^2 \Bigg] \\
&\leq \vartheta \lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^2} \Big[c + \Big(1 + \gamma + \delta + c + 2\vartheta c - \tfrac{\eta}{2}\Big) \|x\|_{\mathbb{R}^2}^2\Big] < \infty.
\end{aligned} \tag{55}
$$

[1] Global boundedness and global Lipschitz continuity of σ is, for example, satisfied in the additive noise case in which there exists $\beta \in (0, \infty)$ such that for all $x \in \mathbb{R}^3$ it holds that $\sigma(x) = \sqrt{\beta} I_{\mathbb{R}^3} \in \mathbb{R}^{3\times 3}$ (see, e.g., Zhou and E [71]).

Moreover, note that Cox, Hutzenthaler and Jentzen [13], Subsection 4.2, ensures that for all $\varepsilon \in (0,\infty)$ it holds that

$$
\begin{aligned}
&\sup_{\substack{x=(x_1,x_2),\\ y=(y_1,y_2)\in\mathbb{R}^2,\\ x\neq y}} \left[\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle_{\mathbb{R}^2}}{\|x-y\|^2_{\mathbb{R}^2}} - \varepsilon\,(U_1(x)+U_1(y))\right] \\
&\leq \sup_{\substack{x=(x_1,x_2),\\ y=(y_1,y_2)\in\mathbb{R}^2,\\ x\neq y}} \left[\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle_{\mathbb{R}^2}}{\|x-y\|^2_{\mathbb{R}^2}} - \varepsilon\,[\alpha-2c\vartheta]\Big((x_1x_2)^2+(y_1y_2)^2\Big)\right] < \infty.
\end{aligned} \tag{56}
$$

Combining this with (55) shows that (49) and (50) are satisfied. We can thus apply Corollary 3.4 to obtain that there exist $C_r \in \mathbb{R}$, $r \in (0,\infty)$, such that for all $r \in (0,\infty)$, $n \in \mathbb{N}$, $\theta = (t_0, t_1, \ldots, t_n) \in \mathcal{P}_T$ it holds that $\sup_{t\in[0,T]} \|X_t - Y_t^\theta\|_{L^r(\Omega;\mathbb{R}^d)} \leq C_r\big[\max_{k\in\{1,2,\ldots,n\}} |t_k - t_{k-1}\big]^{1/2}$.

3.1.4. *Stochastic Duffing–van der Pol oscillator.* In this subsection assume Setting 3.5, let $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_3, c \in (0,\infty)$, let $g\colon \mathbb{R} \to \mathbb{R}^{1\times m}$ be a globally Lipschitz continuous function[2], assume for all $x \in \mathbb{R}$ that $\|g(x)^*\|^2_{\mathbb{R}^m} \leq c(1+x^2)$, assume for all $x = (x_1,x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}^m$ that $d = 2$, $\mu(x) = \big(x_2, \alpha_2 x_2 - \alpha_1 x_1 - \alpha_3 (x_1)^2 x_2 - (x_1)^3\big)$, and $\sigma(x)u = (0, g(x_1)u)$, and let $\vartheta \in (0, \frac{\alpha_3}{c})$, $U_0, U_1 \in C(\mathbb{R}^2,[0,\infty))$ satisfy for all $x = (x_1,x_2) \in \mathbb{R}^2$ that $U_0(x) = \frac{\vartheta}{2}\big[\frac{(x_1)^4}{2} + (x_2)^2\big]$ and $U_1(x) = \vartheta\,[\alpha_3 - c\vartheta]\,(x_1x_2)^2$. Note that

$$
\begin{aligned}
&\lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^2} \Big[(\mathcal{G}_{\mu,\sigma}U_0)(x) + \tfrac{1}{2}\,\|\sigma(x)^*(\nabla U_0)(x)\|^2_{\mathbb{R}^m} + U_1(x) - \eta\,U_0(x)\Big] \\
&= \vartheta \lim_{\eta\to\infty} \sup_{\substack{x=(x_1,x_2)\\ \in\mathbb{R}^2}} \left[\alpha_2\,(x_2)^2 - \alpha_1 x_1 x_2 - \alpha_3\,(x_1x_2)^2 + \tfrac{[1+\vartheta(x_2)^2]\,\|g(x_1)^*\|^2_{\mathbb{R}^m}}{2} + \tfrac{U_1(x)-\eta\,U_0(x)}{\vartheta}\right] \\
&\leq \vartheta \lim_{\eta\to\infty} \sup_{\substack{x=(x_1,x_2)\\ \in\mathbb{R}^2}} \left[[|\alpha_1|+|\alpha_2|]\,\|x\|^2_{\mathbb{R}^2} - \alpha_3\,(x_1x_2)^2 + \tfrac{c\,[1+\vartheta(x_2)^2][1+(x_1)^2]}{2} + \tfrac{U_1(x)-\eta\,U_0(x)}{\vartheta}\right] \\
&\leq \vartheta \lim_{\eta\to\infty} \sup_{\substack{x=(x_1,x_2)\\ \in\mathbb{R}^2}} \left[\tfrac{c}{2} + \Big[|\alpha_1|+|\alpha_2| + \tfrac{c(1+\vartheta)}{2}\Big]\,\|x\|^2_{\mathbb{R}^2} + [c\vartheta-\alpha_3]\,(x_1x_2)^2 + \tfrac{U_1(x)-\eta\,U_0(x)}{\vartheta}\right] \\
&= \vartheta \lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^2} \left[\tfrac{c}{2} + \Big[|\alpha_1|+|\alpha_2| + \tfrac{c(1+\vartheta)}{2}\Big]\,\|x\|^2_{\mathbb{R}^2} - \tfrac{\eta\,U_0(x)}{\vartheta}\right] < \infty.
\end{aligned} \tag{57}
$$

Moreover, note that for all $\varepsilon \in (0,\infty)$ it holds that

$$
\begin{aligned}
&\sup_{x,y\in\mathbb{R}^2,\,x\neq y} \left[\tfrac{\langle x-y,\mu(x)-\mu(y)\rangle_{\mathbb{R}^2}}{\|x-y\|^2_{\mathbb{R}^2}} - \varepsilon\,\big(U_0(x)+U_0(y)\big)\right] \\
&\leq \sup_{x,y\in\mathbb{R}^2,\,x\neq y} \left[\tfrac{\|\mu(x)-\mu(y)\|_{\mathbb{R}^2}}{\|x-y\|_{\mathbb{R}^2}} - \varepsilon\,\big(U_0(x)+U_0(y)\big)\right] < \infty.
\end{aligned} \tag{58}
$$

---

[2] A common choice for the natural number $m \in \mathbb{N}$ and the function $g\colon \mathbb{R} \to \mathbb{R}^{1\times m}$ in the stochastic Duffing–van der Pol oscillator is the choice where there exist $\beta_1, \beta_2 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $u = (u_1,u_2) \in \mathbb{R}^2$ it holds that $m = 2$ and $g(x)u = \beta_1 x u_1 + \beta_2 u_2$ (see, e.g., Schenk-Hoppé [66]).

Combining this with (57) proves that (49) and (50) are fulfilled. We can thus apply Corollary 3.4 to obtain that there exist $C_r \in \mathbb{R}$, $r \in (0,\infty)$, such that for all $r \in (0,\infty)$, $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that $\sup_{t\in[0,T]} \|X_t - Y_t^\theta\|_{L^r(\Omega;\mathbb{R}^d)} \le C_r \big[\max_{k\in\{1,2,\dots,n\}} |t_k - t_{k-1}|\big]^{1/2}$.

3.1.5. *Experimental psychology model*. In this subsection assume Setting 3.5, let $\alpha, \delta \in (0,\infty)$, $\beta \in \mathbb{R}$, assume for all $x = (x_1, x_2) \in \mathbb{R}^2$ that $d = 2$, $m = 1$, $\mu(x_1, x_2) = \big((x_2)^2(\delta + 4\alpha x_1) - \frac{1}{2}\beta^2 x_1, -x_1 x_2(\delta + 4\alpha x_1) - \frac{1}{2}\beta^2 x_2\big)$, and $\sigma(x_1, x_2) = (-\beta x_2, \beta x_1)$, and let $q \in [3,\infty)$, $U_0 \in C(\mathbb{R}^2, \mathbb{R})$ satisfy for all $x \in \mathbb{R}^2$ that $U_0(x) = \|x\|_{\mathbb{R}^2}^q$. Note that

$$\begin{aligned} &\lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^d} \Big[(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_0)(x)\|_{\mathbb{R}^d}^2 - \eta\, U_0(x)\Big] \\ &\le \lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^2} \Big[q\,\|x\|_{\mathbb{R}^2}^{(q-2)} \langle x, \mu(x)\rangle_{\mathbb{R}^2} + \tfrac{q(q-1)}{2}\|x\|_{\mathbb{R}^2}^{(q-2)} \|\sigma(x)\|_{\mathrm{HS}(\mathbb{R}^2)}^2 - \eta\,\|x\|_{\mathbb{R}^2}^q\Big] < \infty. \end{aligned} \tag{59}$$

Moreover, note that for all $\varepsilon \in (0,\infty)$ it holds that

$$\begin{aligned} &\sup_{x,y\in\mathbb{R}^2,\, x\neq y} \Big[\tfrac{\langle x-y, \mu(x)-\mu(y)\rangle_{\mathbb{R}^2}}{\|x-y\|_{\mathbb{R}^2}^2} - \varepsilon\big(U_0(x) + U_0(y)\big)\Big] \\ &\le \sup_{x,y\in\mathbb{R}^2,\, x\neq y} \Big[\tfrac{\|\mu(x)-\mu(y)\|_{\mathbb{R}^2}}{\|x-y\|_{\mathbb{R}^2}} - \varepsilon\big(\|x\|_{\mathbb{R}^2}^q + \|y\|_{\mathbb{R}^2}^q\big)\Big] < \infty. \end{aligned} \tag{60}$$

Combining this with (59) proves that (49) and (50) are fulfilled. We can thus apply Corollary 3.4 to obtain that there exist $C_r \in \mathbb{R}$, $r \in (0,\infty)$, such that for all $r \in (0,\infty)$, $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that $\sup_{t\in[0,T]} \|X_t - Y_t^\theta\|_{L^r(\Omega;\mathbb{R}^d)} \le C_r \big[\max_{k\in\{1,2,\dots,n\}} |t_k - t_{k-1}|\big]^{1/2}$.

3.1.6. *Brownian dynamics* (*Overdamped Langevin dynamics*). In this subsection assume Setting 3.5, let $c, \beta \in (0,\infty)$, $\theta \in [0, 2/\beta)$, $V \in \mathcal{C}_{\mathcal{D}}^3(\mathbb{R}^d, [0,\infty))$, assume for all $x \in \mathbb{R}^d$ that $d = m$, $\limsup_{r\searrow 0} \sup_{z\in\mathbb{R}^d} \frac{\|z\|_{\mathbb{R}^d}^r}{1+V(z)} < \infty$, $\mu(x) = -(\nabla V)(x)$, $\sigma(x) = \sqrt{\beta} I_{\mathbb{R}^d}$, and $(\triangle V)(x) \le c + c\,V(x) + \theta\,\|(\nabla V)(x)\|_{\mathbb{R}^d}^2$, assume for all $\varepsilon \in (0,\infty)$ that

$$\begin{aligned} &\sup_{\substack{x,y\in\mathbb{R}^d,\\ x\neq y}} \Big[\tfrac{\langle x-y, (\nabla V)(y)-(\nabla V)(x)\rangle_{\mathbb{R}^d}}{\|x-y\|_{\mathbb{R}^d}^2} \\ &\qquad - \varepsilon\Big(V(x) + V(y) + \|(\nabla V)(x)\|_{\mathbb{R}^d}^2 + \|(\nabla V)(y)\|_{\mathbb{R}^d}^2\Big)\Big] < \infty, \end{aligned} \tag{61}$$

and let $\vartheta \in (0, \frac{2}{\beta} - \theta)$, $U_0, U_1 \in C(\mathbb{R}^d, \mathbb{R})$ satisfy for all $x \in \mathbb{R}^d$ that $U_0(x) = \vartheta\,V(x)$ and $U_1(x) = \vartheta\,(1 - \frac{\beta}{2}(\theta + \vartheta))\,\|(\nabla V)(x)\|_{\mathbb{R}^d}^2$. Note that

$$\begin{aligned} &\lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^d} \Big[(\mathcal{G}_{\mu,\sigma} U_0)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_0)(x)\|_{\mathbb{R}^d}^2 + U_1(x) - \eta\, U_0(x)\Big] \\ &= \vartheta \lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^d} \Big[-\|(\nabla V)(x)\|_{\mathbb{R}^d}^2 + \tfrac{\beta}{2}(\triangle V)(x) + \tfrac{\vartheta\beta}{2}\|(\nabla V)(x)\|_{\mathbb{R}^d}^2 + \tfrac{U_1(x)}{\vartheta} - \eta\, V(x)\Big] \\ &\le \vartheta \lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^d} \Big[\tfrac{c\beta}{2} + \Big[\tfrac{(\theta+\vartheta)\beta}{2} - 1\Big]\|(\nabla V)(x)\|_{\mathbb{R}^d}^2 + \tfrac{U_1(x)}{\vartheta} + \Big[\tfrac{c\beta}{2} - \eta\Big] V(x)\Big] \\ &= \vartheta \lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^d} \Big[\tfrac{c\beta}{2} + \Big[\tfrac{c\beta}{2} - \eta\Big] V(x)\Big] < \infty. \end{aligned}$$

This and (61) ensure that (49) and (50) are fulfilled. Lemma 2.12 in [34] thus allows us to apply Corollary 3.4 to obtain that there exist $C_r \in \mathbb{R}$, $r \in (0,\infty)$, such that for all $r \in (0,\infty)$, $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that $\sup_{t\in[0,T]} \|X_t - Y_t^\theta\|_{L^r(\Omega;\mathbb{R}^d)} \leq C_r\big[\max_{k\in\{1,2,\dots,n\}} |t_k - t_{k-1}|\big]^{1/2}$.

