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TOTAL VARIATION DISTANCE BETWEEN STOCHASTIC POLYNOMIALS AND INVARIANCE PRINCIPLES

BY VLAD BALLY AND LUCIA CARAMELLINO[1]

Université Paris-Est and Università di Roma Tor Vergata

The goal of this paper is to estimate the total variation distance between two general stochastic polynomials. As a consequence, one obtains an invariance principle for such polynomials. This generalizes known results concerning the total variation distance between two multiple stochastic integrals on one hand, and invariance principles in Kolmogorov distance for multilinear stochastic polynomials on the other hand. As an application, we first discuss the asymptotic behavior of U-statistics associated to polynomial kernels. Moreover, we also give an example of CLT associated to quadratic forms.

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1. Introduction. *Stochastic polynomials.* This paper deals with stochastic polynomials of the following type: given a sequence $X = (X_n)_{n\in\mathbb{N}}$ of independent random variables which have finite moments of any order and, given $N \in \mathbb{N}$

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and $k_* \in \mathbb{N}$, one looks to

$$Q_{N,k_*}(c,X) = \sum_{m=0}^{N} \Phi_m(c,X) \quad \text{with} \tag{1.1}$$

$$\begin{aligned}\Phi_m(c,X) := \sum_{k_1,\dots,k_m=1}^{k_*} \sum_{n_1<\cdots<n_m=1}^{\infty} & c\big((n_1,k_1),\dots,(n_m,k_m)\big) \\ & \times \prod_{j=1}^{m} \big(X_{n_j}^{k_j} - \mathbb{E}(X_{n_j}^{k_j})\big).\end{aligned} \tag{1.2}$$

The coefficients c are symmetric and null on the diagonals (i.e., if $n_i = n_j$ for $i \neq j$) and only a finite number of them are nonnull, so the above sum is finite. Let us mention that here, for notation simplicity, we take $X_n \in \mathbb{R}$, but in the paper we work with $X_n = (X_{n,1},\dots,X_{n,d_*}) \in \mathbb{R}^{d_*}$. Note also that we use the centered random variables $X_n^k - \mathbb{E}(X_n^k)$, $k = 1,\dots,k_*$, but if the polynomial is given in terms of X_n^k, we may always rewrite it in terms of centered random variables.

If $k_* = 1$ then $Q_{N,1}(c,X)$ is a multilinear stochastic polynomial and moreover, if X_n, $n \in \mathbb{N}$, are independent standard Gaussian random variables then $\Phi_m(c,X)$ is an iterated stochastic integral of order m. So, multilinear stochastic polynomials are a natural generalization of elements of the classical Wiener chaoses. However, general stochastic polynomials are of interest, for example, in applications to U-statistics theory (see Section 4.1 and the references cited therein).

Our goal is to estimate the total variation distance between the laws of two such polynomials, and moreover, to establish an invariance principle. The starting point in our approach is the following general invariance principle. Let $Z_n = (Z_{n,1},\dots,Z_{n,k_*})$, $n \in \mathbb{N}$ be a sequence of centered independent random vectors which have finite moments of any order and let

$$S_N(c,Z) = \sum c\big((n_1,k_1),\dots,(n_m,k_m)\big) \prod_{j=1}^{m} Z_{n_j,k_j}$$

with the sum over all $n_1,\dots,n_m \in \mathbb{N}$ and $k_1,\dots,k_m \in \{1,\dots,k_*\}$ with $m \leq N$. So we are back to multilinear polynomials. We prove (roughly speaking) that for every $f \in C_b^3(\mathbb{R})$

$$\big|\mathbb{E}\big(f\big(S_N(c,Z)\big)\big) - \mathbb{E}\big(f\big(S_N(c,G)\big)\big)\big| \leq C\|f'''\|_\infty \delta_*(c), \tag{1.3}$$

where $\delta_*(c)$ is the "low influence factor" (see the definition (1.6) below) and $G_n = (G_{n,1},\dots,G_{n,k_*})$ are centered independent Gaussian random vectors with the same covariance matrix as Z_n (see Theorem 2.2 for the precise statement). We stress that the dependence structure in the vector $Z_n = (Z_{n,1},\dots,Z_{n,k_*})$ is completely general. This allows us to take $Z_n = (X_n - \mathbb{E}(X_n),\dots,X_n^{k_*} - \mathbb{E}(X_n^{k_*}))$ and to come back to our polynomials. Notice that the Gaussian vector $G_n =$

$(G_{n,1},\dots,G_{n,k_*})$ does not keep the structure given by the powers in the original vector Z_n.

The estimate in (1.3) concerns smooth functions. The main contribution of our paper is to replace $\|f'''\|_\infty$ by $\|f\|_\infty$ and so to obtain convergence in total variation distance. In order to precisely describe the nondegeneracy assumptions, we have to use the framework of stochastic polynomials.

Doeblin's condition and Nummelin's splitting. Since the total variation distance concerns measurable functions, a "regularization effect" has to be at work. This leads us to make the following assumption (known as Doeblin's condition): there exists $\varepsilon>0, r>0$ and $x_n\in\mathbb{R}, n\in\mathbb{N}$, such that $\sup_n|x_n|<\infty$ and $\mathbb{P}(X_n\in dx)\geq \varepsilon\,dx$ on the ball $B_r(x_n)$. It is easy to see that this is equivalent with saying that

$$\mathbb{P}(X_n\in dx)=p\psi(x-x_n)\,dx+(1-p)\nu_n(dx), \tag{1.4}$$

where $p\in(0,1]$, ψ is a C^∞ probability density with the support included in $B_r(0)$ and ν_n is a probability measure. The decomposition (1.4) being given, one constructs three independent random variable χ_n, V_n, U_n with $V_n\sim\psi(x-x_n)\,dx$, $U_n\sim\nu_n(dx)$ and χ_n Bernoulli with parameter p and then employs the identity of laws

$$X_n\overset{\text{law}}{=}\chi_nV_n+(1-\chi_n)U_n. \tag{1.5}$$

The density ψ may be chosen (see (3.6)) in order that $\ln\psi$ has nice properties and this allows one to build an abstract Malliavin-type calculus based on $V_n, n\in\mathbb{N}$ and to use this calculus in order to obtain the "regularization effect" which is needed. We have already used this argument in [4–7]. In an independent way, Nourdin and Poly in [33] have used similar arguments in a similar problem: they take $\psi=(1/2r)^{-1}1_{B_r(0)}$ so V_n has a uniform distribution, and they use a chaos type decomposition obtained in [2]. Note also that hypothesis (1.4) is in fact necessary: in his seminal paper [39], Prohorov proved that (1.4) is (essentially) necessary and sufficient in order to obtain convergence in total variation distance in the central limit theorem (see [4] for details).

The decomposition (1.5) has been introduced by Nummelin (see [24, 36]) in order to produce atoms which allow one to use the renewal theory for studying the convergence to equilibrium for Markov chains—this is why it is also known as "the Nummelin splitting method." It has been also used by Poly in his Ph.D. thesis [38] and, to our knowledge, this is the first place where the idea of using the regularization given by the noise V_n appears.

Main results. In order to present our results, we have to introduce some more notation. Given the coefficient c in (1.3) we denote $|c|_m$ and $\delta_{\mathcal{U},*}*(c)$ through

$$\begin{aligned}|c|_m^2&=\sum_{k_1,\dots,k_m=1}^{k_*}\sum_{n_1<\dots<n_m=1}^{\infty}c^2\big((n_1,k_1),(n_2,k_2),\dots,(n_m,k_m)\big),\\ \delta_*^2(c)&=\max_n\sum_{m=1}^{N}\sum_{k_1,\dots,k_m=1}^{k_*}\sum_{n_1<\dots<n_m=1}^{\infty}c^2\big((n_1,k_1),(n_2,k_2),\dots,(n_m,k_m)\big).\end{aligned} \tag{1.6}$$

$|c|_m$ and $\delta_*(c)$ are the quantities which come in, in order to estimate the errors. The quantity $|c|_m$ is essentially equivalent (up to a multiplicative factor) with the variance of $\Phi_m(c, X)$ and $\delta_*(c)$ is essentially equivalent with the "low influence factor" as it has first been defined in [40] and then used in [25] (and we follow several ideas from this paper). Another interpretation, in terms of the Malliavin derivative D, is as follows. Suppose that $k_* = 1$ and let $G = (W(k+1) - W(k))_{k\in\mathbb{N}}$ where W is a Brownian motion (so $\Phi_m(c, G)$ is a stochastic iterated integral of order m). Then

$$\delta_*^2(c) = \sup_{s>0} \mathbb{E}\left(\left|D_s\left(\sum_{m=1}^{\infty} \Phi_m(c, G)\right)\right|^2\right). \tag{1.7}$$

For $f \in C_b^\infty(\mathbb{R}^d)$, we denote by $\|f\|_{k,\infty}$ the supremum norm of f and of its derivatives of order less or equal to k, and, for two random variables F and G, we define the distances

$$d_k(F, G) = \sup\{|\mathbb{E}(f(F)) - \mathbb{E}(f(G))| : \|f\|_{k,\infty} \le 1\}. \tag{1.8}$$

For $k = 0$, $d_0 = d_{\mathrm{TV}}$ is the total variation distance, and, if $F \sim p_F(x)\,dx$ and $G \sim p_G(x)\,dx$ then $d_{\mathrm{TV}}(F, G) = \|p_F - p_G\|_1$. d_1 is the Fortet–Mourier distance which metrizes the convergence in law. We also consider the Kolmogorov distance

$$d_{\mathrm{Kol}}(F, G) = \sup_{x\in\mathbb{R}} |\mathbb{P}(F \le x) - \mathbb{P}(G \le x)|. \tag{1.9}$$

We are now able to give our first result, Theorem 3.3, concerning the distance between two polynomials $Q_{N,k_*}(c, X)$ and $Q_{N,k_*}(d, Y)$. Assume that X and Y satisfy the Doeblin's condition (see (1.4)), and moreover, assume that the nondegeneracy condition $|c|_N > 0$, $|d|_N > 0$ holds. Then we prove that for every $k \in \mathbb{N}$ and $\theta \in (\frac{1}{(1+k)^2}, 1)$,

$$\begin{aligned} &d_{\mathrm{TV}}(Q_{N,k_*}(c, X), Q_{N,k_*}(d, Y)) \\ &\quad\le \mathrm{Const}(c, d) \\ &\qquad \times \left(d_k(Q_{N,k_*}(c, X), Q_{N,k_*}(d, Y))^{\frac{\theta}{2kk_*\overline{N}+1}} + e^{-\frac{|c|_N^2}{C\delta_*^2(c)}} + e^{-\frac{|d|_N^2}{C\delta_*^2(d)}}\right), \end{aligned} \tag{1.10}$$

where $\mathrm{Const}(c, d)$ denote a quantity which depends on the coefficients c and d in an explicit way (see (3.19)). Theorem 3.3 is the main result in our paper (in fact, the statement of this theorem is more general).

In Theorem 3.7 we give a variant of this result in Kolmogorov distance: we prove that, for $k \ge 3$,

$$\begin{aligned} &d_{\mathrm{Kol}}(Q_{N,k_*}(c, X), Q_{N,k_*}(d, Y)) \\ &\quad\le \mathrm{Const}(c, d) \\ &\qquad \times \left(d_k(Q_{N,k_*}(c, X), Q_{N,k_*}(d, Y))^{\frac{\theta}{2Nk+1}} + \delta_*^{\frac{\theta}{2kN+1}}(c) + \delta_*^{\frac{\theta}{2kN+1}}(d)\right), \end{aligned} \tag{1.11}$$

$\mathrm{Const}(c,d)$ is again a positive quantity explicitly depending on c and d (see (3.23)). The estimate (1.11) holds for general laws for X_n and Y_n (without assuming the Doeblin's condition). However, now we have to assume that the covariance matrix of the powers $(X_n, X_n^2, \dots, X_n^{k_*})$ and $(Y_n, Y_n^2, \dots, Y_n^{k_*})$ are both invertible. The proof of (1.11) is a direct consequence of the results of Mossel et al. in [25].

A second result, given in Theorem 3.10, concerns the invariance principle. We consider a sequence of independent centered Gaussian random vectors $G_n = (G_{n,1}, \dots, G_{n,k_*}) \in \mathbb{R}^{k_*}$ and we assume that the covariance matrix of G_n coincides with the covariance matrix of $Z_n = (Z_{n,1}, \dots, Z_{n,k_*})$ where $Z_{n,k} := X_n^k - \mathbb{E}(X_n^k)$. We denote by $S_N(c,G)$ the polynomial $Q_{N,k_*}(c,X)$ in which $Z_n = (Z_{n,1}, \dots, Z_{n,k_*})$ is replaced by $G_n = (G_{n,1}, \dots, G_{n,k_*})$. We stress that $S_N(c,G)$ is multilinear with respect to $G_{n,i}$, $i = 1, \dots, k_*$ in contrast to $Q_{N,k_*}(c,X)$ which is a general polynomial with respect to X_n. In Theorem 3.10, we prove that, if $|c|_N > 0$, then for every $\theta \in (\frac{1}{16}, 1)$,

$$d_{\mathrm{TV}}\big(Q_{N,k_*}(c,X), S_N(c,G)\big) \le \mathrm{Const}(c) \times \delta_*(c)^{\frac{\theta}{6k_* m+1}}, \tag{1.12}$$

$\mathrm{Const}(c)$ being explicitly dependent on c (see (3.24)). A result going in the same direction was previously obtained by Nourdin and Poly in [33]. They take $k_* = 1$, so $Q_N(c,X)$ is a multilinear polynomial, and they assume Doeblin's condition for X_i. Then they prove that, if $c_n, n \in \mathbb{N}$ is a sequence of coefficients such that $\lim_n \delta_*(c_n) = 0$, then $\lim_n d_{\mathrm{TV}}(Q_{N,k_*}(c,X), S_N(c,G)) = 0$. The progress achieved in our paper consists in the fact that we deal with general polynomials on one hand and we obtain an estimate of the error on the other hand.

Applications. An important consequence of (1.12) is that it allows to replace the study of the asymptotic behavior of a sequence $Q_{N,k_*}(c_n, X), n \in \mathbb{N}$ of general stochastic polynomials by the study of $S_N(c_n, G), n \in \mathbb{N}$, which are elements of a finite number of Wiener chaoses. Of course, the central example is the classical CLT, where $N = 1$ and $k_* = 1$, so $S_1(c_n, G) = \sum_{i=1}^{\infty} c_n(i) G_i$ is just a Gaussian random variable. But, starting with the proof of the "fourth moment theorem" by Nualart and Peccati [35] and Nourdin and Peccati [27], a lot of work has been done in order to characterize the convergence to normality of elements of a finite number of Wiener chaoses (see [26, 31, 34, 37] or [28] for an overview). Moreover, convergence to a χ_2 distribution has been treated in [27]. We study the consequences of these results in Theorem 3.12 and Theorem 3.15.

Finally, we give two more applications. The first one concerns the asymptotic behavior of U-statistics written on polynomial kernels. Let us mention that number of results are already known concerning the convergence in Kolmogorov distance for U-statistics: they represent generalizations of the Berry–Esseen theorem (see [23]). But the result in total variation distance, which generalizes Prohorov's theorem for the CLT, seems to be new.

Another subject which is very close, is that of quadratic forms. Here, also the asymptotic behavior in Kolmogorov distance is well understood (see de Jong [15, 16], Rotar et al. [19, 41] and Götze et al. [20]) but we have not found results concerning the convergence in total variation. We present here an interesting example, giving a "change of regime" asymptotic behavior either in Kolmogorov and total variation distance. This example fits in the framework of nonsymmetric U-statistics discussed in [17, 18] (see Remark 4.4). Moreover, similar stochastic series appear in some statistical mechanics models; see [10, 11].

Organization of the paper. In Section 2, we fix our settings and we give some preliminary results. Section 3 is devoted to our main results: we first define the Doeblin's condition and the Nummelin splitting (Section 3.1); then we introduce our main result Theorem 3.3 and its several consequences (Section 3.2); finally, we analyze the Gaussian and Gamma approximation (Section 3.3). The main examples are developed in Section 4: Section 4.1 is devoted to U-statistics and in Section 4.2 we present our example of a quadratic CLT. Finally, Section 5 contains the proof of our main Theorem 3.3, which is given in the last Section 5.4: in Section 5.1, we introduce the abstract Malliavin calculus and state the regularization lemma, Section 5.2 is devoted to proper estimates of the Sobolev norms and Section 5.3 refers to the nondegeneracy of the Malliavin covariance matrix. The paper concludes with the Appendix, where an iterated Hoeffding's inequality for martingales is studied.

2. Notation, basic objects and preliminary results. In this section, we introduce multilinear stochastic polynomials based on a sequence of abstract independent random vectors $Z_n = (Z_{n,1}, \dots, Z_{n,m_*}) \in \mathbb{R}^{m_*}$, $n \in \mathbb{N}$. We stress that no hypothesis concerning the dependence structure of Z_n is needed. In the following, when dealing with general polynomials as in (1.1), we will take $Z_{n,k} = X_n^k - \mathbb{E}(X_n^k)$.

• *The basic noise*. We assume that $\mathbb{E}(Z_{n,i}) = 0$ and that Z_n has finite moments of any order: for every $p \geq 1$ there exists some $M_p(Z) \geq 1$ such that for every $n \in \mathbb{N}$ and $i \in [m_*] = \{1, \dots, m_*\}$

$$\|Z_{n,i}\|_p \leq M_p(Z). \tag{2.1}$$

• *Multi-indexes*. We will use "double" multi-indexes $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_i = (\alpha'_i, \alpha''_i) = (n_i, j_i)$ with $n_i \in \mathbb{N}$ and $j_i \in [m_*]$. We always assume that $n_1 < \cdots < n_m$. So we work with "ordered" multi-indexes. We also denote $\alpha' = (\alpha'_1, \dots, \alpha'_m) = (n_1, \dots, n_m)$, $\alpha'' = (\alpha''_1, \dots, \alpha''_m) = (j_1, \dots, j_m)$ and $|\alpha| = m$. The set of such multi-indexes is denoted by Γ_m and we set $\Gamma = \bigcup_m \Gamma_m$. We stress that we consider also the void multi-index $\alpha = \varnothing$ and in this case we put $|\alpha| = 0$. Moreover, for a sequence $x_n = (x_{n,1}, \dots, x_{n,m_*}) \in \mathbb{R}^{m_*}$, $n \in \mathbb{N}$ we denote $x^\alpha = \prod_{i=1}^m x_{\alpha_i}$, with $x^\alpha = 1$ if $\alpha = \varnothing$.

• *Coefficients*. We consider a Hilbert space $\mathcal{U}$ with norm $|\cdot|_{\mathcal{U}}$ and for a $\mathcal{U}$ valued random variable X, we denote $\|X\|_{\mathcal{U},p} = (\mathbb{E}(|X|_{\mathcal{U}}^p))^{1/p}$. In a first stage, we have just $\mathcal{U} = \mathbb{R}$ but in Section 5, when considering stochastic derivatives, we have to use some general space $\mathcal{U}$. We denote $\mathcal{C}(\mathcal{U}) = \{c = (c(\alpha))_{\alpha \in \Gamma} : c(\alpha) \in \mathcal{U}\}$. These are the coefficients we will use (we stress again that we work with ordered multi-indexes). We define

$$|c|_{\mathcal{U}} = \Big(\sum_{\alpha} |c(\alpha)|^2_{\mathcal{U}}\Big)^{1/2}, \qquad |c|_{\mathcal{U},m} = \Big(\sum_{|\alpha|=m} |c(\alpha)|^2_{\mathcal{U}}\Big)^{1/2}, \tag{2.2}$$

$$\delta_{\mathcal{U},*}(c) = \Big(\sup_n \sum_{\alpha} 1_{\{n\in\alpha'\}} |c(\alpha)|^2_{\mathcal{U}}\Big)^{1/2}. \tag{2.3}$$

The notation $n \in \alpha'$ means that $\alpha'_j = n$ for some $j \in [m]$. When $\mathcal{U} = \mathbb{R}$, we shall omit the subscript $\mathcal{U}$, so we simply write $|c|$, $|c|_m$ and $\delta_*(c)$. For several authors (see, e.g., [25, 30]), $\delta^2_{\mathcal{U},*}(c)$ is called the "influence" factor.

