The Annals of Probability
2019, Vol. 47, No. 6, 4077–4115
https://doi.org/10.1214/19-AOP1354

HEAVY BERNOULLI-PERCOLATION CLUSTERS ARE INDISTINGUISHABLE

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We prove that the heavy clusters are indistinguishable for Bernoulli percolation on quasi-transitive nonunimodular graphs. As an application, we show that the uniqueness threshold of any quasi-transitive graph is also the threshold for connectivity decay. This resolves a question of Lyons and Schramm (*Ann. Probab.* **27** (1999) 1809–1836) in the Bernoulli percolation case and confirms a conjecture of Schonmann (*Comm. Math. Phys.* **219** (2001) 271–322). We also prove that every infinite cluster of Bernoulli percolation on a nonamenable quasi-transitive graph is transient almost surely.

1. Introduction. Let $G = (V, E)$ be a connected, locally finite, quasi-transitive infinite graph and for simplicity we will just say "let G be a quasi-transitive graph" hereafter. We allow multiple edges and loops in G but we will always assume G is locally finite. Fix some parameter $p \in [0, 1]$, and consider Bernoulli(p) percolation process on G. The critical probability is defined as

$$p_c = p_c(G) := \inf\{p \in [0, 1] : \text{a.s. there exists an infinite open cluster}\}.$$

Grimmett and Newman [13] gave an example showing that there are some $p \in (0, 1)$ such that for Bernoulli(p) percolation on $T \times \mathbb{Z}$, a.s. there exist infinite many infinite clusters, where T is a regular tree with high degree. Later Benjamini and Schramm [6] conjectured that if G is a quasi-transitive nonamenable graph, then $p_c < p_u$, where the uniqueness threshold p_u is defined as follows:

$$p_u = p_u(G) := \inf\{p \in [0, 1] : \text{a.s. there exists a unique infinite open cluster}\}.$$

If G is a quasi-transitive amenable graph, then there is at most one infinite cluster for Bernoulli percolation on G; see [7] and [10] for more details.

Recently, for all quasi-transitive graphs whose automorphism group has a quasi-transitive nonunimodular subgroup, the above conjecture has been proved by Hutchcroft [17]. The conjecture has also been shown to hold for many nonamenable unimodular graphs of special types. For details, see the discussion in [17] and references therein.

Historically, many properties for percolation processes on transitive graphs are first proved in the unimodular case [4, 14, 30] while the nonunimodular case are

Received September 2018; revised March 2019.
MSC2010 subject classifications. Primary 60K35, 82B43; secondary 60B99, 60C05.
Key words and phrases. Bernoulli percolation, unimodularity, indistinguishability, heavy clusters, quasi-transitive graph.

proved later [16, 26] or remain open. The reason is that the mass transport principle [3] is a very powerful technique in the unimodular case. One interesting fact about Hutchcroft's result in [17] is that he proved the above conjecture for nonunimodular case first while general unimodular case remains open. The present paper also mainly focuses on nonunimodular quasi-transitive graphs.

If G is a quasi-transitive graph with $p_c < p_u$ and $p \in (p_c, p_u)$, then Bernoulli(p) percolation has infinitely many infinite open clusters. Are these infinite open clusters similar or different? Lyons and Schramm [21] showed for every graph G with a transitive unimodular closed automorphism group $\Gamma \subset \text{Aut}(G)$, every Γ-invariant, insertion-tolerant percolation process on G has indistinguishable infinite clusters.

Suppose G is a quasi-transitive graph and suppose $\Gamma \subset \text{Aut}(G)$ is closed, nonunimodular and acts quasi-transitively on G. Let m denote the Haar measure on Γ (m is unique up to a multiplicative constant). In particular, let $m(x) := m(\Gamma_x)$, where $\Gamma_x := \{\gamma \in \Gamma : \gamma x = x\}$ denotes the stabilizer of $x \in V$. Then there are two types of infinite clusters: for an infinite cluster C, if $\sum_{x \in C} m(x) < \infty$, it is called a ($\Gamma$-)light cluster; otherwise it is called a (Γ-)heavy cluster. For the nonunimodular case, Lyons and Schramm [21] also pointed out that light clusters can be distinguished by Γ-invariant properties and they asked whether heavy clusters are indistinguishable (Question 3.17 there).

Here we give a positive answer and the exact definitions of indistinguishability and Γ-invariant properties are given later in Definition 2.5 and Definition 2.4.

THEOREM 1.1. *Suppose G is a locally finite, connected infinite graph, and suppose that* Aut(G) *has a closed, quasi-transitive and nonunimodular subgroup Γ. If there are infinite many Γ-heavy clusters a.s. for* Bernoulli(p) *percolation on G, then they are indistinguishable by Γ-invariant properties.*

Question 3.17 in [21] was indeed asked for general insertion-tolerant percolation processes. Here we only have a positive answer for the Bernoulli percolation case. The general case is still open. For more discussion on the general case, see the last section.

Let d be the graph distance on G and $B(x, N) := \{z \in V : d(z, x) \le N\}$ be the ball centered at x with radius N. Let $B(x, N) \leftrightarrow B(y, N)$ denote the event that there is an open path connecting some vertex $u \in B(x, N)$ and some vertex $v \in B(y, N)$. In particular, we use $x \leftrightarrow y$ to denote the event that there is an open path connecting the two vertices x, y. Schonmann [26] also proved a criterion of p_u for all quasi-transitive graphs:

$$p_u = \inf\Big\{p \in [0, 1] : \lim_{N\to\infty} \inf_{x,y\in V} \mathbf{P}_p\big(B(x, N) \leftrightarrow B(y, N)\big) = 1\Big\}.$$

For unimodular transitive graphs, Lyons and Schramm [21] gave another characterization of p_u: let G be a unimodular transitive graph and $o \in V$ be a fixed vertex. Then

$$p_u = \inf\Big\{p \in [0, 1] : \inf_{x\in V} \mathbf{P}_p(o \leftrightarrow x) > 0\Big\}.$$

Schonmann then conjectured this is also true for nonunimodular case (Conjecture 3.1 in [28]). Following Schonmann [28], we denote the right-hand side of the above equality as follows.

DEFINITION 1.2.

$$\overline{p}_{\mathrm{conn}} := \inf\Big\{p \in [0,1] : \inf_{x \in V} \mathbf{P}_p(o \leftrightarrow x) > 0\Big\}.$$

Note the right-hand side does not depend on the choice of $o \in V$ by the Harris-FKG inequality. Since $\mathbb{P}_p(o \leftrightarrow x)$ is a nondecreasing function of p, we also have

$$\overline{p}_{\mathrm{conn}} = \sup\Big\{p \in [0,1] : \inf_{x \in V} \mathbf{P}_p(o \leftrightarrow x) = 0\Big\}.$$

An application of the above Theorem 1.1 is the following extension of Theorem 4.1 of [21] in the Bernoulli percolation setting for all quasi-transitive graphs (we use Theorem 1.1 to deal with the nonunimodular case while the unimodular case is already proved in Theorem 4.1 of [21]), and this also confirms Schonmann's Conjecture 3.1 in [28].

THEOREM 1.3. *Suppose $G = (V, E)$ is a quasi-transitive graph and $\mathbf{P}_p$ is a Bernoulli bond percolation on G. If $\mathbf{P}_p$ has more than one infinite cluster a.s., then connectivity decays, that is,*

$$\inf\{\mathbf{P}_p(x \leftrightarrow y) : x, y \in V\} = 0.$$

In particular,

$$p_u(G) = \overline{p}_{\mathrm{conn}}.$$

REMARK 1.4. The above theorems also hold for Bernoulli site percolation by similar arguments.

The proofs of the above two theorems follow similar strategies as the ones of unimodular transitive case. The main new ingredient is that we introduce certain "biased" random walks to replace the role of simple random walk in the study of properties of Bernoulli percolation clusters on nonunimodular quasi-transitive graphs. Mass transport principle in its general form also plays a key role in the proof of the stationarity of the "biased" random walks.

Grimmett, Kesten and Zhang [12] first proved that the infinite cluster for supercritical Bernoulli percolation on $\mathbb{Z}^d$ with $d \geq 3$ is transient for simple random walk. In [21], Lyons and Schramm showed that if G is a locally finite, connected infinite graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$, and $(\mathbf{P}, \omega)$ is a Γ-invariant insertion-tolerant percolation process on G that has almost surely infinitely many infinite clusters, then a.s. each infinite cluster is transient (Proposition 3.11 of [21]). Benjamini, Lyons and Schramm conjectured that if G

is a transient Cayley graph, then a.s. every infinite cluster of Bernoulli percolation on G is transient (Conjecture 1.7 in [5]). One may even conjecture that the same conclusion is true for all transient quasi-transitive graphs. Here we give a positive answer for the nonamenable quasi-transitive case. The case for general amenable quasi-transitive graphs remains open.

THEOREM 1.5. *Suppose G is a quasi-transitive nonamenable graph. Then a.s. every infinite cluster in Bernoulli percolation is transient for simple random walk.*

Benjamini, Lyons and Schramm proved a stronger result (Theorem 1.3 in [5]) for nonamenable Cayley graphs, namely for Bernoulli percolation on such graphs, simple random walk on the infinite clusters has positive speed a.s. Their result can be easily generalized to hold for every quasi-transitive unimodular nonamenable graph. So to prove Theorem 1.5, it suffices to deal with the nonunimodular case; see Proposition 5.12.

The paper is organized as follows: in Section 2, we introduce some preliminary results and notation. In Section 3, we review the tilted mass transport principle introduced in [17]. Some properties of heavy clusters are discussed in Section 4, where their proofs also illustrate the applications of the tilted mass transport principle. We introduce the "biased" random walks in Section 5 and the stationary property for them. We also prove Theorem 1.5 in Section 5. We prove Theorem 1.1 and Theorem 1.3 in Section 6. Finally, in Section 7 we discuss some examples of nonunimodular transitive graphs and also give some further questions.

2. Preliminary. Let $G = (V, E)$ be a locally finite, connected infinite unoriented graph with vertex set V and edge set E. If $e = (u, v) \in E$, we say u, v are adjacent and denote by $u \sim v$. An automorphism of G is a bijection from $V = V(G)$ to itself which preserves adjacency. Let $\mathrm{Aut}(G)$ denote the group of all automorphisms of G. With the topology of pointwise convergence, $\mathrm{Aut}(G)$ is a locally compact Hausdorff topological group. Suppose $\Gamma \subset \mathrm{Aut}(G)$ is a closed subgroup with the induced topology. For any $v \in V$, the orbit of v under Γ is denoted by $\Gamma v := \{\gamma v : \gamma \in \Gamma\}$. We use $G/\Gamma := \{\Gamma v : v \in V\}$ to denote the orbit sets and say that Γ is *quasi-transitive* or Γ acts on G quasi-transitively if there are only finite many orbits, namely G/Γ is a finite set. In particular, if G/Γ has a single element, then we say Γ is *transitive* or Γ acts on G transitively. The graph G is called quasi-transitive (transitive) if $\mathrm{Aut}(G)$ acts on G quasi-transitively (transitively).

An infinite set of vertices $V_0 \subset V$ is called end convergent if for any finite set $K \subset V$, there is a connected component of $G \backslash K$ that contains all but finitely many vertices of V_0. And two end convergent sets V_0, V_1 are said to be equivalent if $V_0 \cup V_1$ is also end convergent. Now an end of G is defined to be an equivalence class of end-convergent sets. For example, suppose G is an infinite tree. Fix some $\rho \in V$ and call it the root of G. Then each end corresponds bijectively to a ray starting from the root ρ. The number of ends of G is equal to the supremum of the number of components of $G\backslash K$ over all finite sets $K \subset V$. For example, $\mathbb{Z}$ has two ends and $\mathbb{Z}^d$ has one end for $d \geq 2$.

Recall that on every locally compact Hausdorff topological group Γ, there is a unique (up to a multiplicative constant) Borel measure $|\cdot|$ that is invariant under left multiplication, namely $|A| = |\gamma A|$ for every Borel set $A \subset \Gamma$ and all $\gamma \in \Gamma$. This measure m is called the (left) Haar measure. If it is also invariant under right multiplication, then Γ is called a *unimodular group*. A quasi-transitive graph G is said to be unimodular if its automorphism group is unimodular.

For each $x \in V$, $\Gamma_x := \{\gamma \in \Gamma : \gamma x = x\}$ is called the stabilizer of x. Let $m(x) = |\Gamma_x|$ denote the Haar measure of the stabilizer of x. And $\Gamma_x y := \{\gamma y : \gamma \in \Gamma_x\}$ is the orbit of y under Γ_x.

Examples of unimodular graphs include transitive amenable graphs [29] and Cayley graphs. Grandparent graphs [33] and Diestel–Leader graphs [36] $\mathrm{DL}(k, n)$ with $k \neq n$ are typical examples of nonunimodular transitive graphs. More examples of nonunimodular transitive graphs can be found in Timár's paper [31].

Even if a quasi-transitive graph G itself is unimodular, there might exist a subgroup $\Gamma \subset \mathrm{Aut}(G)$ that is nonunimodular and acts on G quasi-transitively. Based on [29], such graphs must be nonamenable. A well-known example is the regular tree T_3 with degree 3. The regular tree T_3 is a unimodular transitive graph. Fix an end ξ of T_3. Let Γ be the subgroup of all automorphisms that fix ξ. Then Γ is nonunimodular and acts on T_3 transitively. Other examples will be discussed in more detail in the last section.

Suppose $\Gamma \subset \mathrm{Aut}(G)$ acts on G quasi-transitively. A simple criterion for unimodularity of Γ is provided by the following proposition.

PROPOSITION 2.1 (Trofimov [33]). *Γ is unimodular if and only if for all x, y in the same orbit under Γ,*

$$|\Gamma_x y| = |\Gamma_y x|,$$

where $|\Gamma_x y|$ denotes the number of elements in the set $\Gamma_x y$.

From this proposition, one can easily see that in the above example T_3, the subgroup Γ fixing a specific end ξ is nonunimodular.

A well-known result concerning the Haar measure is the following lemma (for a proof see, e.g., formula (1.28) and Lemma 1.29 in [35]).

LEMMA 2.2. *Suppose $\Gamma \subset \mathrm{Aut}(G)$ acts on $G = (V, E)$ quasi-transitively, then for all $x, y \in V$,*

$$\frac{m(x)}{m(y)} = \frac{|\Gamma_x y|}{|\Gamma_y x|} = \frac{m(\gamma x)}{m(\gamma y)}, \quad \forall \gamma \in \Gamma.$$

A well-known criterion for unimodularity is the following simple application of Lemma 2.2.

LEMMA 2.3. *Suppose $\Gamma \subset \mathrm{Aut}(G)$ acts on $G = (V, E)$ quasi-transitively and m is an associated Haar measure, then Γ is unimodular if and only if $\{m(\Gamma_x) : x \in V\}$ is a finite set.*

PROOF. If Γ is unimodular, then by Proposition 2.1 and Lemma 2.2 $m(\Gamma_x)$ is a constant on each orbit. Since there are only finitely many orbits, $\{m(\Gamma_x), x \in V\}$ is finite.

If Γ is nonunimodular, then by Proposition 2.1 and Lemma 2.2 there exist $x \in V$, $\gamma \in \Gamma$ such that $M := \frac{m(\gamma x)}{m(x)} > 1$. Since by Lemma 2.2, one has $\frac{m(\gamma^n x)}{m(\gamma^{n-1} x)} = \frac{m(\gamma^{n-1} x)}{m(\gamma^{n-2} x)} = \cdots = \frac{m(\gamma x)}{m(x)} = M > 1$, $\frac{m(\gamma^n x)}{m(x)} = M^n \to \infty$ as $n \to \infty$, in particular $\{m(\Gamma_x), x \in V\}$ is not a finite set. $\square$

Now we recall some terminology from percolation theory. Suppose $G = (V, E)$ is a locally finite, connected graph. Let 2^E be the collection of all subsets $\eta \subset E$. Let $\mathcal{F}_E$ be the σ-field generated by sets of the form $\{\eta : e \in \eta\}$ where $e \in E$. A bond percolation on G is a pair $(\mathbf{P}, \omega)$, where ω is a random element in 2^E and $\mathbf{P}$ is the law of ω. For simplicity, sometimes we will just say ω is a bond percolation. One can also define site percolation and mixed percolation on G and the interested reader can refer to [20] or [11] for more background on percolation. If ω is a bond percolation, then $\widehat{\omega} := V \cup \omega$ is a mixed percolation on $2^{V \cup E}$. We will think of ω as a subgraph of G and not bother to distinguish ω and $\widehat{\omega}$.