REMARK 3.1 (Higher order strong convergence rates for SDEs with possibly nonglobally monotone coefficients). Corollary 3.4 applies both to SDEs with additive and nonadditive noise and establishes the strong convergence rate 1/2. We expect that, in the case of SDEs with additive noise (see, e.g., Sections 3.1.6 and 3.1.7) and possibly nonglobally monotone coefficients, an application of the perturbation theory in Section 2 (to be more specific, an application of Proposition 2.9) yields the strong convergence rate 1. Similarly, we expect that Proposition 2.9 can be used to establish higher order strong convergence rates for suitable higher order schemes in the case of SDEs with possibly nonglobally monotone coefficients.

3.1.7. *Langevin dynamics and stochastic Duffing oscillator.* In this subsection,[3] assume Setting 3.5, let $\gamma \in [0,\infty)$, $\beta \in (0,\infty)$, $V \in \mathcal{C}^3_{\mathcal{D}}(\mathbb{R}^m, [0,\infty))$, assume for all $x = (x_1, x_2) \in \mathbb{R}^{2m}$, $u \in \mathbb{R}^m$ that $\limsup_{r\searrow 0} \sup_{z\in\mathbb{R}^m} \frac{\|z\|^r_{\mathbb{R}^m}}{1+V(z)} < \infty$, $d = 2m$, $\mu(x) = (x_2, -(\nabla V)(x_1) - \gamma x_2)$, and $\sigma(x)u = (0, \sqrt{\beta}u)$, assume for all $\varepsilon \in (0,\infty)$ that

$$\sup_{x,y\in\mathbb{R}^m, x\neq y} \Big[\tfrac{\|(\nabla V)(x)-(\nabla V)(y)\|_{\mathbb{R}^m}}{\|x-y\|_{\mathbb{R}^m}} - \varepsilon\Big(\|x\|^2_{\mathbb{R}^m} + \|y\|^2_{\mathbb{R}^m} + V(x) + V(y)\Big)\Big] < \infty, \tag{62}$$

and let $\vartheta \in (0,\infty)$, $U_0 \in C(\mathbb{R}^{2m}, \mathbb{R})$ satisfy for all $x = (x_1, x_2) \in \mathbb{R}^{2m}$ that $U_0(x) = \frac{\vartheta}{2}\|x_1\|^2_{\mathbb{R}^m} + \vartheta\, V(x_1) + \frac{\vartheta}{2}\|x_2\|^2_{\mathbb{R}^m}$. Note that

$$\begin{aligned}
&\lim_{\eta\to\infty} \sup_{x\in\mathbb{R}^d} \Big[(\mathcal{G}_{\mu,\sigma}U_0)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_0)(x)\|^2_{\mathbb{R}^m} - \eta\, U_0(x)\Big] \\
&= \lim_{\eta\to\infty} \sup_{x=(x_1,x_2)\in\mathbb{R}^{2m}} \Big[\vartheta\,\langle x_1, x_2\rangle_{\mathbb{R}^m} - \vartheta\gamma\,\|x_2\|^2_{\mathbb{R}^m} + \tfrac{\vartheta\beta m}{2} + \tfrac{\beta\vartheta^2}{2}\|x_2\|^2_{\mathbb{R}^m} - \eta\, U_0(x)\Big] \qquad (63)\\
&\leq \vartheta \lim_{\eta\to\infty} \sup_{x_1,x_2\in\mathbb{R}^m} \Big[\Big[\tfrac{1}{2} - \tfrac{\eta}{2}\Big]\|x_1\|^2_{\mathbb{R}^m} + \Big[\tfrac{1}{2} + \tfrac{\beta\vartheta}{2} - \gamma - \tfrac{\eta}{2}\Big]\|x_2\|^2_{\mathbb{R}^m} + \tfrac{\beta m}{2}\Big] < \infty
\end{aligned}$$

(cf. Cox, Hutzenthaler and Jentzen [13], Section 4.5). Inequalities (62) and (63) show that (50) and (49) are fulfilled. We can thus apply Corollary 3.4 to obtain that there exist $C_r \in \mathbb{R}$, $r \in (0,\infty)$, such that for all $r \in (0,\infty)$, $n \in \mathbb{N}$, $\theta = (t_0, t_1, \dots, t_n) \in \mathcal{P}_T$ it holds that $\sup_{t\in[0,T]} \|X_t - Y_t^\theta\|_{L^r(\Omega;\mathbb{R}^d)} \leq C_r\big[\max_{k\in\{1,2,\dots,n\}} |t_k - t_{k-1}|\big]^{1/2}$ (cf. also Remark 3.1 above).

3.2. *Galerkin approximations of stochastic partial differential equations* (*SPDEs*). The next result, Corollary 3.6, is useful for the estimation of approximation errors of Galerkin approximations of solutions of SPDEs.

COROLLARY 3.6. *Assume Setting* 1.5, *let* $\varepsilon \in [0,\infty]$, $p \in [2,\infty)$, $P \in L(H)$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U,H))$ *satisfy* $P(\mathcal{O}) \subseteq \mathcal{O}$, *let* $X, Y\colon [0,T]\times\Omega \to \mathcal{O}$, $\chi\colon [0,T]\times\Omega \to \mathbb{R}$ *be predictable stochastic processes*, *assume that for all* $t \in [0,T]$ *it holds* $\mathbb{P}$-*a.s. that* $\int_0^T \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(PX_s)\|_H + \|\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H +$

[3]These assumptions are, for example, satisfied in the case of *stochastic Duffing oscillator with additive noise* (see, e.g., (9) in Datta and Bhattacharjee [16]) in which there exists $\lambda \in (0,\infty)$ such that, for all $x \in \mathbb{R}$, it holds that $m = 1$ and $V(x) = \frac{1}{2}x^2 + \frac{\lambda}{4}x^4$.

$\|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, $Y_t = PX_0 + \int_0^t P\mu(Y_s)\,ds + \int_0^t P\sigma(Y_s)\,dW_s$, *and*

$$\int_0^T \Big[\frac{\langle Y_s - PX_s, P\mu(Y_s) - P\mu(PX_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|P\sigma(Y_s) - P\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)}}{\|Y_s - PX_s\|^2_H} + \chi_s\Big]^+ ds < \infty. \tag{64}$$

Then for all $r, q \in (0,\infty]$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *it holds that*

$$\begin{aligned}
\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^r(\Omega;H)} &\le \sup_{t\in[0,T]} \|(I-P)X_t\|_{L^r(\Omega;H)} \\
&+ \Bigg\| \exp\Bigg(\int_0^T \Big[\frac{\langle Y_s - PX_s, P\mu(Y_s) - P\mu(PX_s)\rangle_H}{\|Y_s - PX_s\|^2_H} \\
&\qquad + \frac{\frac{(p-1)(1+\varepsilon)}{2}\|P\sigma(Y_s) - P\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)}}{\|Y_s - PX_s\|^2_H} + \chi_s\Big]^+ ds\Bigg)\Bigg\|_{L^q(\Omega;\mathbb{R})} \\
&\cdot \Big\| p\,\|Y - PX\|_H^{(p-2)}\big[\langle Y - PX, P\mu(PX) - P\mu(X)\rangle_H \\
&+ \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P\sigma(X) - P\sigma(PX)\|^2_{\mathrm{HS}(U,H)} - \chi\,\|Y - PX\|^2_H\big]^+\Big\|^{1/p}_{L^1([0,T]\times\Omega;\mathbb{R})}.
\end{aligned} \tag{65}$$

Corollary 3.6 is a special case of Corollary 2.11 (choose $D(A) = H$, $A = 0$, $F_1 = \mu$, $B_1 = \sigma$, $F_2 = P\mu$, $B_2 = P\sigma$, $X^1 = X$, $X^2 = Y$, $\hat{X} = P(X)$ in the setting of Corollary 2.11 and Corollary 3.6, respectively). If the processes X and Y in Corollary 3.6 satisfy suitable exponential integrability properties (see Corollary 2.4 in Cox, Hutzenthaler and Jentzen [13]), then the right-hand side of (65) can be further estimated in an appropriate way. This is the subject of the next result.

PROPOSITION 3.7. *Assume Setting* 1.5, *let* $\varepsilon \in [0,\infty]$, $r, q_0, q_1, \hat{q}_0, \hat{q}_1 \in (0,\infty]$, $c, \alpha, \beta, \hat{\alpha}, \hat{\beta} \in [0,\infty)$, $p \in [2,\infty)$, $U_0, \hat{U}_0 \in C^2(O,[0,\infty))$, $U_1, \hat{U}_1 \in C(\mathcal{O},[0,\infty))$, $\varphi \in \mathcal{L}^0(\mathcal{O};\mathbb{R})$, $\mu \in \mathcal{L}^0(\mathcal{O};H)$, $\sigma \in \mathcal{L}^0(\mathcal{O};\mathrm{HS}(U,H))$, $P \in L(H)$, *let* $X, Y\colon [0,T]\times\Omega \to \mathcal{O}$ *be predictable stochastic processes, assume that* $P(\mathcal{O}) \subseteq \mathcal{O}$, $\frac{1}{p} + \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{\hat{q}_0} + \frac{1}{\hat{q}_1} = \frac{1}{r}$, *and* $\mathbb{E}[e^{U_0(X_0)} + e^{\hat{U}_0(Y_0)}] < \infty$, *assume for all* $x \in \mathcal{O}$, $y \in (P(H)\cap\mathcal{O})$ *that*

$$\begin{aligned}
&(\mathcal{G}_{\mu,\sigma}U_0)(x) + \tfrac{1}{2}\|\sigma(x)^*(\nabla U_0)(x)\|^2_U + U_1(x) \le \alpha\, U_0(x) + \beta, \\
&(\mathcal{G}_{P\mu,P\sigma}\hat{U}_0)(y) + \tfrac{1}{2}\|\sigma(y)^* P^*(\nabla \hat{U}_0)(y)\|^2_U + \hat{U}_1(y) \le \hat{\alpha}\,\hat{U}_0(y) + \hat{\beta}, \\
&\langle Px - y, P\mu(Px) - P\mu(y)\rangle_H + \tfrac{(p-1)(1+\varepsilon)}{2}\|P\sigma(Px) - P\sigma(y)\|^2_{\mathrm{HS}(U,H)} \\
&+ \langle y - Px, P\mu(Px) - P\mu(x)\rangle_H + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P\sigma(Px) - P\sigma(x)\|^2_{\mathrm{HS}(U,H)} \\
&\le \tfrac{|\varphi(x)|^2}{2} + \Big[c + \tfrac{U_0(x)}{q_0 T e^{\alpha T}} + \tfrac{\hat{U}_0(y)}{\hat{q}_0 T e^{\hat{\alpha}T}} + \tfrac{U_1(x)}{q_1 e^{\alpha T}} + \tfrac{\hat{U}_1(y)}{\hat{q}_1 e^{\hat{\alpha}T}}\Big]\|Px - y\|^2_H,
\end{aligned} \tag{66}$$

and assume that for all $t \in [0,T]$ *it holds* $\mathbb{P}$-*a.s. that* $\int_0^T \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(PX_s)\|_H + \|\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)}\,ds < \infty$, $X_t = X_0 +$