• *Multilinear polynomials*. Given $c \in \mathcal{C}(\mathcal{U})$, we define

$$\begin{aligned} \Phi_m(c, Z) &= \sum_{|\alpha|=m} c(\alpha) Z^{\alpha} \\ &= \sum_{j_1,\dots,j_m=1}^{m_*} \sum_{n_1<\dots<n_m} c\big((n_1, j_1), \dots, (n_m, j_m)\big) \prod_{i=1}^{m} Z_{n_i, j_i}, \end{aligned} \tag{2.4}$$

$$S_N(c, Z) = \sum_{0\le|\alpha|\le N} c(\alpha) Z^{\alpha} = \sum_{m=0}^{N} \Phi_m(c, Z). \tag{2.5}$$

In the sequel, we use several times Burkholder's inequality for Hilbert space valued martingales: if $M_n \in \mathcal{U}, n \in \mathbb{N}$ is a martingale then for every $p \ge 2$ there exists $b_p \ge 1$ such that

$$\begin{aligned} \|M_n\|_{\mathcal{U},p} &\le b_p \bigg(\mathbb{E}\bigg(\Big(\sum_{k=1}^{n-1} |M_{k+1} - M_k|^2_{\mathcal{U}}\Big)^{p/2}\bigg)\bigg)^{1/p} \\ &\le b_p \Big(\sum_{k=1}^{n-1} \|M_{k+1} - M_k\|^2_{\mathcal{U},p}\Big)^{1/2} \end{aligned} \tag{2.6}$$

the second inequality being obtained by using the triangle inequality with respect to $\|\cdot\|_{p/2}$.

Moreover, as an immediate consequence of (2.1), for every $n \in \mathbb{N}$ and every $d_j \in \mathcal{U}$, $j \in [m_*]$ we have

$$\Big\|\sum_{j=1}^{m_*} d_j \times Z_{n,j}\Big\|_{\mathcal{U},p} \le \sqrt{m_*} M_p(Z) \Big(\sum_{j=1}^{m_*} |d_j|^2_{\mathcal{U}}\Big)^{1/2}. \tag{2.7}$$

Using these two inequalities, we obtain the following.

LEMMA 2.1. *Suppose that* (2.1) *holds and denote* $\overline{M}_p = b_p M_p(Z)\sqrt{m_*}$. *Then*

$$\|\Phi_N(c, Z)\|_{\mathcal{U},p} \le \overline{M}_p^N |c|_{\mathcal{U},N}. \tag{2.8}$$

If $m_* = 1$ *and* $p = 2$, *then* $\overline{M}_2 = 1$ *and the above inequality becomes an equality.*

PROOF. We proceed by recurrence on N. For $N = 0$, we have $\Phi_N(c, Z) = c(\emptyset)$ so (2.8) is obvious. For $\alpha \in \Gamma$ with $|\alpha| = N - 1$, we denote

$$c^{n,j}(\alpha) = c\big(\alpha, (n, j)\big) 1_{\{\alpha'_{N-1} < n\}} \tag{2.9}$$

and we write

$$\Phi_N(c, Z) = \sum_{n=N}^{\infty} \sum_{j=1}^{m_*} Z_{n,j} \Phi_{N-1}\big(c^{n,j}, Z\big). \tag{2.10}$$

Note that, if $n \geq N$, $Z_{n,j}$ and $\Phi_{N-1}(c^{n,j}, Z)$ are independent. So, using (2.6) first and (2.7) then we get

$$\begin{aligned} \|\Phi_N(c, Z)\|_{\mathcal{U},p}^2 &\leq b_p^2 \sum_{n=N}^{\infty} \left\| \sum_{j=1}^{m_*} Z_{n,j} \Phi_{N-1}\big(c^{n,j}, Z\big) \right\|_{\mathcal{U},p}^2 \\ &\leq b_p^2 M_p^2(Z) m_* \sum_{n=N}^{\infty} \sum_{j=1}^{m_*} \big\|\Phi_{N-1}\big(c^{n,j}, Z\big)\big\|_{\mathcal{U},p}^2. \end{aligned}$$

Since $\sum_{n=N}^{\infty} \sum_{j=1}^{m_*} |c^{n,j}|_{\mathcal{U},N-1}^2 = \sum_{|\alpha|=N} |c(\alpha)|_{\mathcal{U}}^2$, (2.8) follows by recurrence. $\square$

We give now the basic invariance principle.

THEOREM 2.2. *Let* $Z = (Z_n)_{n \in \mathbb{N}}, Z_n \in \mathbb{R}^{m_*}$ *be a sequence of centered independent random vectors which verify* (2.1) *and let* $G = (G_n)_{n \in \mathbb{N}}, G_n \in \mathbb{R}^{m_*}$ *be a sequence of independent centered Gaussian random vectors such that* $\mathbb{E}(G_{n,i} G_{n,j}) = \mathbb{E}(Z_{n,i} Z_{n,j})$. *Then, for every* $f \in C_b^3(\mathbb{R})$,

$$\big|\mathbb{E}\big(f\big(S_N(c, Z)\big) - \mathbb{E}\big(f\big(S_N(c, G)\big)\big| \leq \mathcal{K}(Z) \|f'''\|_\infty \times |c|^2 \times \delta_*(c) \tag{2.11}$$

with

$$\mathcal{K}(Z) = \frac{2m_*}{3} \big(M_3^3(Z) + M_3^3(G)\big) \overline{M}_3^{3N},$$

in which $\overline{M}_3 = b_3 \sqrt{m_*} M_3(Z) \vee M_3(G)$.

PROOF. The proof is based on Lindberg's method (we follow the argument from [25]). We fix $J \geq N$, we denote $\Gamma_N(J) = \bigcup_{m=0}^{N} \{\alpha \in \Gamma : |\alpha| = m, \alpha'_m \leq J\}$

and we define $S_{N,J}(c,Z)=\sum_{\alpha\in\Gamma_N(J)}c(\alpha)Z^\alpha$. For $j=1,\dots,J+1$, we define the intermediate sequences $Z^j=(Z_1,\dots,Z_{j-1},G_j,\dots,G_J)$, with $Z^1=(G_1,\dots,G_J)$ and $Z^{J+1}=(Z_1,\dots,Z_J)$, and we write

$$\begin{aligned}&\mathbb{E}\big(f\big(S_{N,J}(c,Z)\big)\big)-\mathbb{E}\big(f\big(S_{N,J}(c,G)\big)\big)\\&\quad=\sum_{j=1}^J\mathbb{E}\big(f\big(S_{N,J}\big(c,Z^{j+1}\big)\big)-\mathbb{E}\big(f\big(S_{N,J}\big(c,Z^j\big)\big)=:\sum_{j=1}^J I_j.\end{aligned}$$

We denote $\Gamma_N(j,J)=\{\alpha\in\Gamma_N(J):j\notin\alpha'\}$ and, for $\beta\in\Gamma_N(j,J)$ with $|\beta|=m$ we define

$$\begin{aligned}c_{j,i}(\beta)=&\sum_{k=2}^m c\big(\beta_1,\dots,\beta_{k-1},(j,i),\beta_k,\dots,\beta_m\big)1_{\{\beta'_{k-1}<j<\beta'_k\}}\\&+c\big((j,i),\beta_1,\dots,\beta_m\big)1_{\{j<\beta'_1\}}+c\big(\beta_1,\dots,\beta_m,(j,i)\big)1_{\{\beta'_m<j\}}.\end{aligned}$$

This means that, if β does not contain j, we insert (j,i) in the convenient position. We put

$$A_j=\sum_{\alpha\in\Gamma_N(j,J)}c(\alpha)\big(Z^j\big)^\alpha,\qquad B_{j,i}=\sum_{\beta\in\Gamma_{N-1}(j,J)}c_{j,i}(\beta)\big(Z^j\big)^\beta$$

and then

$$S_{N,J}\big(c,Z^{j+1}\big)=A_j+\sum_{i=1}^{m_*}Z_{j,i}B_{j,i}.$$

Moreover, with $f_j:\mathbb{R}^{m_*}\to\mathbb{R}$ defined by $f_j(x):=f(A_j+\sum_{i=1}^{m_*}x_iB_{j,i})$ we get

$$I_j=\mathbb{E}\big(f\big(S_{N,J}\big(c,Z^{j+1}\big)\big)-\mathbb{E}\big(f\big(S_{N,J}\big(c,Z^j\big)\big)=\mathbb{E}\big(f_j(Z_j)\big)-\mathbb{E}\big(f_j(G_j)\big).$$

We use now Taylor's expansion of order three around 0 for both $f_j(Z_j)$ and $f_j(G_j)$. Since Z_j and G_j are independent of A_j and $B_{j,\cdot}$ and the first and second moments of $Z_{j,i}$ and $G_{j,i}$ coincide, the first- and second-order terms in the Taylor expansion cancel and we obtain

$$\begin{aligned}|I_j|&\le\frac12\sum_{i_1,i_2,i_3=1}^{m_*}\mathbb{E}(\Xi_{i_1i_2i_3})\quad\text{with}\\ \Xi_{i_1i_2i_3}&=\prod_{r=1}^3\big(|Z_{j,i_r}|+|G_{j,i_r}|\big)\\&\quad\times\int_0^1(1-\lambda)^2\big(\big|\partial^3_{i_1i_2i_3}f_j(\lambda Z_j)\big|+\big|\partial^3_{i_1i_2i_3}f_j(\lambda G_j)\big|\big)\,d\lambda.\end{aligned}$$

We have

$$\left|\partial^3_{i_1 i_2 i_3} f_j(\lambda Z_j)\right| = \left|f_j^{(3)}(\lambda Z_j)\right| \times \prod_{r=1}^{3} |B_{j,i_r}| \leq \|f'''\|_\infty \times \prod_{r=1}^{3} |B_{j,i_r}|.$$

The same is true for $|\partial^3_{i_1 i_2 i_3} f_j(\lambda G_j)|$, so (recall that Z_j and G_j are independent of $B_{j,\cdot}$)

$$|I_j| \leq \frac{1}{3}\|f'''\|_\infty \big(M_3^3(Z) + M_3^3(G)\big) \sum_{i_1,i_2,i_3=1}^{m_*} \mathbb{E}\left(\prod_{r=1}^{3} |B_{j,i_r}|\right). \tag{2.12}$$

Suppose first that $N \geq 2$. Using (2.8),

$$\|B_{j,i}\|_3 \leq \overline{M}_3^N \left(\sum_{\beta \in \Gamma_{N-1}(j,J)} \left|c_{j,i}(\beta)\right|^2\right)^{1/2} \leq \overline{M}_3^N \delta_*(c).$$

If $N = 1$ then

$$|B_{j,i}| = \left|c(i,j))\right| \leq \max_n \left(\sum_{l=1}^{m_*} \left|c\big((n,l)\big)\right|^2\right)^{1/2} = \delta_*(c).$$

This gives

$$\mathbb{E}\left(\prod_{r=1}^{3} |B_{j,i_r}|\right) \leq \prod_{r=1}^{3} \|B_{j,i_r}\|_3 \leq \overline{M}_3^N \delta_*(c)\big(\|B_{j,i_1}\|_3^2 + \|B_{j,i_2}\|_3^2\big).$$

We sum over j and we get

$$\begin{aligned}
\sum_{j=1}^{J} |I_j| &\leq \frac{2m_*}{3}\|f'''\|_\infty \big(M_3^3(Z) + M_3^3(G)\big)\overline{M}_3^N \delta_*(c) \sum_{j=1}^{J}\sum_{i=1}^{m_*} \|B_{j,i}\|_3^2 \\
&\leq \frac{2m_*}{3}\|f'''\|_\infty \big(M_3^3(Z) + M_3^3(G)\big)\overline{M}_3^{3N} \delta_*(c)|c|^2.
\end{aligned}$$

Since the above estimate does not depend on J, we may pass to the limit with $J \to \infty$ and we obtain (2.11). $\square$

We recall now the main result from [25] on the invariance principle in Kolmogorov distance (see (1.9)).

THEOREM 2.3. *Let $Z = (Z_n)_{n \in \mathbb{N}}$, $Z_n \in \mathbb{R}^{m_*}$ be a sequence of centered independent random vectors which verify* (2.1) *and let* $\mathrm{Cov}(Z_n)$ *denote the covariance matrix of Z_n. We assume that there exists* $0 < \underline{\lambda} \leq 1$ *such that for every* $n \in \mathbb{N}$

$$\mathrm{Cov}(Z_n) \geq \underline{\lambda}. \tag{2.13}$$

Let $G = (G_n)_{n\in\mathbb{N}}, G_n \in \mathbb{R}^{m_*}$ *be a sequence of independent centered Gaussian random vectors such that* $\mathrm{Cov}(Z_n) = \mathrm{Cov}(G_n)$. *Then*

$$d_{\mathrm{Kol}}\big(S_N(c, Z), S_N(c, G)\big) \le \mathcal{K}(Z)\underline{\lambda}^{-m_* N} \times \delta_*^{1/(1+3N)}(c), \tag{2.14}$$

where $\mathcal{K}(Z) > 0$ *is a constant depending on* N *and* $M_3(Z)$.

PROOF. We denote $A_n = \mathrm{Cov}^{1/2}(Z_n)$ and we define $\overline{Z}_n = A_n^{-1} \times Z_n$, so that $\overline{Z}_{n,1}, \dots, \overline{Z}_{n,m_*}$ are orthonormal. In the formalism in [25], $\overline{Z}_n$ is called an "orthonormal ensemble." Then we define

$$\begin{aligned}&\overline{c}\big((n_1, k_1), \dots, (n_N, k_N)\big)\\ &\quad = \sum_{i_1,\dots,i_N=1}^{m_*} c\big((n_1, i_1), \dots, (n_N, i_N)\big) A_{n_1}^{i_1,k_1} \cdots A_{n_N}^{i_N,k_N}\end{aligned} \tag{2.15}$$

and we notice that, with this definition,

$$S_N(c, Z) = S_N(\overline{c}, \overline{Z}). \tag{2.16}$$

Moreover, one easily checks that

$$|\overline{c}| \le (m_* M_2)^N |c| \quad \text{and} \quad \delta_*(\overline{c}) \le (m_* M_2)^N \delta_*(c). \tag{2.17}$$

Let us check that $\overline{Z}$ is hypercontractive in the sense of [25]. We notice that $M_p(\overline{Z}) \le \underline{\lambda}^{-m_*} M_p(Z)$ and we take $\eta^{-1} = b_p (b_p \underline{\lambda}^{-m_*} M_p(Z))^N$. Then, for any coefficients $c \in \mathcal{C}(\mathbb{R})$ we have (with $p = 3$)

$$\begin{aligned}\big\|S_N(c, \eta\overline{Z}) - c(\varnothing)\big\|_p &\le b_p\big(b_p M_p(\overline{Z})\big)^N \Big(\sum_{1\le|\alpha|\le N} \eta^{|\alpha|} \big|\overline{c}(\alpha)\big|^2\Big)^{1/2}\\ &\le b_p\big(b_p \underline{\lambda}^{-m_*} M_p(Z)\big)^N \Big(\sum_{1\le|\alpha|\le N} \eta^{|\alpha|} \big|\overline{c}(\alpha)\big|^2\Big)^{1/2}\\ &\le \Big(\sum_{1\le|\alpha|\le N} \big|\overline{c}(\alpha)\big|^2\Big)^{1/2}\\ &= \big\|S_N(c, \overline{Z}) - c(\varnothing)\big\|_2\end{aligned}$$

and this means, in the formalism from [25] that $\overline{Z}$ is $(2, 3, \eta)$-hypercontractive. Now we are able to use Theorem 3.19 in [25] (which is written in terms of $\tau = \delta_*^2(c)$), and this yields (2.14). □

3. Main results.

3.1. *Doeblin's condition and splitting.* We fix $d_* \in \mathbb{N}$ and $k_* \in \mathbb{N}$, we denote $m_* = d_* \times k_*$, and we work with a sequence of independent random vectors $X = (X_n)_{n\in\mathbb{N}}$, $X_n = (X_{n,1}, \dots, X_{n,d_*}) \in \mathbb{R}^{d_*}$. We deal with general polynomials with variables $X_{n,j}$ that is, with linear combinations of monomials $\prod_{i=1}^m X_{n_i,j_i}^{k_i}$, $k_i \le k_*$. Because of the powers k_i, this is no more a multilinear polynomial. In order to come back to multilinear polynomials we define $Z_n(X) \in \mathbb{R}^{m_*}$ by

$$Z_{n,kd_*+j}(X) = X_{n,j}^{k+1} - \mathbb{E}\big(X_{n,j}^{k+1}\big) \quad \text{for } j \in [d_*], k \in \{0, 1, \dots, k_* - 1\}. \tag{3.1}$$

With this definition, if $\alpha = ((n_1, l_1), \dots, (n_m, l_m))$, with $n_1 < \cdots < n_m$ and $l_1, \dots, l_m \in \{1, \dots, m_*\}$, then

$$Z^\alpha(X) = \prod_{i=1}^m \big(X_{n_i,j_i}^{k_i+1} - \mathbb{E}\big(X_{n_i,j_i}^{k_i+1}\big)\big),$$

where $(k_i, j_i) = (k(l_i), j(l_i))$, $i = 1, \dots, m$, with

$$k(l) = \left\lfloor \frac{l-1}{d_*} \right\rfloor \quad \text{and} \quad j(l) = \left\{ \frac{l-1}{d_*} \right\} d_* + 1, \tag{3.2}$$

the symbols $\lfloor x \rfloor$ and $\{x\}$ denoting the integer and the fractional part of $x \ge 0$, respectively. We denote

$$Q_{N,k_*}(c, X) = \sum_{0 \le |\alpha| \le N} c(\alpha) Z^\alpha(X) = S_N\big(c, Z(X)\big), \tag{3.3}$$

that is,

$$\begin{aligned} &Q_{N,k_*}(c, X) \\ &\quad = \sum_{m=0}^N \sum c\big((n_1, (k_1 - 1)d_* + j_1), \dots, (n_m, (k_m - 1)d_* + j_m)\big) \\ &\quad\quad \times \prod_{i=1}^m \big(X_{n_i,j_i}^{k_i} - \mathbb{E}\big(X_{n_i,j_i}^{k_i}\big)\big), \end{aligned}$$

in which the second sum runs over $n_1 < \cdots < n_m$, $k_1, \dots, k_m \in \{0, 1, \dots, k_*\}$ and $j_1, \dots, j_m \in [d_*]$. Notice that it agrees with (1.1)–(1.3) when $d_* = 1$.

The crucial hypothesis in this section is that for every $n \in \mathbb{N}$, the law of X_n is locally lower bounded by the Lebesgue measure—this is Doeblin's condition. Let us be more precise.

HYPOTHESIS $\mathfrak{D}(\varepsilon, r, R)$. *Let $\varepsilon > 0$, $r > 0$ and $R > 0$ be fixed. We say that $X = (X_n)_{n\in\mathbb{N}}$ satisfies hypothesis $\mathfrak{D}(\varepsilon, r, R)$ if there exist $x_n \in \mathbb{R}^{d_*}, n \in \mathbb{N}$ such that for every measurable set $A \subset B_r(x_n)$*

$$\mathbb{P}(X_n \in A) \ge \varepsilon\lambda(A), \tag{3.4}$$

λ denoting the Lebesgue measure on $\mathbb{R}^{d_*}$, and

$$\sup_{n\in\mathbb{N}} |x_n| \le R. \tag{3.5}$$

Note that there is no assumption about X_n, $n \in \mathbb{N}$, being identically distributed, but the fact that the parameters ε, r and R are the same for every n, represents a uniformity assumption. Note also that this property never holds for $Z_n(X)$. This is why the Malliavin-type calculus presented in the following is based on X_n only.