We will mainly focus on Bernoulli bond percolation on G, which we now define. The Bernoulli(p) bond percolation on G is a pair $(\mathbf{P}_p, \omega)$ that satisfies $\mathbf{P}_p$ is a product measure on 2^E and $\mathbf{P}_p[e \in \omega] = p$ for all edge $e \in E$. Sometimes we also write Bernoulli(p) percolation as Bernoulli p-percolation.

If $v \in V$ and ω is a bond percolation on G, the cluster $C_\omega(v)$ (or just $C(v)$) of ω is defined as the connected component of v in ω. We sometimes also use $C(v)$ to denote the set of vertices (or edges) of the connected component of v in ω.

If $(\mathbf{P}, \omega)$ is a bond percolation on G and Γ is a subgroup of $\mathrm{Aut}(G)$, we call $(\mathbf{P}, \omega)$ Γ-invariant if $\mathbf{P}$ is invariant under each $\gamma \in \Gamma$. Bernoulli percolation is obviously $\mathrm{Aut}(G)$-invariant, in particular Bernoulli percolation is Γ-invariant for any subgroup $\Gamma \subset \mathrm{Aut}(G)$.

Next, we recall some definitions for indistinguishability of infinite clusters. The following definition of invariant property is adapted from Definition 1.8 of [19].

DEFINITION 2.4. Suppose $G = (V, E)$ is locally finite, connected infinite graph and Γ is a closed quasi-transitive subgroup of $\mathrm{Aut}(G)$. For a measurable set $\mathcal{A}$ of $V \times \{0, 1\}^E$, we say $\mathcal{A}$ is a Γ-*invariant* property, if $\gamma\mathcal{A} = \mathcal{A}$ for every $\gamma \in \Gamma$ and

$$(v, \omega) \in \mathcal{A} \quad \Rightarrow \quad \forall u \in C_\omega(v), \quad (u, \omega) \in \mathcal{A}.$$

We say that a cluster C of ω has property $\mathcal{A}$ (and abuse notation by writing $C \in \mathcal{A}$) if $(u, \omega) \in \mathcal{A}$ for some (and hence every) vertex $u \in C$.

For example, $\mathcal{A}$ might be the collection of all connected heavy subgraphs of G, or the collection of all recurrent subgraphs of G. For other examples, see [19, 21].

DEFINITION 2.5 (Definition 3.1 of [21]). Suppose $G = (V, E)$ is locally finite, connected infinite graph and Γ is a closed quasi-transitive subgroup of $\mathrm{Aut}(G)$. Let $(\mathbf{P}, \omega)$ be a Γ-invariant bond percolation process on G. We say that $\mathbf{P}$ has (Γ-) *indistinguishable infinite clusters* if for every Γ-invariant property $\mathcal{A}$, almost surely, for all infinite clusters C of ω, the cluster C has property $\mathcal{A}$, or none of the infinite clusters has property $\mathcal{A}$.

DEFINITION 2.6 (Definition 3.2 of [21]). Let $G = (V, E)$ be a graph. Given a configuration $\omega \in 2^E$ and an edge $e \in E$, denote $\Pi_e\omega := \omega \cup \{e\}$. For a set $\mathcal{A} \subset 2^E$, we write $\Pi_e\mathcal{A} := \{\Pi_e\omega : \omega \in \mathcal{A}\}$. A bond percolation process $(\mathbf{P}, \omega)$ on G is said to be *insertion tolerant* if $\mathbf{P}[\Pi_e\mathcal{A}] > 0$ for every $e \in E$ and every measurable $\mathcal{A} \subset 2^E$ with $\mathbf{P}[\mathcal{A}] > 0$. A percolation ω is *deletion tolerant* if $\mathbf{P}[\Pi_{\neg e}\mathcal{A}] > 0$ for every $e \in E$ and every measurable $\mathcal{A} \subset 2^E$ with $\mathbf{P}[\mathcal{A}] > 0$, where $\Pi_{\neg e}\omega := \omega - \{e\}$.

For example, Bernoulli(p) percolation is insertion and deletion tolerant when $p \in (0, 1)$.

For unimodular transitive graphs, Lyons and Schramm showed the following.

THEOREM 2.7 (Theorem 3.3 of [21]). *Let G be a graph with a transitive unimodular closed automorphism group $\Gamma \subset \mathrm{Aut}(G)$. Every Γ-invariant, insertion-tolerant, bond percolation process on G has indistinguishable infinite clusters.*

Martineau's paper [23] explored the link between ergodicity and indistinguishability in percolation theory.

Example 3.15 of [21] showed that a deletion-tolerant bond percolation could have distinguishable infinite clusters. For other examples of indistinguishability for noninsertion tolerant percolation processes, see [19] and [32].

We also need a result from Häggström, Peres and Schonmann [15].

DEFINITION 2.8 (Robust invariant property). Let $G = (V, E)$ be a graph and Γ be a closed quasi-transitive subgroup of $\mathrm{Aut}(G)$. Let $(\mathbf{P}, \omega)$ be a Γ-invariant bond percolation process on G. Suppose $\mathcal{A}$ is a Γ-invariant property. We say that $\mathcal{A}$ is a *robust invariant property* if for every infinite connected subgraph C of G and every edge $e \in C$ we have the equivalence: C has property $\mathcal{A}$ iff there is an infinite connected component of $C \backslash \{e\}$ that has property $\mathcal{A}$.

Even without the unimodularity condition, the robust invariant properties do not distinguish between the infinite clusters of Bernoulli(p) percolation [15].

THEOREM 2.9 (Theorem 4.1.6 of [15]). *Let G be a quasi-transitive graph and $p \in (p_c, p_u]$. If Q is a robust invariant property such that $\mathbf{P}_p(A) > 0$, where A is the event that there exists an infinite cluster satisfying Q, then $\mathbf{P}_p$-a.s., all infinite clusters in G satisfy Q.*

An immediate application of Theorem 2.9 is that almost surely light and heavy infinite clusters cannot coexist for Bernoulli percolation.

3. The tilted mass transport principle. The mass transport principle turns out to be a very useful technique especially on unimodular transitive graphs [3]. Hutchcroft [17] introduced a tilted version of mass transport principle and it turns out be quite useful when applied to nonunimodular quasi-transitive graphs.

Following Hutchcroft's notation [17], we define modular function as below.

DEFINITION 3.1. Let $G = (V, E)$ be a locally finite connected infinite graph and $\Gamma \subset \mathrm{Aut}(G)$ acts on G quasi-transitively. Let $\mathcal{O} = \{o_1, \dots, o_L\}$ be a complete set of representatives in V of the orbits of Γ. Let $a = (a_1, \dots, a_L)$ be a sequence of positive real numbers with $\sum_{i=1}^{L} a_i = 1$. The *modular function* $\Delta_{\Gamma,a} : V^2 \to (0, \infty)$ is defined as follows:

$$\Delta_{\Gamma,a}(x, y) = \frac{a_y m(y)}{a_x m(x)} \overset{\text{Lemma 2.2}}{=} \frac{a_y |\Gamma_y x|}{a_x |\Gamma_x y|},$$

where $a_x := a_i$ if $x \in \Gamma o_i$.

Now we give the tilted mass transport principle (TMTP) (a slight generalization of Proposition 2.2 in Hutchcroft [17]).

PROPOSITION 3.2 (TMTP). *With the same notation as in the above definition of modular function, suppose $f : V^2 \to [0, \infty]$ is invariant under the diagonal action of Γ. Let ρ be sampled from $\mathcal{O}$ with distribution a. Then*

$$\mathbb{E}\left[\sum_{x \in V} f(\rho, x)\right] = \mathbb{E}\left[\sum_{x \in V} f(x, \rho) \Delta_{\Gamma,a}(\rho, x)\right].$$

PROOF. This is easily seen from the Corollary 3.7 of [3]. $\square$

Hutchcroft made a particular choice for a and proved for this particular choice that $\Delta_{\Gamma,a}$ has certain nice properties (Lemma 2.3 in [17]). To be precise, write $[v]$ for the orbit of v under Γ and identify $\mathcal{O}$ with the space of orbits. Let $(X_n)_{n \geq 0}$ be a lazy simple random walk on G, namely $\mathbb{P}[X_{n+1} = u | X_n = v] = \frac{1}{2}\mathbf{1}_{\{u=v\}} + \frac{1}{2\deg(v)}\mathbf{1}_{\{u \sim v\}}$. Let $([X_n])_{n \geq 0}$ be the corresponding Markov chain induced on the finite state space $\mathcal{O}$, which was called the *lazy orbit chain*. Note this chain is irreducible, and hence it has a unique stationary measure $\widetilde{\mu}$ on $\mathcal{O}$. Let μ be the probability measure gotten from $\widetilde{\mu}$ biased by $\deg([v])^{-1}$:

$$\mu([o]) = \frac{\widetilde{\mu}([o]) \deg([o])^{-1}}{\sum_{x\in\mathcal{O}} \widetilde{\mu}([x]) \deg([x])^{-1}}, \quad \forall o \in \mathcal{O}.$$

We also write $\mu_x := \mu([x])$, $\forall x \in V$. Then Lemma 2.3 in [17] can be slightly strengthened to the following.

LEMMA 3.3. *With the same notation as in the above definition of modular function, let $a = (a_1, \dots, a_L)$ be a sequence of positive numbers with $\sum_i^L a_i = 1$. Then the modular function $\Delta = \Delta_{\Gamma,a}$ satisfies the following properties:*

1. *Δ is Γ diagonally invariant.*
2. *Δ satisfies the cocycle identity, that is, for all $x, y, z \in V$,*

$$\Delta(x, y)\Delta(y, z) = \Delta(x, z).$$

3. *Furthermore, $a = \mu$, that is, $a_i = \mu([o_i])$, $i = 1, \dots, L$ if and only if for some fixed $x \in V$ (hence for every $x \in V$), $\Delta(x, y)$ is a harmonic function of $y \in V$, namely*

$$\Delta(x, y) = \frac{1}{\deg(y)} \sum_{z\sim y} \Delta(x, z),$$

where the sum on the right-hand side is taken with multiplicity if there are multiple edges between y and z.

PROOF. (1) For any $x, y \in V$, $\gamma \in \Gamma$, one has $a_{\gamma x} = a_x$, $a_{\gamma y} = a_y$. Hence

$$\Delta(\gamma x, \gamma y) = \frac{a_{\gamma y} m(\gamma y)}{a_{\gamma x} m(\gamma x)} = \frac{a_y m(y)}{a_x m(x)} = \Delta(x, y).$$

(2) For any $x, y, z \in V$, one has

$$\Delta(x, y)\Delta(y, z) = \frac{a_y m(y)}{a_x m(x)} \frac{a_z m(z)}{a_y m(y)} = \frac{a_z m(z)}{a_x m(x)} = \Delta(x, z).$$

(3) The direction that $a = \mu$ implies the harmonicity of $\Delta(x, \cdot)$ was already shown in Lemma 2.3 in [17] and here we provide an alternative proof. If one defines the transition matrix $\widetilde{P}$ on the factor chain on $\mathcal{O}$ by $\widetilde{p}(o_i, o_j) = \frac{1}{\deg(o_i)} \sum_{x\in[o_j]} \mathbf{1}_{\{o_i\sim x\}}$, then $\widetilde{\mu}$ is also a stationary probability measure for the Markov chain on $\mathcal{O}$ determined by $\widetilde{p}$. By Lemma 3.25 of [35], one has that $\nu(x) := \widetilde{\mu}([x])|\Gamma_x| = \widetilde{\mu}([x])m(x)$ is a stationary measure for simple random walk

on G. From this, it follows that for $a = \mu$ the function $\Delta(x, \cdot)$ is harmonic for any fixed $x \in V$.

On the other hand, if for some fixed x, the function $y \mapsto \Delta(x, y)$ is harmonic, then one has the following system of equations:

$$\begin{cases} 0 = \sum_{z \sim y} [a_z m(z) - a_y m(y)], & y \in \mathcal{O}, \\ 1 = a_1 + \cdots + a_L. \end{cases} \tag{3.1}$$

We already showed that when $a = \mu$, the function $y \mapsto \Delta(x, y)$ is harmonic, whence $a = \mu$ satisfies the above equations (3.1) and then it suffices to show $a = \mu$ is also the unique solution of (3.1).

Rewrite (3.1) as the following:

$$\begin{cases} 0 = \sum_{z \sim y} \left[\frac{a_z}{\mu_z} \mu_z m(z) - \frac{a_y}{\mu_y} \mu_y m(y) \right], & y \in \mathcal{O}, \\ 1 = a_1 + \cdots + a_L. \end{cases} \tag{3.2}$$

Since $a = \mu$ satisfies (3.1), one can define a transition matrix p_μ on $\mathcal{O}$ by $p_\mu(y, z) = \frac{1}{\deg(y)} \sum_{w \sim y, w \in [z]} \frac{\mu_w m(w)}{\mu_y m(y)}$. Since G is connected, the Markov chain on the finite space $\mathcal{O}$ determined by p_μ is irreducible. Define $f(x) := \frac{a_x}{\mu_x}$, $x \in V$. Then equation (3.2) implies that f is a harmonic function for p_μ. Since the Markov chain determined by p_μ is an irreducible Markov chain on a finite state space, the harmonic function f must be constant. Together with $a_1 + \cdots + a_L = \mu([o_1]) + \cdots + \mu([o_L]) = 1$, we have $a = \mu$.

This completes the proof. $\square$

REMARK 3.4. If Γ is unimodular, then on each orbit $m(\cdot)$ is a constant and then one obvious solution of (3.1) is $a_i = \frac{m(o_i)^{-1}}{\sum_{x \in \mathcal{O}} m(x)^{-1}}$. By the uniqueness of solution, one has $\mu([v]) = \frac{m(v)^{-1}}{\sum_{x \in \mathcal{O}} m(x)^{-1}}$, $\forall v \in V$. Therefore, in the case $a = \mu$, $\Delta_{\Gamma,\mu}(x, y) = \frac{\mu_y m(y)}{\mu_x m(x)}$ is a constant function of y for fixed x, whence $\Delta_{\Gamma,\mu}(x, \cdot)$ is not only harmonic for simple random walk but also for any random walk on G associated with a Γ-invariant conductance.

Moreover, by Theorem 3.1 of [1], if one takes a random root ρ from $\mathcal{O}$ with probability $\mathbb{P}(\rho = o_i) = \mu([o_i])$, then $[G, \rho]$ is a unimodular random rooted graph. And by Theorem 4.1 of [1] with p corresponding to simple random walk on G and ν corresponding to degree, one has $\widehat{\mu}$ is a stationary probability measure for the trajectories on rooted graphs induced from simple random walk. In particular, the marginal of $\widehat{\mu}$ on the root of the initial rooted graph is a stationary measure for the lazy orbit chain which coincides with previous choice of $\widetilde{\mu}$.

Another natural question to ask is whether for a given $a = (a_1, \ldots, a_L)$ with $a_i > 0$, $\sum_{i=1}^L a_i = 1$, there is a deterministic Γ-invariant conductance function

$c : E \to (0, \infty)$ such that the function $\Delta_{\Gamma,a}(x, \cdot)$ is harmonic for the network (G, c). Previous harmonic function (corresponding to simple random walk without explicit mentioned conductance) is just the case $c \equiv 1$.

For unimodular Γ, we have a complete answer.

LEMMA 3.5. *Suppose Γ is unimodular and $a = (a_i)_{i=1}^L$ is a sequence of positive numbers with $\sum_{i=1}^L a_i = 1$. Then $a = \mu$ if and only if there exists a Γ-invariant conductance $c : E \to (0, \infty)$ such that $\Delta(x, \cdot) = \Delta_{\Gamma,a}(x, \cdot)$ is harmonic for network (G, c). Also in the case $a = \mu$, c can be chosen to be any Γ-invariant conductance.*

PROOF. The if part is already given in the above Remark 3.4.