$\int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, *and* $Y_t = PX_0 + \int_0^t P\mu(Y_s)\,ds + \int_0^t P\sigma(Y_s)\,dW_s$. *Then*

$$\begin{aligned}\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^r(\Omega;H)} &\le T^{(\frac{1}{2}-\frac{1}{p})} \exp\Bigg(\frac{1}{2} - \frac{1}{p} + \int_0^T c + \sum_{i=0}^{1}\left[\frac{\beta}{q_i e^{\alpha s}} + \frac{\hat\beta}{\hat q_i e^{\hat\alpha s}}\right] ds\Bigg) \\ &\quad \cdot \|\varphi(X)\|_{L^p([0,T]\times\Omega;\mathbb{R})} \left|\mathbb{E}\left[e^{U_0(X_0)}\right]\right|^{\left[\frac{1}{q_0}+\frac{1}{q_1}\right]} \left|\mathbb{E}\left[e^{\hat U_0(Y_0)}\right]\right|^{\left[\frac{1}{\hat q_0}+\frac{1}{\hat q_1}\right]} \\ &\quad + \sup_{t\in[0,T]} \|(I-P)X_t\|_{L^r(\Omega;H)}.\end{aligned} \tag{67}$$

PROOF. Throughout this proof let $q \in (0,\infty]$ be given by $\frac{1}{q_0}+\frac{1}{q_1}+\frac{1}{\hat q_0}+\frac{1}{\hat q_1} = \frac{1}{q}$ and let $\chi\colon [0,T]\times\Omega \to \mathbb{R}$ be the stochastic process which satisfies for all $t\in[0,T]$ that

$$\begin{aligned}\chi_t &= c + \frac{U_0(X_t)}{q_0 T e^{\alpha T}} + \frac{\hat U_0(Y_t)}{\hat q_0 T e^{\hat\alpha T}} + \frac{U_1(X_t)}{q_1 e^{\alpha T}} + \frac{\hat U_1(Y_t)}{\hat q_1 e^{\hat\alpha T}} + \frac{(1/2-1/p)}{T} \\ &\quad - \frac{\langle Y_t - PX_t, P\mu(Y_t) - P\mu(PX_t)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|P\sigma(Y_t) - P\sigma(PX_t)\|^2_{\mathrm{HS}(U,H)}}{\|Y_t - PX_t\|_H^2}.\end{aligned} \tag{68}$$

Note that (68), Hölder's inequality, nonnegativity of U_0, $\hat U_0$, U_1 and $\hat U_1$, Jensen's inequality, Cox, Hutzenthaler and Jentzen ([13], Corollary 2.4), the assumption that $\mathbb{E}[e^{U_0(X_0)} + e^{\hat U_0(Y_0)}] < \infty$ and (66) prove that

$$\begin{aligned}&\left\|\exp\left(\int_0^T \left[\frac{\langle Y_s - PX_s, P\mu(Y_s) - P\mu(PX_s)\rangle_H + \frac{(p-1)(1+\varepsilon)}{2}\|P\sigma(Y_s) - P\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)}}{\|Y_s - PX_s\|_H^2} + \chi_s\right]^+ ds\right)\right\|_{L^q(\Omega;\mathbb{R})} \\ &\le \exp\left(\frac{1}{2} - \frac{1}{p} + \int_0^T c + \sum_{i=0}^{1}\left[\frac{\beta}{q_i e^{\alpha s}} + \frac{\hat\beta}{\hat q_i e^{\hat\alpha s}}\right] ds\right) \\ &\quad \cdot \sup_{s\in[0,T]}\left|\mathbb{E}\left[\exp\left(\frac{U_0(X_s)}{e^{\alpha s}} + \int_0^s \frac{U_1(X_u)-\beta}{e^{\alpha u}}\,du\right)\right]\right|^{\left[\frac{1}{q_0}+\frac{1}{q_1}\right]} \\ &\quad \cdot \sup_{s\in[0,T]}\left|\mathbb{E}\left[\exp\left(\frac{\hat U_0(Y_s)}{e^{\hat\alpha s}} + \int_0^s \frac{\hat U_1(Y_u)-\hat\beta}{e^{\hat\alpha u}}\,du\right)\right]\right|^{\left[\frac{1}{\hat q_0}+\frac{1}{\hat q_1}\right]} \\ &\le \exp\left(\frac{1}{2} - \frac{1}{p} + \int_0^T c + \sum_{i=0}^{1}\left[\frac{\beta}{q_i e^{\alpha s}} + \frac{\hat\beta}{\hat q_i e^{\hat\alpha s}}\right] ds\right) \left|\mathbb{E}\left[e^{U_0(X_0)}\right]\right|^{\left[\frac{1}{q_0}+\frac{1}{q_1}\right]} \left|\mathbb{E}\left[e^{\hat U_0(Y_0)}\right]\right|^{\left[\frac{1}{\hat q_0}+\frac{1}{\hat q_1}\right]}.\end{aligned} \tag{69}$$

In addition, observe that Corollary 3.6 yields that

$$\begin{aligned}\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^r(\Omega;H)} &\le \sup_{t\in[0,T]} \|(I-P)X_t\|_{L^r(\Omega;H)} \\ &\quad + \Bigg\|\exp\Bigg(\int_0^T \Bigg[\frac{\langle Y_s - PX_s, P\mu(Y_s) - P\mu(PX_s)\rangle_H}{\|Y_s - PX_s\|_H^2} \\ &\qquad + \frac{\frac{(p-1)(1+\varepsilon)}{2}\|P\sigma(Y_s) - P\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)}}{\|Y_s - PX_s\|_H^2} + \chi_s\Bigg]^+ ds\Bigg)\Bigg\|_{L^q(\Omega;\mathbb{R})} \\ &\quad \cdot \Big\| p\,\|Y - PX\|_H^{(p-2)}\big[\langle Y - PX, P\mu(PX) - P\mu(X)\rangle_H \\ &\qquad + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P\sigma(X) - P\sigma(PX)\|^2_{\mathrm{HS}(U,H)} - \chi\,\|Y - PX\|_H^2\big]^+\Big\|^{1/p}_{L^1([0,T]\times\Omega;\mathbb{R})}.\end{aligned} \tag{70}$$

Moreover, note that (66), (68), the fact that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $Y_t \in (P(H) \cap \mathcal{O})$, and Young's inequality show that

$$\begin{aligned} &\Big\| p \, \|Y - PX\|_H^{(p-2)} \big[\langle Y - PX, P\mu(PX) - P\mu(X) \rangle_H \\ &\quad + \tfrac{(p-1)\,(1+1/\varepsilon)}{2} \, \|P\sigma(X) - P\sigma(PX)\|^2_{\mathrm{HS}(U,H)} - \chi \, \|Y - PX\|_H^2 \big]^+ \Big\|^{1/p}_{L^1([0,T]\times\Omega;\mathbb{R})} \\ &\leq p \Big\| \Big[\tfrac{(2T)^{(1-2/p)}}{2} \, |\varphi(X)|^2 \, \tfrac{\|Y-PX\|_H^{(p-2)}}{(2T)^{(1-2/p)}} - \tfrac{(1/2-1/p)}{T} \, \|Y - PX\|_H^p \Big]^+ \Big\|^{1/p}_{L^1([0,T]\times\Omega;\mathbb{R})} \\ &\leq T^{(1/2-1/p)} \, \|\varphi(X)\|_{L^p([0,T]\times\Omega;\mathbb{R})} \, . \end{aligned} \tag{71}$$

Putting this and (69) into (70) establishes (67). The proof of Proposition 3.7 is thus complete. □

In a number of cases the functions U_0 and $\hat{U}_0$ in Proposition 3.7 satisfy that there exists $\rho \in (0,\infty)$ such that for all $x \in O$ it holds that $U_0(x) = \hat{U}_0(x) = \frac{\rho}{2} \, \|x\|_H^2$. This special case of Proposition 3.7 is the subject of the next result, Corollary 3.8. Corollary 3.8 follows immediately from Proposition 3.7.

COROLLARY 3.8. *Assume Setting 1.5, let* $\varepsilon \in [0,\infty]$, $r,\rho \in (0,\infty)$, $q \in (0,\infty]$, $c,\beta \in [0,\infty)$, $p \in [2,\infty)$, $\mathcal{U} \in C(\mathcal{O},[0,\infty))$, $\mu \in \mathcal{L}^0(\mathcal{O};H)$, $\sigma \in \mathcal{L}^0(\mathcal{O};\mathrm{HS}(U,H))$, $\varphi \in \mathcal{L}^0(\mathcal{O};\mathbb{R})$, $P \in L(H)$, *let* $X, Y\colon [0,T]\times\Omega \to \mathcal{O}$ *be predictable stochastic processes, assume that* $P^2 = P = P^*$, $\|P\|_{L(H)} \leq 1$, $P(\mathcal{O}) \subseteq \mathcal{O}$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, *and* $\mathbb{E}[e^{\frac{\rho}{2}\|X_0\|_H^2}] < \infty$, *assume for all* $x \in \mathcal{O}$, $y \in (P(H) \cap \mathcal{O})$ *that*

$$\begin{aligned} &\langle x, \mu(x) \rangle_H + \tfrac{1}{2} \, \|\sigma(x)\|^2_{\mathrm{HS}(U,H)} + \tfrac{\rho}{2} \, \|\sigma(x)^* x\|_U^2 + \mathcal{U}(x) \leq \beta, \\ &\langle Px - y, \mu(Px) - \mu(y) \rangle_H + \tfrac{(p-1)\,(1+\varepsilon)}{2} \, \|\sigma(Px) - \sigma(y)\|^2_{\mathrm{HS}(U,H)} \\ &\quad + \langle y - Px, P\mu(Px) - P\mu(x) \rangle_H + \tfrac{(p-1)\,(1+1/\varepsilon)}{2} \, \|\sigma(Px) - \sigma(x)\|^2_{\mathrm{HS}(U,H)} \\ &\qquad \leq \tfrac{|\varphi(x)|^2}{2} + \big[c + \tfrac{\rho}{2q} \mathcal{U}(x) + \tfrac{\rho}{2q} \mathcal{U}(y) \big] \, \|Px - y\|_H^2 \, , \end{aligned} \tag{72}$$

and assume that for all $t \in [0,T]$ *it holds* $\mathbb{P}$*-a.s. that* $\int_0^T \|\mu(X_s)\|_H + \|\sigma(X_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(PX_s)\|_H + \|\sigma(PX_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(Y_s)\|_H + \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)} \, ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$, *and* $Y_t = PX_0 + \int_0^t P\mu(Y_s)\,ds + \int_0^t P\sigma(Y_s)\,dW_s$. *Then*

$$\begin{aligned} &\sup_{t\in[0,T]} \|X_t - Y_t\|_{L^r(\Omega;H)} \\ &\leq \|\varphi(X)\|_{L^p([0,T]\times\Omega;\mathbb{R})} \, T^{(\frac{1}{2}-\frac{1}{p})} \, e^{\left[\frac{1}{2}-\frac{1}{p}+cT+\frac{\beta\rho T}{q}\right]} \big| \mathbb{E}\big[e^{\frac{\rho}{2}\|X_0\|_H^2}\big] \big|^{\frac{1}{q}} \\ &\quad + \sup_{t\in[0,T]} \|(I-P)X_t\|_{L^r(\Omega;H)} \, . \end{aligned} \tag{73}$$

We now apply Corollary 3.8 and Proposition 3.7, respectively, to two semilinear example SPDEs with nonglobally monotone nonlinearities. In both example SPDEs, the verification of assumption (66) in Proposition 3.7 is partially based on Cox, Hutzenthaler and Jentzen [13], Section 5.

3.2.1. *Setting.* We frequently employ the following setting.