HYPOTHESIS $\mathfrak{M}(\varepsilon, r, R)$. *We say that $X = (X_n)_{n\in\mathbb{N}}$ satisfies hypothesis $\mathfrak{M}(\varepsilon, r, R)$ if $\mathfrak{D}(\varepsilon, r, R)$ holds and for every $p \ge 1$ one has* $\sup_{n\in\mathbb{N}} \|X_n\|_p < \infty$.

Note that if Assumption $\mathfrak{M}(\varepsilon, r, R)$ holds then $Z_n(X)$ verifies (2.1).

The interesting point about random vectors which verity $\mathfrak{D}(\varepsilon, r, R)$ is that one may use a splitting method in order to obtain a nice representation for X_n (in law). We introduce the auxiliary functions $\theta_r, \psi_r : \mathbb{R} \to \mathbb{R}_+$ defined by

$$\theta_r(t) = 1 - \frac{1}{1 - (\frac{t}{r} - 1)^2}, \qquad \psi_r(t) = 1_{\{|t|\le r\}} + 1_{\{r<|t|\le 2r\}} e^{\theta_r(|t|)} \tag{3.6}$$

and we denote

$$\mathfrak{m}_r = \int_{\mathbb{R}} \psi_r(|z|^2)\, dz. \tag{3.7}$$

Let $V_n, U_n \in \mathbb{R}^{d_*}$ and $\chi_n \in \{0, 1\}$ be independent random variables with laws

$$\begin{aligned}
&\mathbb{P}(\chi_n = 1) = \varepsilon \mathfrak{m}_r^{d_*}, \qquad \mathbb{P}(\chi_n = 0) = 1 - \varepsilon \mathfrak{m}^{d_*}, \\
&\mathbb{P}(V_n \in dx) = \frac{1}{\mathfrak{m}_r^{d_*}} \prod_{k=1}^{d_*} \psi_r(|x_k - x_{n,k}|^2)\, dx_1 \cdots dx_{d_*}, \\
&\mathbb{P}(U_n \in dx) = \frac{1}{1 - \mathfrak{m}_r^{d_*}} \left(\mathbb{P}(X_n \in dx) - \varepsilon \prod_{k=1}^{d_*} \psi_r(|x_k - x_{n,k}|)^2 \right) dx.
\end{aligned} \tag{3.8}$$

$\mathfrak{D}(\varepsilon, r, R)$ ensures that $\mathbb{P}(X_n \in dx) - \varepsilon \prod_{k=1}^{d_*} \psi_r(|x_k - x_{n,k})|^2)\, dx \ge 0$, so that the law of U_n is well defined. It is easy to check that $\chi_n V_n + (1 - \chi_n)U_n$ has the same law as X_n. Since all our statements concern only the law of X_n, now on we assume that

$$X_n = \chi_n V_n + (1 - \chi_n)U_n. \tag{3.9}$$

Let us mention two nice properties for the function ψ_r. First, it is easy to check that for each $k \in \mathbb{N}$, $p \ge 1$ there exists a universal constant $C_{k,p} \ge 1$ such that

$$\psi_r(t)\big|\theta_r^{(k)}(|t|)\big|^p \le \frac{C_{k,p}}{r^{kp}}, \tag{3.10}$$

where $\theta_r^{(k)}$ denotes the derivative of order k of θ_r. (3.10) will be useful in order to give estimates for the Ornstein–Uhlenbeck operator (see Remark 5.1). Second, $\theta_r(|t|)$ is concave for $r < |t| < 2r$ (direct computation), so that the law of V_n is log-concave. This allows one to use the Cherbery–Wright inequality (see Lemma 5.6).

We discuss now some nondegeneracy properties (see (3.12) and (3.13) below) which hold under the hypothesis (3.5). We define the random vector $\widetilde{V}_n = Z_n(V)$ in $\mathbb{R}^{m_*}$, that is,

$$\widetilde{V}_{n,l} = V_{n,j(l)}^{k(l)+1} - \mathbb{E}\big(V_{n,j(l)}^{k(l)+1}\big), \quad l = 1, \dots, m_*, \tag{3.11}$$

where $k(l)$ and $j(l)$ are given in (3.2). Then, one has the following result.

LEMMA 3.1. *Let $R > 0$ be such that (3.5) holds and let $\operatorname{Cov}(\widetilde{V}_n)$ denote the covariance matrix of $\widetilde{V}_n$. Then there exists $\lambda_R > 0$ such that for every $\xi \in \mathbb{R}^{m_*}$ and $n \in \mathbb{N}$,*

$$\big\langle \operatorname{Cov}(\widetilde{V}_n)\xi, \xi\big\rangle \ge \lambda_R |\xi|^2. \tag{3.12}$$

PROOF. For $y \in \mathbb{R}^{d_*}$ and $\xi \in \mathbb{R}^{m_*}$, we define

$$e_l(y) = \frac{1}{\mathfrak{m}_r^{d_*}} \int x_{j(l)}^{k(l)} \prod_{i=1}^{d_*} \psi_r\big(|x_i - y_i|^2\big)\, dx, \quad l \in [m_*] \quad \text{and}$$

$$I_\xi(y) = \frac{1}{\mathfrak{m}_r^{d*}} \int \left(\sum_{l=1}^{m_*} \big(x_{j(l)}^{k(l)} - e_l(y)\big)\xi_l \right)^2 \prod_{i=1}^{d_*} \psi_r\big(|x_i - y_i|^2\big)\, dx.$$

If $I_\xi(y) = 0$, then $\sum_{l=1}^{m_*} (x_{j(l)}^{k(l)} - e_l(y))\xi_l = 0$ for x in an open set, and this imply that $\xi = 0$. Since $\xi \mapsto I_\xi(y)$ is continuous, it follows that $\lambda(y) := \inf_{|\xi|=1} I_\xi(y) > 0$. And since $y \mapsto \lambda(y)$ is continuous, it follows that one may find $\lambda_R > 0$ such that $\inf_{|y| \le R} \lambda(y) \ge \lambda_R$. Now, we note that $e_l(x_n) = \mathbb{E}(V_{n,j(l)}^{k(l)}) = \mathbb{E}(\widetilde{V}_{n,l})$ and $I_\xi(x_n) = \langle \operatorname{Cov}(\widetilde{V}_n)\xi, \xi\rangle$. Thus, if $|\xi| = 1$ then $\inf_n \langle \operatorname{Cov}(\widetilde{V}_n)\xi, \xi\rangle = \inf_n \inf_{|\xi|=1} I_\xi(x_n) \ge \lambda_R$. $\square$

We conclude with an inequality which will be useful later on.

LEMMA 3.2. *Let $R > 0$ be such that (3.5) holds and let λ_R be given in Lemma 3.1. Let $\widetilde{V} = Z(V)$ be defined in (3.11) and $S_N(d, \widetilde{V})$ given in (2.5). Then for every $d \in \mathcal{C}(\mathbb{R})$,*

$$\mathbb{E}\big(\big|S_N(d, \widetilde{V})\big|^2\big) \ge \lambda_R^N \sum_{m=0}^{N} |d|_m^2 = \lambda_R^N |d|^2. \tag{3.13}$$

PROOF. We first fix an integer m, $n_1 < \cdots < n_m$ and we consider an arbitrary family of numbers $d(l_1, \dots, l_m)$, $l_i \in [m_*]$. We prove that

$$\mathbb{E}\Biggl(\Biggl(\sum_{l_1,\dots,l_m=1}^{m_*} d(l_1, \dots, l_m) \prod_{i=1}^{m} \widetilde{V}_{n_i,l_i}\Biggr)^2\Biggr) \geq \lambda_R^m \sum_{l_1,\dots,l_m=1}^{m_*} d^2(l_1, \dots, l_m). \tag{3.14}$$

We define the random variable

$$\widehat{d}(l_m) = \sum_{l_1,\dots,l_{m-1}=1}^{m_*} d(l_1, \dots, l_m) \prod_{i=1}^{m-1} \widetilde{V}_{n_i,l_i}.$$

We notice that $\widehat{d}(k)$, $k \in [m_*]$ are independent of $\widetilde{V}_{n_m,l}$, $l \in [m_*]$ and that

$$\sum_{l_1,\dots,l_m=1}^{m_*} d(l_1, \dots, l_m) \prod_{i=1}^{m} \widetilde{V}_{n_i,l_i} = \sum_{l_m=1}^{m_*} \widehat{d}(l_m) \widetilde{V}_{n_m,l_m}.$$

So,

$$\begin{aligned}
&\mathbb{E}\Biggl(\Biggl(\sum_{l_1,\dots,l_m=1}^{m_*} d(l_1, \dots, l_m) \prod_{i=1}^{m} \widetilde{V}_{n_i,l_i}\Biggr)^2\Biggr) \\
&\quad \geq \lambda_R \mathbb{E}\Biggl(\sum_{l_m=1}^{m_*} \widehat{d}(l_m)^2\Biggr) \\
&\quad = \lambda_R \sum_{l_m=1}^{m_*} \mathbb{E}\Biggl(\Biggl(\sum_{l_1,\dots,l_{m-1}=1}^{m_*} d(l_1, \dots, l_{m-1}, l_m) \prod_{i=1}^{m-1} \widetilde{V}_{n_i,l_i}\Biggr)^2\Biggr),
\end{aligned}$$

the above lower bound following from (3.12). By iteration, one gets (3.14).

Consider now the general case. We recall that, for any two multi-indexes α and $\overline{\alpha}$, $\mathbb{E}(\widetilde{V}^\alpha \widetilde{V}^{\overline{\alpha}}) = 0$ if $|\alpha| \neq |\alpha'|$ or $\alpha \neq \alpha'$. This gives

$$\begin{aligned}
\mathbb{E}(|S_N(d, \widetilde{V})|^2) &= \sum_{m=0}^{N} \sum_{|\alpha|=|\overline{\alpha}|=m, \alpha'=\overline{\alpha}'} d(\alpha) d(\overline{\alpha}) \mathbb{E}(\widetilde{V}^\alpha \widetilde{V}^{\overline{\alpha}}) \\
&= \sum_{m=0}^{N} \sum_{n_1<\cdots<n_m} \mathbb{E}\Biggl(\Biggl(\sum_{l_1,\dots,l_m \in [m_*]} d_{n_1,\dots,n_m}(l_1, \dots, l_m) \prod_{i=1}^{m} \widetilde{V}_{n_i,l_i}\Biggr)^2\Biggr),
\end{aligned}$$

where we have set $d_{n_1,\dots,n_m}(l_1, \dots, l_m) = d((n_1, l_1), \dots, (n_m, l_m))$. The statement now follows from (3.14). □

3.2. *Main results.* Our goal is to estimate the total variation distance between two polynomials of type $Q_{N,k_*}(c, X)$, which we write as in (3.3), that is,

$$Q_{N,k_*}(c, X) = \sum_{0 \leq |\alpha| \leq N} c(\alpha) Z^\alpha(X),$$

where $Z(X)$ is defined in (3.1) and $\alpha = (\alpha', \alpha'')$ with $\alpha_i'' \in [m_*]$, $m_* = d_* k_*$.

We will use the following quantities related to the coefficients c. We work first with the Hilbert space $\mathcal{U}=\mathbb{R}$ (so, we drop $\mathcal{U}$ from the notation) and we recall that $|c|=|c|_{\mathcal{U}}$, $|c|_m=|c|_{\mathcal{U},m}$ and $\delta_*(c)=\delta_{\mathcal{U},*}(c)$ are defined in (2.2) and in (2.3). Moreover, for $m\leq N$, we define

$$|c|_{m,N}=\Big(\sum_{m\leq|\alpha|\leq N}c^2(\alpha)\Big)^{1/2}.$$

Finally, we assume that X verifies $\mathfrak{D}(\varepsilon,r,R)$ and we denote

$$e_{m,N}(c)=\exp\biggl(-\Bigl(\frac{\varepsilon\mathfrak{m}_r}{2}\Bigr)^{2m}\frac{|c|_m^2}{\delta_*^2(c)}\biggr). \tag{3.15}$$

Notice that one may find a constant C, depending on ε, $\mathfrak{m}_r$ and on m such that

$$e_{m,N}(c)\leq C\frac{\delta_*^2(c)}{|c|_m^2}. \tag{3.16}$$

If X and Y satisfy $\mathfrak{D}(\varepsilon,r,R)$, respectively, $\mathfrak{D}(\varepsilon',r',R')$, then they both satisfy $\mathfrak{D}(\varepsilon\wedge\varepsilon',r\wedge r',R\vee R')$ so we may assume that ε, r and R are the same.

For $k\in\mathbb{N}$, we recall the distance $d_k(F,G)$ in (1.8). We give now our first result.

THEOREM 3.3. *Suppose that X and Y verify Hypothesis $\mathfrak{M}(\varepsilon,r,R)$ and let $c,d\in\mathcal{C}(\mathbb{R})$ be two families of coefficients. We fix k, k_* and N and we take $m\leq N$ and $m'\leq N$ such that $|c|_m>0$ and $|d|_{m'}>0$. We denote $\overline{m}=m\vee m'$. We set*

$$\mathfrak{d}_k=d_k\bigl(Q_{N,k_*}(c,X),Q_{N,k_*}(d,Y)\bigr) \tag{3.17}$$

and we assume that

$$\mathfrak{a}_k:=\mathfrak{d}_k\vee\bigl(|c|_{m+1,N}^2+|d|_{m'+1,N}^2\bigr)^{\frac{2kk_*\overline{m}+1}{k_*\overline{m}}}\leq 1. \tag{3.18}$$

Let $\theta\in((\frac{1}{1+k})^2,1)$. Then there exist $C>0$ and $a\in(\frac{1}{1+k},1]$, which depend on the parameters $\varepsilon,r,R,k,k_*,N,m,m',\theta$ and the moment bounds $M_p(X)$, $M_p(Y)$ for a suitable $p>1$, but independent of the coefficients $c,d\in\mathcal{C}(\mathbb{R})$, such that

$$\begin{aligned}&\bigl|\mathbb{E}\bigl(f\bigl(Q_{N,k_*}(c,X)\bigr)\bigr)-\mathbb{E}\bigl(f\bigl(Q_{N,k_*}(d,Y)\bigr)\bigr|\\&\quad\leq C\max\bigl(1,\bigl(|c|_m^{-\frac{2}{k_*m}}+|d|_{m'}^{-\frac{2}{k_*m'}}\bigr)^a\bigr)\|f\|_\infty\bigl(1+|c|+|d|\bigr)^{5k}\\&\qquad\times\bigl(\mathfrak{d}_k^{\frac{\theta}{1+2kk_*\overline{m}}}+e_{m,N}^a(c)+e_{m',N}^a(d)+|c|_{m+1,N}^{\frac{2\theta}{k_*\overline{m}}}+|d|_{m'+1,N}^{\frac{2\theta}{k_*\overline{m}}}\bigr),\end{aligned} \tag{3.19}$$

$e_{m,N}(c)$ and $e_{m',N}(d)$ being defined in (3.15).

The proof of Theorem 3.3 is done by using a Malliavin-type calculus based on V_n, $n\in\mathbb{N}$ which we present in Section 5, so we postpone it for Section 5.4. It represents the main effort in our paper.

We investigate now the consequences of Theorem 3.3. First, we give the following estimate of the total variation distance between two multiple stochastic integrals. We consider a m_* dimensional Brownian motion $W=(W^1,\dots,W^{m_*})$, we fix $\kappa=(k_1,\dots,k_m)\in[m_*]^m$, and, for a symmetric kernel $f\in L^2[0,1]^m$, we denote

$$I_\kappa(f)=m!\int_0^1 dW_{s_m}^{k_m}\int_0^{s_m} dW_{s_{m-1}}^{k_{m-1}}\cdots\int_0^{s_2} f(s_1,\dots,s_m)\,dW_{s_1}^{k_1}.$$

THEOREM 3.4. *Let $m\in\mathbb{N}_*$ and $f,g\in L^{2p}[0,1]^m$, $p>1$. Then, for every $k\in\mathbb{N}_*$ and $\theta\in((\frac{1}{1+k})^2,1)$ there exist $C>0$ and $a\in(\frac{1}{1+k},1)$ (both depending on θ, m and k) such that*

$$\begin{aligned}d_{\mathrm{TV}}\big(I_\kappa(f),I_\kappa(g)\big)\le{}& C\max\big(1,\big(\|f\|_2^{-2/m}+\|g\|_2^{-2/m}\big)^a\big)\\&\times\big(1+\|f\|_2+\|g\|_2\big)^{5k}d_k\big(I_\kappa(f),I_\kappa(g)\big)^{\theta/(1+2km)}.\end{aligned}\tag{3.20}$$

REMARK 3.5. In the case $k=1$, the above result has first been announced in [14] (see also the recent paper [13]) with the power $\frac{1}{m}$ instead of $\frac{\theta}{2m+1}$ above, but the proof was only sketched. It has rigorously been proved in [32] with power $\frac{1}{2m+1}$ and recently improved in [9] where the power $\frac{1}{m}\times(\ln m)^d$ is obtained, d being a suitable constant. So (3.20) is not the best possible estimate. This also indicates that the power in (3.19) is not optimal (but the approach in [9] does not seem to work in our general framework, so for the moment we are not able to improve it).

REMARK 3.6. Theorem 3.4, with exactly the same proof, extends to general random variables which live in a finite sum of Wiener chaoses: let F and G be two random variables belonging to $\bigoplus_{m=0}^N\mathcal{W}_m$ where $\mathcal{W}_m$ is the chaos of order m. We denote by P_m the projection on $\mathcal{W}_m$ and we put $m(F)=\max\{m: P_mF\neq 0\}$ and $\alpha(F)=\|P_{m(F)}F\|_2^{-2/m(F)}$. Then, with $N=m(F)\vee m(G)$,

$$\begin{aligned}d_{\mathrm{TV}}(F,G)\le{}& C\max\big(1,\big(\alpha(F)+\alpha(G)\big)^a\big)\\&\times\big(1+\|F\|_2+\|G\|_2\big)^{5k}d_k^{\theta/(1+2kN)}(F,G),\end{aligned}\tag{3.21}$$

where $a\in(\frac{1}{1+k},1)$ and $C>0$ depend on θ, k, N.