Now suppose a Γ-invariant function $c : E \to (0, \infty)$ is such that $\Delta(x, \cdot) = \Delta_{\Gamma,a}(x, \cdot)$ is harmonic for network (G, c). Hence a satisfies the following equations:

$$\begin{cases} 0 = \sum_{z \sim y} [a_z m(z) c(z, y) - a_y m(y) c(z, y)], & y \in \mathcal{O}, \\ 1 = a_1 + \cdots + a_L, \end{cases} \tag{3.3}$$

where $c(z, y) = c(y, z)$ denote the conductance of the unoriented edge $e = (y, z)$.

Consider the Markov chain on $\mathcal{O}$ with transition probability q_c given by

$$q_c(o_i, o_j) = \frac{\sum_{z \sim o_i, z \in [o_j]} c(o_i, z)}{\sum_{x \sim o_i} c(o_i, x)}, \quad \forall o_i, o_j \in \mathcal{O}.$$

Since Γ is unimodular, $m(\cdot)$ is a constant function on each orbit, and then we can define $f : \mathcal{O} \to (0, \infty)$ by $f([z]) := a_z m(z)$. Then by (3.3) f is a harmonic function for the Markov chain determined by q_c. Since G is connected, this Markov chain is irreducible. Moreover, this Markov chain has finite state space $\mathcal{O}$, thus f must be a constant function. Thus one has $a_z \propto m(z)^{-1}$. From Remark 3.4, one has $a = \mu$. $\square$

For the case Γ is nonunimodular, we do not have a complete answer except the transitive case.

For example, let T be the infinite regular tree with degree 3 and let G be the graph obtained from T by adding a new vertex at the midpoint of each edge of T. Fix an end ξ of T and let Γ be the subgroup fixing this end ξ. Then Γ is nonunimodular and quasi-transitive on G. Fix $x \in T$ and let $x_1 \in T$ denote the neighbor of x such that x_1 is closer to ξ. And let x_2, x_3 be the other two neighbors of x in T. Suppose y_i is the midpoint on edge (x, x_i) for $i = 1, 2, 3$. Then $\mathcal{O} = \{x, y_1\}$ is a complete set of representatives. Write $a_1 = a_x, a_2 = a_{y_1}$ and let $\lambda := \frac{a_1}{a_2}$.

Since $a_1 + a_2 = 1$, $a_1 = \frac{\lambda}{1+\lambda}$, $a_2 = \frac{1}{1+\lambda}$. Solving (3.3) yields that for any $\lambda \in (\frac{1}{2}, 1)$, there exists a unique (up to multiplicative constant) Γ-invariant conductance such that $\Delta_{\Gamma,a}(x,\cdot)$ is harmonic w.r.t. this conductance. For other λ, the corresponding function $\Delta_{\Gamma,a}(x,\cdot)$ is not harmonic w.r.t. any Γ-invariant conductance.

Note in the transitive case $L = 1$, $a_1 = 1 = \mu$, and thus the choice of a does not matter. And the function $\Delta_{\Gamma,a}(x,\cdot) = \frac{m(\cdot)}{m(x)}$ in this case.

LEMMA 3.6. *Suppose Γ is nonunimodular and transitive. Then $\Delta(x,\cdot) = \frac{m(\cdot)}{m(x)}$ is harmonic for any Γ-invariant conductance $c : E \to (0,\infty)$.*

PROOF. Let $c : E \to (0,\infty)$ be a Γ-invariant conductance. It suffices to show for some fixed $y \in V$, one has

$$\sum_{z\sim y} c(z,y)m(z) = \sum_{z\sim y} c(z,y)m(y).$$

Define $f : V^2 \to [0,\infty]$ by $f(u,v) = c(u,v)\mathbf{1}_{\{u\sim v\}}$. Then f is invariant under Γ-diagonal action since c is Γ-invariant. Suppose $\mathcal{O} = \{y\}$ and then the random vertex ρ with law μ is just y. By TMTP, one has

$$\sum_{z\in V} f(\rho,z) = \sum_{z\in V} f(z,\rho)\Delta(\rho,z).$$

That is just $\sum_{z\sim y} c(y,z) = \sum_{z\sim y} c(y,z)\frac{m(z)}{m(y)}$ and we are done. $\square$

4. Some properties of heavy percolation clusters. In this section, we review some important properties for heavy percolation clusters. Some of their proofs are also typical applications of TMTP. We begin with a definition of level set.

DEFINITION 4.1. Suppose G is a connected, locally finite graph and $\Gamma \subset \mathrm{Aut}(G)$ is closed and quasi-transitive. A *level* of (G,Γ) is a maximal set X of vertices such that for any $x, y \in X$, $|\Gamma_x y| = |\Gamma_y x|$. Or equivalently, a level is a maximal set X of vertices such that for any $x, y \in X$, $m(x) = m(y)$.

Note the level set depends on the subgroup Γ. In the following, without explicit mention, we will always fix some subgroup Γ.

The next proposition was implicitly used in the proof of Corollary 5.10 in [31] and we provide its proof for completeness.

PROPOSITION 4.2. *Suppose G is a connected, locally finite graph and $\Gamma \subset \mathrm{Aut}(G)$ is closed, quasi-transitive and nonunimodular, then every level is an infinite set.*

PROOF. Let $L_x := \{y \in V : m(y) = m(x)\}$ denote the level set containing x. By Lemma 2.2, one has

$$L_{\gamma x} = \gamma L_x := \{\gamma y : y \in L_x\}, \quad \forall \gamma \in \Gamma. \tag{4.1}$$

Choose a complete set of representatives $\mathcal{O} = \{o_1, \dots, o_L\}$ for the orbit under Γ. Denote $L_i := L_{o_i}$, let $n_i = |L_i|$ denote the cardinality of L_i and define $N = \max\{n_1, \dots, n_L\}$.

First we show if there exists one index $i \in \{1, \dots, L\}$ such that $n_i = \infty$, then $n_j = \infty$ for every $j \in \{1, \dots, L\}$. Without loss of generality, we assume $n_1 = \infty$, note $L_1 = \bigcup_{j=1}^L L_1 \cap \Gamma o_j$, whence there exists at least one $j \in \{1, \dots, L\}$ such that $|L_1 \cap \Gamma o_j| = \infty$. Thus there exists an infinite set $\{\gamma_k : k \in \mathbb{N}\} \subset \Gamma$ such that $m(o_1) = m(\gamma_k o_j)$, $\forall k \in \mathbb{N}$ and $\{\gamma_k o_j : k \in \mathbb{N}\}$ is an infinite set. Then using Lemma 2.2 again for every $i \in \{1, \dots, L\}$, one has that $m(\gamma_0 o_i) = m(\gamma_1 o_i) = m(\gamma_2 o_i) = \cdots$. Since γ_k preserves the graph distance, $\{\gamma_k o_i : k \in \mathbb{N}\}$ is also an infinite set. By the fact that $L_{\gamma_0 o_i} \supset \{\gamma_k o_i : k \in \mathbb{N}\}$, one has that $L_{\gamma o_i}$ is also an infinite set, whence by (4.1) $n_i = \infty$ for every $i \in \{1, \dots, L\}$.

Now by the above result and (4.1) it suffices to show $N = \infty$. For any $M > 0$, since Γ is nonunimodular, from the proof of Lemma 2.3 one has that there exists some $x_0, y_0 \in V$ such that $M_0 := \frac{m(y_0)}{m(x_0)} > LM$. Define $f : V^2 \to [0, \infty]$ as

$$f(x, y) = \mathbf{1}_{\{d(x,y)=d(x_0,y_0), \frac{m(y)}{m(x)}=M_0\}}.$$

Obviously, f is Γ-diagonally invariant. Take $a = (a_1, \dots, a_L)$ such that $a_i = \frac{1}{L}$, $\sum_{i=1}^L a_i = 1$, then $\Delta_{\Gamma,a}(x, y) = \frac{m(y)}{m(x)}$. Let k_y denote the number of vertices x such that $d(x, y) = d(x_0, y_0)$ and $\frac{m(y)}{m(x)} = M_0$, and $k := \max\{k_{o_i} : o_i \in \mathcal{O}\}$. Applying the tilted mass transport principle, one has

$$\frac{1}{L} \le \sum_{i=1}^L \frac{1}{L} \sum_{x \in V} f(o_i, x) = \sum_{i=1}^L \frac{1}{L} \sum_{x \in V} f(x, o_i) \frac{1}{M_0} \le \frac{k}{M_0}.$$

Thus $k \ge \frac{M_0}{L} > M$. Suppose $k = k_{o_i}$, then this means there exists at least $k > M$ vertices x such that $\frac{m(o_i)}{m(x)} = M_0$, in particular all these k vertices belong to the same level set, whence $N \ge k > M$. Since this is true for arbitrary M, one has $N = \infty$. $\square$

The following proposition was mentioned and used in [31] without a proof. We provide its proof for the reader's convenience and later use. For an automorphism γ and a level A, if $\gamma A = A$, then we say γ fixes the level A.

PROPOSITION 4.3. *Suppose G is a connected, locally finite graph and $\Gamma \subset \mathrm{Aut}(G)$ is closed, quasi-transitive and nonunimodular, let $\Gamma' \subset \Gamma$ be the subgroup that fixes a level (hence fixes every level). Suppose H is a connected component of* *some finite union of levels and Γ'_H is the subgroup of Γ' that preserves H. Then Γ'_H acts quasi-transitively on H, and it is closed and unimodular.*

PROOF. *Step 1*: First we show that for any $\gamma \in \Gamma$, if there exists $x \in V(G)$ such that $m(x) = m(\gamma x)$, then $\gamma A = A$ for every level A. Actually, this is immediate from Lemma 2.2:

$$m(x) = m(\gamma x) \quad \text{and} \quad \frac{m(x)}{m(y)} = \frac{m(\gamma x)}{m(\gamma y)} \quad \Rightarrow \quad m(y) = m(\gamma y).$$

In particular, one has $\Gamma_x \subset \Gamma'$, $\forall x \in V$.

Step 2: Suppose $L_{x_i}, i = 1, \dots, n$ are n distinct levels and $G_n := \bigcup_{i=1}^n L_{x_i}$ be the union of these n levels, viewed as the subgraph induced by the vertices in these levels. G_n might be disconnected, however it has finitely many types of connected components up to isomorphisms. Indeed, for any $y_1 \neq y_2 \in L_{x_i} \cap \Gamma o_j$, there exists some $\gamma_1, \gamma_2 \in \Gamma$ such that $y_1 = \gamma_1 o_j$, $y_2 = \gamma_2 o_j$ and $m(x) = m(y_1) = m(y_2)$. Thus $\gamma_2\gamma_1^{-1} y_1 = y_2$. Since $m(y_1) = m(y_2)$, $\gamma_2\gamma_1^{-1} \in \Gamma'$ and is an isomorphism from the connected component of y_1 to the one of y_2. In particular, there are at most Ln types of connected components up to isomorphism, where L is the number of orbits for the action of Γ on G.

Step 3: For any connected component $H \subset G_n$, let Γ'_H be the subgroup of Γ' consisting of those maps which fix this component H. In particular, $\Gamma'_x = \Gamma_x \subset \Gamma'_H$, $\forall x \in H$. We now show that Γ'_H acts on H quasi-transitively and it is closed and unimodular. Notice m is also a nonzero Haar measure restricted on Γ'_H and $(\Gamma'_H)_x = \Gamma_x$. Also $m(\Gamma_x)$ is bounded on H since G_n has only finitely many levels. Therefore, Γ'_H is unimodular by Lemma 2.3.

It is easy to see that Γ' is closed. Indeed for any $\gamma \notin \Gamma'$, there exists $x \in V$ such that $m(\gamma x) \neq m(x)$. Therefore, as an open neighborhood of γ, the set $\{\beta : \beta(x) = \gamma(x)\}$ is in the complement of Γ', whence Γ' is closed. Similarly, one can show that Γ'_H is also closed.

For any $y_1 \neq y_2 \in H \cap L_{x_i} \cap \Gamma o_j$, as in step 2 there exists some $\gamma \in \Gamma'$ such that $\gamma y_1 = y_2$. Since γ preserves graph distance, it maps the connected component of y_1 to the connected component of y_2, that is, γ maps H to H, whence $\gamma \in \Gamma'_H$. Therefore, under Γ'_H there are at most Ln different orbits for H (here L denotes the number of orbits for Γ acting on G), whence Γ'_H acts quasi-transitively on H. □

The following proposition is well known ([17, 31]) and its proof will be omitted. We stress that one needs only the assumption of Γ-invariance.

PROPOSITION 4.4. *Suppose G is a connected, locally finite graph and $\Gamma \subset \mathrm{Aut}(G)$ is closed, quasi-transitive and nonunimodular, let $(\mathbf{P}, \omega)$ be a Γ-invariant percolation on G. If $\mathbf{P}$-a.s. there is a unique infinite cluster, then the unique infinite cluster must be heavy.*

DEFINITION 4.5. Häggström, Peres and Schonmann [15] introduced the heaviness transition:

$$p_h(G,\Gamma) := \inf\{p \in [0,1] : \mathbf{P}_p\text{-a.s. there exists a heavy cluster}\}.$$

In the case $\Gamma = \mathrm{Aut}(G)$, one denote $p_h = p_h(G) = p_h(G, \mathrm{Aut}(G))$.

Note that if Γ is closed, quasi-transitive and unimodular, $p_h(G,\Gamma) = p_c$. Using the canonical coupling, one can see that for all $p > p_h$, $\mathbf{P}_p$-a.s. there exists a heavy cluster. By Theorem 2.9 a.s., all infinite clusters are heavy for every $p > p_h(G,\Gamma)$. An immediate consequence of Proposition 4.4 is that $p_h(G,\Gamma) \le p_u$. Hutchcroft [17] proved that $p_c(G) < p_h(G,\Gamma)$ if $\Gamma \subset \mathrm{Aut}(G)$ is closed, quasi-transitive and nonunimodular.

The next proposition is important for later use. It is proved for the transitive case and the proof can be easily adapted to quasi-transitive case.

PROPOSITION 4.6 (Corollary 5.6 of [31]). *Suppose G is a connected, locally finite graph, and $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive and nonunimodular. Let $(\mathbf{P}_p, \omega)$ be a* Bernoulli(p) *percolation on G. If $\mathbf{P}_p$-a.s. there are infinitely many heavy clusters, then there exists some finite union of levels L_{x_i}, $i = 1, \dots, n$ such that there exists a connected component H_n of $G_n := \bigcup_{i=1}^n L_{x_i}$ on which* Bernoulli(p) *percolation induces infinitely many infinite open components.*

Suppose $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive and nonunimodular. m is a Haar measure on Γ. Let $N(p)$ denote the number of infinite clusters for Bernoulli (p) percolation on G. It is well known that $N(p) = 0, 1$ or ∞ a.s.; see, for example, Theorem 7.5 of [20]. By Hutchcroft [17], one has $p_c < p_h(G,\Gamma)$. Thus $N(p_h(G,\Gamma)) = 1$ or ∞. We now clarify whether these clusters are heavy or light.

PROPOSITION 4.7. *Suppose $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive and nonunimodular. If $N(p_h(G,\Gamma)) = \infty$ a.s., then all these infinite clusters are light a.s. If $N(p_h(G,\Gamma)) = 1$ a.s., then the unique infinite cluster is heavy a.s.*

PROOF. If $N(p_h(G,\Gamma)) = 1$, then the unique infinite cluster is heavy a.s. by Proposition 4.4.

If $N(p_h(G,\Gamma)) = \infty$, we proceed by contradiction. Suppose there is at least one heavy infinite cluster, then by Theorem 2.9 one has that all these infinite clusters are heavy a.s. Then by Proposition 4.6, there exist finitely many levels L_{x_i}, $i = 1, \dots, n$ such that $G_n := \bigcup_{i=1}^n L_{x_i}$ induces infinitely many infinite open components for Bernoulli percolation at $p_h(G,\Gamma)$.