SETTING 3.9. Let $k, l \in \mathbb{N}$, $T \in (0,\infty)$, $D = (0,1)$, $\theta \in [0,1)$, $\varrho \in \mathbb{R}$, $\vartheta \in (\theta - 1, 0]$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathbb{F}_t)_{t\in[0,T]})$ be a filtered probability space which fulfills the usual conditions, $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2(D;\mathbb{R}^k), \langle\cdot,\cdot\rangle_{L^2(D;\mathbb{R}^k)}, \|\cdot\|_{L^2(D;\mathbb{R}^k)})$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U) = (L^2(D;\mathbb{R}^l), \langle\cdot,\cdot\rangle_{L^2(D;\mathbb{R}^l)}, \|\cdot\|_{L^2(D;\mathbb{R}^l)})$, let $(W_t)_{t\in[0,T]}$ be an Id_U-cylindrical $(\mathbb{F}_t)_{t\in[0,T]}$-Wiener process, let $A\colon D(A) \subseteq H \to H$ be a generator of a strongly continuous analytic semigroup, assume that $\varrho - A$ is strictly positive, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\varrho - A$, let $x_0 \in H_\theta$, $F \in C(H_\theta, H_\vartheta)$, $B \in C(H, \mathrm{HS}(U,H))$, let $P_N \in L(H_\vartheta, D(A))$, $N \in \mathbb{N}$, assume for all $N \in \mathbb{N}$ that $\dim(P_N(H)) < \infty$, let $X\colon [0,T]\times\Omega \to H_\theta$ be an adapted stochastic process with c.s.p., assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}F(X_s)\,ds + \int_0^t e^{A(t-s)}B(X_s)\,dW_s, \tag{74}$$

let $\mu_N\colon P_N(H) \to P_N(H)$, $N \in \mathbb{N}$, and $\sigma_N\colon P_N(H) \to \mathrm{HS}(U, P_N(H))$, $N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}$, $v \in P_N(H)$, $u \in U$ that $\mu_N(v) = P_N(Av + F(v))$ and $\sigma_N(v)u = P_N\big(B(v)u\big)$, and let $X^N\colon [0,T]\times\Omega \to P_N(H)$, $N \in \mathbb{N}$, be adapted stochastic processes with c.s.p., assume that for all $t \in [0,T]$, $N \in \mathbb{N}$ it holds $\mathbb{P}$-a.s. that

$$X_t^N = P_N(X_0) + \int_0^t \mu_N(X_s^N)\,ds + \int_0^t \sigma_N(X_s^N)\,dW_s. \tag{75}$$

3.2.2. *Cahn–Hilliard–Cook-type equations.* In the following result, Corollary 3.10 below, we establish strong convergence rates for spatial spectral Galerkin approximations for certain Cahn–Hilliard–Cook-type equations. In our proof of Corollary 3.10 we apply Proposition 3.7 to these spatial spectral Galerkin approximations and for this application of Proposition 3.7 we construct in the proof of Corollary 3.10 suitable functions U_0, $\hat{U}_0$, U_1 and $\hat{U}_1$ such that (66) is satisfied (cf. (85) and (91) in the proof of Corollary 3.10 below).

COROLLARY 3.10. *Assume Setting 3.9, assume $\theta \in (1/12, 1/2)$, $k = 1$, $\vartheta = -1/2$, $\varrho \in (0,\infty)$, let $c \in (0,\infty)$, $\eta \in [0,\infty)$, $(\varsigma_\varepsilon)_{\varepsilon\in(0,\infty)} \subseteq [0,\infty)$, let $L\colon D(L) \subseteq H \to H$ be the Laplacian with the standard Neumann boundary conditions on H, assume for all $v \in D(A)$ that $D(A) = D(L^2)$ and $Av = -L^2 v$, let $e_n \in H$, $n \in \mathbb{N}$, satisfy for all $n \in \mathbb{N}$ that $e_n = \{2^{\min\{n-1,1\}/2}\cos((n-1)\pi x)\}_{x\in(0,1)}$, and assume for all $v \in H_\theta$, $N \in \mathbb{N}$, $\varepsilon \in (0,\infty)$ that $P_N(v) = \sum_{n=1}^N \langle e_n, v\rangle_H\, e_n$, $F(v) = c\,\Delta(v^3 - v)$, $\eta = \sup_{v,w\in H, v\neq w} \frac{\|B(v)-B(w)\|_{\mathrm{HS}(U,H)}}{\|v-w\|_H}$, $x_0 \in H_{1/2}$, and*

$$\varsigma_\varepsilon = \sup_{v\in H_\theta}\Big[\|(I-P_1)B(v)\|^2_{\mathrm{HS}(U,H)} - \varepsilon\,\|((I-P_1)v)^2\|_H^2 - \varepsilon\,\|(I-P_1)v\|_H^2\,\|v\|_H^2\Big]. \tag{76}$$

Then for every $r \in (0,\infty)$, $\alpha \in (-\infty, 2)$ there exists $C \in \mathbb{R}$ such that for every $N \in \mathbb{N}$ it holds that

$$\sup\nolimits_{t\in[0,T]} \|X_t - X_t^N\|_{L^r(\Omega;H)} \leq C\,N^{-\alpha}. \tag{77}$$

PROOF. Throughout this proof let $\tilde{P}, \tilde{L} \in L(H)$ satisfy for all $v \in H$ that $\tilde{P}v = (I - P_1)v = v - P_1(v) = v - e_1\,\langle e_1, v\rangle_H$ and $\tilde{L}v = -\sum_{n=2}^\infty (n-1)^{-2}\pi^{-2}\,\langle e_n, v\rangle_H\, e_n$. Note that for all $v \in D(L)$ it holds that

$$\tilde{L}Lv = L\tilde{L}v = \tilde{P}v. \tag{78}$$

Moreover, observe that Young's inequality proves that for all $\delta \in [3/4, \infty)$, $M \in \mathbb{N}$, $x \in P_M(H_\vartheta)$ it holds that

$$
\begin{aligned}
-c\langle \tilde{P}x, x^3\rangle_H &= -c\langle \tilde{P}x, (\tilde{P}x + P_1 x)^3\rangle_H \\
&= -c\langle \tilde{P}x, (\tilde{P}x)^3\rangle_H - 3c\langle \tilde{P}x, (\tilde{P}x)^2(P_1 x)\rangle_H - 3c\langle \tilde{P}x, (\tilde{P}x)(P_1 x)^2\rangle_H \\
&\quad - c\langle \tilde{P}x, (P_1 x)^3\rangle_H \\
&= -c\|(\tilde{P}x)^2\|_H^2 - 3c\langle \tilde{P}x, (\tilde{P}x)^2\rangle_H \langle e_1, x\rangle_H - 3c\|\tilde{P}x\|_H^2 |\langle e_1, x\rangle_H|^2 \\
&\quad - c\langle \tilde{P}x, e_1\rangle_H (\langle e_1, x\rangle_H)^3 \\
&\le -c\|(\tilde{P}x)^2\|_H^2 + [\sqrt{2c}\,\delta^{1/2}\|(\tilde{P}x)^2\|_H]\Big[\tfrac{3\sqrt{c}}{\sqrt{2}}\delta^{-1/2}\|\tilde{P}x\|_H |\langle e_1, x\rangle_H|\Big] \\
&\quad - 3c\|\tilde{P}x\|_H^2 |\langle e_1, x\rangle_H|^2 \\
&\le -c(1-\delta)\|(\tilde{P}x)^2\|_H^2 - 3c\Big(1 - \tfrac{3}{4\delta}\Big)\|\tilde{P}x\|_H^2 |\langle e_1, x\rangle_H|^2 \\
&= -c(1-\delta)\|(\tilde{P}x)^2\|_H^2 - 3c\Big(1 - \tfrac{3}{4\delta}\Big)\|\tilde{P}x\|_H^2 [\|x\|_H^2 - \|\tilde{P}x\|_H^2] \\
&\le c\Big[\delta + 3\Big(1 - \tfrac{3}{4\delta}\Big) - 1\Big]\|(\tilde{P}x)^2\|_H^2 - 3c\Big(1 - \tfrac{3}{4\delta}\Big)\|\tilde{P}x\|_H^2 \|x\|_H^2.
\end{aligned}
\tag{79}
$$

In the next step observe that for all $M \in \mathbb{N}$, $x \in P_M(H)$ it holds that

$$
\begin{aligned}
\langle \tilde{L}x, \mu_M(x)\rangle_H &= \langle \tilde{L}x, P_M(Ax + F(x))\rangle_H \\
&= \langle \tilde{L}P_M x, Ax + F(x)\rangle_H = \langle \tilde{L}x, Ax + F(x)\rangle_H \\
&= -\langle \tilde{L}x, L^2 x\rangle_H + \langle \tilde{L}x, F(x)\rangle_H = -\langle \tilde{P}x, Lx\rangle_H + c\langle \tilde{P}x, x^3 - x\rangle_H \\
&= \langle (-L)^{1/2}\tilde{P}x, (-L)^{1/2}\tilde{P}x\rangle_H + c\langle \tilde{P}x, x^3\rangle_H - c\langle \tilde{P}x, x\rangle_H \\
&= \|(-L)^{1/2}\tilde{P}x\|_H^2 + c\langle \tilde{P}x, x^3\rangle_H - c\|\tilde{P}x\|_H^2.
\end{aligned}
\tag{80}
$$

Hence, we obtain that for all $M \in \mathbb{N}$, $\rho, \hat{\rho} \in (0, \infty)$, $U_0 \in C^2(P_M(H), [0, \infty))$, $x \in P_M(H)$ with $\forall\, y \in P_M(H)\colon U_0(y) = \frac{\rho}{2}\|(-\tilde{L})^{1/2}y\|_H^2 + \frac{\hat{\rho}}{2}\|\tilde{P}y\|_H^2$ it holds that

$$
\begin{aligned}
&(\mathcal{G}_{\mu_M,\sigma_M} U_0)(x) + \frac{1}{2}\|\sigma_M(x)^*(\nabla U_0)(x)\|_U^2 \\
&\quad = \Big[-\rho\langle \tilde{L}x, \mu_M(x)\rangle_H + \tfrac{\rho}{2}\|(-\tilde{L})^{1/2}\sigma_M(x)\|_{\mathrm{HS}(U,P_M(H))}^2\Big] \\
&\qquad + \Big[\hat{\rho}\langle \tilde{P}x, \mu_M(x)\rangle_H + \tfrac{\hat{\rho}}{2}\|\tilde{P}\sigma_M(x)\|_{\mathrm{HS}(U,P_M(H))}^2\Big] \\
&\qquad + \tfrac{1}{2}\|\sigma_M(x)^*[\rho(-\tilde{L})x + \hat{\rho}\tilde{P}x]\|_U^2 \\
&\quad \le \rho\Big[c\|\tilde{P}x\|_H^2 - \|x'\|_H^2 - c\langle \tilde{P}x, x^3\rangle_H + \tfrac{1}{2}\|(-\tilde{L})^{1/2}B(x)\|_{\mathrm{HS}(U,H)}^2\Big] \\
&\qquad + \hat{\rho}\Big[c\|x'\|_H^2 - \|x''\|_H^2 - c\langle x', (x^3)'\rangle_H + \tfrac{1}{2}\|\tilde{P}B(x)\|_{\mathrm{HS}(U,H)}^2\Big] \\
&\qquad + \tfrac{1}{2}\|B(x)^*[\hat{\rho}\tilde{P} - \rho\tilde{L}]x\|_U^2.
\end{aligned}
\tag{81}
$$

Combining this with (79) and the fact that $\forall\, v \in H\colon \|(-\tilde{L})^{1/2}v\|_H^2 \leq \|\tilde{P}v\|_H^2$ proves that for all $\delta \in [3/4,\infty)$, $M \in \mathbb{N}$, $\rho,\hat{\rho} \in (0,\infty)$, $U_0 \in C^2(P_M(H),[0,\infty))$, $x \in P_M(H)$ with $\forall\, y \in P_M(H)\colon U_0(y) = \frac{\rho}{2}\,\|(-\tilde{L})^{1/2}y\|_H^2 + \frac{\hat{\rho}}{2}\,\|\tilde{P}y\|_H^2$ it holds that

$$
\begin{aligned}
&(\mathcal{G}_{\mu_M,\sigma_M}U_0)(x) + \tfrac{1}{2}\|\sigma_M(x)^*(\nabla U_0)(x)\|_H^2 \\
&\quad\leq \rho\bigg[c\|\tilde{P}x\|_H^2 - \|x'\|_H^2 + c\bigg[\delta + 3\bigg(1-\tfrac{3}{4\delta}\bigg) - 1\bigg]\|(\tilde{P}x)^2\|_H^2 \\
&\qquad - 3c\bigg(1-\tfrac{3}{4\delta}\bigg)\|\tilde{P}x\|_H^2\|x\|_H^2\bigg] \\
&\qquad + \hat{\rho}[c\|x'\|_H^2 - \|x''\|_H^2 - 3c\|x'x\|_H^2] \\
&\qquad + \tfrac{(\rho+\hat{\rho})\|\tilde{P}B(x)\|_{\mathrm{HS}(U,H)}^2}{2} + \tfrac{\|B(x)\|_{\mathrm{HS}(U,H)}^2\|\hat{\rho}\tilde{P}-\rho\tilde{L}\|_{L(H)}^2\|\tilde{P}x\|_H^2}{2}.
\end{aligned}
\tag{82}
$$