PROOF OF THEOREM 3.4. Let $n\in\mathbb{N}$. For $\alpha'=(\alpha'_1,\dots,\alpha'_m)\in[n-1]^m$, we denote $I_{\alpha'}=\prod_{j=1}^m[\frac{\alpha'_j}{n},\frac{\alpha'_j+1}{n})$ and we define

$$f_n(s)=\sum_{\alpha'}d_{n,f}(\alpha')1_{I_{\alpha'}}(s)\quad\text{with } d_{n,f}(\alpha')=\int_{I_{\alpha'}}f(u)\,du.$$

Note that f_n is the conditional expectation of f with respect to the partition $I_{\alpha'}$ and to the uniform law on $[0,1]^m$. Take now $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_i=(\alpha_i',\alpha_i'')$ and $(\alpha_1'',\dots,\alpha_m'')\in[m_*]^m$. We denote

$$c_{n,f}(\alpha)=m!n^{-m/2}d_{n,f}(\alpha')1_{\alpha_1'<\cdots<\alpha_m'<n}\prod_{i=1}^m 1_{\alpha_i''=k_i},$$

$$G_{\alpha_i',\alpha_i''}=n^{1/2}\times\left(W^{\alpha_i''}\left(\frac{\alpha_i'+1}{n}\right)-W^{\alpha_i''}\left(\frac{\alpha_i'}{n}\right)\right)$$

so that

$$I_\kappa(f_n)=\sum_\alpha c_{n,f}(\alpha)G^\alpha=\Phi_m(c_{n,f},G).$$

We are now in the framework of Theorem 3.3 and we compare $\Phi_m(c_{n,f},G)$ and $\Phi_m(c_{n,g},G)$. We take $k_*=1, d_*=m_*$ and $N=m=m'$. Then $|c_{n,f}|_{m+1,N}=|c_{n,g}|_{m+1,N}=0$. Let us estimate the parameters associated to $c_{n,f}$. By the convergence theorem for martingales, $|c_{n,f}|_m^2=m!\|f_n\|_2^2\to m!\|f\|_2^2>0$. We estimate now $\delta_*(c_{n,f})$. By using Hölder's inequality,

$$\begin{aligned}
\delta_*^2(c_{n,f})&=\max_{i\in[n]}\sum_{j=1}^m\sum_{\alpha':\alpha_j'=i}(m!)^2n^{-m}\left(n^m\int_{I_{\alpha'}}f(s)\,ds\right)^2\\
&=\max_{i\in[n]}(m!)^2\sum_{j=1}^m\sum_{\alpha'}n^{-m}\left(n^m\int_{I_{\alpha'}}f(s)1_{s_j\in[\frac{i}{n},\frac{i+1}{n})}\,ds\right)^2\\
&\le\max_{i\in[n]}(m!)^2\sum_{j=1}^m\sum_{\alpha'}\int_{I_{\alpha'}}f^2(s)1_{s_j\in[\frac{i}{n},\frac{i+1}{n})}\,ds\\
&\le\max_{i\in[n]}m!\max_{j\in[m]}\int_{[0,1]^m}f^2(s)1_{s_j\in[\frac{i}{n},\frac{i+1}{n})}\,ds\\
&\le m!\|f\|_{2p}^2\frac{1}{n^{1-1/p}}\to0
\end{aligned}$$

so that $e_{m,m}(c_{n,f})\to0$ and $e_{m,m}(c_{n,g})\to0$ as $n\to\infty$. Now, (3.19) gives, for $\theta<1$, and $n,n'\in\mathbb{N}$,

$$\begin{aligned}
&d_{\mathrm{TV}}\big(I_\kappa(f_n),I_\kappa(g_{n'})\big)\\
&\quad\le C(m!)^{5k/2}\max\big(1,(\|f_n\|_2^{-\frac{2}{m}}+\|g_{n'}\|_2^{-\frac{2}{m}})^a\big)\big(1+\|f_n\|_2+\|g_{n'}\|_2\big)^{5k}\\
&\qquad\times\big(e_{m,m}^a(c_{n,f})+e_{m,m}^a(c_{n',g})+d_k^{\theta/(1+2km)}\big(I_\kappa(f_n),I_\kappa(g_{n'})\big)\big),
\end{aligned}\tag{3.22}$$

where $a\in(\frac{1}{1+k},1)$. We take $n'>n$ and we notice that $d_k(I_\kappa(f_n),I_\kappa(f_{n'}))\le\|f_n-f_{n'}\|_2\to0$ so that the above inequality gives $d_{\mathrm{TV}}(I_\kappa(f_n),I_\kappa(f_{n'}))\to0$ as $n,n'\to$

∞. It follows that the sequences $I_\kappa(f_n)$ and $I_\kappa(g_n), n \in \mathbb{N}$ are Cauchy in d_{TV} and we may pass to the limit in (3.22) in order to obtain (3.20). □

We give now the analogous of Theorem 3.3 but in terms of Kolmogorov distance. Here, one needs no more Doeblin's condition nor nondegeneracy conditions.

THEOREM 3.7. *Suppose that X and Y verify (2.1) and are such that $Z(X)$ and $Z(Y)$ both satisfy (2.13). Let $c, d \in \mathcal{C}(\mathbb{R})$ be two families of coefficients such that $|c|_N > 0$ and $|d|_N > 0$ and such that $\delta_*(c), \delta_*(d) \leq 1$. Then, for every $k \geq 3$, and $\theta \in ((\frac{1}{1+k})^2, 1)$ there exist $C > 0$ and $a \in (\frac{1}{1+k}, 1)$ such that*

$$\begin{aligned}
&d_{\mathrm{Kol}}\big(Q_{N,k_*}(c, X), Q_{N,k_*}(d, Y)\big) \\
&\quad \leq C\big(1 + |c|_N^{-2N} + |d|_N^{-2N}\big)\big(1 + |c| + |d|\big)^{5k+1} \\
&\qquad \times \big(\delta_*^{\theta/(2kN+1)}(c) + \delta_*^{\theta/(2kN+1)}(d) + \mathfrak{d}_k^{\theta/(2kN+1)}\big)
\end{aligned} \tag{3.23}$$

$\mathfrak{d}_k$ being defined in (3.17), where $C > 0$ denotes a constant depending on N, suitable moments of X and Y and on the lower bounds $\underline{\lambda}$ in (2.13) applied to $Z(X)$ and $Z(Y)$.

REMARK 3.8. Note that (3.23) is in terms of $\delta_*^{\theta/(2kN+1)}(c)$ whereas in (3.19) it appears $e_{m,N}(c)$, which is much smaller. But we need that X_n and Y_n satisfy Doeblin's condition $\mathfrak{D}(\varepsilon, r, R)$.

REMARK 3.9. Another tempting approach to inequalities of type (3.23) is the following (we thank to the referee for this suggestion). One would like to use the classical inequality $d_{\mathrm{Kol}}(A, B) \leq \sqrt{C d_1(A, B)}$ where A and B are random variables and $B \sim p(x)\,dx$ with the density p bounded by C (see [1] for a proof). In our framework, one has to take $B = Q_{N,k_*}(d, Y)$ where $Y = (Y_k)_{k\in\mathbb{N}}$ is a sequence of independent standard normal random variables. Although it is known that the law of multiple stochastic integrals is absolutely continuous, it is not clear that the density is bounded, so we are not able to use this approach directly here. The counterpart of this regularity assumption is hidden in the influence factor which represents a bound for the Malliavin derivative (see (1.7)).

PROOF OF THEOREM 3.7. We consider the Gaussian random vectors G_X and G_Y corresponding to $Z(X)$ and $Z(Y)$, respectively, and we use Theorem 2.3 (see (2.14)) in order to obtain

$$\begin{aligned}
&d_{\mathrm{Kol}}\big(Q_{N,k_*}(c, X), Q_{N,k_*}(d, Y)\big) \\
&\quad \leq C\big(\delta_*^{1/(1+3N)}(c) + \delta_*^{1/(1+3N)}(d)\big) + d_{\mathrm{Kol}}\big(S_N(c, G_X), S_N(d, G_Y)\big).
\end{aligned}$$

Using the same argument as in the proof of Theorem 2.3 ($\mathrm{Cov}(G_X)$ and $\mathrm{Cov}(G_Y)$ are invertible), we may assume that G_X and G_Y are standard Gaussian random vectors so that $S_N(c, G_X)$ and $S_N(d, G_Y)$ are multiple stochastic integrals. By $d_{\mathrm{Kol}} \leq d_{\mathrm{TV}}$ and by (3.21)) first and (2.11) (recall that $Q_{N,k_*}(c, X) = S_N(c, Z_n(X)$ and $d_k \leq d_3$), then

$$\begin{aligned}
&d_{\mathrm{Kol}}\big(S_N(c, G_X), S_N(d, G_Y)\big) \\
&\quad \leq d_{\mathrm{TV}}\big(S_N(c, G_X), S_N(d, G_Y)\big) \\
&\quad \leq C\big(1 + |c|_N^{-2N} + |d|_N^{-2N}\big)\big(1 + |c| + |d|\big)^{5k} \\
&\qquad \times d_k^{\theta/(2kN+1)}\big(S_N(c, G_X), S_N(d, G_Y)\big) \\
&\quad \leq C\big(1 + |c|_N^{-2N} + |d|_N^{-2N}\big)\big(1 + |c| + |d|\big)^{5k+1} \\
&\qquad \times (\delta_*^{\theta/(2kN+1)}(c) + \delta_*^{\theta/(2kN+1)}(d) \\
&\qquad + d_k^{\theta/(2kN+1)}\big(Q_{N,k_*}(c, X), Q_{N,k_*}(d, Y)\big). \qquad \square
\end{aligned}$$

We give now the invariance principle.

THEOREM 3.10. *Let $X = (X_n)_{n\in\mathbb{N}}$ be a sequence of independent and centered $\mathbb{R}^{d_*}$ valued random vectors which verify Hypothesis $\mathfrak{M}(\varepsilon, r, R)$ and $G_X = (G_{n,X})_{n\in\mathbb{N}}, G_{n,X} \in \mathbb{R}^{m_*}$ a sequence of independent and centered Gaussian random vectors such that* $\mathrm{Cov}(G_{n,X}) = \mathrm{Cov}(Z_n(X))$. *Suppose that for some $m \leq N$, one has $|c|_m > 0$. Let $\theta \in (\frac{1}{16}, 1)$. Then there exist $C > 0$ and $a \in (\frac{1}{4}, 1]$, which depend on the parameters $\varepsilon, r, R, k_*, N, m, \theta$ and the moment bounds $M_p(X)$, $M_p(Y)$ for a suitable $p > 1$ but independent of the coefficients $c \in \mathcal{C}(\mathbb{R})$, such that*

$$\begin{aligned}
&d_{\mathrm{TV}}\big(Q_{N,k_*}(c, X), S_N(c, G_X)\big) \\
&\quad \leq C \max\big(1, |c|_m^{-\frac{2(k_*m+1)}{k_*m}}\big)^a \big(1 + |c|\big)^{19/2} \qquad (3.24)\\
&\qquad \times \big(\delta_*^{\frac{\theta}{6k_*m+1}}(c) + |c|_{m+1,N}^{\frac{2\theta}{k_*m}}\big).
\end{aligned}$$

PROOF. This is an immediate consequence of Theorem 3.3 with $k = 3$ and of Theorem 2.2. We have also used (3.16) in order to replace $e_{m,N}(c)$ by $\delta_*^2(c)|c|_m^{-2}$. $\square$

In a number of concrete applications (see, e.g., Theorem 4.1), one takes $S_N(c, Z(X)) = \sum_{n=m}^N \Phi_n(c, Z(X))$ and, asymptotically, $\Phi_m(c, Z(X))$ represents the principal term. In particular, we focus on $c(\alpha)$ with $|\alpha| = m$. So we use the notation $c_{(m)}(\alpha) = c(\alpha)1_{\{|\alpha|=m\}}$. Having this in mind, we can state the following result.

THEOREM 3.11. *Let $c \in \mathcal{C}(\mathbb{R})$ be such that $c(\alpha) = 0$ for $|\alpha| \leq m - 1$ and $|c|_m > 0$. Suppose $|c|_{m+1,N} \leq 1$.*

A. *If* $G=(G_n)_{n\in\mathbb{N}}$ *denote independent centered Gaussian random vectors then, for every* $\theta\in(\frac{1}{4},1)$ *there exists* $a\in(\frac{1}{2},1]$ *such that*

$$\begin{aligned} &d_{\mathrm{TV}}\big(S_N(c,G),\Phi_m(c,G)\big)\\ &\quad\le C\max\big(1,|c|_m^{-\frac{2}{m}}\big)^a\big(1+|c|\big)^5\big(|c|_{m+1,N}^{\frac{\theta}{2m+1}}+e_{m,N}(c_{(m)})^a\big).\end{aligned} \tag{3.25}$$

B. *Let* X *satisfy* $\mathfrak{M}(\varepsilon,r,R)$ *and let* $G=(G_n)_{n\in\mathbb{N}}, G_n\in\mathbb{R}^{m_*}$, *be a sequence of independent and centered Gaussian random vectors such that* $\mathrm{Cov}(G_n)=\mathrm{Cov}(Z_n(X))$. *Then for every* $\theta\in(\frac{1}{4},1)$ *there exists* $a\in(\frac{1}{2},1]$ *such that*

$$\begin{aligned} &d_{\mathrm{TV}}\big(Q_{N,k_*}(c,X),\Phi_m(c,G)\big)\\ &\quad\le C\max\big(1,|c|_m^{-\frac{2(k_*m+1)}{k_*m}}\big)^a\big(1+|c|\big)^{\frac{19}{2}}\big(\delta_*^{\frac{\theta}{6k_*m+1}}(c_{(m)})+|c|_{m+1,N}^{\frac{2\theta}{k_*m}\wedge\frac{\theta}{2m+1}}\big).\end{aligned} \tag{3.26}$$

C. *If* $Z(X)$ *satisfies* (2.13), *then for every* $\theta\in(\frac{1}{4},1)$ *there exists* $a\in(\frac{1}{2},1]$ *such that*

$$\begin{aligned} &d_{\mathrm{Kol}}\big(Q_{N,k_*}(c,X),\Phi_m(c,G)\big)\\ &\quad\le C\max\big(1,|c|_m^{-\frac{2(m+1)}{m}}\big)^a\big(1+|c|\big)^5\big(\delta_*^{\frac{1}{1+3N}}(c_{(m)})+|c|_{m+1,N}^{\frac{\theta}{2m+1}}\big).\end{aligned} \tag{3.27}$$

In the above estimates (3.25), (3.26) *and* (3.27), $C>0$ *denotes a constant independent of the coefficients* $c\in\mathcal{C}(\mathbb{R})$.

PROOF. One has $d_1(S_N(c,G),\Phi_m(c,G))\le\|S_N(c,G)-\Phi_m(c,G)\|_2\le|c|_{m+1,N}$, so (3.25) follows from Theorem 3.3 (see (3.19)). Using (3.25) and (3.24), we obtain (3.26), and (3.27) follows from (3.25) and (2.14). □

3.3. *Gaussian and Gamma approximation.* Theorem 3.11 has the following interesting application: if one considers a sequence of coefficients $c_n\in\mathcal{C}(\mathbb{R}), n\in\mathbb{N}$, the study of the asymptotic behavior of $Q_{N,k_*}(c_n,X), n\in\mathbb{N}$ reduces to the study of the asymptotic behavior of $\Phi_m(c_n,G), n\in\mathbb{N}$, where $G=(G_n)_{n\in\mathbb{N}}, G_n\in\mathbb{R}^{m_*}$, is a sequence of independent and centered Gaussian random vectors such that $\mathrm{Cov}(G_n)=\mathrm{Cov}(Z_n(X))$. Since $\Phi_m(c_n,G)$ is (nearly) a multiple Wiener stochastic integral of order m, this problem is already treated at least in two significant cases: the convergence to normality and the convergence to a Gamma distribution. In fact, the convergence to normality of the law of $\Phi_m(c_n,G)$ is controlled by the fourth moment theorem due to Nualart and Peccati [35] and Nourdin and Peccati [27], and the convergence to a Gamma distribution (and in particular to a χ_2 distribution) is treated in [27]. In order to give the consequences of these results in our framework, we have to identify the link between the notation in our paper and in the above mentioned works. Note that the coefficients $c\in\mathcal{C}(\mathbb{R})$ have been defined as $c(\alpha)$ with $\alpha=(\alpha_1,\dots,\alpha_m)$, $\alpha_i=(\alpha_i',\alpha_i'')$, with α' on the simplex

$\alpha'_1 < \cdots < \alpha'_m$. We extend them by symmetry on the whole $(\mathbb{N} \times [m_*])^m$ and we denote by c_s this extension. So we will have

$$\Phi_m(c, G) = \sum_{|\alpha|=m} c(\alpha) G^\alpha = \frac{1}{m!} \sum_{|\alpha|=m} c_s(\alpha) G^\alpha.$$

The second point is to write the sequence of multidimensional random vectors $G_n = (G_{n,1}, \dots, G_{n,m_*}) \in \mathbb{R}^{m_*}$, $n \in \mathbb{N}$ as a sequence of one-dimensional random variables $\overline{G}_n \in \mathbb{R}$, $n \in \mathbb{N}$ and to reindicate the coefficients in a corresponding way. But we have to note first that $G_{n,1}, \dots, G_{n,m_*}$ are not a priori independent, because $\mathrm{Cov}(G_n) = \mathrm{Cov}(Z_n(X))$ is not the identity matrix. So we have to assume that $\mathrm{Cov}(Z_n(X))$ is invertible and we first use (2.16) in order to write

$$\Phi_m(c, G) = \frac{1}{m!} \sum_{|\alpha|=m} \overline{c}_s(\alpha) \overline{G}^\alpha$$

with $\overline{c}$ defined in (2.15). Now $\overline{G}_{n,1}, \dots, \overline{G}_{n,m_*}$ are independent and we are ready to write them as a sequence. We define $I : \mathbb{N} \times [m_*] \to \mathbb{N}$ by $I(n, j) = n \times m_* + j$. Setting $\lfloor x \rfloor$ and $\{x\}$ the integer respectively the fractional part of x, the inverse function $J = I^{-1} : \mathbb{N} \to \mathbb{N} \times [m_*]$ is then defined as follows: $J(n) = (\lfloor n/m_* \rfloor, \{n/m_*\} m_*)$ if $\{n/m_*\} > 0$ and $J(n) = (\lfloor n/m_* \rfloor - 1, m_*)$ if $\{n/m_*\} = 0$. We extend this definition to multi-indexes: if $\beta = (n_1, \dots, n_m) \in \mathbb{N}^m$ then $J(\beta) = (J(n_1), \dots, J(n_m)) \in (\mathbb{N} \times [m_*])^m$; and to coefficients: if $f : (\mathbb{N} \times [m_*])^m \to \mathbb{R}$ we define $\widehat{f} : \mathbb{N}^m \to \mathbb{R}^m$ by $\widehat{f}(\beta) = f(J(\beta))$. Moreover, we consider the sequence $\widehat{G}_n = \overline{G}_{J(n)}$, $n \in \mathbb{N}$. Then

$$\Phi_m(c, G) = \frac{1}{m!} \sum_{|\alpha|=m} \overline{c}_s(\alpha) \overline{G}^\alpha = \frac{1}{m!} \sum_{|\alpha|=m} \widehat{c}_s(\alpha) \widehat{G}^\alpha$$

with the convention that now we work with the multi-index $\alpha \in \mathbb{N}^m$. Note that $\Phi_m(\widehat{c}_s, \widehat{G})$ is a multiple stochastic integral of order m.

We introduce now the "contraction operators." For $0 \le r \le m$ and $\alpha, \beta \in \Gamma_{m-r}$, one denotes $\widehat{c}_s \otimes_r \widehat{c}_s(\alpha, \beta) = \sum_{\gamma \in \Gamma_r} \widehat{c}_s(\alpha, \gamma) \widehat{c}_s(\beta, \gamma)$ with the convention that for $r = 0$ we put $\widehat{c}_s \otimes_0 \widehat{c}_s(\alpha, \beta) = \widehat{c}_s(\alpha) \widehat{c}_s(\beta)$ and for $r = m$, $\widehat{c}_s \otimes_m \widehat{c}_s = \sum_{\gamma \in \Gamma_m} \widehat{c}_s(\gamma) \widehat{c}_s(\gamma)$. Note that, even if $\widehat{c}_s$ is symmetric, $\widehat{c}_s \otimes_r \widehat{c}_s$ is not symmetric, so we introduce $\widehat{c}_s \widetilde{\otimes}_r \widehat{c}_s$ to be the symmetrization of $\widehat{c}_s \otimes_r \widehat{c}_s$.

We introduce now

$$\kappa_{4,m}(\widehat{c}_s) = \sum_{r=1}^{m-1} m!^2 \binom{m}{r}^2 \left\{ |\widehat{c}_s \otimes_r \widehat{c}_s|^2_{2(m-r)} + \binom{2m-2r}{m-r} |\widehat{c}_s \widetilde{\otimes}_r \widehat{c}_s|^2_{2(m-r)} \right\}.$$

It is known (see [27]) that $\kappa_{4,m}(\widehat{c}_s)$ is equal to the fourth cumulant of $\Phi_m(\widehat{c}_s, \widehat{G})$, and moreover, it is proved in [27] that, if $\mathcal{N}$ is a standard normal random variable, then

$$d_{\mathrm{TV}}\big(\Phi_m(\widehat{c}_s,\widehat{G}),\mathcal{N}\big)\le C\kappa_{4,m}^{1/2}(\widehat{c}_s). \tag{3.28}$$

Using this and Theorem 3.11, we immediately obtain the following.