Define

$$p_c(G_n) := \min\{p_c(H) : H \text{ is a connented component of } G_n\},$$

where if H is a finite connected component we set $p_c(H) = 1$. The minimum is achieved because there are finitely many types of connected components up to isomorphisms by Proposition 4.3.

Since there are infinitely many infinite open components for Bernoulli percolation at $p_h(G, \Gamma)$ on G_n, by Proposition 4.6 there exists a connected component H_n such that Bernoulli $p_h(G, \Gamma)$-percolation on H_n has infinitely many infinite open components. In particular, $p_c(H_n) \leq p_h(G, \Gamma)$.

First consider the case $p_c(H_n) = p_h(G, \Gamma)$. By Proposition 4.3, the subgroup Γ'_{H_n} is quasi-transitive on H_n and unimodular. Since Bernoulli $p_h(G, \Gamma)$-percolation on H_n has infinitely many infinite open components, H_n must be nonamenable. By [4] or [16], there are no infinite clusters at $p_c(H_n)$, which contradicts $p_c(H_n) = p_h(G, \Gamma)$ and Bernoulli $p_h(G, \Gamma)$-percolation on H_n has infinitely many infinite open components.

If $p_c(H_n) < p_h(G, \Gamma)$, take some $p \in (p_c(H_n), p_h(G, \Gamma))$, then Bernoulli p-percolation has at least one infinite cluster on H_n a.s. This infinite cluster must be heavy since $\{m(x) : x \in H_n\}$ is bounded. Hence Bernoulli p-percolation on G would also have at least one heavy cluster a.s., which contradicts with the definition of $p_h(G, \Gamma)$.

Therefore, if $N(p_h(G, \Gamma)) = \infty$ a.s., then all these infinite clusters are light a.s. $\square$

COROLLARY 4.8. *Suppose $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive and nonunimodular. Suppose that $(G_n)_{n\in\mathbb{N}}$ is an increasing exhausting sequence of finite union of levels in the sense that for each level L_x there exists $N_x > 0$ such that $L_x \subset G_n$ whenever $n \geq N_x$. Then $\lim_{n\to\infty} p_c(G_n)$ exists and*

$$\lim_{n\to\infty} p_c(G_n) \geq p_h(G, \Gamma). \tag{4.2}$$

Moreover, the following are equivalent:

(1) *There exists some $p \in (0, 1)$ such that* Bernoulli(p) *percolation on G has infinitely many heavy clusters*;

(2) $p_h(G, \Gamma) < p_u$;

(3) $\lim_{n\to\infty} p_c(G_n) < p_u$.

If $\lim_{n\to\infty} p_c(G_n) \leq p_u$ (*slightly weaker than the above three conditions*), *then*

$$\lim_{n\to\infty} p_c(G_n) = p_h(G, \Gamma).$$

PROOF. For any $p > p_c(G_n)$, Bernoulli(p) percolation has at least one infinite cluster on G_n a.s., which must be heavy. Hence Bernoulli(p) percolation on G would also have at least one heavy cluster a.s., whence $p \geq p_h(G, \Gamma)$. Therefore, $p_c(G_n) \geq p_h(G, \Gamma)$ for all n. Note $p_c(G_n)$ is decreasing, thus the limit exists and satisfies (4.2).

(1) $\Rightarrow$ (2): Suppose (1) holds but (2) does not hold, then $p_h(G,\Gamma) = p_u$. For any $p < p_h(G,\Gamma)$, there is no heavy cluster a.s. by the definition of $p_h(G,\Gamma)$. At $p = p_h(G,\Gamma)$, if $N(p_h(G,\Gamma)) = \infty$, then Proposition 4.7 yields all these infinite clusters are light a.s. If $N(p_h(G,\Gamma)) = 1$, then there exists only one heavy cluster a.s. For $p > p_h(G,\Gamma) = p_u$, there is a unique infinite cluster a.s. [26]. Thus for all $p \in [0,1]$, there cannot exist infinitely many heavy clusters, contradicting (1).

(2) $\Rightarrow$ (3): For any $p \in (p_h(G,\Gamma), p_u)$, $N(p) = \infty$ since $p < p_u$. Since $p > p_h(G,\Gamma)$, almost surely all the infinite clusters are heavy by the definition of $p_h(G,\Gamma)$ and Theorem 2.9. Then by Proposition 4.6, there exists a finite union of levels G_n such that some connected component H_n of G_n has the property that Bernoulli(p) percolation on H_n has infinitely many infinite clusters. In particular, $p \geq p_c(H_n) \geq p_c(G_n) \geq \lim_{n\to\infty} p_c(G_n)$. Let $p \downarrow p_h(G,\Gamma)$ to obtain that

$$p_h(G,\Gamma) \geq \lim_{n\to\infty} p_c(G_n). \tag{4.3}$$

Combining with $p_h(G,\Gamma) < p_u$, one has (3).

(3) $\Rightarrow$ (1): By (4.2), one has $p_h(G,\Gamma) < p_u$, whence for $p \in (p_h(G,\Gamma), p_u)$, Bernoulli$(p)$ percolation on G has infinitely many heavy clusters.

Now suppose $\lim_{n\to\infty} p_c(G_n) \leq p_u$. If $p_h(G,\Gamma) < p_u$, that is, condition (2) holds, then one has (4.3), this combining with (4.2) yields $\lim_{n\to\infty} p_c(G_n) = p_h(G,\Gamma)$. If $p_h(G,\Gamma) = p_u$ then by assumption $p_h(G,\Gamma) = p_u \geq \lim_{n\to\infty} p_c(G_n)$, this combining with (4.2) also yields $\lim_{n\to\infty} p_c(G_n) = p_h(G,\Gamma)$. $\square$

A necessary condition for $p_h(G,\Gamma) < p_u$ is provided by Timár [31], Corollary 5.8, and Hutchcroft conjectured that it is also sufficient [17], Conjecture 8.5.

REMARK 4.9. The inequality in (4.2) can be strict. For example, consider the regular tree $\mathbb{T}_d$ with degree $d \geq 3$ and fix an end ξ of $\mathbb{T}_d$. Let $\Gamma_\xi \subset \mathrm{Aut}(\mathbb{T}_d)$ be the subgroup consisting of automorphisms that fixes this end ξ. Let $G = \mathbb{T}_d \times \mathbb{Z}$ and $\Gamma = \Gamma_\xi \times \mathrm{Aut}(\mathbb{Z})$. Then Γ is transitive and nonunimodular. Notice $p_h(G,\Gamma) = p_u(G) < 1$, where the first equality is due to Corollary 5.8 of [31]. However, any union of finite consecutive levels consists of infinitely many copies of a cartesian product of a finite tree and $\mathbb{Z}$, whence $\lim_{n\to\infty} p_c(G_n) = 1 > p_h(G,\Gamma) = p_u$ in this example.

5. The "biased" two-sided random walks. We start with some notation from Lyons and Schramm [22]. Suppose V is a countable infinite set and Γ is a locally compact group acting on V (on the left) and that all stabilizers of elements of V have finite Haar measure. We also suppose that the quadruple $(\Xi, \mathcal{F}, \mathbf{P}, \Gamma)$ is a measure-preserving dynamical system, namely Γ acts measurably on the measure space $(\Xi, \mathcal{F}, \mathbf{P})$ and preserves the measure $\mathbf{P}$.

The space of trajectories of random walk is $V^{\mathbb{N}}$. Let $(\Xi, \mathcal{F})$ be a measurable space. Define the shift operator $\mathcal{S} : V^{\mathbb{N}} \to V^{\mathbb{N}}$ by

$$(\mathcal{S}w)(n) := w(n+1),$$

and let $\mathcal{S}$ act on $\Xi \times V^{\mathbb{N}}$ as

$$\mathcal{S}(\xi, w) := (\xi, \mathcal{S}w), \quad \forall (\xi, w) \in \Xi \times V^{\mathbb{N}}.$$

For $\gamma \in \Gamma$, we define its action on $\Xi \times V^{\mathbb{N}}$ by

$$\gamma(\xi, w) := (\gamma\xi, \gamma w),$$

where $(\gamma w)(n) := \gamma(w(n))$.

Let $\mathcal{T} : V^{\mathbb{Z}} \to V^{\mathbb{Z}}$ be the natural extension of $\mathcal{S}$, namely

$$\mathcal{T}\widehat{w}(n) = \widehat{w}(n+1), \quad \forall n \in \mathbb{Z}, \widehat{w} \in V^{\mathbb{Z}},$$

and as before let $\mathcal{T}$ act on $\{0,1\}^V \times V^{\mathbb{Z}}$ as $\mathcal{T}(\xi, \widehat{w}) = (\xi, \mathcal{T}\widehat{w})$.

Define the projection maps $\pi : V^{\mathbb{Z}} \to V^{\mathbb{N}}$ as follows:

$$\pi(\widehat{w})(n) = \widehat{w}(n), \quad n \geq 0, \forall \widehat{w} \in V^{\mathbb{Z}},$$

and define $\pi^- : V^{\mathbb{Z}} \to V^{\mathbb{N}}$ as follows:

$$\pi(\widehat{w})(n) = \widehat{w}(-n), \quad n \geq 0, \forall \widehat{w} \in V^{\mathbb{Z}}.$$

We call a measurable function $q : \Xi \times V \times V \to [0,1]$, written as $q : (\xi, x, y) \mapsto q_\xi(x, y)$, a random environment (from Ξ) if for all $\xi \in \Xi$, $x \in V$, we have $\sum_{y \in V} q_\xi(x, y) = 1$. The natural action of Γ on q is given by $(\gamma q)(\xi, x, y) := q(\gamma^{-1}\xi, \gamma^{-1}x, \gamma^{-1}y)$. Given $x \in V$ and a measurable map $\xi \mapsto \nu_\xi(x)$ from Ξ to $[0, \infty)$, let $\widehat{\mathbf{P}}_x$ denote the joint distribution on $\Xi \times V^{\mathbb{N}}$ of ξ biased by $\nu_\xi(x)$ and the trajectory of the Markov chain determined by q_ξ starting at x. Let $\mathcal{I}$ denote the σ-field of Γ-invariant events in $\Xi \times V^{\mathbb{N}}$. For examples, one can refer to [22] or see our application of the following general theorem.

THEOREM 5.1 (Theorem 3 of [22]). *Let V be a countable set with a quasi-transitive action by a locally compact group Γ. Let $\{o_1, \dots, o_L\}$ be a complete set of representatives of $\Gamma \backslash V$ and write $m_i = m(o_i)$. Let $(\Xi, \mathcal{F}, \mathbf{P}, \Gamma)$ be a measure-preserving dynamical system and q be a Γ-invariant random environment from Ξ. Suppose that $\nu : (\xi, x) \mapsto \nu_\xi(x)$ is a Γ-invariant measurable mapping from $\Xi \times V \to [0, \infty)$ such that for each $\xi \in \Xi$, $x \mapsto m(x)\nu_\xi(x)$ is a stationary distribution for the Markov chain determined by q_ξ. Write*

$$\widehat{\mathbf{P}} := \sum_{i=1}^{L} \widehat{\mathbf{P}}_{o_i}.$$

Then the restriction of $\widehat{\mathbf{P}}$ to the Γ-invariant σ-field $\mathcal{I}$ is an $\mathcal{S}$-invariant measure. If

$$\sum_{i=1}^{L} \mathbf{E}\big[\nu_\xi(o_i)\big] = 1$$

then $\widehat{\mathbf{P}}$ is a probability measure.

Now we give a natural extension of Theorem 5.1 for a two-sided random walk.

THEOREM 5.2. *With the same notation as in Theorem 5.1, let*

$$q_\xi^{\leftarrow}(x_1, x_2) := \frac{\nu_\xi(x_2)m(x_2)}{\nu_\xi(x_1)m(x_1)} q_\xi(x_2, x_1).$$

Since $x \mapsto \nu_\xi(x)m(x)$ is a stationary measure for the Markov chain determined by q_ξ, $q_\xi^{\leftarrow}$ is a transition probability. Moreover, $q_\xi^{\leftarrow}$ is also Γ-invariant.

For $(\xi, x) \in \Xi \times V$, let θ_ξ^x denote the law of a two-sided random walk $\widehat{w} \in V^{\mathbb{Z}}$ starting from x, namely $w = \pi\widehat{w}$ is a random walk starting from x determined by q_ξ, and $w^- := \pi^-(\widehat{w})$ is an independent random walk starting from x determined by transition probability $q_\xi^{\leftarrow}$. Let Θ_x denote the joint law of $(\xi, \widehat{w})$ biased by $\nu_\xi(x)$. Write

$$\mathbf{\Theta} := \sum_{i=1}^{L} \Theta_{o_i}.$$

Let $\mathcal{I}_{\mathbb{Z}}$ denote the Γ-invariant σ-field. Then the restriction of $\mathbf{\Theta}$ to $\mathcal{I}_{\mathbb{Z}}$ is a $\mathcal{T}$-invariant measure. If

$$\sum_{i=1}^{L} \mathbf{E}\big[\nu_\xi(o_i)\big] = 1,$$

then $\mathbf{\Theta}$ is a probability measure.

PROOF. This is just an adaptation of the proof of Theorem 3 of [22]. Let F be a nonnegative Γ-invariant measurable function on $\Xi \times V^{\mathbb{Z}}$. It suffices to show

$$\int_{\Xi \times V^{\mathbb{Z}}} F \circ \mathcal{T}\, d\mathbf{\Theta} = \int_{\Xi \times V^{\mathbb{Z}}} F\, d\mathbf{\Theta}.$$

A key observation is that

$$d\theta_\xi^x(\widehat{w}) = \sum_{y \in V} \frac{\nu_\xi(y)m(y)}{\nu_\xi(x)m(x)} \mathbf{1}_{\{\mathcal{T}\widehat{w}(-1)=x\}}\, d\theta_\xi^y(\mathcal{T}\widehat{w}).$$

Define

$$f(x, y; \xi) := \nu_\xi(y)\frac{m(y)}{m(x)} \int_{V^{\mathbb{Z}}} \mathbf{1}_{\{\widehat{w}(-1)=x\}} F(\xi, \widehat{w})\, d\theta_\xi^y(\widehat{w}).$$

Then we have

$$\int_{\Xi \times V^{\mathbb{Z}}} F \circ \mathcal{T}\, d\mathbf{\Theta} = \int_{\Xi \times V^{\mathbb{Z}}} F\, d\mathbf{\Theta} \circ \mathcal{T}^{-1} = \sum_{j=1}^{L} \sum_{y \in V} \int_{\Xi} d\mathbf{P}(\xi) f(o_j, y; \xi).$$

It is easy to verify that f is Γ-invariant, and hence so is $\mathbf{E}[f(x,y;\cdot)]$. Then mass transport principle (e.g., Lemma 1 of [22]) yields that

$$\begin{aligned}\int_{\Xi\times V^{\mathbb{Z}}} F\circ\mathcal{T}\,d\mathbf{\Theta} &= \sum_{j=1}^{L}\sum_{y\in V}\int_{\Xi} d\mathbf{P}(\xi)\frac{m(y)}{m(o_j)}f(y,o_j;\xi)\\ &= \sum_{j=1}^{L}\int_{\Xi} d\mathbf{P}(\xi)\int_{V^{\mathbb{Z}}}\sum_{y\in V}\nu_\xi(o_j)\mathbf{1}_{\{y=\widehat{w}(-1)\}}F(\xi,\widehat{w})\,d\theta_\xi^{o_j}(\widehat{w})\\ &= \sum_{j=1}^{L}\int_{\Xi} d\mathbf{P}(\xi)\int_{V^{\mathbb{Z}}}\nu_\xi(o_j)F(\xi,\widehat{w})\,d\theta_\xi^{o_j}(\widehat{w})\\ &= \int_{\Xi\times V^{\mathbb{Z}}} F\,d\mathbf{\Theta},\end{aligned}$$

which completes the proof. $\square$

Now we summarize some properties of the backward random walk determined by $q_\xi^{\leftarrow}$. The proof is standard for reversed Markov chains and we omit it.