The fact that B is globally Lipschitz continuous and (76) therefore imply that for all $\delta \in [3/4,\infty)$, $M \in \mathbb{N}$, $\varepsilon,\rho,\hat{\rho} \in (0,\infty)$, $U_0 \in C^2(P_M(H),[0,\infty))$, $x \in P_M(H)$ with $\forall y \in P_M(H) : U_0(y) = \frac{\rho}{2}\|(-\tilde{L})^{1/2}y\|_H^2 + \frac{\hat{\rho}}{2}\|\tilde{P}y\|_H^2$ it holds that

$$
\begin{aligned}
&(\mathcal{G}_{\mu_M,\sigma_M}U_0)(x) + \tfrac{1}{2}\|\sigma_M(x)^*(\nabla U_0)(x)\|_U^2 \\
&\quad\leq \rho\bigg[-\|x'\|_H^2 + c\bigg[\delta + 3\bigg(1-\tfrac{3}{4\delta}\bigg) - 1\bigg]\|(\tilde{P}x)^2\|_H^2 - 3c\bigg(1-\tfrac{3}{4\delta}\bigg)\|\tilde{P}x\|_H^2\|x\|_H^2\bigg] \\
&\qquad + \hat{\rho}[c\|x'\|_H^2 - \|x''\|_H^2 - 3c\|x'x\|_H^2] \\
&\qquad + \tfrac{(\rho+\hat{\rho})[\varsigma_\varepsilon + \varepsilon\|(\tilde{P}x)^2\|_H^2 + \varepsilon\|\tilde{P}x\|_H^2\|x\|_H^2]}{2} \\
&\qquad + \tfrac{\|B(x)-B(0)+B(0)\|_{\mathrm{HS}(U,H)}^2\|\hat{\rho}\tilde{P}-\rho\tilde{L}\|_{L(H)}^2\|\tilde{P}x\|_H^2}{2} + \rho c\|\tilde{P}x\|_H^2 \\
&\quad\leq \bigg[\tfrac{(\rho+\hat{\rho})\varepsilon}{2} + \rho c\bigg[\delta + 2 - \tfrac{9}{4\delta}\bigg]\bigg]\|(\tilde{P}x)^2\|_H^2 \\
&\qquad + [\rho c + \|B(0)\|_{\mathrm{HS}(U,H)}^2\|\hat{\rho}\tilde{P} - \rho\tilde{L}\|_{L(H)}^2]\|\tilde{P}x\|_H^2 + [\hat{\rho}c - \rho]\|x'\|_H^2 \\
&\qquad - \hat{\rho}[\|x''\|_H^2 + 3c\|x'x\|_H^2] \\
&\qquad + \bigg[\tfrac{(\rho+\hat{\rho})\varepsilon}{2} + \eta^2\|\hat{\rho}\tilde{P} - \rho\tilde{L}\|_{L(H)}^2 - \rho c\bigg(3 - \tfrac{9}{4\delta}\bigg)\bigg]\|\tilde{P}x\|_H^2\|x\|_H^2 + \tfrac{\varsigma_\varepsilon(\rho+\hat{\rho})}{2}.
\end{aligned}
\tag{83}
$$

This implies that there exist $\rho,\hat{\rho},\tilde{\rho} \in (0,\infty)$ such that for all $U_0, U_1 \in C^2(D(A),[0,\infty))$ with $\forall\, x \in D(A)\colon U_0(x) = \frac{\rho}{2}\,\|(-\tilde{L})^{1/2}x\|_H^2 + \frac{\hat{\rho}}{2}\,\|\tilde{P}x\|_H^2$ and $\forall\, x \in D(A)\colon U_1(x) = \hat{\rho}\,\|x''\|_H^2 + \tilde{\rho}\,\|x\|_H^2\,\|\tilde{P}x\|_H^2$ it holds that

$$
\sup_{M\in\mathbb{N}}\sup_{x\in P_M(H)}\Big[(\mathcal{G}_{\mu_M,\sigma_M}U_0|_{P_M(H)})(x) + \tfrac{1}{2}\,\|\sigma_M(x)^*(\nabla U_0)(x)\|_U^2 + U_1(x)\Big] < \infty.
\tag{84}
$$

This allows us to choose $\beta \in \mathbb{R}$, $\rho,\hat{\rho},\tilde{\rho} \in (0,\infty)$, $U_0, U_1 \in C^2(D(A),[0,\infty))$ which satisfy $\forall\, x \in D(A)\colon U_0(x) = \frac{\rho}{2}\,\|(-\tilde{L})^{1/2}x\|_H^2 + \frac{\hat{\rho}}{2}\,\|\tilde{P}x\|_H^2$, $\forall\, x \in D(A)\colon U_1(x) = \hat{\rho}\,\|x''\|_H^2 + \tilde{\rho}\,\|x\|_H^2\,\|\tilde{P}x\|_H^2$ and

$$
\beta = \sup_{M\in\mathbb{N}}\sup_{x\in P_M(H)}\Big[(\mathcal{G}_{\mu_M,\sigma_M}U_0|_{P_M(H)})(x) + \tfrac{1}{2}\,\|\sigma_M(x)^*(\nabla U_0)(x)\|_U^2 + U_1(x)\Big] < \infty.
\tag{85}
$$

Next note that for all $\varepsilon \in [0,\infty)$, $p \in [2,\infty)$, $M, N \in \mathbb{N}$, $x \in P_M(H)$, $y \in P_N(H)$ with $M > N$ it holds that

$$\begin{aligned}
&\langle P_N x - y, P_N \mu_M(P_N x) - P_N \mu_M(y)\rangle_H \\
&\quad + \tfrac{(p-1)\,(1+\varepsilon)}{2}\, \|P_N \sigma_M(P_N x) - P_N \sigma_M(y)\|^2_{\mathrm{HS}(U,P_N(H))} \\
&\quad + \langle y - P_N x, P_N \mu_M(P_N x) - P_N \mu_M(x)\rangle_H \\
&\quad + \tfrac{(p-1)\,(1+1/\varepsilon)}{2}\, \|P_N \sigma_M(P_N x) - P_N \sigma_M(x)\|^2_{\mathrm{HS}(U,P_N(H))} \\
&\le \langle P_N x - y, F(P_N x) - F(y)\rangle_H - \|L(P_N x - y)\|^2_H \\
&\quad + \tfrac{(p-1)\,(1+\varepsilon)}{2}\, \|B(P_N x) - B(y)\|^2_{\mathrm{HS}(U,H)} \\
&\quad + \langle y - P_N x, F(P_N x) - F(x)\rangle_H + \tfrac{(p-1)\,(1+1/\varepsilon)}{2}\, \|B(P_N x) - B(x)\|^2_{\mathrm{HS}(U,H)} \\
&\le c\, \|(-L)^{1/2}(P_N x - y)\|^2_H + c \Big\langle P_N x - y, L\Big[(P_N x)^3 - y^3\Big]\Big\rangle_H - \|L(P_N x - y)\|^2_H \\
&\quad + \tfrac{(p-1)\,(1+\varepsilon)\,\eta^2}{2}\, \|P_N x - y\|^2_H + c \Big\langle L(y - P_N x), (P_N x)^3 - x^3 - (P_N x - x)\Big\rangle_H \\
&\quad + \tfrac{(p-1)\,(1+1/\varepsilon)\,\eta^2}{2}\, \|(I - P_N)x\|^2_H .
\end{aligned} \tag{86}$$

This implies that for all $\varepsilon \in [0,\infty)$, $p \in [2,\infty)$, $M, N \in \mathbb{N}$, $x \in P_M(H)$, $y \in P_N(H)$ with $M > N$ it holds that

$$\begin{aligned}
&\langle P_N x - y, P_N \mu_M(P_N x) - P_N \mu_M(y)\rangle_H \\
&\quad + \tfrac{(p-1)(1+\varepsilon)}{2} \|P_N \sigma_M(P_N x) - P_N \sigma_M(y)\|^2_{\mathrm{HS}(U,P_N(H))} \\
&\quad + \langle y - P_N x, P_N \mu_M(P_N x) - P_N \mu_M(x)\rangle_H \\
&\quad + \tfrac{(p-1)(1+1/\varepsilon)}{2} \|P_N \sigma_M(P_N x) - P_N \sigma_M(x)\|^2_{\mathrm{HS}(U,P_N(H))} \\
&\le c\|(P_N x - y)'\|^2_H - c\langle (P_N x - y)', [(P_N x - y)((P_N x)^2 + (P_N x)y + y^2)]'\rangle_H \\
&\quad - \tfrac{\|L(y - P_N x)\|^2_H}{2} + \tfrac{(p-1)(1+\varepsilon)\eta^2}{2} \|P_N x - y\|^2_H + c^2 \|(P_N x)^3 - x^3\|^2_H \\
&\quad + \Big[c^2 + \tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}\Big] \|(I - P_N)x\|^2_H \\
&\le c\|(P_N x - y)'\|^2_H - c\langle [(P_N x - y)']^2, (P_N x)^2 + (P_N x)y + y^2\rangle_H \\
&\quad - \tfrac{1}{2}\|L(y - P_N x)\|^2_H - c\langle (P_N x - y)', (P_N x - y)[(P_N x)^2 + (P_N x)y + y^2]'\rangle_H \\
&\quad + \tfrac{(p-1)(1+\varepsilon)\eta^2}{2} \|P_N x - y\|^2_H + c^2 \|[x - P_N x][x^2 + (P_N x)^2 + (P_N x)x]\|^2_H \\
&\quad + \Big[c^2 + \tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}\Big] \|(I - P_N)x\|^2_H \\
&\le c\|(P_N x - y)'\|^2_H - \tfrac{c}{2}\langle [(P_N x - y)']^2, (P_N x)^2 + y^2\rangle_H - \tfrac{1}{2}\|L(y - P_N x)\|^2_H \\
&\quad - c\langle (P_N x - y)', (P_N x - y)[2(P_N x)'(P_N x) + (P_N x)'y + (P_N x)y' + 2y'y]\rangle_H \\
&\quad + \tfrac{(p-1)(1+\varepsilon)\eta^2 \|P_N x - y\|^2_H}{2} + c^2 \|[x - P_N x][x^2 + (P_N x)^2 + (P_N x)x]\|^2_H \\
&\quad + \Big[c^2 + \tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}\Big] \|(I - P_N)x\|^2_H \\
&\le c\|(P_N x - y)'\|^2_H - \tfrac{c}{2}\langle [(P_N x - y)']^2, (P_N x)^2 + y^2\rangle_H - \tfrac{1}{2}\|L(y - P_N x)\|^2_H
\end{aligned} \tag{87}$$

$$
\begin{aligned}
&+ 2c\big\|(P_N x - y)'\big(|P_N x| + |y|\big)\big\|_H \big\||P_N x - y|\big(|(P_N x)'| + |y'|\big)\big\|_H \\
&+ \tfrac{(p-1)(1+\varepsilon)\eta^2}{2}\|P_N x - y\|_H^2 + c^2\big\|[x - P_N x][x^2 + (P_N x)^2 + (P_N x)x]\big\|_H^2 \\
&+ \Big[c^2 + \tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}\Big]\|(I - P_N)x\|_H^2.
\end{aligned}
$$