THEOREM 3.12. *Let $\mathcal{N}$ be a standard normal random variable and let $c\in\mathcal{C}(\mathbb{R})$ be such that $c(\alpha)=0$ for $|\alpha|\le m-1$ and $|c|_m>0$. Suppose $|c|_{m+1,N}\le 1$.*

A. *If X satisfies $\mathfrak{M}(\varepsilon,r,R)$ and, for every $n\in\mathbb{N}$, $\mathrm{Cov}(Z_n(X)$ is invertible, then for every $\theta\in(\frac{1}{4},1)$ there exists $a\in(\frac{1}{2},1]$ such that*

$$\begin{aligned}&d_{\mathrm{TV}}\big(Q_{N,k_*}(c,X)\big),\mathcal{N}\big)\\ &\quad\le C\max\big(1,|c|_m^{-\frac{2(k_*m+1)}{k_*m}}\big)^a\big(1+|c|\big)^{\frac{19}{2}}\\ &\qquad\times\big(\delta_*^{\frac{\theta}{6k_*m+1}}(c_{(m)})+|c|_{m+1,N}^{\frac{2\theta}{k_*m}\wedge\frac{\theta}{2m+1}}+\kappa_{4,m}^{1/2}(\widehat{c}_s)\big).\end{aligned} \tag{3.29}$$

B. *If $Z(X)$ satisfies (2.1) and (2.13), then for every $\theta\in(\frac{1}{4},1)$ there exists $a\in(\frac{1}{2},1]$ such that*

$$\begin{aligned}&d_{\mathrm{Kol}}\big(Q_{N,k_*}(c,X)\big),\mathcal{N}))\\ &\quad\le C\max\big(1,|c|_m^{-\frac{2(m+1)}{m}}\big)^a\big(1+|c|\big)^{5}\\ &\qquad\times\big(\delta_*^{\frac{1}{1+3N}}(c_{(m)})+|c|_{m+1,N}^{\frac{\theta}{2m+1}}+\kappa_{4,m}^{1/2}(\widehat{c}_s)\big).\end{aligned} \tag{3.30}$$

In the above estimates (3.29) and (3.30), $C>0$ denotes a constant independent of the coefficients $c\in\mathcal{C}(\mathbb{R})$.

This is a generalization of the "fourth moment theorem" to stochastic polynomials. However, there is a difference because the influence factor $\delta_*(c)$ appears in (3.29). One may ask if it is possible to control the distance between stochastic polynomials and the normal distribution in terms of $\kappa_{4,m}(\widehat{c}_s)$ only. The following useful remark from [30] (see (1.9)) allows to do it:

$$\begin{aligned}\kappa_{4,m}(\widehat{c}_s)&\ge|c\otimes_{m-1}c|_2^2\\ &=\sum_{|\alpha|=|\beta|=1}\Big(\sum_{|\gamma|=m-1}c(\alpha,\gamma)c(\beta,\gamma)\Big)^2\\ &\ge\sum_{|\alpha|=1}\Big(\sum_{|\gamma|=m-1}c^2(\alpha,\gamma)\Big)^2\\ &\ge\max_{|\alpha|=1}\Big(\sum_{|\gamma|=m-1}c^2(\alpha,\gamma)\Big)^2=\delta_*^2(c_{(m)}).\end{aligned}$$

This gives the following.

THEOREM 3.13. *Under the hypothesis of Theorem* 3.12A, *one has*

$$d_{\mathrm{TV}}\big(Q_{N,k_*}(c,X),\mathcal{N}\big)\leq C\max\big(1,|c|_m^{-\frac{2}{k_*m}}\big)^a\big(1+|c|\big)^{\frac{19}{2}}$$
$$\times\big(\kappa_{4,m}^{\frac{\theta}{6k_*m+1}}(\widehat{c}_s)+|c|_{m+1,N}^{\frac{2\theta}{k_*m}\wedge\frac{\theta}{2m+1}}\big)$$

and under the hypothesis of Theorem 3.12B, *one has*

$$d_{\mathrm{Kol}}\big(Q_{N,k_*}(c,X),\mathcal{N}\big)\leq C\max\big(1,|c|_m^{-\frac{2}{m}}\big)^a\big(1+|c|\big)^5$$
$$\times\big(\kappa_{4,m}^{\frac{1}{1+3N}}(c_s)+|c|_{m+1,N}^{\frac{\theta}{2m+1}}\big).$$

REMARK 3.14. Notice that the power of $\kappa_{4,m}(c_s)$ is smaller than $\frac{1}{2}$, so there is a loss with respect to the classical fourth moment theorem. The estimate with the right power $\frac{1}{2}$ has recently been obtained in the following particular framework: assume that $d_*=k_*=1$ so that $\Phi_m(c,X)$ is a multi-linear polynomial. Assume also that the random variables X_n, $n\in\mathbb{N}$ are identically distributed. Then, if $\mathbb{E}(X_1^4)\geq 3$, the convergence to normality is controlled by $\kappa_{4,m}(\widehat{c}_s))$ only (see Theorem 2.3 in [29]).

We discuss now the convergence to a Gamma distribution. For $\nu\geq 1$, we consider $F(\nu)$ a centered Gamma distribution of parameter ν: $F(\nu)=2G(\nu/2)-\nu$, where $G(\nu/2)$ has a Gamma law with parameter $\nu/2$ (i.e., with density $g_{\nu/2}(x)\propto x^{\nu/2-1}e^{-x}1_{x>0}$). If ν is an integer, then $F(\nu)$ is a centered chi-square distribution with ν degrees of freedom. We introduce

$$\eta_{\nu,m}(\widehat{c}_s)=\big(\nu-m!|\widehat{c}_s|_m^2\big)^2+4m!|\theta_m\times\widehat{c}_s\,\widetilde{\otimes}_{m/2}\,\widehat{c}_s-\widehat{c}_s|_{2m-r}^2$$
$$+m^2\sum_{\substack{r\in\{1,\dots,m-1\}\\ r\neq m/2}}(2m-2r)!(r-1)!^2\binom{m-1}{r-1}^4|\widehat{c}_s\otimes_r\widehat{c}_s|_{2m-r}^2$$

with $\theta_m=\frac{1}{4}(m/2)!\binom{m}{m/2}$. Combining Theorem 3.11 and Proposition 3.13 from [27], one obtains

$$d_1\big(\Phi_m(c,Z),F(\nu)\big)\leq C\eta_{\nu,m}^{1/2}(\widehat{c}_s).$$

If ν is an integer, then $F(\nu)$ has a centered $\chi^2(\nu)$ distribution, so may be represented as a polynomial of degree two of Gaussian random variables. Then, using Theorem 5.9 in [9], one obtains

$$d_{\mathrm{TV}}\big(\Phi_m(c,Z),F(\nu)\big)\leq d_1^{\frac{1}{m+1}}\big(\Phi_m(c,Z),F(\nu)\big)\leq C\eta_{\nu,m}^{1/2(m+1)}(\widehat{c}_s).$$

Then, using Theorem 3.11, we obtain the following.

THEOREM 3.15. *Let $\mathcal{X}_\nu$ be a random variable with a centered χ^2 distribution with ν degrees of freedom.*

A. *If X satisfies $\mathfrak{M}(\varepsilon, r, R)$ and, for every $n \in \mathbb{N}$, $\mathrm{Cov}(Z_n(X)$ is invertible, then for every $\theta \in (\frac{1}{4}, 1)$ there exists $a \in (\frac{1}{2}, 1]$ such that*

$$\begin{aligned} d_{\mathrm{TV}}&\big(Q_{N,k_*}(c, X), \mathcal{X}_\nu\big) \\ &\le C \max\big(1, |c|_m^{-\frac{2}{k_* m}}\big)^a \big(1 + |c|\big)^{\frac{19}{2}} \\ &\quad \times \big(\delta_*^{\frac{\theta}{6k_* m+1}}(c) + e_{m,N}(c)^a + |c|_{m+1,N}^{\frac{2\theta}{k_* m} \wedge \frac{\theta}{2m+1}} + \eta_{\nu,m}^{1/2(m+1)}(\widehat{c}_s)\big). \end{aligned} \tag{3.31}$$

B. *If $Z(X)$ satisfies (2.1) and (2.13), then for every $\theta \in (\frac{1}{4}, 1)$ there exists $a \in (\frac{1}{2}, 1]$ such that*

$$\begin{aligned} d_{\mathrm{Kol}}&\big(Q_{N,k_*}(c, X), \mathcal{X}_\nu\big) \\ &\le C \max\big(1, |c|_m^{-\frac{2}{m}}\big)^a \big(1 + |c|\big)^5 \\ &\quad \times \big(\delta_*^{\frac{1}{1+3N}}(c) + |c|_{m+1,N}^{\frac{\theta}{2m+1}} + e_{m,N}(c)^a + \eta_{\nu,m}^{1/2(m+1)}(\widehat{c}_s)\big). \end{aligned} \tag{3.32}$$

In the above estimates (3.31) and (3.32), $C > 0$ denotes a constant independent of the coefficients $c \in \mathcal{C}(\mathbb{R})$.

4. Examples.

4.1. *U-statistics associated to polynomial kernels.* Let us first shortly recall how U-statistics appear. One considers a class of distributions $\mathcal{M}$ and aims to estimate a functional $\theta(\mu)$ with $\mu \in \mathcal{M}$. In order to do it, one has at hand a sequence of independent random variables $X_1, \dots, X_n$ with law $\mu \in \mathcal{M}$, but does not know which is this law. The goal is to construct an unbiased estimator, that is, a sequence of functions $f_n : \mathbb{R}^n \to \mathbb{R}$, such that the estimator $U_n = f_n(X_1, \dots, X_n)$ converges to $\theta(\mu)$, and moreover, $\mathbb{E}(U_n) = \theta(\mu)$ for every $\mu \in \mathcal{M}$. This means that the estimator is unbiased—and this is the origin of the name U-statistics. In 1948, Halmos [21] asked the question if such an unbiased estimator exists and if it is unique. It turns out that the necessary and sufficient condition in order to be able to construct such an estimator is that $\theta(\mu)$ has the following particular form: there exists $N \in \mathbb{N}$ and a measurable function $\psi : \mathbb{R}^N \to \mathbb{R}$ such that

$$\theta(\mu) = \int_{\mathbb{R}^N} \psi(x_1, \dots, x_N)\, d\mu(x_1) \cdots d\mu(x_N). \tag{4.1}$$

In this case, one may construct the symmetric unbiased estimator f_n (and if $\mathcal{M}$ is sufficiently large, this estimator is unique in the class of the symmetric estimators) in the following way:

$$U_n^\psi = \frac{(n-N)!}{n!} \sum_{(n,N)} \psi(X_{i_1}, \dots, X_{i_N}), \tag{4.2}$$

where the sum $\sum_{(n,N)}$ is taken over all the subsets $\{i_1,\dots,i_N\}\subset\{1,\dots,n\}$ such that $i_k\neq i_p$ for $k\neq p$. It is clear that ψ may be taken to be symmetric (if not, one takes its symmetrization).

When $\psi(x_1,\dots,x_N)$ is a polynomial, this fits in our framework and our results apply but, for example, $\psi(x_1,\dots,x_N)=\max\{|x_1|,\dots,|x_N|\}$, is out of reach.

We fix $k_*, N\in\mathbb{N}$, we denote $\mathcal{K}_N=\{0,1,\dots,k_*\}^N$, and we define

$$\psi(x_1,\dots,x_N)=\sum_{\kappa\in\mathcal{K}_N}a(\kappa)x^\kappa\quad\text{with } x^\kappa=\prod_{j=1}^N x_j^{k_j}\tag{4.3}$$

with symmetric coefficients $a(\kappa)$ which are null on the diagonals. So ψ is a general symmetric polynomial of order k_* in the variables $x_1,\dots,x_N$. We associate to ψ the U-statistic U_n^ψ defined in (4.2):

$$\begin{aligned}U_n^\psi&=\frac{(n-N)!}{n!}\sum_{i_1,\dots,i_N}\psi(X_{i_1},\dots,X_{i_N})\\&=\binom{n}{N}^{-1}\sum_{i_1<\cdots<i_N}\sum_{\kappa\in\mathcal{K}_N}a(\kappa)\prod_{j=1}^N X_{i_j}^{k_j}.\end{aligned}\tag{4.4}$$

The above quantity is linked with the stochastic polynomials defined in the previous sections in the following way. One takes $d_*=1$ and $m_*=k_*$ and constructs coefficients c_n such that $U_n^\psi=Q_{N,k_*}(c_n,X)=S_N(c_n,Z(X))$ with $Z(X)$ associated to X in (3.1): $Z_{i,k}(X)=X_i^k-\mathbb{E}(X_i^k), k=1,\dots,k_*$. The problem is that $Z_{i,k}(X)$ is centered whereas X_i^k, which appears in (4.4), is not. It turns out that the operation which consists in centering X_i^k in (4.4) is exactly the Hoeffding decomposition [22], which plays a crucial role in U-statistics theory. Let us recall it. For $1\le j\le N$, one defines the kernels

$$h_j(x_1,\dots,x_j)=\int\cdots\int\psi(u_1,\dots,u_N)\prod_{i=1}^j(\delta_{x_i}-\mu)(du_i)\prod_{i=j+1}^N\mu(du_i).$$

Then Hoeffding's decomposition (Theorem 1 in Section 1.6 in [23]) is the following:

$$U_n^\psi=\theta(\mu)+\sum_{j=1}^N\binom{N}{j}U_n^{h_j},\tag{4.5}$$

where $U_n^{h_j}$ is the U-statistic associated to h_j in the first equality from (4.4) (with N replaced by j).

We denote $m_k=\mathbb{E}(X^k)$ and we compute

$$\int\cdots\int\prod_{l=1}^N u_l^{k_l}\prod_{i=1}^j(\delta_{x_i}-\mu)(du_i)\prod_{i=j+1}^N\mu(du_i)=\prod_{i=1}^j(x_i^{k_i}-m_{k_i})\times\prod_{i=j+1}^N m_{k_i},$$

so we obtain

$$h_j(x_1,\dots,x_j)=\sum_{\kappa\in\mathcal{K}_j}a_j(\kappa)\prod_{i=1}^{j}\big(x_i^{k_i}-m_{k_i}\big)\quad\text{with}$$

$$a_j(\kappa)=\sum_{k_{j+1},\dots,k_N=1}^{k_*}a(\kappa,k_{j+1},\dots,k_N)\prod_{i=j+1}^{N}m_{k_i}.$$

We conclude that

$$U_n^{\psi}=\theta(\mu)+\sum_{j=1}^{N}\binom{N}{j}\binom{n}{j}^{-1}\sum_{i_1<\cdots<i_j}\sum_{\kappa\in\mathcal{K}_j}a_j(\kappa)\prod_{l=1}^{j}\big(X_{i_l}^{k_l}-\mathbb{E}\big(X_{i_l}^{k_l}\big)\big).$$

The U-statistics U_n^{ψ} is called "degenerated" at order $m\in[N]$ if $h_j=0$ for $j\le m-1$ and $h_m\neq 0$, that is,

$$\sum_{\kappa\in\mathcal{K}_j}a_j^2(\kappa)=0,\quad 1\le j\le m-1\quad\text{and}\quad\sum_{\kappa\in\mathcal{K}_m}a_m^2(\kappa)>0. \tag{4.6}$$

We assume that (4.6) holds and we write

$$V_m(n):=n^{m/2}\big(U_n^{\psi}-\theta(\mu)\big)=\sum_{j=m}^{N}n^{m/2}\binom{N}{j}U_n^{h_j}=\sum_{j=m}^{N}\sum_{|\alpha|=j}c_n(\alpha)Z^{\alpha}(X)$$

with

$$c_n\big((i_1,k_1),\dots,(i_j,k_j)\big)=n^{m/2}\binom{N}{j}\times\binom{n}{j}^{-1}a_j(k_1,\dots,k_j).$$

By (4.6), the U-statistic U_n^{ψ} is degenerated at order $m\in[N]$ if and only if

$$|c_n|_j=0\quad\text{for } j\le m-1\quad\text{and}\quad|c_n|_m>0,$$

which is the same nondegeneracy condition we are interested in.

We recall that $X_i\sim\mu$ and that in (2.13) we have introduced the covariance matrix $\mathrm{Cov}(Z(X))=\mathrm{Cov}(\mu)$, that is,

$$\mathrm{Cov}^{i,j}(\mu)=\mathbb{E}\big(\big(X^i-\mathbb{E}(X^i)\big)\big(X^j-\mathbb{E}(X^j)\big)\big),\quad i,j=1,\dots,k_*.$$

We consider a correlated Brownian motion $W=(W^1,\dots,W^{k_*})$ in $\mathbb{R}^{k_*}$ with $\langle W^i,W^j\rangle_t=\mathrm{Cov}^{i,j}(\mu)t$, $i,j=1,\dots,k_*$. For $\kappa=(k_1,\dots,k_m)\in\mathcal{K}_m$, we define the multiple stochastic integral

$$I_\kappa^{\mu}(1)=\int_0^1 dW_{s_m}^{k_m}\int_0^{s_m}dW_{s_{m-1}}^{k_{m-1}}\cdots\int_0^{s_2}1\,dW_{s_1}^{k_1}$$

and we denote

$$V_m=\binom{N}{N-m}\sum_{\kappa\in\mathcal{K}_m}a_m(\kappa)I_\kappa^{\mu}(1).$$

THEOREM 4.1. A. *If X verifies $\mathfrak{M}(\varepsilon, r, R)$ and (4.6) holds, then for every $\theta \in (\frac{1}{4}, 1)$,*

$$d_{\mathrm{TV}}\big(V_m(n), V_m\big) \leq \frac{C}{n^{\theta\beta(m,k_*)}} \quad \text{with } \beta(m,k_*) = \frac{1}{2(6k_*m+1)}. \tag{4.7}$$

B. *Suppose that X has finite moments of any order and that* $\mathrm{Cov}(Z(X)) = \mathrm{Cov}(\mu) \geq \underline{\lambda} > 0$. *If (4.6) holds then, for every $\theta \in (\frac{1}{4}, 1)$,*

$$d_{\mathrm{Kol}}\big(V_m(n), V_m\big) \leq \frac{C}{n^{\theta\alpha(N)}} \quad \text{with } \alpha(N) = \frac{1}{2(3N+1)}. \tag{4.8}$$

PROOF. In order to use Theorem 3.11, we estimate

$$|c_n|^2_{m+1,N} = \sum_{m+1 \leq |\alpha| \leq N} c_n^2(\alpha) \leq Cn^m \times \sum_{j=m+1}^{N} n^{-2j} \times n^j \times \|a\|_\infty \leq \frac{C}{n},$$

$$|c_n|^2_m = \sum_{|\alpha|=m} c_n^2(\alpha) \geq \frac{1}{C} \times \sum_{\kappa \in \mathcal{K}_m} a_m^2(\kappa) > 0.$$

Finally, we study the influence factor:

$$\delta_*^2(c_n) = \max_r \sum_{m \leq |\alpha| \leq N} c_n^2(\alpha) 1_{\{r \in \alpha'\}} \leq Cn^m \times \sum_{j=m}^{N} n^{-2j} \times n^{j-1} = \frac{C}{n}.$$

Then (3.26) gives

$$\begin{aligned} d_{\mathrm{TV}}\big(Q_{N,k_*}(c,X), \Phi_m(c,G)\big) &\leq C\bigg(\Big(\frac{1}{\sqrt{n}}\Big)^{\frac{\theta}{6k_*m+1}} + \Big(\frac{1}{\sqrt{n}}\Big)^{\frac{2\theta}{k_*m} \wedge \frac{\theta}{2m+1}}\bigg) \\ &\leq C\frac{1}{n^{\frac{\theta}{2(6k_*m+1)}}}. \end{aligned}$$

And by employing (3.27), one has

$$\begin{aligned} d_{\mathrm{Kol}}\big(Q_{N,k_*}(c,X), \Phi_m(c,G)\big) &\leq C\bigg(\Big(\frac{1}{\sqrt{n}}\Big)^{\frac{1}{1+3N}} + \Big(\frac{1}{\sqrt{n}}\Big)^{\frac{\theta}{2m+1}}\bigg) \\ &\leq C\frac{1}{n^{\frac{\theta}{2(1+3N)}}}. \end{aligned}$$