PROPOSITION 5.3. *With the same notation as in Theorem 5.2, one has the following properties*:

(1) *$x\mapsto\nu_\xi(x)m(x)$ is also a stationary distribution for the Markov chain determined by $q_\xi^{\leftarrow}$, and $(q_\xi^{\leftarrow})^{\leftarrow}=q_\xi$.*

(2) *q_ξ, $q_\xi^{\leftarrow}$ induces same communicating classes. In particular, q_ξ is irreducible iff $q_\xi^{\leftarrow}$ is irreducible. Moreover, on each communicating class q_ξ is transient if and only if $q_\xi^{\leftarrow}$ is transient.*

Next, we return to the Bernoulli percolation setting. First we consider simple random walk on a quasi-transitive graph $G=(V,E)$.

COROLLARY 5.4. *Let $G=(V,E)$ be a quasi-transitive graph with automorphism group $\Gamma=\mathrm{Aut}(G)$. Let $\{o_1,\dots,o_L\}$ be a complete set of representatives of $\Gamma\backslash V$. Suppose that $(\mathbf{P}_p,\xi)$ is* Bernoulli(p) *bond percolation process on G, and let $\widetilde{\mathbf{P}}_o:=\mathbf{P}_p\times\mathbf{P}_o$, where $\mathbf{P}_o$ is the law of simple random walk on G starting at the point o. Then there exist positive constants c_i, $i=1,\dots,L$, summing to* 1 *such that $\widehat{\mathbf{P}}:=\sum_{i=1}^{L}c_i\widetilde{\mathbf{P}}_{o_i}$ is a probability measure and the restriction of $\widehat{\mathbf{P}}$ to the Γ-invariant σ-field $\mathcal{I}$ is an $\mathcal{S}$-invariant measure.*

PROOF. In Theorem 5.1, we take $(\Xi,\mathcal{F},\mathbf{P},\Gamma)=(\{0,1\}^E,\mathcal{F}_E,\mathbf{P}_p,\mathrm{Aut}(G))$. $\mathbf{E}$ will be the corresponding expectation operator with respect to this measure $\mathbf{P}=\mathbf{P}_p$.

Take $q(\xi, x, y) = q_\xi(x, y) = \frac{\mathbf{1}_{\{x\sim y\}}}{\deg(x)}$, then the Markov chain associated with transition probability q_ξ is just simple random walk on G and obviously q is Γ-invariant.

Fix some $u \in V$. We take $\nu : \{0,1\}^V \times V \to [0,\infty)$ to be

$$\nu_\xi(x) = c\frac{\Delta(u,x)\deg(x)}{m(x)},$$

where $c > 0$ is a constant to be determined later and $a = \mu$ as in Lemma 3.3 such that Δ is harmonic for simple random walk. By Lemma 2.2,

$$\begin{aligned}\nu_\xi(x) &= c\frac{\widetilde{\mu}([x])\deg(u)|\Gamma_x u|}{\widetilde{\mu}([u])\deg(x)|\Gamma_u x|}\frac{\deg(x)}{m(x)}\\ &= c\frac{\deg(u)}{\widetilde{\mu}([u])m(u)}\widetilde{\mu}([x]).\end{aligned}$$

From this, we see that ν is also Γ-invariant and is constant on each orbit.

From the harmonicity of modular function (Lemma 3.3), we know that $x \mapsto m(x)\nu_\xi(x)$ is a stationary distribution for the Markov chain (simple random walk now) determined by q_ξ. We choose the normalizing constant $c > 0$ to be determined by

$$\sum_{i=1}^{L}\mathbf{E}\bigl[\nu_\xi(o_i)\bigr] = 1.$$

Now take $c_i = \nu_\xi(o_i)$ (this does not depend on ξ) for $i = 1, \dots, L$. Then $c_i\widetilde{\mathbf{P}}_{o_i} = \widehat{\mathbf{P}}_{o_i}$, where $\widehat{\mathbf{P}}_x$ is defined earlier as the joint distribution on $\Xi \times V^{\mathbb{N}}$ of ξ biased by $\nu_\xi(x)$ and the trajectory of the Markov chain determined by q_ξ starting at x. Therefore, Theorem 5.1 yields the desired result. $\square$

Next, we consider the corresponding two-sided random walk as in Theorem 5.2. Since $q_\xi(x, y) = \frac{\mathbf{1}_{\{x\sim y\}}}{\deg(x)}$, one has

$$q_\xi^{\leftarrow}(x,y) = \frac{\nu_\xi(y)m(y)}{\nu_\xi(x)m(x)}q_\xi(y,x) = \frac{\mu([y])m(y)}{\mu([x])m(x)}\frac{\mathbf{1}_{\{x\sim y\}}}{\deg(x)}.$$

Note by Remark 3.4 in the unimodular case $q_\xi^{\leftarrow} = q_\xi$ is just the transition probabilities for simple random walk. As in Theorem 5.2 fix some $x \in V$ and let w be a simple random walk on G started from x, and w^- be an independent random walk determined by $q_\xi^{\leftarrow}$ also started from x. Set θ^x to be the law of the two-sided random walk $\widehat{w}$ started from x determined by $w = \pi\widehat{w}$ and $w^- = \pi^-\widehat{w}$.

Let $\mathbf{P}_\rho^{\mathbb{Z}} := \sum_{i=1}^{L} c_i\theta^{o_i}$ denote the law of a two-sided random walk defined above starting from the independently random chosen vertex ρ, where c_i is from Corollary 5.4.

With this notation and $\widehat{\mathbf{P}}$ as in Corollary 5.4, we have the following extension.

COROLLARY 5.5. *Let $\mathcal{I}_{\mathbb{Z}}$ denote the Γ-invariant σ field of $\{0,1\}^E \times V^{\mathbb{Z}}$, then $(\{0,1\}^E \times V^{\mathbb{Z}}, \mathcal{I}_{\mathbb{Z}}, \mathbf{P}_p \times \mathbf{P}_\rho^{\mathbb{Z}}, \mathcal{T})$ is an invertible measure preserving dynamical system. Moreover, $\widehat{\mathbf{P}} = \mathbf{P}_p \times \mathbf{P}_\rho^{\mathbb{Z}} \circ \pi^{-1}$.*

PROOF. This is just an application of Theorem 5.2 in this particular setting. It is straightforward to verify that $\widehat{\mathbf{P}} = \mathbf{P}_p \times \mathbf{P}_\rho^{\mathbb{Z}} \circ \pi^{-1}$ and we omit the details. $\square$

Notice that in the above "simple random walk on G" setting, q_ξ actually is independent of the percolation configuration ξ. In the following, we shall consider random walk on the percolation clusters of ξ.

We first recall the delayed two-sided simple random walk on percolation clusters from [22]: let $(\mathbf{P}, \xi)$ be a bond percolation process on G and $\xi \in 2^E$ be a percolation configuration. Let $x \in V$ be some fixed vertex, called base point. Let $w(0) = x$. For $n \geq 0$, conditioned on $\langle w(0), \dots, w(n)\rangle$ and ξ, let $w'(n+1)$ be chosen uniformly from the neighbors of $w(n)$ in G with equal probability. If the edge $(w(n), w'(n+1))$ belongs to ξ, then we set $w(n+1) = w'(n+1)$, otherwise we set $w(n+1) = w(n)$. This w is a delayed simple random walk on percolation cluster $C_\xi(x)$. Given ξ, let w, w^- be a two independent delayed simple random walk. Set $\widehat{w}$ such that $w = \pi(\widehat{w})$, $w^- = \pi^-(\widehat{w})$. Then $\widehat{w}$ is called a two-sided delayed simple random walk. If G is transitive and unimodular, the two-sided delayed simple random walk $\widehat{w}$ is shift invariant on the Γ-invariant σ-field, which is a key ingredient in the proof of Theorem 2.7.

Now for general quasi-transitive graphs, we introduce the following "biased" two-sided random walk inspired by Example 6 in [22]. Suppose $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive and m is an associated Haar measure on Γ. Let $(\mathbf{P}, \xi)$ be a bond percolation process on G. For each edge $e = (x, y) \in E$, set conductance $c(e) = \sqrt{m(x)m(y)}$. Now given a percolation configuration ξ, for each edge $e \in E$, if $e \notin \xi$, delete the edge e from G and add a loop at x with conductance $c(e)$ and a loop at y also with conductance $c(e)$. In the resulting network, one has a corresponding random walk according the conductance, which is our desired "biased" random walk. To be specific, we define $q_\xi(x, y)$ as follows:

$$q(\xi, x, y) = q_\xi(x, y) = \begin{cases} \dfrac{\sqrt{m(x)m(y)}}{\sum_{z\sim x}\sqrt{m(z)m(x)}} & \text{if } (x, y) \in \xi, y \neq x, \\ 0 & \text{if } y \neq x \text{ and } (x, y) \notin \xi, \\ 1 - \displaystyle\sum_{z\sim x, z\neq x} q_\xi(x, z) & \text{if } y = x. \end{cases} \tag{5.1}$$

Obviously, q and $\nu_\xi(x) := c\sum_{z\sim x}\sqrt{m(z)/m(x)}$ are both Γ-invariant by Lemma 2.2, where $c > 0$ is a constant. Moreover, it is obvious that $x \mapsto \nu_\xi(x)m(x) = c\sum_{z\sim x}\sqrt{m(z)m(x)}$ is a stationary measure for q_ξ. Let $q_\xi^{\leftarrow}$ be given as in Theorem 5.2, then

$$q_{\xi}^{\leftarrow}(x,y) = \frac{\nu_{\xi}(y)m(y)}{\nu_{\xi}(x)m(x)} q_{\xi}(y,x) = q_{\xi}(x,y).$$

DEFINITION 5.6. Fix $x \in V$. Let w be a "square-root biased" random walk determined by q_ξ in formula (5.1) started from x. Let w^- be an independent "square-root biased" random walk determined by $q_{\xi}^{\leftarrow} = q_\xi$ started from x. Let $\widehat{w} \in V^{\mathbb{Z}}$ be such that $w = \pi\widehat{w}$, $w^- = \pi^-\widehat{w}$. Then $\widehat{w}$ is called a two-sided "square-root biased" random walk started from x.

Notice in the case Γ is transitive and unimodular, $\widehat{w}$ is just the two-sided delayed simple random walk we recalled earlier.

COROLLARY 5.7. *Suppose $G = (V, E)$ is a connected, locally finite graph and $\Gamma \subset \mathrm{Aut}(G)$ is closed and quasi-transitive. Let m be a Haar measure on Γ. Let $\{o_1, \dots, o_L\}$ be a complete set of representatives of $\Gamma \backslash V$. Suppose that $(\mathbf{P}, \xi)$ is a Γ-invariant bond percolation process on G, and let q_ξ be given by formula (5.1), $c > 0$ be a constant, and $\nu_\xi(x) = c\sum_{z\sim x}\sqrt{m(z)/m(x)}$. For $\xi \in 2^E$, $x \in V$, let $\widehat{w}$ be a "square-root biased" two-sided random walk started from x as in Definition 5.6. Let Θ_x denote the joint law of $(\xi, \widehat{w})$ biased by $\nu_\xi(x)$. Write*

$$\boldsymbol{\Theta} := \sum_{i=1}^{L} \Theta_{o_i}.$$

Let $\mathcal{I}_{\mathbb{Z}}$ denote the Γ-invariant σ-field. Then the restriction of $\boldsymbol{\Theta}$ to $\mathcal{I}_{\mathbb{Z}}$ is an $\mathcal{T}$-invariant measure, where $\mathcal{T}$ is the natural extension on $\Xi \times V^{\mathbb{Z}}$ of $\mathcal{S}$. If the constant $c > 0$ satisfies $\sum_{i=1}^{L} \mathbf{E}[\nu_\xi(o_i)] = 1$, then $\boldsymbol{\Theta}$ is a probability measure.

PROOF. This is immediate from Theorem 5.2. $\square$

PROPOSITION 5.8. *With the same notation as in Corollary 5.7, if C is a light cluster of ξ, then the "square-root biased" random walk determined by q_ξ on C is positive recurrent.*

PROOF. Since Γ is quasi-transitive, there exists a constant $M > 0$ such that for any $x \sim y$, $\frac{1}{M} \le \frac{m(y)}{m(x)} \le M$. Let D be the maximal degree of G. Then consider the induced network on C (described just before formula (5.1)), the total conductance is finite:

$$\sum_{e\in C} c(e) \le \sum_{x\in C} D\sqrt{M} m(x) < \infty,$$

where in the last inequality we use the fact C is light. Therefore, C is positive recurrent w.r.t. q_ξ. $\square$

PROPOSITION 5.9. *Suppose* $\Gamma \subset \mathrm{Aut}(G)$ *is quasi-transitive and nonunimodular*, *if* Bernoulli(p) *percolation on* G *has infinitely many heavy clusters a.s.*, *then these heavy clusters are transient for both simple random walk and the "square-root biased" random walk determined by* q_ξ *in the formula* (5.1).

PROOF. By Proposition 4.6, there exists some finite union of levels $L_{x_i}, i = 1, \dots, n$ such that there exists a connected component H_n of $G_n := \bigcup_{i=1}^n L_{x_i}$ with the property that Bernoulli(p) percolation on H_n has infinitely many infinite clusters. Since Γ'_{H_n} is quasi-transitive and unimodular, simple random walk is transient on infinite clusters of Bernoulli(p) percolation on H_n; see, for example, Proposition 3.11 of [21] (its proof can be easily adapted to quasi-transitive case). Since conductance $c(e) = \sqrt{m(e^-)m(e^+)}$ is bounded on H_n, the random walk determined by q_ξ is also transient on infinite clusters of Bernoulli p-percolation on H_n. Therefore, by Rayleigh's monotonicity principle, there exists some infinite cluster of Bernoulli(p) percolation on G such that simple random walk and the "square-root biased" random walk determined by q_ξ are both transient. Since transience for simple random walk and the "square-root biased" random walk are both Γ-invariant robust properties, Theorem 2.9 yields the desired conclusion. □

REMARK 5.10. For general insertion-and-deletion tolerant percolation processes, by Corollary 5.6 and Remark 5.11 of [31] we can show that with positive probability there is at least one heavy cluster that is transient for both simple random walk and the "square-root biased" random walk.

If $p_h(G, \Gamma) = p_u$, then there is no $p \in [0, 1]$ such that there are infinitely many heavy clusters for Bernoulli(p) percolation. We conjecture that in this case if there is a unique infinite cluster for Bernoulli(p) percolation a.s., then the unique infinite cluster is also transient for the "square-root biased" random walk determined by q_ξ. Moreover, the unique infinite cluster is transient for simple random walk; see Proposition 5.12.

In Remark 3.12 of [21], it was conjectured that if there are almost surely infinitely many infinite clusters for Bernoulli percolation, then almost surely every infinite cluster is transient (for simple random walk). Moreover, there were examples of Γ-invariant insertion-tolerant percolation processes showing that infinite clusters can be recurrent for simple random walk in the case Γ is nonunimodular. An explicit example by Russell Lyons is as follows.

EXAMPLE 5.11. Let T be a regular tree with degree 3 and ξ be a distinguished end of T. Let Γ be the group of automorphisms that fixes the end ξ. Let every vertex of T be connected to precisely one of its offspring (as measured from ξ), each with probability $1/2$. Let ω_1 denote the configuration. Then every cluster of ω_1 is a ray. Now for each edge $e = (x, y) \in E(T) \backslash \omega_1$ where y is an offspring of x, let n be the graph distance from x to the highest vertex of its ray in ω_1. Next, we insert the edge e with probability $\frac{1}{2^{n+1}}$. Let ω_2 be the configuration gotten from ω_1 by applying the above procedure independently for each edge $e = (x, y) \in E(T)\backslash\omega_1$. Then ω_2 is a Γ-invariant insertion-tolerant percolation process. It is easy to see that every cluster of ω_2 is infinite and recurrent for simple random walk.

We now show that the conjecture in Remark 3.12 of [21] holds. Unlike the short proof of Proposition 5.9, the proof of transience for all infinite Bernoulli percolation clusters is much longer. Moreover, this transience result is not needed for the proof of the main theorems.