Young's inequality therefore shows that for all $\varepsilon \in [0, \infty)$, $p \in [2, \infty)$, $M, N \in \mathbb{N}$, $x \in P_M(H)$, $y \in P_N(H)$ with $M > N$ it holds that

$$
\begin{aligned}
&\big\langle P_N x - y, P_N \mu_M(P_N x) - P_N \mu_M(y)\big\rangle_H \\
&\quad + \tfrac{(p-1)(1+\varepsilon)}{2}\|P_N \sigma_M(P_N x) - P_N \sigma_M(y)\|_{\mathrm{HS}(U, P_N(H))}^2 \\
&\quad + \big\langle y - P_N x, P_N \mu_M(P_N x) - P_N \mu_M(x)\big\rangle_H \\
&\quad + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_N \sigma_M(P_N x) - P_N \sigma_M(x)\|_{\mathrm{HS}(U, P_N(H))}^2 \\
&\le c\big\|(P_N x - y)'\big\|_H^2 - \tfrac{\|L(y - P_N x)\|_H^2}{2} + 4c\big\||P_N x - y|\big(|(P_N x)'| + |y'|\big)\big\|_H^2 \\
&\quad + \tfrac{(p-1)(1+\varepsilon)\eta^2\|P_N x - y\|_H^2}{2} \\
&\quad + c^2\|x - P_N x\|_H^2\|x^2 + (P_N x)^2 + (P_N x)x\|_{L^\infty(D;\mathbb{R})} \qquad (88)\\
&\quad + \Big[c^2 + \tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}\Big]\|(I - P_N)x\|_H^2 \\
&\le c\big\|(P_N x - y)'\big\|_H^2 - \tfrac{\|L(y - P_N x)\|_H^2}{2} \\
&\quad + 8c\|P_N x - y\|_H^2\big[\|(P_N x)'\|_{L^\infty(D;\mathbb{R})}^2 + \|y'\|_{L^\infty(D;\mathbb{R})}^2\big] \\
&\quad + \tfrac{(p-1)(1+\varepsilon)\eta^2\|P_N x - y\|_H^2}{2} + \tfrac{3c^2}{2}\|x - P_N x\|_H^2\|x^2 + (P_N x)^2\|_{L^\infty(D;\mathbb{R})} \\
&\quad + \Big[c^2 + \tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}\Big]\|(I - P_N)x\|_H^2.
\end{aligned}
$$

In the next step observe that the Sobolev embedding theorem together with interpolation shows that there exist $\hat{\kappa} \in [0, \infty)$ and $(\kappa_q)_{q \in (0,\infty)} \subseteq [0, \infty)$ which satisfy for all $x \in D(A)$, $q \in (0, \infty)$ that

$$
c\,\|x'\|_H^2 \le \hat{\kappa}\,\|x\|_H^2 + \tfrac{1}{2}\,\|x''\|_H^2 \qquad \text{and} \qquad 8\,c\,\|x'\|_{L^\infty(D;\mathbb{R})}^2 \le \tfrac{\kappa_q}{2} + \tfrac{1}{2q}\,U_1(x) \tag{89}
$$

(cf., e.g., Sell and You [67], Theorem 37.2). Putting this into (88) proves that for all $\varepsilon \in [0, \infty)$, $p \in [2, \infty)$, $q \in (0, \infty)$, $M, N \in \mathbb{N}$, $x \in P_M(H)$, $y \in P_N(H)$ with $M > N$ it holds that

$$
\begin{aligned}
&\big\langle P_N x - y, P_N \mu_M(P_N x) - P_N \mu_M(y)\big\rangle_H \\
&\quad + \tfrac{(p-1)(1+\varepsilon)}{2}\|P_N \sigma_M(P_N x) - P_N \sigma_M(y)\|_{\mathrm{HS}(U, P_N(H))}^2 \\
&\quad + \big\langle y - P_N x, P_N \mu_M(P_N x) - P_N \mu_M(x)\big\rangle_H \\
&\quad + \tfrac{(p-1)(1+1/\varepsilon)}{2}\|P_N \sigma_M(P_N x) - P_N \sigma_M(x)\|_{\mathrm{HS}(U, P_N(H))}^2 \qquad (90)\\
&\le \Big[\hat{\kappa} + \kappa_q + \tfrac{(p-1)(1+\varepsilon)\eta^2}{2}\Big]\|P_N x - y\|_H^2 + \tfrac{1}{2q}\|P_N x - y\|_H^2\big[U_1(x) + U_1(y)\big]
\end{aligned}
$$

$$+\left[c^2+\tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}+\tfrac{3c^2}{2}\|x\|^2_{L^\infty(D;\mathbb{R})}+\tfrac{3c^2}{2}\|P_N(x)\|^2_{L^\infty(D;\mathbb{R})}\right]\|(I-P_N)x\|^2_H.$$

Combining this and (85) with Proposition 3.7 (with $c=\hat{\kappa}+\kappa_q+\frac{1}{2}(p-1)(1+\varepsilon)\eta^2$, $q_0=\infty$, $\hat{q}_0=\infty$, $q_1=2q$, $\hat{q}_1=2q$, $\alpha=0$, $\hat{\alpha}=0$, $U_0=U_0|_{P_M(H)}$, $\hat{U}_0=U_0|_{P_M(H)}$, $U_1=U_1|_{P_M(H)}$, $\hat{U}_1=U_1|_{P_M(H)}$ for $M\in\mathbb{N}$ in the notation of Proposition 3.7) shows that for all $\varepsilon\in[0,\infty)$, $p\in[2,\infty)$, $q,r\in(0,\infty)$, $M,N\in\mathbb{N}$ with $M>N$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ it holds that

$$\begin{aligned}\sup_{t\in[0,T]}&\|X^M_t-X^N_t\|_{L^r(\Omega;H)}\\&\leq\sqrt{2}T^{(\frac{1}{2}-\frac{1}{p})}\exp\biggl(\tfrac{1}{2}-\tfrac{1}{p}+\left[\hat{\kappa}+\kappa_q+\tfrac{(p-1)(1+\varepsilon)\eta^2}{2}\right]T+\frac{\beta T}{q}\biggr)\\&\quad\cdot\biggl\|\sqrt{c^2+\tfrac{(p-1)(1+1/\varepsilon)\eta^2}{2}+\tfrac{3c^2}{2}\|X^M\|^2_{L^\infty(D;\mathbb{R})}+\tfrac{3c^2}{2}\|P_N(X^M)\|^2_{L^\infty(D;\mathbb{R})}}\\&\qquad\cdot\|(I-P_N)X^M\|_H\biggr\|_{L^p([0,T]\times\Omega;\mathbb{R})}\\&\quad\cdot\bigl|\mathbb{E}[e^{U_0(X^M_0)}]\mathbb{E}[e^{U_0(X^N_0)}]\bigr|^{\frac{1}{2q}}+\sup_{t\in[0,T]}\|(I-P_N)X^M_t\|_{L^r(\Omega;H)}.\end{aligned}\tag{91}$$

The fact that for all $N\in\mathbb{N}$, $v\in H_{\alpha/4}$, $\alpha\in(1/2,\infty)$ it holds that

$$\begin{aligned}\|P_Nv\|_{L^\infty(D;\mathbb{R})}&\leq\sum_{n=0}^{\infty}|\langle e_{n+1},v\rangle_H|\,\|e_{n+1}\|_{L^\infty(D;\mathbb{R})}\\&\leq\sqrt{2}\left[\sum_{n=0}^{\infty}\left(\varrho+\pi^4n^4\right)^{-\frac{\alpha}{4}}\left(\left(\varrho+\pi^4n^4\right)^{\frac{\alpha}{4}}|\langle e_{n+1},v\rangle_H|\right)\right]\\&\leq\sqrt{2}\left[\sum_{n=0}^{\infty}\left(\varrho+\pi^4n^4\right)^{-\frac{\alpha}{2}}\right]^{\frac{1}{2}}\left[\sum_{n=0}^{\infty}\left|\left(\varrho+\pi^4n^4\right)^{\frac{\alpha}{4}}\langle e_{n+1},v\rangle_H\right|^2\right]^{\frac{1}{2}}\\&=\sqrt{2}\left[\sum_{n=0}^{\infty}\left(\varrho+\pi^4n^4\right)^{-\frac{\alpha}{2}}\right]^{\frac{1}{2}}\left[\sum_{n=1}^{\infty}\left|\left\langle e_n,(\varrho-A)^{\frac{\alpha}{4}}v\right\rangle_H\right|^2\right]^{\frac{1}{2}}\\&=\sqrt{2}\left[\sum_{n=0}^{\infty}\left(\varrho+\pi^4n^4\right)^{-\frac{\alpha}{2}}\right]^{\frac{1}{2}}\|v\|_{H_{\alpha/4}}\end{aligned}\tag{92}$$

and

$$\begin{aligned}\|(I-P_N)v\|_H&\leq\left\|(I-P_N)(\varrho-A)^{-\alpha/4}\right\|_{L(H)}\|v\|_{H_{\alpha/4}}\leq\left(\varrho+\pi^4N^4\right)^{-\alpha/4}\|v\|_{H_{\alpha/4}}\\&\leq\left[N^4\pi^4\right]^{-\alpha/4}\|v\|_{H_{\alpha/4}}=N^{-\alpha}\pi^{-\alpha}\|v\|_{H_{\alpha/4}}\end{aligned}\tag{93}$$

hence proves that for all $p\in[2,\infty)$, $q,r\in(0,\infty)$, $\alpha\in(1/2,\infty)$, $M,N\in\mathbb{N}$ with $M>N$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ it holds that

$$\sup_{t\in[0,T]}\left\|X^M_t-X^N_t\right\|_{L^r(\Omega;H)}\leq N^{-\alpha}\pi^{-\alpha}\sqrt{2T}\exp\left(\tfrac{1}{2}-\tfrac{1}{p}+\left[\hat{\kappa}+\kappa_q+(p-1)\eta^2+\tfrac{\beta}{q}\right]T\right)$$

$$\cdot \left[\eta \sqrt{p-1} + c + \sqrt{6}\, c \sqrt{\textstyle\sum_{n=0}^{\infty} \left(\varrho + \pi^4 n^4 \right)^{-\alpha/2}} \right] \max\Big(1, \sup_{t \in [0,T]} \|X_t^M\|^2_{L^{2p}(\Omega; H_{\alpha/4})}\Big)$$

$$\cdot \left| \mathbb{E}\Big[\exp\Big(\tfrac{\varrho}{2} \, \|(-\tilde{L})^{1/2} X_0\|_H^2 + \tfrac{\hat{\varrho}}{2} \, \|\tilde{P} X_0\|_H^2 \Big) \Big] \right|^{1/q} + N^{-\alpha} \pi^{-\alpha} \left[\sup_{t \in [0,T]} \left\| X_t^M \right\|_{L^r(\Omega; H_{\alpha/4})} \right].$$

Combining this with the fact that $\forall\, p \in [2, \infty)$, $\alpha \in (0, \infty)$: $\pi^{-\alpha} \sqrt{2T} \exp(\frac{1}{2} - \frac{1}{p}) \leq \sqrt{2T} \exp(\frac{1}{2}) \leq \sqrt{2} \exp(\frac{T}{2})$ implies that for all $p \in [2, \infty)$, $q, r \in (0, \infty)$, $\alpha \in (1/2, \infty)$, $M, N \in \mathbb{N}$ with $M > N$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ it holds that

$$\begin{aligned} &\sup_{t \in [0,T]} \left\| X_t^M - X_t^N \right\|_{L^r(\Omega; H)} \\ &\leq N^{-\alpha} \sqrt{2} \exp\Big(\big[\tfrac{1}{2} + \hat{\kappa} + \kappa_q + (p-1)\, \eta^2 + \tfrac{\beta}{q} \big] T \Big) \left| \mathbb{E}\Big[\exp\Big(\tfrac{(\varrho + \hat{\varrho})}{2} \, \|X_0\|_H^2 \Big) \Big] \right|^{1/q} \\ &\quad \cdot \left[1 + \eta \sqrt{p-1} + c + \sqrt{6}\, c \, \big[\textstyle\sum_{n=0}^{\infty} \left(\varrho + \pi^4 n^4 \right)^{-\alpha/2} \big]^{1/2} \right] \\ &\quad \cdot \max\Big(1, \sup_{t \in [0,T]} \|X_t^M\|^2_{L^{2p}(\Omega; H_{\alpha/4})}\Big). \end{aligned} \tag{94}$$

Fatou's lemma together with Cox, Hutzenthaler and Jentzen [12], Corollary 3.5 (with $H = H$, $U = U$, $A = A - \varrho$, $\mathbb{H} = \{e_1, e_2, \ldots\}$, $\mathcal{P}_N = \mathrm{Id}_U$, $\gamma = \theta$, $\alpha = \theta - \vartheta$, $\beta = \theta$, $F = F$, $B = B|_{H_\theta}$, $X^0 = X$, $X^M = X^M$ for $N \in \mathbb{N}_0$, $M \in \mathbb{N}$ in the notation of Cox, Hutzenthaler and Jentzen [12], Corollary 3.5) hence shows that for all $p \in [2, \infty)$, $q, r \in (0, \infty)$, $\alpha \in (1/2, \infty)$, $N \in \mathbb{N}$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ it holds that

$$\begin{aligned} &\sup_{t \in [0,T]} \left\| X_t - X_t^N \right\|_{L^r(\Omega; H)} \\ &\leq N^{-\alpha} \sqrt{2} \exp\Big(\big[\tfrac{1}{2} + p\, \eta^2 + \hat{\kappa} + \kappa_q + \tfrac{\beta}{q} \big] T \Big) \left| \mathbb{E}\Big[\exp\Big(\tfrac{(\varrho + \hat{\varrho})}{2} \, \|X_0\|_H^2 \Big) \Big] \right|^{1/q} \\ &\quad \cdot \left[1 + \eta \sqrt{p} + c + \sqrt{6}\, c \, \big[\textstyle\sum_{n=0}^{\infty} \left(\varrho + \pi^4 n^4 \right)^{-\alpha/2} \big]^{1/2} \right] \\ &\quad \cdot \max\Big\{1, \liminf_{M \to \infty} \sup_{t \in [0,T]} \|X_t^M\|^2_{L^{2p}(\Omega; H_{\alpha/4})}\Big\}. \end{aligned} \tag{95}$$