□

4.2. *A quadratic central limit theorem.* For $p \in (0, \frac{1}{2}]$, we look to the quadratic form

$$S_{n,p}(Z) = \begin{cases} \dfrac{1}{n^{1-p}} \displaystyle\sum_{i,j=1}^{n} 1_{\{i \neq j\}} \frac{1}{|i-j|^p} Z_i Z_j & \text{if } 0 < p < \dfrac{1}{2}, \\ \dfrac{1}{(2n \ln n)^{1/2}} \displaystyle\sum_{i,j=1}^{n} 1_{\{i \neq j\}} \frac{1}{|i-j|^{1/2}} Z_i Z_j & \text{if } p = \dfrac{1}{2}, \end{cases} \tag{4.9}$$

where $Z_i, i \in \mathbb{N}$ are centered independent random variables which have finite moments of any order. We prove here that that if $p < \frac{1}{2}$ then $S_{n,p}(Z)$ converges to a double stochastic integral whereas for $p = \frac{1}{2}$ the limit is a standard Gaussian random variable. In our notation, we have $d_* = 1$, $k_* = 1$, $N = 2$ and

$$S_{n,p}(Z) = Q_{2,1}(c_{n,p}, Z) = S_2(c_{n,p}, Z),$$

where $c_{n,p}(\alpha) = 0$ for $|\alpha| \neq 2$ and if $|\alpha| = 2$,

$$c_{n,p}(\alpha) = \begin{cases} \dfrac{2}{n^{1-p}|\alpha'_1 - \alpha'_2|^p} 1_{\{1 \leq \alpha'_1 < \alpha'_2 \leq n\}} & \text{if } 0 < p < \dfrac{1}{2}, \\ \dfrac{2}{(2n \ln n)^{1/2}|\alpha'_1 - \alpha'_2|^{1/2}} 1_{\{1 \leq \alpha'_1 < \alpha'_2 \leq n\}} & \text{if } p = \dfrac{1}{2}. \end{cases} \tag{4.10}$$

THEOREM 4.2. *Let $Z_i, i \in \mathbb{N}$ be a sequence of independent and centered random variables, with $\mathbb{E}(Z_i^2) = 1$ and which have finite moments of any order.*

A. *Let $p < \frac{1}{2}$. We denote $I_2(\psi_p) = \int_0^1 \int_0^1 \psi_p(s, t)\, dW_s\, dW_t$, W being a Brownian motion and $\psi_p(s, t) = |s - t|^{-p}$. Then for every $\theta \in (\frac{1}{4}, 1)$ there exists n_* and C such that for $n \geq n_*$*

$$d_{\mathrm{Kol}}\big(S_{n,p}, I_2(\psi_p)\big) \leq \frac{C}{n^{\frac{\theta(1-2p)}{15}}}. \tag{4.11}$$

Suppose moreover that $\mathfrak{M}(\varepsilon, r, R)$ holds. Then for every $\theta \in (\frac{1}{4}, 1)$ there exists n_* and C such that for $n \geq n_*$

$$d_{\mathrm{TV}}\big(S_{n,p}, I_2(\psi_p)\big) \leq \frac{C}{n^{\frac{\theta}{26} \wedge \frac{\theta(1-2p)}{15}}}. \tag{4.12}$$

B. *Let $p = \frac{1}{2}$. We denote Δ a standard normal random variable. There exists n_* and C such that for $n \geq n_*$*

$$d_{\mathrm{Kol}}(S_{n,1/2}, \Delta) \leq \frac{C}{(\ln n)^{1/2}}. \tag{4.13}$$

Suppose moreover that $\mathfrak{M}(\varepsilon, r, R)$ holds. Then (4.13) holds with d_{TV} instead of d_{Kol}.

REMARK 4.3. The nice feature of the above result is that there is a change of regime if $p < 1/2$ or $p = 1/2$: if $p = 1/2$ the singularity in $|i - j|^{-p}$ is sufficiently strong in order to pass from the second-order chaos to the first one (Gaussian), whereas if $p < 1/2$ then one remains in the second chaos.

PROOF OF THEOREM 4.2. We extend by symmetry the coefficients $c_{n,p}(\alpha)$ to all indexes $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 \neq \alpha_2$. We denote $t_i = \frac{i}{n}$ and we define

$$\psi_{n,p}(s, t) = c_{n,p}(i, j) 1_{[t_i, t_{i+1})}(s) 1_{[t_j, t_{j+1})}(t).$$

Let us prove that

$$\int_0^1\int_0^1 \left|\psi_p(s,t)-\psi_{n,p}(s,t)\right|^2 ds\,dt \le \frac{C}{n^{\frac{2}{3}(1-2p)}}. \tag{4.14}$$

We take $q=\frac{2}{3}$ and we write

$$\int_0^1\int_0^1 \left|\psi_p(s,t)-\psi_{n,p}(s,t)\right|^2 ds\,dt \le I+J+J'$$

with

$$I=\int_{|s-t|\ge 1/n^q} \left|\psi_p(s,t)-\psi_{n,p}(s,t)\right|^2 ds\,dt,$$
$$J=\int_{|s-t|<1/n^q} \left|\psi_p(s,t)\right|^2 ds\,dt, \qquad J'=\int_{|s-t|<1/n^q} \left|\psi_{n,p}(s,t)\right|^2 ds\,dt.$$

Note that if $|s-t|\ge 1/n^q$ then

$$\left|\psi_p(s,t)-\psi_{n,p}(s,t)\right| \le \frac{C}{n}\times\frac{1}{|s-t|^{p+1}} \le \frac{C}{n^{1-q(p+1)}}$$

so that

$$I\le \frac{C}{n^{2(1-q(p+1))}}.$$

Moreover,

$$J=2\int_0^1 dt\int_0^{t+\frac{1}{n^q}} \frac{ds}{|s-t|^{2p}} = \frac{C}{n^{q(1-2p)}}.$$

Finally, by comparing Riemann sums with the corresponding integral,

$$\begin{aligned} J' &= \frac{1}{n^2}\sum_{i=1}^n \sum_{0<|t_i-t_j|\le 1/n^q} \frac{1}{|t_i-t_j|^{2p}} \\ &\le \frac{1}{n^2}\left(2n+\sum_{i=1}^n \sum_{\substack{0<|t_i-t_j|\le 1/n^q\\ |i-j|\ge 2}} \frac{1}{|t_i-t_j|^{2p}}\right) \le \frac{1}{n^2}(2n+J) \le \frac{C}{n^{q(1-2p)}}. \end{aligned}$$

Since $q=\frac{2}{3}$, we obtain (4.14). It follows that, for sufficiently large n,

$$\begin{aligned} \frac{1}{2}\int_0^1\int_0^1 \left|\psi_p(s,t)\right|^2 ds\,dt \le |c_n|^2 &= \int_0^1\int_0^1 \left|\psi_{n,p}(s,t)\right|^2 ds\,dt \\ &\le 2\int_0^1\int_0^1 \left|\psi_p(s,t)\right|^2 ds\,dt. \end{aligned}$$

We also have

$$\delta_*^2(c_{n,p}) = \max_{i\le n} \sum_{j\ne i} \frac{4}{n^{2(1-p)}} \frac{1}{|i-j|^{2p}} 1_{\{i\ne j\}} \le \frac{C}{n}.$$

Note that $S_{n,p}(Z) = S_2(c_n, Z)$ and $S_2(c_n, G) = I_2(\psi_{n,p})$. Using Theorem 2.3 (with $N = 2$), Theorem 3.4 (see (3.20) with $k = 1, m = 2, \frac{1}{4} < \theta < 1$) and (4.14), we obtain

$$\begin{aligned} &d_{\mathrm{Kol}}\big(S_{n,p}(Z), I_2(\psi_p)\big) \\ &\quad \le d_{\mathrm{Kol}}\big(S_2(c_{n,p}, Z), S_2(c_{n,p}, G)\big) + d_{\mathrm{Kol}}\big(I_2(\psi_{n,p}), I_2(\psi_p)\big) \\ &\quad \le C\big(\delta_*^{1/7}(c_{n,p}) + \|\psi_p - \psi_{n,p}\|_2^{\theta/5}\big) \le C\left(\frac{1}{n^{1/14}} + \frac{1}{n^{\frac{\theta(1-2p)}{15}}}\right) \end{aligned}$$

so (4.11) is proved for d_{Kol}.

We suppose now that Z verifies $\mathfrak{M}(\varepsilon, r, R)$ and we use Theorem 3.10 (see (3.24) with $N = 2$) in order to obtain

$$\begin{aligned} d_{\mathrm{TV}}\big(S_{n,p}(Z), I_2(\psi_p)\big) &\le C(\delta_*^{\theta/13}(c_n) + \|\psi_p - \psi_{n,p}\|_2^{\theta/5} \\ &\le C\left(\frac{1}{n^{\theta/26}} + \frac{1}{n^{\frac{\theta}{15}(1-2p)}}\right) \end{aligned}$$

so (4.12) is proved for d_{TV} also.

B. We have $S_{n,1/2}(Z) = S_2(c_n, Z)$ with (recall that $t_i = i/n$)

$$c_n(i,j) = \frac{1}{\sqrt{2n\ln n}} 1_{i\ne j} \frac{1}{|i-j|^{1/2}} = \frac{1}{\sqrt{2\ln n}} 1_{i\ne j} \frac{1}{|t_i - t_j|^{1/2}}.$$

We note first that

$$\ln i + \ln(n-i) \le \sum_{j=1}^{n} 1_{i\ne j}|i-j|^{-1} \le 2 + \ln i + \ln(n-i).$$

These inequalities are easily obtained by comparing $\sum_{j=1}^n 1_{i\ne j}|i-j|^{-1}$ with $\int_{\{|t_i - y|>1/n\}} |t_i - t|^{-1}\,dt$. It immediately follows that

$$1 - \frac{1}{\ln n} \le |c_n|^2 \le 1 + \frac{1}{\ln n}$$

and $\delta_*(c_n) \le \frac{\sqrt{2}}{\sqrt{n}}$. Now, using Theorem 2.3

$$d_{\mathrm{Kol}}\big(S_2(c_n, Z), S_2(c_n, G)\big) \le \frac{C}{n^{1/14}}$$

and, if Z_i satisfies $\mathfrak{D}(\varepsilon, r, R)$, we use Theorem 3.10 and we obtain

$$d_{\mathrm{TV}}\big(S_2(c_n, Z), S_2(c_n, G)\big) \le \frac{C}{n^{\theta/26}}.$$

Now we have to estimate the total variation distance between $S_2(c_n, G) = \Phi_2(c_n, G)$ and the normal random variable Δ. In order to do it, we use (3.28), so we have to estimate the kurtosis $\kappa(c_n)$. We denote $a(i,j) = 1_{i\neq j}|i-j|^{-1/2}$ and we write

$$\begin{aligned}
a \otimes_1 a(i,j) &= \sum_k 1_{k\neq i} 1_{k\neq j} \frac{1}{\sqrt{|t_i - t_k||t_j - t_k|}} \times \frac{1}{n} \\
&\leq 2 + \sum_{k<\lfloor \frac{i+j}{2} \rfloor} \frac{1}{\sqrt{|t_i - t_k||t_j - t_k|}} 1_{k\neq i,j} \\
&\quad + \sum_{k>\lfloor \frac{i+j}{2} \rfloor + 1} \frac{1}{\sqrt{|t_i - t_k||t_j - t_k|}} 1_{k\neq i,j} \\
&\leq 2 + \int_0^1 \frac{dt}{\sqrt{|t_i - t||t_j - t|}}.
\end{aligned}$$

In order to obtain the last inequality, one just looks to the graphs of the functions $t \mapsto (|t_i - t||t_j - t|)^{-1/2}$ and to the graph of the step approximation of this function. The step approximation is below the function in these regions. Moreover (see [5] Lemma B1 for a complete computation),

$$\int_0^1 \frac{dt}{\sqrt{|t_i - t||t_j - t|}} = \pi + 2 \ln \frac{\sqrt{1 - t_i} + \sqrt{1 - t_j}}{|\sqrt{t_i} - \sqrt{t_j}|}.$$

It follows that

$$\begin{aligned}
\kappa^2(c_n) &= |c_n \otimes_1 c_n|^2 = \frac{1}{4n^2 \ln^2 n} \sum_{i\neq j} (a \otimes_1 a)^2(i,j) \\
&\leq \frac{2(\pi + 2)}{\ln^2 n} + \frac{2}{n^2 \ln^2 n} \sum_{i\neq j} \ln^2 \frac{\sqrt{1 - t_i} + \sqrt{1 - t_j}}{|\sqrt{t_i} - \sqrt{t_j}|} \leq \frac{C}{\ln^2 n}.
\end{aligned} \tag{4.15}$$

The statement now follows from Theorem 3.12. $\square$

REMARK 4.4. Let us point out that the polynomial $S_{n,p}(Z)$ in (4.9) represents a particular example of general (nonsymmetric) U-statistics which are discussed in [17, 18] but there one discusses convergence in Wasserstein distance.

5. Stochastic calculus of variation under the Doeblin's condition. We assume that the sequence $X = (X_n)_{n\in\mathbb{N}}$, $X_n = (X_{n,1}, \dots, X_{n,d_*}) \in \mathbb{R}^{d_*}$, of independent random vectors satisfies Hypothesis $\mathfrak{M}(\varepsilon, r, R)$, that is, the Doeblin's condition $\mathfrak{D}(\varepsilon, r, R)$ and the moment finiteness one. We strongly use here the representation (3.9) discussed in Section 3.1, that is, $X_n = \chi_n V_n + (1 - \chi_n) U_n$, $n \in \mathbb{N}$, where χ_n, V_n, U_n are independent with laws given in (3.8). The goal of this section is to present a differential calculus based on $V_n, n \in \mathbb{N}$ which has been introduced in [4, 6] (and which is inspired by the standard Malliavin calculus).

5.1. *A regularization lemma.* To begin, we introduce the space of the simple functionals. We denote by Λ_m the set of multi-indexes $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_i = (n_i, j_i) \in \mathbb{N} \times [d_*]$ (in contrast with the definition of Γ_m, we do not impose that $n_1 < \cdots < n_m$). We also consider the finite dimensional Hilbert space $\mathcal{U} = \mathbb{R}^d$ (for some $d \in \mathbb{N}$). We work with polynomials with random coefficients

$$P_N(x) = \sum_{m=0}^{N} \sum_{\alpha \in \Lambda_m} d(\alpha) x^\alpha,$$

where $x = (x_n)_{n \in \mathbb{N}}$ with $x_n = (x_{n,1}, \dots, x_{n,d_*}) \in \mathbb{R}^{d_*}$ and $x^\alpha = \prod_{i=1}^m x_{\alpha_i}$. The coefficients $d(\alpha) \in \mathcal{U}$ are random variables which are measurable with respect to $\sigma(\chi_n, U_n, n \in \mathbb{N})$ and so, in particular, are independent of $(V_n)_{n \in \mathbb{N}}$. We define $\mathcal{P}_N(\mathcal{U})$ to be the space of the polynomials of order less or equal to N, computed in $x_n = V_n$ that is $F \in \mathcal{P}_N(\mathcal{U})$ if

$$F = P_N(V) = \sum_{m=0}^{N} \sum_{\alpha \in \Lambda_m} d(\alpha) V^\alpha.$$

In particular, our polynomials $Q_{N,k_*}(c, X)$ belong to $\mathcal{P}_N(\mathcal{U})$. We set $\mathcal{P}(\mathcal{U}) = \bigcup_N \mathcal{P}_N(\mathcal{U})$ and we note that $\mathcal{P}(\mathcal{U})$ is dense in $L^p(\Omega, \mathcal{F}, P)$ with $\mathcal{F} = \sigma(X_n, n \in \mathbb{N})$. So we will define first our differential operators on $\mathcal{P}(\mathcal{U})$, then we extend them in the canonical way to their domains in $L^p(\Omega, \mathcal{F}, P)$.

Let $F \in \mathcal{P}(\mathcal{U})$, so $F = P_N(V)$. For $n \in \mathbb{N}$ and $i \in [d_*]$ we define the first-order derivatives

$$D_{n,i} F = \chi_n \times \partial_{n,i} P_N(V).$$

Notice that

$$D_{n,i} Q_{N,k_*}(c, X) = \frac{\partial Q_{N,k_*}(c, X)}{\partial V_{n,i}}.$$

We look to $DF = (D_{n,i} F)_{n \in \mathbb{N}, i \in [d_*]}$ as to a random element of the following Hilbert space $\mathcal{H}(\mathcal{U})$:

$$\mathcal{H}(\mathcal{U}) = \left\{ x \in \bigotimes_{n=1}^{\infty} \mathcal{U}^{d_*} : |x|_{\mathcal{H}(\mathcal{U})}^2 := \sum_{n=1}^{\infty} \sum_{i=1}^{d_*} |x_{n,i}|_{\mathcal{U}}^2 < \infty \right\}. \tag{5.1}$$

So $D : \mathcal{P}_N(\mathcal{U}) \to \mathcal{P}_{N-1}(\mathcal{H}(\mathcal{U}))$. The Malliavin covariance matrix of $F \in \mathcal{P}(\mathcal{U})^d$ is defined by

$$\sigma_F^{i,j} = \langle DF^i, DF^j \rangle_{\mathcal{H}(\mathcal{U})} = \sum_{n=1}^{\infty} \sum_{l=1}^{d_*} D_{n,l} F^i \times D_{n,l} F^j, \quad i, j = 1, \dots, d. \tag{5.2}$$

Moreover, we define the higher order derivatives in the following way. Let $m \in \mathbb{N}$ be fixed and let $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_i = (n_i, j_i) \in \mathbb{N} \times [d_*]$. For $F = P_N(X) \in \mathcal{P}(\mathcal{U})$, we define

$$
\begin{aligned}
D_\alpha^{(m)} F = D_{\alpha_m} \cdots D_{\alpha_1} F &= \left(\prod_{j=1}^{m} \chi_{n_j}\right)(\partial_{n_m, j_m} \cdots \partial_{n_1, j_1} P_N)(V) \\
&= \left(\prod_{j=1}^{m} \chi_{n_j}\right)\partial_\alpha P_N(V).
\end{aligned} \tag{5.3}
$$

We look to $D^{(m)} F = (D_\alpha^{(m)} F)_{\alpha \in \Lambda_m}$ as to a random element of $\mathcal{H}_m(\mathcal{U}) := \mathcal{H}^{\otimes m}(\mathcal{U})$, so $D^{(m)} : \mathcal{P}_N(\mathcal{U}) \to \mathcal{P}_{N-m}(\mathcal{H}^{\otimes m}(\mathcal{U}))$. For $m = 1$, we have $D^{(1)} F = DF$.

We define now the divergence operator (recall that θ_r is defined in (3.6))

$$
LF = -\sum_{n=1}^{\infty}\sum_{i=1}^{d_*}(D_{n,i} D_{n,i} F + D_{n,i} F \times \Theta_{n,i}) \quad \text{with} \tag{5.4}
$$

$$
\Theta_{n,i} = 2\chi_n \theta_r'(|X_{n,i} - x_{n,i}|^2)(X_{n,i} - x_{n,i}). \tag{5.5}
$$

Standard integration by parts on $\mathbb{R}$ gives the following duality relation: for every $F, G \in \mathcal{P}(\mathcal{U})$,

$$
\mathbb{E}(\langle DF, DG\rangle_{\mathcal{H}(\mathcal{U})}) = \mathbb{E}(\langle F, LG\rangle_{\mathcal{U}}) = \mathbb{E}(\langle G, LF\rangle_{\mathcal{U}}). \tag{5.6}
$$

We define now the Sobolev norms. For $q \geq 1$, we set

$$
|F|_{1,q,\mathcal{U}} = \sum_{n=1}^{q} |D^{(n)} F|_{\mathcal{H}^{\otimes n}(\mathcal{U})} \quad \text{and} \quad |F|_{q,\mathcal{U}} = |F| + |F|_{1,q,\mathcal{U}}. \tag{5.7}
$$

Moreover, we define

$$
\|F\|_{1,q,p,\mathcal{U}} = \left(\mathbb{E}(|F|_{1,q,\mathcal{U}}^p)\right)^{1/p}, \qquad \|F\|_{q,p,\mathcal{U}} = \left(\mathbb{E}(|F|_{q,\mathcal{U}}^p)\right)^{1/p}. \tag{5.8}
$$

Finally, we define the Sobolev spaces

$$
\mathbb{D}^{q,p}(\mathcal{U}) = \overline{\mathcal{P}}^{\|\cdot\|_{q,p,\mathcal{U}}}(\mathcal{U}), \qquad \mathbb{D}^\infty(\mathcal{U}) = \bigcap_{q=1}^{\infty}\bigcap_{p=1}^{\infty} \mathbb{D}^{q,p}(\mathcal{U}). \tag{5.9}
$$

As usual, we will drop the notation $\mathcal{U}$ when $\mathcal{U} = \mathbb{R}$.