First, we briefly review existing results. Suppose G is a nonamenable transitive graph. For p close to 1, the anchored expansion constants of the infinite clusters of Bernoulli(p) percolation are positive a.s. [8], whence the liminf speeds of simple random walk on the infinite clusters are positive a.s. [34]. If moreover G is unimodular, then the speed of simple random walk on an infinite cluster exists and is positive ([5], Theorem 4.4) for every Bernoulli(p) percolation on G with $p > p_c$. In particular, this implies that the infinite clusters of Bernoulli(p) percolation on G with $p > p_c$ are transient. Proposition 3.11 of [21] shows that the same is true for any Aut(G)-invariant insertion-tolerant percolation process that has infinitely many infinite clusters a.s.

Now we consider the case G is nonunimodular and quasi-transitive. For $p > p_c$ and a delayed simple random walk on the infinite clusters of every Bernoulli(p) percolation on G, we do not know whether the speed of the delayed simple random walk exists since it is not shift invariant as in the unimodular case. However, the infinite clusters of Bernoulli(p) percolation on G with $p > p_c$ are still transient.

PROPOSITION 5.12. *Suppose G is nonunimodular and quasi-transitive and $p > p_c$. Then the infinite clusters of* Bernoulli(p) *percolation on G are transient.*

The proof of Proposition 5.12 is a simple modification of the proof of Theorem 4.3 of [31].

First we point out that Lemma 4.1 of [31] can be slightly strengthened to (the conditions are the same while the conclusion is slightly stronger).

LEMMA 5.13. *Consider a random rooted tree with the following properties. Fix some $p > 0$ and define $O_0 := \{o\}$, where o is the root. If a generation O_m is already given, then the number of children that the vertices in O_m will have depends only on O_m (and not on the past). Each vertex of O_m has at least k children with probability $\geq p$ and 0 children otherwise. Furthermore, there is a positive integer α such that given any generation $O_m = \{v_1, \dots, v_n\}$, if we let X_i be the number of children of v_i, then for each $i \in \{1, \dots, n\}$, X_i is independent of X_j for all but at most α of them. Denote by O_{m+1} the set of children of the vertices in O_m.*

Then the tree is transient with positive probability whenever $kp > 1$.

PROOF. We may assume that every vertex has exactly k children with probability p and 0 children otherwise. Pick some $q \in (0,1)$ sufficiently close to 1 such that $kpq^k > 1$. Let T be a random tree as stated in the lemma. Consider a Bernoulli(q) percolation on T and construct a new random rooted tree T' as follows: Let V' be the set of vertices x with the property that it has exactly k children $x_1, \dots, x_k$ and every edge (x, x_i) is open in the Bernoulli(q) percolation on T. T' is the subtree of T induced by $V' \cup \{o\}$. Notice T' is also a random rooted tree satisfies the property of Lemma 4.1 of [31], whence T' is infinite with positive probability. This implies that $p_c(T) < 1$ with positive probability, whence $\mathrm{br}(T) > 1$ with positive probability ([20], Theorem 5.15) and in particular T is transient with positive probability ([20], Theorem 3.5). $\square$

We now adopt some notation from [17, 31]. Suppose $G = (V, E)$ is a locally finite, connected graph and $\Gamma \subset \mathrm{Aut}(G)$ is nonunimodular and acts quasi-transitively on G. Let $\mathcal{O} = \{o_1, \dots, o_L\}$ be a complete set of representatives of G/Γ. Let $a = \mu$ be as in Lemma 3.3 and then the modular function $\Delta(x, y) := \Delta_{\Gamma,a}(x, y)$ is harmonic for simple random walk. Let ρ be a random vertex picked from $\mathcal{O}$ with distribution a.

For each $s \le t$ and $v \in V$, we define the slab

$$S_{s,t}(v) := \{x \in V : s \le \log \Delta(v, x) \le t\}.$$

We also define

$$t_0 := \sup\{\log \Delta(v, u) : u, v \in V, u \sim v\}.$$

We define the separating layers

$$L_n(v) := \{x \in V : (n-1)t_0 \le \log \Delta(v, x) \le nt_0\}$$

and half-spaces $H_n^+(v) := \bigcup_{m \ge n} L_m(v)$ and $H_n^-(v) := \bigcup_{m \le n} L_m(v)$. We also define $L_{m,n}(v) := \bigcup_{k=m}^{n} L_k(v)$.

For each $v \in V$, $-\infty \le m \le k \le n \le \infty$, we define

$$X_k^{m,n}(v) := \left|\left\{x \in L_k(v) : v \overset{L_{m,n}(v)}{\longleftrightarrow} x\right\}\right|$$

and

$$\widetilde{X}_k^{m,n}(v) := \left|\left\{x \in L_k(v) : v \overset{L_{m,n}(v)}{\longleftrightarrow} x \text{ by an open path with only } x \in L_k(v)\right\}\right|,$$

where $\{v \overset{L_{m,n}(v)}{\longleftrightarrow} x\}$ denotes the event that v is connected to x by an open path in the subgraph $L_{m,n}(v)$.

We also need a modification of Lemma 4.2 from [31] as follows.

LEMMA 5.14. *Let G be a nonunimodular quasi-transitive graph and v be a vertex of G. Consider* Bernoulli(p) *percolation on G that has light infinite clusters a.s. Then given the event that $C(v)$ is an infinite light cluster, $\widetilde{X}_{-n}^{-n,\infty}(v) \to \infty$ as $n \to \infty$.*

PROOF. The proof is almost the same as the one of Lemma 4.2 in [31]. Just replace the event $E = E(k)$ there by the event that $C(v)$ is infinite and light and there are infinitely many n such that $\widetilde{X}_{-n}^{-n,\infty}(v) \le k$. $\square$

PROOF OF PROPOSITION 5.12. Denote by Γ the group $\mathrm{Aut}(G)$ in this proof. Since Γ is nonunimodular and acts quasi-transitively on G, one has $p_c < p_h$ by [17]. Moreover, there is no infinite cluster at p_c; see [16]. For $p \in (p_c, p_h)$ Bernoulli(p) percolation has infinitely many infinite light clusters. Since for all $p_c < p_1 < p_2 \le 1$, every infinite p_2-cluster contains an infinite p_1-cluster [15], Theorem 4.1.3, by Rayleigh's monotonicity principle it suffices to show for any fixed $p \in (p_c, p_h)$ the light infinite clusters of Bernoulli(p) percolation are transient for simple random walk. In the following, we fix some $p \in (p_c, p_h)$ and some $o \in V$ such that there exists a neighbor o' of o such that $\log \Delta(o, o') = -t_0$. Recall $[o] := \Gamma o$ denotes the orbit of o.

First, by quasi-transitivity and maximum principle for the harmonic function $\Delta(v, \cdot)$, there exists $n_0 > 0$ such that for each $v \in V$. there exists a path $P_v = (v_0 = v, v_1, \dots, v_n)$ such that $n \le n_0$, $\log \Delta(v, v_n) \le -t_0$, $v_n \in [o]$ and $\Delta(v, v_i)$ is decreasing. In particular, in the transitive case for $v = o$, we can choose $n_0 = 1$ and $P_v = (o, o')$. Indeed, we just need to choose one such path for each $x \in \mathcal{O}$. For every vertex $y \notin \mathcal{O}$, suppose y is in the same orbit as x under Γ, namely for some (arbitrarily fixed) $\gamma = \gamma_{x,y} \in \Gamma$, $y = \gamma x$. Then we let $P_y = \gamma P_x$ (the choice of γ does not matter although it may change the choice of P_y). Denote the end point v_n of P_v by v'.

For $x \in \mathcal{O}$, define the graph $G'(x)$ to be the union of the path P_x and the subgraph induced by the vertices in half-space $H_0^-(x')$. For $x \in \mathcal{O}$ and $y \in [x]\backslash\{x\}$, let $G'(y) := \gamma G'(x)$, where $\gamma = \gamma_{x,y}$ is the one fixed as in the above definition of P_y.

We call the cluster $C(x)$ *nice* if $C(x) = C(x) \cap G'(x)$ and $C(x') \cap H_0^-(x') = C(x') \cap G'(x')$. Let $F(x)$ be the event that $C(x)$ is infinite, light and nice. By insertion and deletion tolerance and $p \in (p_c, p_h)$, we have $q_x := \mathbf{P}_p(F(x)) > 0$. Moreover, q_x depends only on the orbit of x, whence $q := \inf\{q_x : x \in V\} = \min\{q_x : x \in \mathcal{O}\} > 0$.

Given a vertex $x \in L_{-j}(o)$ (if there are two such j's, we take the larger one), the vertex x' and $r, i \ge 1$, define $B_x(i; r) := B(x, r) \cap G'(x) \cap L_{-j-i,-j}(o)$. One can see Figure 1 on page 2354 of [31] for an illustration of $B_x(i; r)$ in the transitive case. We say a vertex $v \in B_x(i; r)\backslash P_x$ belongs to the *side boundary* of $B_x(i; r)$ if there is some edge (v, w) such that $w \in H_{-j-i}^+(o)$ and $w \notin B_x(i; r)$. Let $C(x)|_{B_x(i;r)}$ denote the open cluster of x for percolation restricted to the finite graph $B_x(i; r)$. Let $Y_{-i}(o, x) := \{v \in L_{-j-i}(o) : v \overset{L_{-i,0}(x)}{\longleftrightarrow} x$ by an open path with only $v \in L_{-j-i}(o)\}$ and $\widetilde{X}_{-i}(o, x) := |Y_{-i}(o, x)|$. Given $k \ge 1$, we say that $B_x(i; r)$ is *k-good* if $\widetilde{X}_{-i}(o, x) \ge k$ and the side boundary of $B_x(i; r)$ is disjoint from $C(x)|_{B_x(i;r)}$.

CLAIM. *For any given k there is a uniform choice of i, r such that $B_x(i;r)$ is k-good with probability at least $q/2$ for every $x \in H_0^-(o)$.*

Suppose $x \in L_{-j}(o)$. From Lemma 5.14 for any given $\lambda > 1$, there exists a positive integer $i_x = i_x(\lambda)$ such that given the event $F(x)$ occurs, with probability at least $\frac{3}{4}$, $\widetilde{X}_{-i+1}^{-i+1,\infty}(x) \geq \lambda k$ if $i = i_x$. Let $Z_i(x)$ denote the set of vertices that contribute to $\widetilde{X}_{-i+1}^{-i+1,\infty}(x)$. Now condition on $\widetilde{X}_{-i+1}^{-i+1,\infty}(x) \geq \lambda k$, we have $\widetilde{X}_{-i}(o,x) \geq k$ with probability at least $\frac{3}{4}$. Indeed, if $|Z_i(x) \cap L_{-j-i}(o)| \geq k$, then $\widetilde{X}_{-i}(o,x) \geq k$. Otherwise, $|Z_i(x) \cap L_{-j-i+1}(o)| \geq (\lambda - 1)k$. For each $v \in Z_i(x) \cap L_{-j-i+1}(o)$, consider the part of the path P_v stopped at its first hitting time of $L_{-j-i}(o)$ and call it $P_v(o,x)$. If $P_v(o,x)$ is also open, then the endpoint (other than v) belongs to $Y_{-i}(o,x)$. Notice the event $\widetilde{X}_{-i+1}^{-i+1,\infty}(x) \geq \lambda k$ is independent of $P_v(o,x)$ is open for every $v \in Z_i(x) \cap L_{-j-i+1}(o)$ and two such paths are disjoint if $d(v,v') \geq 2n_0$. Therefore, if λ is sufficiently large, with probability at least $\frac{3}{4}$ there are at least k such paths are open. Thus $\widetilde{X}_{-i}(o,x) \geq k$ with probability at least $\frac{3}{4}$ conditioned on $\widetilde{X}_{-i+1}^{-i+1,\infty}(x) \geq \lambda k$. Combining the above, we have $\widetilde{X}_{-i}(o,x) \geq k$ has probability at least $(\frac{3}{4})^2 q_x \geq \frac{9q}{16}$. Now since $C(x)$ is light a.s., $C(x)$ intersect each slab $L_{-i}(x)$ with at most finitely many vertices a.s. Thus there exists a large integer $r = r_x$ such that probability of the event that $C(x)$ intersect the boundary of $B(x,r)$ at some vertex $v \in H_{-i}^+(x)$ is at most $\frac{q}{16}$.

Notice if $F(x)$ occurs, $\widetilde{X}_{-i}(o,x) \geq k$ and $C(x)$ does not intersect the boundary of $B(x,r)$ at some vertex $v \in H_{-i}^+(x)$, then $B_x(i;r)$ is k-good. Hence $B_x(i;r)$ is k-good with probability at least $\frac{q}{2}$. Notice i, r depend only on the orbit of x and this proves the Claim above.

Now we can proceed to construct a random tree as in the proof of Theorem 4.3 of [31]. We fix k such that $\frac{kq}{2D^{n_0}} > 1$ and then some i, r such that $B_x(i;r)$ is k-good with probability at least $q/2$ for every $x \in H_0^-(o)$. The vertex $o \in V(G)$ corresponds to the root $\widehat{o}$ of T, the 0 generation O_0 of T. If $B_o(i;r)$ is k-good, then it contains at least k vertices in $L_{-i}(o)$ that can be connected to o by an open path with only one endpoint lying in $L_i(o)$. For each of these vertices x, add a child $\widehat{x}$ to $\widehat{o}$ in T and let these vertices $\widehat{x}$ constitute the first generation O_1 of T. If $B_o(i;r)$ is not k-good, let $\widehat{o}$ have 0 children and $T = \{\widehat{o}\}$.

Suppose we have defined the gth generation O_g of T such that each vertex $\widehat{x} \in O_g$ corresponds to a vertex $x \in L_{-gi}(o)$. We can partition O_g such that $\widehat{x}$, $\widehat{y}$ are in the same class of the partition iff for the corresponding $x, y \in L_{-gi}(o)$, we have $x' = y'$. Each set of the partition has at most D^{n_0} elements, where D the maximum degree of a vertex in G. Now choose one vertex in each class of the partition uniformly and independently; call the set of chosen vertices *parental* vertices. If $\widehat{x}$ is not parental, then let it have 0 children.

If $\widehat{x}$ is parental, assign f children to it iff $B_x(i;r)$ is k-good and $f := \widetilde{X}_{-i}(o,x) \geq k$; assign 0 children to $\widehat{x}$ otherwise. Assigning children in this way for each $\widehat{x} \in O_g$ and these children constitute O_{g+1}. Note that a vertex has at least k children with probability at least $\frac{q}{2D^{n_0}}$. Notice different vertices in O_{g+1} will also correspond to different vertices in $L_{-(g+1)i}(o)$.

It is straightforward to verify that the tree T constructed above satisfies the condition of Lemma 5.13 and the interested reader can refer to the proof of Theorem 4.3 in [31] for a similar verification. Hence T is transient with positive probability. Notice the open cluster restricted in the subgraph induced by the corresponding vertices x and $B_x(i, r)$ is roughly isometric to T, whence it is also transient. By Rayleigh's monotonicity principle, $C(o)$ is transient with positive probability. Since transience is a robust invariant property, Theorem 2.9 yields the desired conclusion. □

6. Proofs of the main theorems. The proof of Theorem 1.1 follows a similar strategy as the proof of Theorem 2.7 in [21], and the "square-root biased" two-sided random walk in Definition 5.6 will play the role of two-sided delayed simple random walk in the proof of Theorem 2.7.

DEFINITION 6.1. Suppose $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive and $(\mathbf{P}, \xi)$ is a Γ-invariant bond percolation process on G. Suppose $\mathcal{A}$ is a Γ-invariant property. An infinite cluster C of ξ is called of type $\mathcal{A}$ if $C \in \mathcal{A}$; otherwise, C is called of type $\neg\mathcal{A}$.

Suppose that there is an infinite cluster C of ξ and $e \in E \backslash C$ such that the connected component C' of $\xi \cup \{e\}$ that contains C has a type different from the one of C. Then e is called a pivotal edge for (C, ξ).

The following lemma is proved for transitive graphs in [21] and the proof can be easily adapted to quasi-transitive ones.