Combining this with, for exmaple, Cox, Hutzenthaler and Jentzen [13], Corollary 2.4, a standard bootstrap argument (cf., e.g., [41], Lemma 3.1 and Lemma 3.2), the fact that for all $\rho \in (0, \infty)$ it holds that $\mathbb{E}[\exp(\rho \|X_0\|_H^2)] < \infty$, and the hypothesis that $x_0 \in H_{1/2}$ demonstrates that for every $r \in (0, \infty)$, $\alpha \in (-\infty, 2)$ there exists $C \in \mathbb{R}$ such that for every $N \in \mathbb{N}$ it holds that

$$\sup_{t \in [0,T]} \|X_t - X_t^N\|_{L^r(\Omega; H)} \leq C\, N^{-\alpha}. \tag{96}$$

The proof of Corollary 3.10 is thus complete. □

Note that if $Q \in L(U)$ is a trace class operator (see, e.g., Prévôt and Röckner [61], Appendix B), if $k = l = 1$ in Setting 3.9, and if $B\colon H \to \mathrm{HS}(U, H)$ in Setting 3.9 satisfies that for all $u, v \in H$ it holds that $B(v)u = \{(\sqrt{Q}u)(x)\}_{x \in (0,1)}$, then $B\colon H \to \mathrm{HS}(U, H)$ in Setting 3.9 fulfills $\sup_{v, w \in H, v \neq w} \frac{\|B(v) - B(w)\|_{\mathrm{HS}(U,H)}}{\|v - w\|_H} = 0$ and

$$\begin{aligned} &\forall\, \varepsilon \in (0, \infty): \\ &\sup_{v \in H_\theta} \Big[\|(I - P_1) B(v)\|^2_{\mathrm{HS}(U,H)} - \varepsilon \, \|((I - P_1) v)^2\|_H^2 - \varepsilon \, \|(I - P_1) v\|_H^2 \, \|v\|_H^2 \Big] < \infty \end{aligned} \tag{97}$$

and in that case (74) reduces in the setting of Corollary 3.10 to the Cahn–Hilliard–Cook-type SPDE

$$dX_t(x) = \Big[-\tfrac{\partial^4}{\partial x^4} X_t(x) + c\,\tfrac{\partial^2}{\partial x^2}\{(X_t(x))^3 - X_t(x)\}\Big]\,dt + \sqrt{Q}\,dW_t(x) \tag{98}$$

for $x \in (0,1)$, $t \in [0,T]$ equipped with the standard Neumann and the nonflux boundary conditions $X_t'(0) = X_t'(1) = X_t'''(0) = X_t'''(1) = 0$ for $t \in [0,T]$ (cf., e.g., [2, 14]). Corollary 3.10 hence, in particular, ensures that for every arbitrarily small $\varepsilon \in (0,\infty)$ it holds that the spectral Galerkin approximations X^N, $N \in \mathbb{N}$, in (75) converge with the strong convergence order $2-\varepsilon$ to the solution process X of the Cahn–Hilliard–Cook-type equation in (98). Lower bounds for strong approximation errors for SPDEs in the literature demonstrate that the strong convergence rate $2-\varepsilon$ in Corollary 3.10 is essentially optimal and can, in general, not be improved (see, e.g., [11], Lemma 7.2). Further lower bounds for strong approximation errors for SPDEs can, for example, be found in [17, 40, 56–58].

3.2.3. *Stochastic Burgers equation.* In the following result, Corollary 3.11 below, we establish strong convergence rates for spatial spectral Galerkin approximations for certain stochastic Burgers equations. In our proof of Corollary 3.11 we apply Corollary 3.8 to these spatial spectral Galerkin approximations and for this application of Corollary 3.8 we construct in the proof of Corollary 3.11 a suitable function U such that (72) is satisfied (cf. (103) in the proof of Corollary 3.11 below).

COROLLARY 3.11. *Assume Setting* 3.9, *assume that* $D = (0,1)$, $k = 1$, $\varrho = 0$, $\theta = 1/4$, $\vartheta = -1/2$, $x_0 \in H_{1/2}$, *assume that A is the Laplacian with the standard Dirichlet boundary conditions on D, assume that* $B \colon H \to \mathrm{HS}(U,H)$ *is globally Lipschitz continuous, let* $e_n \in H$, $n \in \mathbb{N}$, *satisfy for all* $n \in \mathbb{N}$ *that* $e_n = \{\sqrt{2}\sin(n\pi(x))\}_{x\in(0,1)}$, *let* $c \in \mathbb{R}\backslash\{0\}$, $\eta \in (0,\infty)$, *and assume for all* $v \in H_{1/4} \subseteq L^4(D;\mathbb{R})$, $N \in \mathbb{N}$ *that* $P_N(v) = \sum_{n=1}^N \langle e_n, v\rangle_H\, e_n$, $F(v) = \frac{c}{2}\,(v^2)'$, *and* $\eta = \sup_{x\in H} \|B(x)\|^2_{\mathrm{HS}(U,H)}$. *Then for every* $r \in (0,\infty)$, $\alpha \in (-\infty, 1)$ *there exists* $C \in \mathbb{R}$ *such that for every* $N \in \mathbb{N}$ *it holds that*

$$\sup_{t\in[0,T]} \|X_t - X_t^N\|_{L^r(\Omega;H)} \le C\,N^{-\alpha}. \tag{99}$$

PROOF. Throughout this proof let $\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \in \mathbb{R}$ be the real number which satisfies that $\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} = \sup_{v,w\in H, v\neq w} \frac{\|B(v)-B(w)\|_{\mathrm{HS}(U,H)}}{\|v-w\|_H} < \infty$. Note that for all $M \in \mathbb{N}$, $x \in P_M(H)$, $\rho \in (0,\infty)$ it holds that

$$\begin{aligned} &\langle x, \mu_M(x)\rangle_H + \tfrac{1}{2}\,\|\sigma_M(x)\|^2_{\mathrm{HS}(U,P_M(H))} + \tfrac{\rho}{2}\,\|\sigma_M(x)^* x\|_U^2 \\ &\le \langle x, Ax\rangle_H + \tfrac{1}{2}\,\|B(x)\|^2_{\mathrm{HS}(U,H)} + \tfrac{\rho}{2}\,\|B(x)^*x\|_U^2 \le \tfrac{\eta}{2} + \tfrac{\rho\eta}{2}\,\|x\|_H^2 - \|x'\|_H^2 \\ &= \tfrac{\eta}{2} + \tfrac{\rho\eta}{2}\,\|x\|_H^2 - \tfrac{\rho\eta}{2\pi^2}\,\|x'\|_H^2 - \Big[1 - \tfrac{\rho\eta}{2\pi^2}\Big]\,\|x'\|_H^2 \le \tfrac{\eta}{2} - \Big[1 - \tfrac{\rho\eta}{2\pi^2}\Big]\,\|x'\|_H^2. \end{aligned} \tag{100}$$

Hence, we obtain that for all $N, M \in \mathbb{N}$, $x \in P_M(H)$, $y \in P_N(H)$, $p \in [2,\infty)$, $\varepsilon \in (0,\infty)$ with $N < M$ it holds that

$$\begin{aligned} &\langle P_N x - y, \mu_M(P_N x) - \mu_M(y)\rangle_H + \tfrac{(p-1)\,(1+\varepsilon)}{2}\,\|\sigma_M(P_N x) - \sigma_M(y)\|^2_{\mathrm{HS}(U,P_M(H))} \\ &\quad + \langle y - P_N x, P_N \mu_M(P_N x) - P_N \mu_M(x)\rangle_H \\ &\quad + \tfrac{(p-1)\,(1+1/\varepsilon)}{2}\,\|\sigma_M(P_N x) - \sigma_M(x)\|^2_{\mathrm{HS}(U,P_M(H))} \\ &\le -\|(P_N x - y)'\|_H^2 + \tfrac{c}{4}\Big\langle (P_N x - y)^2, (P_N x + y)'\Big\rangle_H \\ &\quad + \tfrac{(p-1)\,(1+\varepsilon)\,\|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))}\,\|P_N x - y\|_H^2}{2} \end{aligned} \tag{101}$$

$$- \tfrac{c}{2} \big\langle (y - P_N x)', ((P_N - I)x)\,(P_N x + x) \big\rangle_H$$
$$+ \tfrac{(p-1)\,(1+1/\varepsilon)}{2} \, \|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \, \|(I - P_N)x\|^2_H \,.$$

Next let $\kappa\colon (0,\infty) \to (0,\infty)$ be a strictly decreasing function which satisfies for all $v \in D(A)$, $r \in (0,\infty)$ that $\frac{1}{32r} \, \|v\|^2_{L^\infty((0,1);\mathbb{R})} \le \kappa(r) \, \|v\|^2_H + \frac{1}{2} \, \|v'\|^2_H$ (cf., e.g., Sell and You [67], Theorem 37.2). Note that Young's inequality shows that for all $N, M \in \mathbb{N}$, $x \in P_M(H)$, $y \in P_N(H)$, $p \in [2,\infty)$, $\varepsilon, \delta \in (0,\infty)$ with $N < M$ it holds that

$$\begin{aligned}
&\langle P_N x - y, \mu_M(P_N x) - \mu_M(y) \rangle_H + \tfrac{(p-1)\,(1+\varepsilon)}{2} \, \|\sigma_M(P_N x) - \sigma_M(y)\|^2_{\mathrm{HS}(U,P_M(H))} \\
&\quad + \langle y - P_N x, P_N \mu_M(P_N x) - P_N \mu_M(x) \rangle_H \\
&\quad + \tfrac{(p-1)\,(1+1/\varepsilon)}{2} \, \|\sigma_M(P_N x) - \sigma_M(x)\|^2_{\mathrm{HS}(U,P_M(H))} \\
&\le \Big[\tfrac{c^2 \delta}{2} \, \|(P_N x + y)'\|^2_H + \tfrac{(p-1)\,(1+\varepsilon)}{2} \, \|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \Big] \, \|P_N x - y\|^2_H \\
&\quad + \tfrac{1}{32\delta} \, \|P_N x - y\|^2_{L^\infty((0,1);\mathbb{R})} - \tfrac{1}{2} \, \|(P_N x - y)'\|^2_H \\
&\quad + \Big[\tfrac{c^2}{8} \, \|P_N x + x\|^2_{L^\infty((0,1);\mathbb{R})} + \tfrac{(p-1)\,(1+1/\varepsilon)}{2} \, \|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \Big] \, \|(I - P_N)x\|^2_H \\
&\le \Big[\kappa(\delta) + c^2 \delta \, \|x'\|^2_H + c^2 \delta \, \|y'\|^2_H + \tfrac{(p-1)\,(1+\varepsilon)}{2} \, \|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \Big] \, \|P_N x - y\|^2_H \\
&\quad + \tfrac{1}{2} \Big[\tfrac{c^2}{4} \, \|P_N x + x\|^2_{L^\infty((0,1);\mathbb{R})} + \tfrac{(p-1)\,(1+1/\varepsilon)}{2} \|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \Big] \, \|(I - P_N)x\|^2_H \,.
\end{aligned} \tag{102}$$