The duality relation (5.6) implies that the operators $D^{(n)}$ and L are closable so we may extend them to $\mathbb{D}^{q,p}$ in a standard way. We recall now the basic computational rules. For $\phi \in C^1_{\mathrm{pol}}(\mathbb{R}^M)$ and $F \in \mathcal{P}^M$, we have

$$
D\phi(F) = \sum_{j=1}^{M} \partial_j \phi(F) DF^j, \tag{5.10}
$$

and for $\phi \in C^2_{\mathrm{pol}}(\mathbb{R}^M)$

$$L\phi(F) = \sum_{j=1}^{M} \partial_j \phi(F) LF^j - \frac{1}{2} \sum_{i,j=1}^{M} \partial_i \partial_j \phi(F) \langle DF^i, DF^j \rangle_{\mathcal{H}(\mathcal{U})}. \tag{5.11}$$

REMARK 5.1. For any n, j and $k \le k_*$, we take $F = X^k_{n,j}$ and we get

$$\mathbb{E}(LX^k_{n,j}) = 0 \quad \text{and} \quad \|LX^k_{n,j}\|_{q,p} \le \frac{C}{r^{q+1}} M^{k_*}_{2k_* p}(X), \tag{5.12}$$

the above constant $C > 0$ depending just on k_*, p, q. In fact, the first equality follows from the duality formula (5.6) (take $G = 1$). Moreover, by (5.10),

$$LX^k_{n,j} = kX^{k-1}_{n,j} LX_{n,j} + 2k(k-1) X^{k-2}_{n,j} \chi_n,$$

so that

$$\|LX^k_{n,j}\|_{q,p} \le k \|X^{k-1}_{n,j}\|_{q,2p} \|LX_{n,j}\|_{q,2p} + 2k(k-1) \|X^{k-2}_{n,j}\|_{q,p}.$$

It is easy to check that $\|X^l_{n,j}\|_{q,2p} \le l! M^{k_*}_{2k_* p}(X)$ for $l \le k_*$. Moreover, by (5.4) and (3.10) there exists a universal constant C such that $\|LX_{n,j}\|_{q,2p} \le \frac{C}{r^{q+1}}$. Equation (5.12) is now proved.

We give now the "regularization lemma," firstly studied in [3]. To this purpose, we recall that a super kernel $\phi : \mathbb{R}^d \to \mathbb{R}$ is a function which belongs to the Schwartz space $\mathbb{S}(\mathbb{R}^d)$ (infinitely differentiable functions which decrease to infinity faster than any polynomial), $\int \phi(x)\,dx = 1$, and such that for every $m \ge 1$ and for every multi-index α with $|\alpha| = m$ one has

$$\int y^\alpha \phi(y)\,dy = 0 \quad \text{and} \quad \int |y|^m |\phi(y)|\,dy < \infty.$$

Super kernels are used in several approximation problems in the literature. In order to construct a super kernel, one just takes the inverse Fourier transform of a function in $\mathbb{S}(\mathbb{R}^d)$.

For $\delta \in (0,1)$, we define $\phi_\delta(y) = \delta^{-d} \phi(\delta^{-1} y)$ and for a function $f : \mathbb{R}^d \to \mathbb{R}$ we denote $f_\delta = f * \phi_\delta$, the symbol $*$ denoting convolution. Then we have the following result.

LEMMA 5.2. *Let $F \in \mathcal{P}(\mathbb{R})^d$ and $q \in \mathbb{N}$. There exists some constant $C \ge 1$, depending on d and q only, such that for every $f \in C_b(\mathbb{R}^d)$, every $\eta, \delta > 0$ and $a < 1$,*

$$\big|\mathbb{E}(f(F)) - \mathbb{E}(f_\delta(F))\big| \le C\|f\|_\infty \left(\mathbb{P}^a(\det \sigma_F \le \eta) + \frac{\delta^q}{\eta^{2q}} \|\mathcal{K}_{q,0}(F)\|_2 \right), \tag{5.13}$$

with

$$\mathcal{K}_{q,0}(F) = (|F|_{1,q+1} + |LF|_q)^q (1 + |F|_{1,q+1})^{4dq}. \tag{5.14}$$

PROOF. If $f \in C_b^q(\mathbb{R}^d)$, then the statement is proved in Lemma 5.3 of [6] (see (5.19) therein with $m = 0$). For f just continuous, the assertion follows by a standard density argument. □

5.2. *Estimates of the Sobolev norms*. Throughout this section, we assume that X verifies $\mathfrak{M}(\varepsilon, r, R)$ and we estimate the Sobolev norms of $Q_{N,k_*}(c, X)$ and of $LQ_{N,k_*}(c, X)$. To begin, we give an abstract lemma.

LEMMA 5.3. *Let $k \in N$ and let $B_{n,i}, \Lambda_n \in \mathbb{D}^\infty(\mathcal{U})$, $\Theta_{n,i} \in \mathbb{D}^\infty(\mathbb{R})$, $n \in \mathbb{N}$, $i = 1, \dots, k$ be such that $B_{n,i}$ is $\sigma(X_1, \dots, X_n)$ measurable and $\Theta_{n,i}$ is $\sigma(X_n)$ measurable with $\mathbb{E}(\Theta_{n,i}) = 0$. We consider the process*

$$Y_J = \sum_{n=1}^{J} \sum_{i=1}^{k} B_{n-1,i} \Theta_{n,i} + \Lambda_J. \tag{5.15}$$

Then for every $q \in \mathbb{N}$ and $p \geq 2$,

$$\begin{aligned} &\max_{n \leq J} \|Y_n\|_{\mathcal{U},q,p} \\ &\quad \leq q b_p \max_{n \leq J} \|\Theta_n\|_{q,p} \left(\sum_{n=0}^{J-1} \|B_n\|_{\mathcal{U},q,p}^2 \right)^{1/2} + \max_{n \leq J} \|\Lambda_n\|_{\mathcal{U},q,p}, \end{aligned} \tag{5.16}$$

with

$$\|\Theta_n\|_{q,p}^2 = \sum_{i=1}^{k} \|\Theta_{n,i}\|_{q,p}^2, \qquad \|B_n\|_{\mathcal{U},q,p}^2 = \sum_{i=1}^{k} \|B_{n,i}\|_{\mathcal{U},q,p}^2.$$

PROOF. We take first $q = 0$. Since Θ_n and B_{n-1} are independent, one has

$$\left\| \sum_{i=1}^{k} \Theta_{n,i} B_{n-1,i} \right\|_{\mathcal{U},p}^2 \leq \left(\sum_{i=1}^{k} \|B_{n-1,i}\|_{\mathcal{U},p} \times \|\Theta_{n,i}\|_p \right)^2 \leq \|\Theta_n\|_p^2 \|B_{n-1}\|_{\mathcal{U},p}^2.$$

Moreover, since $\mathbb{E}(\Theta_{n,i}) = 0$, $M_m := \sum_{n=1}^{m} \sum_{i=1}^{k_*} B_{n-1,i} \Theta_{n,i}$ is a martingale. So, by (2.6)

$$\begin{aligned} \|M_m\|_{\mathcal{U},p}^2 &\leq b_p^2 \sum_{n=1}^{m} \|\Theta_n\|_p^2 \|B_{n-1}\|_{\mathcal{U},p}^2 \\ &\leq b_p^2 \max_{n \leq m} \|\Theta_n\|_p^2 \left(\sum_{n=1}^{m} \|B_{n-1}\|_{\mathcal{U},p}^2 \right) \end{aligned}$$

and (5.16) is proved for $q = 0$. Take now $q = 1$. We have

$$\overline{Y}_m := DY_m = \sum_{n=1}^{m} \sum_{i=1}^{k_*} \overline{B}_{n-1,i} \Theta_{n,i} + \overline{\Lambda}_m,$$

where $\overline{\Lambda}_m = \sum_{n=1}^m \sum_{i=1}^{k_*}(D\Theta_{n,i})B_{n-1,i} + D\Lambda_m$ and $\overline{B}_{n,i} = DB_{n,i}$ is measurable w.r.t. $\sigma(X_1, \dots, X_{n-1})$. Notice that $\overline{Y}_m$, $\overline{B}_{k,i}$ and $\overline{\Lambda}_m$ take values in $\mathcal{H}(\mathcal{U})$ (defined in (5.1)). So, by applying the step above, we get

$$\max_{n\le J} \|DY_n\|_{\mathcal{H}(\mathcal{U}),p}$$
$$\le b_p \max_{n\le J} \|D\Theta_n\|_p \left(\sum_{k=1}^J \|\overline{B}_k\|^2_{\mathcal{H}(\mathcal{U}),p}\right)^{1/2} + \max_{m\le J} \|\overline{\Lambda}_m\|_{\mathcal{H}(\mathcal{U}),p}.$$

Notice first that $\|\overline{B}_k\|_{\mathcal{H}(\mathcal{U}),p} \le \|B_k\|_{\mathcal{U},1,p}$, $\|D\Lambda_k\|_{\mathcal{H}(\mathcal{U}),p} \le \|\Lambda_k\|_{\mathcal{U},1,p}$ and $\|D\Theta_n\|_p \le \|\Theta_n\|_{1,p}$. So it remains to estimate

$$I_m(n', i') = \sum_{n=1}^m \sum_{i=1}^{k_*}(D_{(n',i')}\Theta_{n,i})B_{n-1,i}.$$

For $n \ne n'$, one has $D_{(n',i')}\Theta_{n,i} = 0$ so that

$$I_m(n', i') = 1_{\{n'\le m\}} \sum_{i=1}^k (D_{(n',i')}\Theta_{n',i})B_{n'-1,i}.$$

It follows that

$$|I_m|^2_{\mathcal{H}(\mathcal{U})} = \sum_{n'=1}^m \sum_{i'=1}^{d_*} \left|\sum_{i=1}^k D_{(n',i')}\Theta_{n',i}B_{n'-1,i}\right|^2_{\mathcal{U}}$$
$$\le \sum_{n'=1}^m \sum_{i'=1}^{d_*} |D_{(n',i')}\Theta_{n'}|^2 |B_{n'-1}|^2_{\mathcal{U}} \le \sum_{n'=1}^m |\Theta_{n'}|_1^2 |B_{n'-1}|^2_{\mathcal{U}}.$$

Then, using the triangle inequality,

$$\|I_m\|^2_{\mathcal{H}(\mathcal{U}),p} = \big\||I_m|^2_{\mathcal{H}(\mathcal{U})}\big\|_{p/2} \le \sum_{n'=1}^m \big\||\Theta_{n'}|_1^2 |B_{n'-1}|^2_{\mathcal{U}}\cdot\big\|_{p/2}$$
$$= \sum_{n'=1}^m \big\||\Theta_{n'}|_1^2\big\|_{p/2}\big\||B_{n'-1}|^2_{\mathcal{U}}\big\|_{p/2} \le \max_{n\le m} \|\Theta_n\|^2_{1,p} \sum_{n=1}^m \|B_{n-1}\|^2_{\mathcal{U},p}.$$

So the proof is completed for $q = 1$. For general q, this follows by recurrence. □

PROPOSITION 5.4. *For every $q, N \in \mathbb{N}$ and $p \ge 2$, one has*

$$\big\|Q_N(c, X)\big\|_{q,p} \le M^N_{q,p} \times |c|, \tag{5.17}$$

$$\big\|LQ_N(c, X)\big\|_{q,p} \le C_{q,p}(N) \times |c|, \tag{5.18}$$

with $M_{q,p} = b_p q k_*! M_{k_* p}(X)$ and $C_{q,p}(N)$ is a constant which depends on N, q, p, k_*, d_*, $r^{-(q+1)}$ and on $M_{2k_* p}(X)$.

As a consequence, there exists $C>0$ depending only on N, q, p, $r^{-(q+1)}$, k_*, d_* and on $M_{2k_*p}(X)$ such that

$$\|\mathcal{K}_{q,0}(Q_{N,k_*}(c,X))\|_2 \leq C|c|^q(1+|c|)^{4q}, \tag{5.19}$$

$\mathcal{K}_{q,0}(Q_{N,k_*}(c,X))$ being defined in (5.14).

PROOF. We prove this by recurrence on N. The case $N=1$ is straightforward, so we suppose $N>1$. For a multi-index β with $|\beta|=m$, we define $c^{n,j}(\beta)=1_{\beta'_m<n}c(\beta,(n,j))$ and we write, with $Z=Z(X)$,

$$Q_{N,k_*}(c,X)=c(\varnothing)+\sum_{n=1}^{\infty}\sum_{j=1}^{m_*}Z_{n,j}Q_{N-1,k_*}(c^{n,j},X).$$

We first prove (5.17). We will use (5.16) with $B_{n,j}=Q_{N-1,k_*}(c^{n,j},X)$, $\Theta_{n,j}=Z_{n,j}$ and $\Lambda_n=0$. Notice that $\|Z_{n,j}\|_{q,p}\leq k_*!M_{k_*p}(X)$. Using the recurrence hypothesis, we obtain

$$\begin{aligned}\|Q_{N,k_*}(c,X)\|_{q,p} &\leq b_pqk_*!M_{k_*p}(X)\left(\sum_{n=1}^{\infty}\sum_{j=1}^{m_*}\|Q_{N-1,k_*}(c^{n,j},X)\|_{q,p}^2\right)^{1/2}\\ &\leq M_{q,p}^N\left(\sum_{n=1}^{\infty}\sum_{j=1}^{m_*}\sum_{|\alpha|\leq N-1}|c^{n,j}(\alpha)|^2\right)=M_{q,p}^N|c|\end{aligned}$$

so (5.17) is proved.

We prove now (5.18). Since $\langle DZ_{n,j},DQ_{N-1,k_*}(c^{n,j},X)\rangle_{\mathcal{H}(\mathcal{U})}=0$, we get (see (5.11))

$$\begin{aligned}LQ_{N,k_*}(c,X)=&\sum_{n=1}^{\infty}\sum_{j=1}^{m_*}Q_{N-1,k_*}(c^{n,j},X)LZ_{n,j}\\ &+\sum_{n=1}^{\infty}\sum_{j=1}^{m_*}Z_{n,j}LQ_{N-1}(c^{n,j},X).\end{aligned}$$

So we are in the framework of Lemma 5.3 with $B_{n-1,j}=Q_{N-1,k_*}(c^{n,j},X)$, $\Theta_{n,j}=LZ_{n,j}$ and

$$\Lambda_m=\sum_{n=1}^{m}\sum_{j=1}^{d_*}Z_{n,j}LQ_{N-1,k_*}(c^{n,j},X).$$

We compute first $\|\Lambda_m\|_{p,q}$. In order to do it, we use once again Lemma 5.3 with $\Theta_{n,j}=Z_{n,j}$ and $B_{n,j}=LQ_{N-1,k_*}(c^{n,j},X)$. We get

$$\|\Lambda_m\|_{p,q}\leq M_{q,p}\left(\sum_{n=1}^{m}\sum_{j=1}^{d_*}\|LQ_{N-1,k_*}(c^{n,j},X)\|_{q,p}\right)^{1/2}$$

$$\leq M_{q,p}C_{q,p}(N-1)\left(\sum_{n=1}^{m}\sum_{j=1}^{d_*}|c^{n,j}|^2\right)^{1/2} = M_{q,p}C_{q,p}(N-1)|c|,$$

the last inequality being a consequence of the recurrence hypothesis. We come now back to $LQ_{N,k_*}(c,X)$ and we use the previous lemma:

$$\|LQ_{N,k_*}(c,X)\|_{q,p} \leq qb_p \max_{n<\infty}\|LZ_{n,j}\|_{q,p}\left(\sum_{n=1}^{m}\sum_{j=1}^{d_*}\|Q_{N-1,k_*}(c^{n,j},X)\|_{q,p}\right)^{1/2} + M_{q,p}C_{q,p}(N-1)|c|.$$

By (5.12), we have $\|LZ_{n,j}\|_{q,p} \leq Cr^{-(q+1)}M_{2k_*p}^{k_*}(X)$ so we obtain

$$\|LQ_{N,k_*}(c,X)\|_{q,p} \leq \left(qb_pM_{q,p}^{N-1}Cr^{-(q+1)}M_{2k_*p}^{k_*}(X) + M_{q,p}C_{q,p}(N-1)\right)|c|. \qquad \square$$

5.3. *Estimate of the covariance matrix.* In this section, we give estimates for the Malliavin covariance matrix of $Q_{N,k_*}(c,X)$ which we shortly denote by σ_N. We restrict ourselves to the scalar case, so that $Q_{N,k_*}(c,X) \in \mathbb{R} = \mathcal{U}$ and σ_N is just a scalar. We start from a precise formula of the Malliavin derivative of $Q_{N,k_*}(c,X)$: straightforward computations give

$$\begin{aligned} D_{n_0,j_0}Q_{N,k_*}(c,X) &= D_{n_0,j_0}S_N(c,Z(X)) \\ &= \sum_{m=0}^{N-1}\sum_{k=0}^{k_*-1}\sum_{\beta\in\Lambda_{n_0,j_0}(m,k)}(k+1)(Dc)_{n_0,j_0,k}(\beta)\chi_{n_0}V_{n_0,j_0}^{k}Z^{\beta}(X), \end{aligned} \tag{5.20}$$

where $\Lambda_{n_0,j_0}(m,k)$ denotes the multi-indexes of length m which do not contain the pair (n_0, kd_*+j_0) and where $(Dc)_{n_0,j_0,k}(\beta) = c((n_0,kd_*+j_0))$ if $|\beta|=0$ and for $|\beta| = m \geq 1$,

$$\begin{aligned} (Dc)_{n_0,j_0,k}(\beta) &= \sum_{i=1}^{m-1}c(\beta_1,\ldots,\beta_i,(n_0,kd_*+j_0),\beta_{i+1},\ldots,\beta_m)1_{\{\beta_i'<n_0<\beta_{i+1}'\}} \\ &\quad + c((n_0,kd_*+j_0),\beta_1,\ldots,\beta_m)1_{\{n_0<\beta_1'\}} \\ &\quad + c(\beta_1,\ldots,\beta_m,(n_0,kd_*+j_0))1_{\{n_0>\beta_m'\}}. \end{aligned} \tag{5.21}$$

The aim of this section is to prove the nondegeneracy estimate (5.25) in the next Lemma 5.6, but we first need to study the conditional expectation of σ_N given the randomness from χ_n and U_n.

LEMMA 5.5. *Assume $\mathfrak{D}(\varepsilon, r, R)$. We denote by $\mathbb{E}_{U,\chi}$ the conditional expectation with respect to $\sigma(U_n, \chi_n, n \in \mathbb{N})$. Then*

$$\mathbb{E}_{U,\chi}(\sigma_N) \geq \lambda_R^N \sum_{|\alpha|=N} c^2(\alpha)\chi^{\alpha'}, \tag{5.22}$$

where $\lambda_R > 0$ is given in Lemma 3.1 and for $\alpha = ((\alpha_1', \alpha_1''), \dots, (\alpha_m', \alpha_m''))$, we set $\alpha' = (\alpha_1', \dots, \alpha_m')$ and $\chi^{\alpha'} = \prod_{i=1}^m \chi_{\alpha_i'}$.