LEMMA 6.2 (Lemma 3.5 of [21]). *Suppose* $\Gamma \subset \mathrm{Aut}(G)$ *is quasi-transitive.* $(\mathbf{P}, \xi)$ *is an insertion-tolerant* Γ*-invariant bond percolation process on* G*. Suppose* $\mathcal{A} \in \mathcal{F}_E$ *is a* Γ*-invariant property. Assume that there is positive probability for coexistence of infinite clusters of type* $\mathcal{A}$ *and* $\neg\mathcal{A}$*. Then with positive probability, there is an infinite cluster* C *of* ξ *that has a pivotal edge.*

PROOF OF THEOREM 1.1. We proceed by contradiction. Suppose there exists some Γ-invariant property $\mathcal{A}$ such that there is positive probability for coexistence of heavy clusters in $\mathcal{A}$ and $\neg\mathcal{A}$. Let $\mathcal{O} = \{o_1, \ldots, o_L\}$ be a complete set of representatives of G/Γ. Fix a Haar measure m on Γ. By Lemma 6.2, we may assume with positive probability there are pivotal edges of heavy clusters of type $\mathcal{A}$ since otherwise one can replace $\mathcal{A}$ by $\neg\mathcal{A}$. For every $x \in V$, fix some $r_x > 0$ such that with positive probability, the cluster $C(x)$ is heavy, of type $\mathcal{A}$ and there is an edge e at graph distance at most r_x from x that is pivotal for $C(x)$. Notice we can choose r_x only depending on the orbit of x. Let $r := \max\{r_x : x \in \mathcal{O}\}$.

Fix $\varepsilon > 0$. Define A_x to be the event that the cluster of x in ξ is heavy and of type $\mathcal{A}$. Let A'_x be an event that depends on only finitely many edges such that $\mathbf{P}_p[A_x \Delta A'_x] < \varepsilon$. Let R_x be large enough such that A'_x only depends on edges in the ball $B(x, R_x)$. Let $R := \max\{R_x : x \in \mathcal{O}\}$.

Take a random root $\rho \in \mathcal{O}$ independent of the Bernoulli percolation $(\mathbf{P}_p, \xi)$ with distribution $\mathbf{P}(\rho = o_i) = \nu_\xi(o_i)$, where $\nu_\xi(x) = c \sum_{z \sim x} \sqrt{\frac{m(z)}{m(x)}}$ and c is such that $\sum_{i=1}^{L} \mathbf{E}[\nu_\xi(o_i)] = 1$.

Let W be a "square-root biased" two-sided random walk given by Definition 5.6 started from the random root ρ. For $n \in \mathbb{Z}$, let $e_n \in E$ be an edge chosen uniformly among the edges within distance at most r from $W(n)$. Recall in Corollary 5.7 $\mathbf{\Theta}$ is the joint law of (ξ, W). Write $\widehat{\mathbf{P}}$ for the joint law of ρ, ξ, W and $\langle e_n : n \in \mathbb{Z}\rangle$.

Given $e \in E$, let $\mathscr{P}_e$ be the event that $\xi \in A_\rho$ and e is pivotal for $C(\rho)$. Let $\mathscr{E}_e^n$ be the event that $e_n = e$ and $W(j)$ is not an endpoint of e whenever $-\infty < j < n$. Recall the insertion operation Π_e in Definition 2.6. For any measurable event $\mathscr{B}$, $e \in E$ and $n \geq 1$, one has that

$$\widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap (\mathscr{B} \backslash \Pi_e \mathscr{B})\big] = \frac{1-p}{p} \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap \Pi_e(\mathscr{B} \backslash \Pi_e \mathscr{B})\big] \leq \frac{1-p}{p} \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap \Pi_e \mathscr{B}\big],$$

where the first equality uses the definition of $\mathscr{E}_e^n$ and the fact that $n \geq 1$.

Then for all measurable event $\mathscr{B}$ one has

$$\begin{aligned}
\widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap \mathscr{B}\big] &= \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap (\mathscr{B} \backslash \Pi_e \mathscr{B})\big] + \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap (\mathscr{B} \cap \Pi_e \mathscr{B})\big] \\
&\leq \frac{1-p}{p} \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap \Pi_e \mathscr{B}\big] + \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap \Pi_e \mathscr{B}\big] \\
&= \frac{1}{p} \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap \Pi_e \mathscr{B}\big].
\end{aligned}$$

Applying the above inequality with $\mathscr{B} = A'_\rho \cap \mathscr{P}_e$, one has

$$\begin{aligned}
\widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap \Pi_e A'_\rho \cap \Pi_e \mathscr{P}_e\big] &\geq p \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap A'_\rho \cap \mathscr{P}_e\big] \\
&= p \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap A'_\rho \cap \mathscr{P}_{e_n}\big].
\end{aligned} \tag{6.1}$$

Define $\mathscr{E}^n := \bigcup_{e \in E} \mathscr{E}_e^n$, and $\mathscr{E}_R^n := \bigcup_{e \in E \backslash B(\rho, R)} \mathscr{E}_e^n$ and note that these are disjoint unions.

By definition of $\mathscr{P}_e$, $\Pi_e \mathscr{P}_e \subset \neg A_\rho$ since the insertion of pivotal edge e would change the type of $C(\rho)$. Also $\Pi_e A'_\rho \subset A'_\rho$ for any edge $e \in E \backslash B(\rho, R)$ since A'_ρ only depends edges within graph distance R from ρ. Thus

$$\begin{aligned}
\widehat{\mathbf{P}}\big[A'_\rho \cap \neg A_\rho\big] &\geq \widehat{\mathbf{P}}\big[\mathscr{E}_R^n \cap A'_\rho \cap \neg A_\rho\big] \\
&= \sum_{e \in E \backslash B(\rho, R)} \widehat{\mathbf{P}}\big[\mathscr{E}_e^n \cap A'_\rho \cap \neg A_\rho\big]
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{e \in E \setminus B(\rho,R)} \widehat{\mathbf{P}}\bigl[\mathscr{E}_e^n \cap \Pi_e A'_\rho \cap \Pi_e \mathscr{P}_e\bigr] \\
&\overset{(6.1)}{\geq} p \sum_{e \in E \setminus B(\rho,R)} \widehat{\mathbf{P}}\bigl[\mathscr{E}_e^n \cap A'_\rho \cap \mathscr{P}_{e_n}\bigr] \\
&= p\widehat{\mathbf{P}}\bigl[\mathscr{E}_R^n \cap A'_\rho \cap \mathscr{P}_{e_n}\bigr] \\
&\geq p\widehat{\mathbf{P}}\bigl[\mathscr{E}_R^n \cap A_\rho \cap \mathscr{P}_{e_n}\bigr] - p\varepsilon.
\end{aligned} \tag{6.2}
$$

Since there are infinitely many heavy clusters a.s., Proposition 5.9 yields that W is transient on the event that $C(\rho)$ is heavy, whence one can fix n sufficiently large such that the probability that $C(\rho)$ is heavy and $e_n \in B(\rho, R)$ is smaller than ε, whence $\widehat{\mathbf{P}}[A_\rho \cap (\mathscr{E}^n - \mathscr{E}_R^n)] \leq \varepsilon$. Then by (6.2), one has for n large

$$
\varepsilon \geq \widehat{\mathbf{P}}\bigl[A'_\rho \Delta A_\rho\bigr] \geq \widehat{\mathbf{P}}\bigl[A'_\rho \cap \neg A_\rho\bigr] \geq p\widehat{\mathbf{P}}\bigl[\mathscr{E}^n \cap A_\rho \cap \mathscr{P}_{e_n}\bigr] - 2p\varepsilon. \tag{6.3}
$$

Note by our choice of r, e_n, $\widehat{\mathbf{P}}[A_\rho \cap \mathscr{P}_{e_0}] > 0$. Conditioned on $A_\rho \cap \mathscr{P}_{e_0}$, transience of W implies that there exists $m \leq 0$ such that $W(m)$ is at graph distance r to e_0 and $W(j)$ is at graph distance more than r to e_0 for all $j < m$, in particular $W(j)$ is not incident to e_0, whence by choice of r, we have $\widehat{\mathbf{P}}[\mathscr{E}^m \cap A_\rho \cap \mathscr{P}_{e_m}] > 0$.

Define $\mathscr{B}_m := \bigcup_{x \in V} \mathscr{E}^m \cap A_\rho \cap \mathscr{P}_{e_m} \cap \{\rho = x\}$. Although event $\{\rho = x\}$ has zero probability under $\widehat{\mathbf{P}}$ for $x \in V \setminus \mathcal{O}$, the event $\mathscr{E}^m \cap A_\rho \cap \mathscr{P}_{e_m} \cap \{\rho = x\}$ is well defined just as the case $x \in \mathcal{O}$.

Notice $\mathscr{B}_m$ is Γ-invariant and $\widehat{\mathbf{P}}[\mathscr{B}_m] = \widehat{\mathbf{P}}[\mathscr{E}^m \cap A_\rho \cap \mathscr{P}_{e_m}]$ for any $m \in \mathbb{Z}$.

Let $\beta_{\xi,W}$ denote the law of $(e_n)_{n\in\mathbb{Z}}$ given ξ, W, then one has

$$
\begin{aligned}
\widehat{\mathbf{P}}[\mathscr{B}_m] &= \int_{2^E \times V^{\mathbb{Z}} \times E^{\mathbb{Z}}} \mathbf{1}_{[(\xi, W, (e_n)_{n\in\mathbb{Z}}) \in \mathscr{B}_m]} \, d\widehat{\mathbf{P}} \\
&= \int_{2^E \times V^{\mathbb{Z}}} \int_{E^{\mathbb{Z}}} \mathbf{1}_{[(\xi, W, (e_n)_{n\in\mathbb{Z}}) \in \mathscr{B}_m]} \, d\beta_{\xi,W}\bigl((e_n)_{n\in\mathbb{Z}}\bigr) \, d\mathbf{\Theta}(\xi, W).
\end{aligned} \tag{6.4}
$$

Define $F(\xi, W) := \int_{E^{\mathbb{Z}}} \mathbf{1}_{[(\xi, W, (e_n)_{n\in\mathbb{Z}}) \in \mathscr{B}_m]} \, d\beta_{\xi,W}((e_n)_{n\in\mathbb{Z}})$. It is straightforward to check that F is a Γ-invariant measurable function, whence Corollary 5.7 yields that $\widehat{\mathbf{P}}[\mathscr{B}_m]$ does not depend on m. Thus $\widehat{\mathbf{P}}[\mathscr{E}^n \cap A_\rho \cap \mathscr{P}_{e_n}] = \widehat{\mathbf{P}}[\mathscr{B}_n]$ does not depend on n. Hence when $\varepsilon > 0$ is sufficiently small, (6.3) gives a contradiction. This completes the proof. $\square$

Given a trajectory $w \in V^{\mathbb{Z}}$, for a set $C \subset V$ and $m, n \in \mathbb{Z}$, $m < n$, write

$$
\alpha_m^n(C)(w) := \frac{1}{n-m} \sum_{k=m}^{n-1} \mathbf{1}_{\{w(k) \in C\}}
$$

and

$$
\alpha(C)(w) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{\{w(k) \in C\}}
$$

when the limit exists, for the frequency of visits to C by the trajectory w on G. We do not need to define $\alpha(C)$ when the limit does not exist due to the following generalization of Lemma 4.2 of [21] in the quasi-transitive setting.

LEMMA 6.3. *Suppose $\Gamma \subset \mathrm{Aut}(G)$ is quasi-transitive. Let $\widehat{\mathbf{P}}$ be the probability measure given in Corollary 5.4. Then there is a Γ-invariant measurable function $f : 2^V \to [0, 1]$ with the following property: $\widehat{\mathbf{P}}$-a.s. $\alpha(C)$ exists and is equal to $f(C)$ for every cluster C. The function f is called the frequency function.*

PROOF. The proof follows a similar strategy as the one of Lemma 4.2 of [21]. Let λ be a probability measure on $\{o_1, \dots, o_L\}$ with $\lambda(\{o_i\}) = c_i$, where these c_i's come from Corollary 5.4. Let ρ be sampled from λ. Let $\mathbf{P}_\rho = \sum_{i=1}^{L} c_i \mathbf{P}_{o_i}$ denote the law of simple random walk starting from random vertex ρ. Then we have $\widehat{\mathbf{P}} = \mathbf{P}_p \times \mathbf{P}_\rho$.

For every $\alpha \in [0, 1]$, let

$$\mathcal{Z}_\alpha := \Big\{C \subset V : \lim_{n\to\infty} \alpha_0^n(C) = \alpha, \mathbf{P}_\rho\text{-a.s.}\Big\}.$$

Note this definition does not depend on the choice of basepoint ρ.

Define $f(C) := \alpha$ when $C \in \mathcal{Z}_\alpha$ for some $\alpha \in [0, 1]$. If $C \notin \mathcal{Z} := \bigcup_{\alpha\in[0,1]} \mathcal{Z}_\alpha$, put $f(C) := 0$. As shown in the proof of Lemma 4.2 of [21], f is measurable and Γ-invariant and for $\widehat{\mathbf{P}}$-a.e. $(\xi, w) \in \{0, 1\}^V \times V^{\mathbb{N}}$,

$$\lim_{n\to\infty} \max\big\{\big|\alpha_0^m(C)(w) - \alpha_0^k(C)(w)\big| : k, m \geq n, C \text{ is a cluster of } \xi\big\} = 0. \tag{6.5}$$

It remains to show that for $\mathbf{P}_p$-a.s. ξ, every cluster of ξ is in $\mathcal{Z}$. Actually, let $A := \{\xi : C \in \mathcal{Z}, \forall \text{ cluster } C \text{ of } \xi\}$, it's easy to see $A \in \mathcal{F}_V$.

Corollary 5.5 and (6.5) then yield that for $\mathbf{P}_p \times \mathbf{P}_\rho^{\mathbb{Z}}$-a.s. $(\xi, \widehat{w})$,

$$\lim_{n\to\infty} \max\big\{\big|\alpha_0^m(C)(\widehat{w}) - \alpha_0^k(C)(\widehat{w})\big| : k, m \geq n, C \text{ is a cluster of } \xi\big\} = 0. \tag{6.6}$$

By the shift invariance of $\mathcal{T}$,

$$\begin{aligned} &2\max\big\{\big|\alpha_0^{2n}(C) - \alpha_0^n(C)\big| : C \text{ is a cluster}\big\} \\ &\quad = \max\big\{\big|\alpha_n^{2n}(C) - \alpha_0^n(C)\big| : C \text{ is a cluster}\big\} \end{aligned}$$

has the same law as $\max\{|\alpha_0^n(C) - \alpha_{-n}^0(C)| : C \text{ is a cluster}\}$.

Therefore by (6.6), we have

$$\max\big\{\big|\alpha_0^n(C) - \alpha_{-n}^0(C)\big| : C \text{ is a cluster}\big\} \to 0 \quad \text{in probability}. \tag{6.7}$$

By (6.6), we also have $\lim_{n\to\infty} \alpha_0^n(C)$ exists. Similar argument shows that $\lim_{n\to\infty} \alpha_{-n}^0(C)$ also exists. Combining with the above equation (6.7) one has that a.s. $\lim_{n\to\infty} \alpha_0^n(C) = \alpha(C) = \lim_{n\to\infty} \alpha_{-n}^0(C)$ for every cluster C.

However, given ξ and a cluster C of ξ, $\alpha_0^n(C)$ is independent of $\alpha_{-n}^0(C)$ conditioned on $w(0)$, but both converge to $\alpha(C)$. Therefore, $\alpha(C)$ is a $\mathbf{P}_p \times \mathbf{P}_\rho^{\mathbb{Z}}$-a.s. constant. This completes the proof. □

PROOF OF THEOREM 1.3. Write $\mathbf{P} = \mathbf{P}_p$ and $\Gamma = \mathrm{Aut}(G)$. If Γ is unimodular, this is just Theorem 4.1 of [21]. In the following, we assume that Γ is nonunimodular.