Combining (100) and (102) allows us to apply Corollary 3.8 (with $\beta = \frac{\eta}{2}$, $\mathcal{U}(x) = [1 - \frac{\rho\eta}{2\pi^2}]\|x'\|^2_H$, $c = \kappa(\delta) + \frac{(p-1)(1+\varepsilon)}{2} \|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))}$ for $x \in P_M(H)$, $M \in \mathbb{N}$ in the notation of Corollary 3.8) to obtain that for all $N, M \in \mathbb{N}$, $r, q, \varepsilon, \delta, \rho \in (0,\infty)$, $p \in [2,\infty)$ with $N < M$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $c^2 \delta \le \frac{\rho}{2q} \left[1 - \frac{\rho\eta}{2\pi^2} \right]$ it holds that

$$\begin{aligned}
&\sup_{t \in [0,T]} \left\| X^M_t - X^N_t \right\|_{L^r(\Omega;H)} \le \sup_{t \in [0,T]} \left\| (I - P_N) X^M_t \right\|_{L^r(\Omega;H)} \\
&\quad + T^{(\frac{1}{2} - \frac{1}{p})} \exp\Big(\tfrac{1}{2} - \tfrac{1}{p} + \Big[\kappa(\delta) + \tfrac{(p-1)\,(1+\varepsilon)}{2} \, \|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \Big] T + \tfrac{\eta\rho T}{2q} \Big) \\
&\quad \cdot \Big| \mathbb{E}\Big[e^{\frac{\rho}{2} \|X^M_0\|^2_H} \Big] \Big|^{1/q} \\
&\quad \cdot \bigg\| \Big[\tfrac{|c|}{2} \left\| P_N X^M + X^M \right\|_{L^\infty(D;\mathbb{R})} + \sqrt{(p-1)\,(1 + \tfrac{1}{\varepsilon})} \, \|B\|_{\mathrm{Lip}(H,\mathrm{HS}(U,H))} \Big] \\
&\qquad \cdot \left\| (I - P_N) X^M \right\|_H \bigg\|_{L^p(\llbracket 0,T \rrbracket;\mathbb{R})} .
\end{aligned} \tag{103}$$

The fact that for all $N \in \mathbb{N}$, $\alpha \in (1/2,\infty)$, $v \in H_{\alpha/2}$ it holds that

$$\begin{aligned}
&\|P_N v\|_{L^\infty(D;\mathbb{R})} \le \sum_{n=1}^{N} |\langle e_n, v \rangle_H| \, \|e_n\|_{L^\infty(D;\mathbb{R})} \le \frac{\sqrt{2}}{\pi^\alpha} \left[\sum_{n=1}^{N} n^{-\alpha} \left(\pi^\alpha \, n^\alpha \, |\langle e_n, v \rangle_H| \right) \right] \\
&\le \frac{\sqrt{2}}{\pi^\alpha} \left[\sum_{n=1}^{\infty} n^{-2\alpha} \right]^{1/2} \left[\sum_{n=1}^{\infty} \pi^{2\alpha} \, n^{2\alpha} \, |\langle e_n, v \rangle_H|^2 \right]^{1/2} = \frac{\sqrt{2} \, \|v\|_{H_{\alpha/2}}}{\pi^\alpha} \left[\sum_{n=1}^{\infty} n^{-2\alpha} \right]^{1/2} \\
&\le \left[\textstyle\sum_{n=1}^{\infty} n^{-2\alpha} \right]^{1/2} \, \|v\|_{H_{\alpha/2}},
\end{aligned} \tag{104}$$

the fact that for all $N \in \mathbb{N}$, $\alpha \in (0, \infty)$, $v \in H_{\alpha/2}$ it holds that

$$\begin{aligned}
&\|(I - P_N)v\|_H = \left\|(-A)^{-\alpha/2}(I - P_N)(-A)^{\alpha/2}v\right\|_H \\
&\leq \left\|(-A)^{-\alpha/2}(I - P_N)\right\|_{L(H)} \left\|(-A)^{\alpha/2}v\right\|_H = \left\|(-A)^{-\alpha/2}(I - P_N)\right\|_{L(H)} \|v\|_{H_{\alpha/2}} \\
&= \left[(N+1)^2\pi^2\right]^{-\alpha/2} \|v\|_{H_{\alpha/2}} = (N+1)^{-\alpha}\,\pi^{-\alpha}\,\|v\|_{H_{\alpha/2}} \leq N^{-\alpha}\,\pi^{-\alpha}\,\|v\|_{H_{\alpha/2}},
\end{aligned} \tag{105}$$

and the fact that $T^{1/2} \leq \exp(\frac{(T-1)}{2})$ hence imply that for all $N, M \in \mathbb{N}$, $r, q, \delta, \rho \in (0, \infty)$, $\alpha \in (1/2, \infty)$, $p \in [2, \infty)$ with $N < M$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and $c^2\delta \leq \frac{\rho}{2q}\left[1 - \frac{\rho\eta}{2\pi^2}\right]$ it holds that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\|X_t^M - X_t^N\right\|_{L^r(\Omega;H)} \\
&\leq N^{-\alpha} \exp\Big(\Big[\tfrac{q+\eta\rho}{2q} + \kappa(\delta) + p\,\|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))}\Big]\,T\Big) \left|\mathbb{E}\Big[e^{\frac{\rho}{2}\|X_0\|_H^2}\Big]\right|^{1/q} \\
&\quad \cdot \Big[1 + |c|\Big[\textstyle\sum_{n=1}^\infty n^{-2\alpha}\Big]^{1/2} + \sqrt{p}\,\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(U,H))}\Big] \\
&\quad \cdot \max\Big(1, \sup_{t\in[0,T]} \|X_t^M\|^2_{L^{2p}(\Omega;H_{\alpha/2})}\Big).
\end{aligned} \tag{106}$$

Combining Fatou's lemma and, for example, Cox, Hutzenthaler and Jentzen [12], Corollary 3.5, therefore shows that for all $N \in \mathbb{N}$, $r, q \in (0, \infty)$, $\alpha \in (1/2, \infty)$, $\rho \in (0, \frac{2\pi^2}{\eta})$, $p \in [2, \infty)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ it holds that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\|X_t - X_t^N\right\|_{L^r(\Omega;H)} \\
&\leq \exp\Big(\tfrac{(q+\eta\rho)T}{2q} + \kappa\big(\tfrac{\rho\,[2\pi^2-\rho\eta]}{4qc^2\pi^2}\big)T + pT\,\|B\|^2_{\mathrm{Lip}(H,\mathrm{HS}(U,H))}\Big) \left|\mathbb{E}\Big[e^{\frac{\rho}{2}\|X_0\|_H^2}\Big]\right|^{1/q} \\
&\qquad \cdot \left[1 + |c|\sqrt{\textstyle\sum_{n=1}^\infty \frac{1}{n^{2\alpha}}} + \sqrt{p}\,\|B\|_{\mathrm{Lip}(H,\mathrm{HS}(U,H))}\right] \\
&\qquad \cdot \max\left(1, \liminf_{M\to\infty} \sup_{t\in[0,T]} \|X_t^M\|^2_{L^{2p}(\Omega;H_{\alpha/2})}\right) N^{-\alpha}.
\end{aligned} \tag{107}$$

Combining this with, for example, [13], Corollary 2.4, a standard bootstrap argument (cf., e.g., [41], Lemmas 3.1–3.2), the fact that $\inf_{\rho\in(0,\infty)} \mathbb{E}[\exp(\rho\|X_0\|_H^2)] < \infty$, and the hypothesis that $x_0 \in H_{1/2}$ demonstrates that for every $r \in (0, \infty)$, $\alpha \in (-\infty, 1)$ there exists $C \in \mathbb{R}$ such that for every $N \in \mathbb{N}$ it holds that

$$\sup_{t\in[0,T]} \|X_t - X_t^N\|_{L^r(\Omega;H)} \leq C\,N^{-\alpha}. \tag{108}$$

The proof of Corollary 3.11 is thus complete. □

Note that if $b\colon (0, 1) \times \mathbb{R} \to \mathbb{R}$ is a globally bounded function with a globally bounded continuous derivative, if $Q \in L(U)$ is a trace class operator (see, e.g., Prévôt and Röckner [61], Appendix B), if $k = l = 1$ in Setting 3.9, and if $B\colon H \to \mathrm{HS}(U, H)$ in Setting 3.9 satisfies that for all $u, v \in H = U$ it holds that $B(v)u = \{b(x, v(x))\,(\sqrt{Q}u)(x)\}_{x\in D}$, then $B\colon H \to \mathrm{HS}(U, H)$ in Setting 3.9 fulfills the assumption in Corollary 3.11 that $\sup_{v\in H} \|B(v)\|^2_{\mathrm{HS}(U,H)} + \sup_{v,w\in H, v\neq w} \frac{\|B(v)-B(w)\|_{\mathrm{HS}(U,H)}}{\|v-w\|_H} < \infty$ and in that case (74) reduces in the setting of Corollary 3.11 to the stochastic Burgers equation

$$dX_t(x) = \Big[\tfrac{\partial^2}{\partial x^2} X_t(x) + c\,X_t(x)\,\tfrac{\partial}{\partial x} X_t(x)\Big]\,dt + b(x, X_t(x))\sqrt{Q}\,dW_t(x) \tag{109}$$

for $x \in (0,1)$, $t \in [0,T]$ equipped with the standard Dirichlet boundary conditions $X_t(0) = X_t(1) = 0$ for $t \in [0,T]$. Corollary 3.11 hence, in particular, ensures that for every arbitrarily small $\varepsilon \in (0,\infty)$ it holds that the spectral Galerkin approximations X^N, $N \in \mathbb{N}$, in (75) converge with the strong convergence order $1-\varepsilon$ to the solution process X of the stochastic Burgers equation in (109). Lower bounds for strong approximation errors for SPDEs in the literature demonstrate that the strong convergence rate $1-\varepsilon$ in Corollary 3.11 is essentially optimal and can, in general, not be improved (see, e.g., [11], Lemma 7.2). Further lower bounds for strong approximation errors for SPDEs can, for example, be found in [17, 40, 56–58].

3.3. *Analysis of SDEs with small noise.* In this subsection, we use Corollary 2.12 to study perturbations of deterministic ordinary differential equations and deterministic partial differential equations by a small noise term.

COROLLARY 3.12. *Assume Setting 1.5, let $\varepsilon \in [0,\infty)$, $\mu \in \mathcal{L}^0(\mathcal{O}; H)$, $\sigma \in \mathcal{L}^0(\mathcal{O}; \mathrm{HS}(U,H))$, let $\tau\colon \Omega \to [0,T]$ be a stopping time, let $X, Y\colon [0,T]\times\Omega \to \mathcal{O}$ be adapted stochastic processes with continuous sample paths, assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $\int_0^\tau \frac{1}{\|X_s - Y_s\|_H^2}[\langle X_s - Y_s, \mu(X_s) - \mu(Y_s)\rangle_H]^+ \, ds + \int_0^T \|\sigma(Y_s)\|^2_{\mathrm{HS}(U,H)} + \|\mu(X_s)\|_H + \|\mu(Y_s)\|_H \, ds < \infty$, $X_t = X_0 + \int_0^t \mu(X_s)\,ds$, and $Y_t = X_0 + \int_0^t \mu(Y_s)\,ds + \int_0^t \varepsilon\,\sigma(Y_s)\,dW_s$. Then for all $\rho, r \in (0,\infty)$, $p \in [2,\infty)$, $q \in (0,\infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ it holds that*

$$
\begin{aligned}
&\|X_\tau - Y_\tau\|_{L^r(\Omega;H)} \\
&\le \varepsilon\,\rho^{(\frac{1}{2}-\frac{1}{p})}\sqrt{p-1}\,\|\sigma(Y)\|_{L^p([\![0,\tau]\!];\mathrm{HS}(U,H))} \\
&\quad\cdot \left\| \exp\left(\int_0^\tau \left[\frac{\langle X_s - Y_s, \mu(X_s) - \mu(Y_s)\rangle_H}{\|X_s - Y_s\|_H^2} + \frac{(\frac{1}{2}-\frac{1}{p})}{\rho} \right]^+ ds \right) \right\|_{L^q(\Omega;\mathbb{R})}.
\end{aligned} \tag{110}
$$

Corollary 3.12 follows immediately from Corollary 2.12 (with $\sigma(x) = 0$, $a_s = \mu(Y_s)$, $b_s = \varepsilon\sigma(Y_s)$, $\varepsilon = \infty$ for $x \in \mathcal{O}$, $s \in [0,T]$ in the notation of Corollary 2.12). If the processes X and Y in Corollary 3.12 enjoy suitable exponential integrability properties (see, e.g., Cox, Hutzenthaler and Jentzen [13], Corollary 2.4), then the right-hand side of (110) can be further estimated in an appropriate way. Corollary 3.12 can be applied to a number of nonlinear ordinary and partial differential equation perturbed by a small noise term such as the examples in Sections 3.1.2–3.1.7 as well as the examples in Sections 3.2.2–3.2.3.

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