Since there is a.s. more than one infinite cluster, $N(p) = \infty$ a.s. Since light and heavy infinite clusters cannot coexist a.s., we consider two cases separately. If these infinitely many infinite clusters are light a.s., then $\inf\{\mathbf{P}(x \leftrightarrow y) : x, y \in V\} = 0$. Indeed, if $\inf\{\mathbf{P}(x \leftrightarrow y) : x, y \in V\} = c_1 > 0$, fix some $o \in V$ and let $L_n := L_n(o)$. Then $\mathbf{P}[o \leftrightarrow L_n] \geq \inf\{\mathbf{P}(x \leftrightarrow y) : x, y \in V\} = c_1 > 0$. Notice $\{o \leftrightarrow L_n\}$ is decreasing. Thus

$$\mathbf{P}\big[C(o) \text{ is heavy}\big] \geq \mathbf{P}\left[\bigcap_{n=1}^{\infty}\{o \leftrightarrow L_n\}\right] \geq \liminf_{n\to\infty} \mathbf{P}[o \leftrightarrow L_n] \geq c_1 > 0.$$

This contradicts the assumption that these infinite clusters are light.

Now in the following we assume that there are infinitely many heavy clusters a.s. Let f be the frequency function given in Lemma 6.3. Since f is Γ-invariant, for each $\alpha \in [0, 1]$, $Q_\alpha := \{C \subset 2^V : f(C) \leq \alpha\}$ is a Γ-invariant property, where for a cluster C we also use C to denote its vertex set.

Let A_α be the event that there exists an infinite cluster that satisfies the property Q_α. If $\mathbf{P}(A_\alpha) > 0$, then $\mathbf{P}(A_\alpha) = 1$ by ergodicity of $\mathbf{P}$.

Let $c := \inf\{\alpha \in [0, 1] : \mathbf{P}(A_\alpha) > 0\}$. By definition of A_α and c, we have $f(C) \geq c$ for all infinite clusters C $\mathbf{P}$-a.s. On the other hand, for each $t > c$, $\mathbf{P}(A_t) > 0$ and then $\mathbf{P}(A_t) = 1$ by ergodicity. Thus $\mathbf{P}$-a.s. there exists an infinite cluster C such that $f(C) \leq t$, that is, C satisfies Q_t. Since there are infinitely many heavy clusters a.s., Theorem 1.1 then implies that all these infinite clusters satisfy property Q_t. Since this is true for arbitrary $t > c$, it follows that $\mathbf{P}$-a.s.

$$f(C) = c, \quad \text{for every infinite cluster } C.$$

The rest proof of the first conclusion is almost the same as the one of Theorem 4.1 of [21] except we use Lemma 6.3 to get the a.s. equality $f(C) = \alpha(C)$ in this quasi-transitive nonunimodular setting.

And deriving $p_u = \overline{p}_{\mathrm{conn}}$ from the first conclusion is easy so we leave it to the reader. □

REMARK 6.4. At $p = p_u$, $\inf_{x\in V} \mathbb{P}_{p_u}(o \leftrightarrow x)$ can be equal to zero or be positive. For example, let G be a nonamenable planar quasi-transitive graph with one end. Then $0 < p_c < p_u < 1$ (see Theorem 8.24 of [20]) and there is a unique infinite cluster $\mathbb{P}_{p_u}$-a.s. In this case by Harris-FKG inequality, we know $\inf_{x\in V} \mathbb{P}_{p_u}(o \leftrightarrow x) > 0$. Other examples include $\mathbb{T}_b$ with $b \geq 3$ or $\mathbb{Z}^2 * \mathbb{Z}_2$, where

$*$ means the free product. They are transitive graphs with infinitely many ends, and hence $p_u = 1$.

In the following, we will use $X \times Y$ to denote the Cartesian product of graphs X and Y. Consider $\mathbb{T}_b \times \mathbb{Z}$ where $\mathbb{T}_b$ is a regular tree with degree $b \geq 3$. We know $0 < p_c < p_u < 1$ from [2, 17]. Schonmann [27] and Peres [24] showed that at $p = p_u$ there are infinitely many infinite clusters a.s. Now from Theorem 1.3 we know $\inf_{x \in V} \mathbb{P}_{p_u}(o \leftrightarrow x) = 0$ in this case. For the case $G = \mathbb{T}_1 \times \cdots \times \mathbb{T}_n$, $n \geq 2$, where $\mathbb{T}_i$ are regular trees with degree at least 3, Theorem 1.3 implies that $\inf_{x \in V} \mathbb{P}_{p_u}(o \leftrightarrow x) = 0$. In this case, Hutchcroft [18], Question 1.9, conjectured that $\mathbb{P}_{p_u}(o \leftrightarrow x)$ even decays exponentially in the graph distance $d(o, x)$.

7. Examples and questions.

7.1. *Examples* 1. Consider the regular tree $\mathbb{T}_b$ with degree $b \geq 3$. Let ξ be a fixed end of $\mathbb{T}_b$ and $\Gamma_\xi \subset \mathrm{Aut}(\mathbb{T}_b)$ be the subgroup that fixes the end ξ. Then Γ_ξ is transitive and nonunimodular. And $p_h(G, \Gamma_\xi) = 1$. This can be seen by direct calculation (comparing to a branching process) or by Corollary 4.8. Indeed since $p_u = 1$, $\lim_{n\to\infty} p_c(G_n) \leq p_u$ holds and then $p_h(G, \Gamma_\xi) = \lim_{n\to\infty} p_c(G_n)$. Since G_n are finite unions of levels, all its connected components are finite, whence $p_c(G_n) = 1$ and then $p_h(G, \Gamma_\xi) = \lim_{n\to\infty} p_c(G_n) = 1$. More information about this example can be found in Section 8.1 of [17]. This gives an example with $p_c < p_h(G, \Gamma) = p_u = 1$.

Now consider the Cartesian product $G := \mathbb{T}_b \times \mathbb{Z}^d$ with $b \geq 3$, $d \geq 1$. Let $\Gamma = \Gamma_\xi \times \mathrm{Aut}(\mathbb{Z}^d)$. Corollary 5.8 of [31] implies that $p_h(G, \Gamma) = p_u$ since the subgraph induced by any finite unions of levels is amenable. Peres [24] showed that $N(p_h(G, \Gamma)) = \infty$ a.s. Proposition 4.7 yields all these infinite clusters are light. This gives an example that $p_c < p_h(G, \Gamma) = p_u < 1$ and $N(p_h(G, \Gamma)) = \infty$. Here, $p_c < p_h(G, \Gamma)$ is due to [17].

QUESTION 7.1. *Is there a graph G and $\Gamma \subset \mathrm{Aut}(G)$ such that Γ is quasi-transitive and nonunimodular, $p_c < p_h(G, \Gamma) = p_u < 1$ and $N(p_h(G, \Gamma)) = 1$ a.s.?*

It was asked in [15] whether there is any explicit example of transitive graph satisfying $p_c < p_h < p_u < 1$. The first such examples are provided in [17]; see Section 8.2 there. The following examples are also devoted to this question. Examples 2 are slightly simpler in the sense that one does not need to consider anisotropic percolation. However, Examples 2 exhibit $p_h(G, \Gamma) < p_u(G)$ with proper subgroup Γ. One might modify them to obtain examples with $\Gamma = \mathrm{Aut}(G)$. We will not do that and instead we point out Examples 3 satisfying the restriction $\Gamma = \mathrm{Aut}(G)$.

7.2. *Examples* 2. To be consistent with the notation in [20], we consider regular trees $\mathbb{T}_{b+1}$ with $b \geq 3$. Suppose n_1, n_2 are two positive integers such that $n_1 + n_2 = b$. We define a $(1, n_1, n_2)$-*orientation* of $\mathbb{T}_{b+1}$ like the $n_1 = 1$, $n_2 = 2$, $b = 3$ case in [17]. To be precise, give a partial orientation of the edge set of $\mathbb{T}_{b+1}$ such that every vertex is incident to exactly one unoriented edge, has n_1 oriented edges emanating from it, and has n_2 oriented edges pointing into it. From now on, we fix such an orientation.

Denote by $\Gamma_{(1,n_1,n_2)} \subset \mathrm{Aut}(\mathbb{T}_{b+1})$ the subgroup of automorphisms that preserve this orientation. It is easy to see that this subgroup $\Gamma_{(1,n_1,n_2)}$ acts transitively on $\mathbb{T}_{b+1}$. Moreover, if $q := \frac{n_2}{n_1} \neq 1$, then $\Gamma_{(1,n_1,n_2)}$ is nonunimodular. Indeed, define $h(u, v)$ to be the *height difference* as in [17]: for $u, v \in \mathbb{T}_{b+1}$ there is a unique simple path r connecting them, suppose there are m_1 edges on r that are crossed in the forward direction when moving from u to v along r and m_2 edges that are crossed in the opposite direction, then $h(u, v) := m_1 - m_2$. Using Lemma 2.2, it is easy to see $\Delta_\Gamma(u, v) = \frac{|\Gamma_v u|}{|\Gamma_u v|} = q^{h(u,v)}$ for $u \sim v$, where $\Gamma := \Gamma_{(1,n_1,n_2)}$. Then by cocycle identity, $\Delta_\Gamma(u, v) = q^{h(u,v)}$ holds for all pairs $u, v \in \mathbb{T}_{b+1}$.

One has $p_h(\mathbb{T}_4, \Gamma_{(1,1,2)}) = \frac{2\sqrt{2}+1-\sqrt{4\sqrt{2}-3}}{6}$ by Proposition 8.1 of [17] and formula (8.1) there. The following proposition is a slight generalization of this result.

PROPOSITION 7.2. *Suppose positive integers n_1, n_2 satisfy $n_1 + n_2 = b$, then*

$$p_h(\mathbb{T}_{b+1}, \Gamma_{(1,n_1,n_2)}) = \frac{1 + 2\sqrt{n_1 n_2} - \sqrt{(2\sqrt{n_1 n_2} + 1)^2 - 4(n_1 + n_2)}}{2(n_1 + n_2)}. \tag{7.1}$$

We will not provide its proof but point out two ways to do it in the following two remarks.

REMARK 7.3. If one defines the *tiltability threshold* p_t as in [17], using the same method of obtaining formula (8.1) in [17], one can calculate the exact value of p_t and it is just the right-hand side of (7.1). Then one can use similar argument as in the proof of Proposition 8.1 of [17] to obtain $p_h(\mathbb{T}_{b+1}, \Gamma_{(1,n_1,n_2)}) = p_t(\mathbb{T}_{b+1}, \Gamma_{(1,n_1,n_2)})$.

REMARK 7.4. Another way to prove Proposition 7.2 is using Corollary 4.8. Notice that the finite union of consecutive levels G_n are infinitely many copies of periodic trees H_n. These periodic trees are directed covers ([20], Section 3.3) of certain finite oriented graphs D_n. Notice $p_c(H_n)^{-1} = \mathrm{br}(H_n) = \mathrm{gr}(H_n)$ for periodic trees. Moreover, let A_n denote the adjacency matrix of D_n, then $\mathrm{gr}(H_n) = \lambda_*(A_n)$ (see the discussion on pages 83–84 of [20]), where $\lambda_*(A_n)$ denotes the largest positive eigenvalue of the matrix A_n. Then the reciprocal of $p_h(\mathbb{T}_{b+1}, \Gamma_{(1,n_1,n_2)})$ equals the limit of $\lambda_*(A_n)$. Calculating the limit of $\lambda_*(A_n)$ then gives Proposition 7.2. The calculation is a little bit long and we omit it here.

Russell Lyons pointed out to me that the adjacency matrices A_n are parts of a block Toeplitz matrix A and that [9] may be relevant. Indeed the formula from [9] does give exactly the reciprocal of (7.1). However, the theorem in [9] does not include our matrix A. One might expect to extend the theorem in [9] in order to have a simple way to find the limit of $\lambda_*(A_n)$.

Now we consider $\mathbb{T}_{b+1} \times \mathbb{Z}$ and $\Gamma_b := \Gamma_{(1,n_1,n_2)} \times \mathrm{Aut}(\mathbb{Z})$. From Theorem 6.10, Proposition 7.35 and Theorem 7.37 of [20], one has the following lower bound for p_u:

$$p_u(\mathbb{T}_{b+1} \times \mathbb{Z}) \geq \frac{1}{\mathrm{cogr}(\mathbb{T}_{b+1} \times \mathbb{Z})} = \frac{1}{\sqrt{b}+1+\sqrt{2\sqrt{b}-1}}. \tag{7.2}$$

Note $p_h(\mathbb{T}_{b+1} \times \mathbb{Z}, \Gamma_b) \leq p_h(\mathbb{T}_{b+1}, \Gamma_{(1,n_1,n_2)})$. Simple calculation shows that when $n_1 \geq 2$, $n_2 \geq 2$ and b large enough, the value of $p_h(\mathbb{T}_{b+1}, \Gamma_{(1,n_1,n_2)})$ (Proposition 7.2) is strictly less than the above lower bound of p_u, whence such graphs are explicit examples exhibiting $p_c(\mathbb{T}_{b+1} \times \mathbb{Z}) < p_h(\mathbb{T}_{b+1} \times \mathbb{Z}, \Gamma_b) < p_u(\mathbb{T}_{b+1} \times \mathbb{Z}) < 1$. However, here Γ_b is not the whole automorphism group of $\mathbb{T}_{b+1} \times \mathbb{Z}$.

7.3. *Examples* 3. The following family of examples are motivated by [15]. Recall for a quasi-transitive graph G, $p_h := p_h(G, \mathrm{Aut}(G))$.

DEFINITION 7.5 (Definition 1.3 in [25]). A graph G is called *prime* w.r.t. Cartesian product if G is nontrivial (not the graph U with a single vertex and no edge) and if $G \cong Y \times Z$ then either $Y \cong U$ or $Z \cong U$, where $A \cong B$ means that graph A is isomorphic to B. Two distinct graphs G, G' are called *relatively prime* if $G \cong X \times Z$ and $G' \cong Y \times Z$ implies that $Z \cong U$.

Now fix G_0 to be a nonunimodular transitive graph and that is relatively prime to regular trees (e.g., G_0 can be the grand-parent graph). Then for any regular tree $\mathbb{T}_k$ by Corollary 3.2 of [25] $\mathrm{Aut}(G_0 \times \mathbb{T}_k) = \mathrm{Aut}(G_0) \times \mathrm{Aut}(\mathbb{T}_k)$, whence it is nonunimodular.

For k large enough, one has $p_h(G_0 \times \mathbb{T}_k) < p_u(G_0 \times \mathbb{T}_k)$ (see the inequality (4.9.2) on page 87 of [15]). By Hutchcroft [17] and the fact that $\mathrm{Aut}(G_0 \times \mathbb{T}_k)$ is nonunimodular and transitive, one has $p_c(G_0 \times \mathbb{T}_k) < p_h(G_0 \times \mathbb{T}_k)$, whence we get another family of graphs exhibiting $p_c < p_h < p_u < 1$.

Last but not least, Theorem 1.1 and Theorem 1.3 are only proved for Bernoulli percolation. However, corresponding theorems in [21] are proved to hold for general insertion tolerant percolation process.

Remark 5.10 implies for insertion-and-deletion tolerant percolation process the following weaker conclusion hold: there exists some heavy cluster that is transient for the "square-root biased" random walk. Notice in the proof of Theorem 1.1 we need that with positive probability the cluster $C(\rho)$ is heavy, transient for the "square-root biased" random walk and has some pivotal edges. However, the weak conclusion and Lemma 6.2 do not guarantee the existence of a pivotal edge for $C(\rho)$.

In Timár's proof of Proposition 4.6, deletion-tolerance was used in the proof of Lemma 5.2 and Lemma 5.3 in [31]. Lemma 5.2 can be extended to percolation processes with just insertion-tolerance property but we do not know whether Lemma 5.3 can be extended to such percolation processes.

QUESTION 7.6. *Does Theorem 1.1 hold if one just assumes Γ-invariance and insertion-and-deletion tolerance? What if just Γ-invariance and insertion-tolerance?*

Acknowledgments. We thank Russell Lyons for many helpful discussions, detailed comments on a first draft and financial support by the National Science Foundation under grant DMS-1612363. We also thank Ádám Timár for comments on the manuscript. We thank the referee for the detailed and helpful comments which improved the quality of writing in this paper.

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