PROOF. We set here $Z = Z(X)$. We recall that $X_{n,j} = \chi_n V_{n,j} + (1 - \chi_n)U_{n,j}$ and we define (with $k(l)$ and $j(l)$ defined in (3.2))

$$\widetilde{V}_{n,l} = V_{n,j(l)}^{k(l)} - \mathbb{E}(V_{n,j(l)}^{k(l)}),$$
$$\overline{U}_{n,l} = (1 - \chi_n)U_{n,j(l)}^{k(l)} + \chi_n \mathbb{E}(V_{n,j(l)}^{k(l)}) - \mathbb{E}(X_{n,j(l)}^{k(l)}).$$

Then

$$Z_{n,l} = \chi_n V_{n,j(l)}^{k(l)} + (1 - \chi_n)U_{n,j(l)}^{k(l)} - \mathbb{E}(X_{n,j(l)}^{k(l)}) = \chi_n \widetilde{V}_{n,l} + \overline{U}_{n,l}.$$

So, we have

$$Z^\alpha = \overline{Z}^\alpha + \chi^{\alpha'} \widetilde{V}^\alpha \quad \text{where } \overline{Z}^\alpha = \sum_{\substack{(\beta,\gamma)=\alpha, \\ \gamma \neq \varnothing}} \chi^{\beta'} \widetilde{V}^\beta \times \overline{U}^\gamma.$$

For every α, θ s.t. $|\alpha| \leq |\theta|$, one has

$$\mathbb{E}_{U,\chi}(\overline{Z}^\alpha \widetilde{V}^\theta) = \sum_{\substack{(\beta,\gamma)=\alpha, \\ \gamma \neq \varnothing}} \chi^\beta \mathbb{E}_{U,\chi}(\widetilde{V}^\beta \widetilde{V}^\theta) \times \overline{U}^\gamma = 0. \tag{5.23}$$

This is because $|\beta| < |\alpha| \leq |\theta|$, so there is at least one $\theta_i \notin \beta$ and $\mathbb{E}_{U,\chi}(\widetilde{V}^{\theta_i}) = 0$. For the same reason, one has

$$\mathbb{E}_{U,\chi}(\widetilde{V}^\alpha \widetilde{V}^\theta) = 0 \quad \text{for every } \alpha, \theta \text{ s.t. } |\alpha| < |\theta|. \tag{5.24}$$

We recall that $V_{n_0,j_0}^k = \widetilde{V}_{n_0,kd_*+j_0} + \mathbb{E}(V_{n_0,j_0}^k)$ and we use (5.20) in order to we write

$$D_{n_0,j_0} S_N(c, Z) = \sum_{m=0}^{N-1} (A_{m,1}^{n_0,j_0} + A_{m,2}^{n_0,j_0} + A_{m,3}^{n_0,j_0}) \quad \text{where}$$

$$A_{m,1}^{n_0,j_0} = \sum_{k=0}^{k_*-1} \sum_{\beta \in \Lambda_{n_0,j_0}(m,k)} (k+1)(Dc)_{n_0,j_0,k}(\beta) \chi_{n_0} \widetilde{V}_{n_0,kd_*+j_0} \chi^{\beta'} \widetilde{V}^\beta,$$

$$A_{m,2}^{n_0,j_0} = \sum_{k=0}^{k_*-1} \sum_{\beta \in \Lambda_{n_0,j_0}(m,k)} (k+1)(Dc)_{n_0,j_0,k}(\beta) \chi_{n_0} \widetilde{V}_{n_0,kd_*+j_0} \overline{Z}^\beta,$$

$$A_{m,3}^{n_0,j_0} = \sum_{k=0}^{k_*-1} \sum_{\beta \in \Lambda_{n_0,j_0}(m,k)} (k+1)(Dc)_{n_0,j_0,k}(\beta)\chi_{n_0}\mathbb{E}(V_{n_0,j_0}^k)Z^\beta,$$

$\Lambda_{n_0,j_0}(m,k)$ denoting the multi-indexes of length m which do not contain the pair $(n_0, kd_* + j_0)$. By (5.23) and (5.24), one has $\mathbb{E}_{U,\chi}(A_{N-1,1}^{n_0,j_0} A_{m,i}^{n_0,j_0}) = 0$ for every $m \le N-1$ and $i = 2, 3$ and $\mathbb{E}_{U,\chi}(A_{N-1,1}^{n_0,j_0} A_{m,1}^{n_0,j_0}) = 0$ for every $m < N-1$. Thus, $A_{N-1,1}^{n_0,j_0}$ is orthogonal (in $L^2(\mathbb{P}_{U,\chi})$) to $D_{n_0,j_0}S_N(c,Z) - A_{N-1,1}^{n_0,j_0}$, so that

$$\mathbb{E}_{U,\chi}(|D_{n_0,j_0}S_N(c,Z)|^2) \ge \mathbb{E}_{U,\chi}(|A_{N-1,1}^{n_0,j_0}|^2).$$

Therefore,

$$\mathbb{E}_{U,\chi}(\sigma_N) = \sum_{n_0=1}^{\infty}\sum_{j_0=1}^{d_*} \mathbb{E}(|D_{n_0,j_0}S_N(c,Z)|^2) \ge \sum_{n_0=1}^{\infty}\sum_{j_0=1}^{d_*} \mathbb{E}_{U,\chi}(|A_{N-1,1}^{n_0,j_0}|^2).$$

Now, we write

$$A_{N-1,1}^{n_0,j_0} = \prod_{i=1}^{N} \chi_{n_i} \sum_{|\alpha|=N} d_{n_0,j_0}(\alpha)\widetilde{V}^\alpha, \quad \text{with}$$

$$d_{n_0,j_0}(\alpha) = \sum_{i=1}^{N}\sum_{k=0}^{k_*-1} (k+1)c(\alpha)1_{\alpha_i=(n_0,kd_*+j_0)}.$$

For every α, there exists at most one (k,i) such that $\alpha_i = (n_0, kd_* + j_0)$ so that

$$d_{n_0,j_0}^2(\alpha) = \sum_{i=1}^{N}\sum_{k=0}^{k_*-1} (k+1)c^2(\alpha)1_{\alpha_i=(n_0,kd_*+j_0)}.$$

By using (3.13),

$$\begin{aligned}\sum_{n_0=1}^{\infty}\sum_{j_0=1}^{d_*} \mathbb{E}_{U,\chi}(|A_{N-1,1}^{n_0,j_0}|^2) &\ge \sum_{n_0=1}^{\infty}\sum_{j_0=1}^{d_*} \lambda_R^N \sum_{|\alpha|=N} d_{n_0,j_0}^2(\alpha)\chi^{\alpha'} \\ &\ge \lambda_R^N \sum_{|\alpha|=N} c^2(\alpha)\chi^{\alpha'}\end{aligned}$$

and the statement holds. $\square$

We can now prove the main result of this section.

LEMMA 5.6. *Assume $\mathfrak{D}(\varepsilon, r, R)$. Let $c \in \mathcal{C}(\mathbb{R})$ with $|c|_N > 0$. For every $\eta > 0$,*

$$\mathbb{P}(\sigma_N \le \eta) \le \frac{2e^3}{9} N \exp\left(-\left(\frac{\varepsilon \mathfrak{m}_r}{2}\right)^{2N} \frac{|c|_N^2}{\delta_*^2(c)}\right) + \frac{2Kk_*N}{\lambda_R \varepsilon \mathfrak{m}_r |c|_N^{2/(k_*N)}} \eta^{1/(k_*N)}, \tag{5.25}$$

where K a universal constant (the one in the Carbery–Wright inequality) and λ_R is given in Lemma 3.1.

REMARK 5.7. Sometimes $|c|_N$ is small and we would like to use $|c|_m$ instead, with $m < N$. We denote $|c|_{m+1,N}^2 = \sum_{k=m+1}^N c^2(\alpha)$. Then for every $h \geq 1$ there exists $C > 0$ such that

$$\begin{aligned} \mathbb{P}(\sigma_N \leq \eta) \leq C\frac{|c|_{m+1,N}^{2h}}{\eta^h} + \frac{2e^3}{9} m \exp\left(-\left(\frac{\varepsilon \mathfrak{m}_r}{2}\right)^{2m} \frac{|c|_m^2}{\delta_*^2(c)}\right) \\ + \frac{2Kk_*m}{\lambda_R \varepsilon \mathfrak{m}_r |c|_m^{2/(k_*m)}} (4\eta)^{1/(k_*m)}. \end{aligned} \tag{5.26}$$

Indeed, we denote $Q_{m+1,N,k_*}(c, X) = Q_{N,k_*}(c, X) - Q_{m,k_*}(c, X)$ and we use the inequality

$$\sigma_N = \left|DQ_{N,k_*}(c, X)\right|_{\mathcal{H}}^2 \geq \frac{1}{2}\left|DQ_{m,k_*}(c, X)\right|_{\mathcal{H}}^2 - \left|DQ_{m+1,N,k_*}(c, X)\right|_{\mathcal{H}}^2$$

in order to obtain

$$\begin{aligned} \mathbb{P}(\sigma_N \leq \eta) &\leq \mathbb{P}\left(\left|DQ_{m+1,N,k_*}(c, X)\right|_{\mathcal{H}}^2 \geq \eta\right) + \mathbb{P}\left(\left|DQ_{m,k_*}(c, X)\right|_{\mathcal{H}}^2 \leq 4\eta\right) \\ &\leq \mathbb{P}\left(\left|DQ_{m+1,N,k_*}(c, X)\right|_{\mathcal{H}}^2 \geq \eta\right) + \frac{2e^3}{9} m \exp\left(-\left(\frac{\varepsilon \mathfrak{m}_r}{2}\right)^{2m} \frac{|c|_m^2}{\delta_*^2(c)}\right) \\ &\quad + \frac{2Kk_*m}{\lambda_R \varepsilon \mathfrak{m}_r |c|_m^{2/(k_*m)}} (4\eta)^{1/(k_*m)}. \end{aligned}$$

Using Chebyshev's inequality and Proposition 5.4, for every h,

$$\begin{aligned} \mathbb{P}\left(\left|DQ_{m+1,N,k_*}(c, X)\right|_{\mathcal{H}}^2 \geq \eta\right) &\leq \eta^{-h} \left\|DQ_{m+1,N,k_*}(c, X)\right\|_{\mathcal{H}(\mathcal{U}),2h}^{2h} \\ &\leq C\eta^{-h} |c|_{m+1,N}^{2h}, \end{aligned}$$

so the proof of (5.26) is completed.

PROOF OF LEMMA 5.6. We will use the Carbery–Wright inequality that we recall here (see Theorem 8 in [12]). Let μ be a probability law on $\mathbb{R}^J$ which is absolutely continuous with respect to the Lebesgue measure and has a log-concave density. There exists a universal constant K such that for every polynomial $Q(x)$ of order k_*N and for every $\eta > 0$ one has

$$\begin{aligned} \mu\left(x : \left|Q(x)\right| \leq \eta\right) &\leq Kk_*N\left(\frac{\eta}{V_\mu(Q)}\right)^{1/(k_*N)}, \\ V_\mu(Q) &= \left(\int Q^2(x)\, d\mu(x)\right)^{1/2}. \end{aligned} \tag{5.27}$$

We will use this result in the following framework. We recall that the coefficients $c(\alpha)$ are null except a finite number of them. So we may find M such that, if $|\alpha| = m$ and $\alpha'_m > M$ then $c(\alpha) = 0$. It follows that we may write (see (5.20))

$$\sigma_N = q_{\chi,\overline{U}}(V),$$

where $q_{q,\overline{U}}(V)$ is a polynomial of order k_*N with unknowns $V_{n,j}, n \le M, j \le d_*$ and coefficients depending on χ_n and $\overline{U}_{n,j,k}$. Moreover, we recall that $\mathbb{P}_{U,\chi}$ is the conditional probability with respect to $\sigma(U_i, \chi_i, i \in \mathbb{N})$. We denote by μ the law of $(V_{n,j}, n \le M, j \le d_*)$ under $\mathbb{P}_{U,\chi}$: this is a product of laws of the form $c\psi_r(|x-\overline{x}|^2)\,dx$ so it is log-concave. Then we can use (5.27). Using (5.22),

$$V_\mu(q_{\chi,\overline{U}}) \ge \int \left|q_{\chi,\overline{U}}(x)\right| d\mu(x) = \mathbb{E}_{U,\chi}(\sigma_N) \ge \lambda_R^N \sum_{|\beta|=N} c^2(\beta)\chi^{\beta'}.$$

We take now $\theta > 0$ (to be chosen in a moment) and we use (5.27) in order to obtain

$$\begin{aligned} \mathbb{P}(\sigma_N \le \eta) &= \mathbb{P}\big(q_{\chi,\overline{U}}(V) \le \eta\big) \\ &\le \mathbb{P}\big(V_\mu(q_{\chi,\overline{U}}) \le \theta\big) + \mathbb{E}\big(\mathbb{P}_{V,\chi}\big(q_{\chi,\overline{U}}(V) \le \eta\big)1_{\{V_\mu(q_{\chi,\overline{U}})\ge\theta\}}\big) \\ &\le \mathbb{P}\bigg(\sum_{\beta\in\Lambda_N} c^2(\beta)\chi^{\beta'} \le \frac{\theta}{\lambda_R^N}\bigg) + Kk_*N(\eta/\theta)^{1/(k_*N)}. \end{aligned} \tag{5.28}$$

The first term in the above inequality is estimated in the Appendix. In order to fit in the notation used there, we denote $\Lambda_N(\beta') = \{\alpha : |\alpha| = N \text{ and } \alpha' = \beta'\}$ and $\overline{c}^2(\beta') = \sum_{\alpha\in\Lambda_N(\beta')} c^2(\alpha)$. Then

$$\sum_{\beta\in\Lambda_N} c^2(\beta)\chi^{\beta'} = \sum_{|\beta'|=N} \overline{c}^2(\beta')\chi^{\beta'} = \Psi_N\big(\overline{c}^2\big).$$

Now we apply Lemma A.1 with $x = \theta/\lambda_R^N$. Recall that $p = \varepsilon\mathfrak{m}_r$ and we have the restriction

$$\theta = \lambda_R^N x < \lambda_R^N \bigg(\frac{p}{2}\bigg)^N \sum_{|\beta'|=N} \overline{c}^2(\beta') = \lambda_R^N \bigg(\frac{\varepsilon\mathfrak{m}_r}{2}\bigg)^N |\overline{c}|_N^2. \tag{5.29}$$

We have $|\overline{c}|_N^2 = |c|_N^2$ and

$$\delta_N^2(\overline{c}) = \max_n \sum_{n\in\beta',|\beta'|=N} \overline{c}^2(\alpha') = \max_n \sum_{n\in\alpha',|\alpha|=N} c^2(\alpha) = \delta_*^2(c).$$

Then (A.2) gives

$$\mathbb{P}\bigg(\Psi_N\big(\overline{c}^2\big) \le \frac{\theta}{\lambda_R^N}\bigg) \le \frac{2e^3}{9} N \exp\bigg(-\frac{(\theta/\lambda_R^N)^2}{\delta_*^2(c)|c|_N^2}\bigg).$$

Inserting this in (5.28), we obtain

$$\mathbb{P}(\sigma_N \le \eta) \le \frac{2e^3}{9} N \exp\bigg(-\frac{(\theta/\lambda_R)^2}{\delta_*^2(c)|c|_N^2}\bigg) + Kk_*N(\eta/\theta)^{1/(k_*N)}.$$

Now, θ is any constant satisfying the restriction (5.29). So, by letting $\theta \uparrow \lambda_R^N((\varepsilon\mathfrak{m}_r)/2)^N|\overline{c}|_N^2 = \lambda_R^N((\varepsilon\mathfrak{m}_r)/2)^N|c|_N^2$, we finally obtain (5.25). $\square$

5.4. *Proof of Theorem* 3.3. The goal of this section is to give the proof of Theorem 3.3 so we use the notation from Section 3.

We take $q \in \mathbb{N}$, $q \geq 1$, and we consider the sequence $\lambda_q = \frac{q}{q+k}$. Since $\lambda_q^2 \uparrow 1$ as $q \to \infty$, we can find q such that such that $\lambda_q^2 < \theta \leq \lambda_{q+1}^2$, and since $\lambda_{q+1}^2 \leq \lambda_q$, we get $\lambda_q^2 < \theta \leq \lambda_q$. We work with this value of q and we write simply λ in place of λ_q. Moreover, in the following, $C > 0$ stands for a constant which may vary from line to line and which depends on the parameters in the statements but not on the coefficients $c, d \in \mathcal{C}(\mathbb{R})$.

We define $a = \theta/\lambda$, so $\frac{1}{1+k} \leq \lambda < a \leq 1$. We consider $\eta, \delta \in (0, 1)$, to be chosen in the sequel, and we use the regularization Lemma 5.2 (see (5.13)) with the above choice of q and a. This gives

$$\begin{aligned}
&\big|\mathbb{E}\big(f\big(Q_{N,k_*}(c,X)\big)\big) - \mathbb{E}\big(f_\delta\big(Q_{N,k_*}(c,X)\big)\big)\big| \\
&\quad \leq C\|f\|_\infty\bigg(\mathbb{P}^a(\det\sigma_{Q_{N,k_*}(c,X)} \leq \eta) + \frac{\delta^q}{\eta^{2q}}\big\|\mathcal{K}_{q,0}\big(Q_{N,k_*}(c,X)\big)\big\|_2\bigg) \\
&\quad \leq C\|f\|_\infty\bigg(\mathbb{P}^a(\det\sigma_{Q_{N,k_*}(c,X)} \leq \eta) + \frac{\delta^q}{\eta^{2q}}|c|^q\big(1+|c|\big)^{4q}\bigg),
\end{aligned}$$

the latter inequality following from (5.19). Moreover, by (5.26) (therein, $\sigma_N = \det\sigma_{Q_{N,k_*}}$), for every $h \geq 1$ (recall that $\overline{m} = m \vee m'$)

$$\mathbb{P}(\det\sigma_{Q_{N,k_*}(c,X)} \leq \eta) \leq C\bigg(\frac{|c|_{m+1,N}^{2h}}{\eta^h} + e_{m,N}(c) + \frac{1}{|c|_m^{2/(k_*m)}}\eta^{1/(k_*\overline{m})}\bigg).$$

So,

$$\begin{aligned}
&\big|\mathbb{E}\big(f\big(Q_{N,k_*}(c,X)\big)\big) - \mathbb{E}\big(f_\delta\big(Q_{N,k_*}(c,X)\big)\big)\big| \\
&\quad \leq C\|f\|_\infty\bigg(\frac{|c|_{m+1,N}^{2ha}}{\eta^{ha}} + e_{m,N}^a(c) + \frac{\eta^{a/(k_*\overline{m})}}{|c|_m^{2a/(k_*m)}} + |c|^q\big(1+|c|\big)^{4q}\frac{\delta^q}{\eta^{2q}}\bigg).
\end{aligned}$$

A similar estimate holds for $Q_{N,k_*}(d,Y)$. We use now $\mathfrak{a}_k$ defined in (3.18). Since $\|f_\delta\|_{k,\infty} \leq \delta^{-k}\|f\|_\infty$, one has

$$\big|\mathbb{E}\big(f_\delta\big(Q_{N,k_*}(c,X)\big)\big) - \mathbb{E}\big(f_\delta\big(Q_{N,k_*}(d,Y)\big)\big| \leq k\delta^{-k}d_k\|f\|_\infty.$$

Putting this together, we get

$$\begin{aligned}
&\big|\mathbb{E}\big(f\big(Q_{N,k_*}(c,X)\big)\big) - \mathbb{E}\big(f\big(Q_{N,k_*}(d,Y)\big)\big| \\
&\quad \leq C\max\big(1, \big(|c|_m^{-\frac{2}{k_*m}} + |d|_{m'}^{-\frac{2}{k_*m'}}\big)^a\big)\|f\|_\infty \\
&\qquad \times\bigg(\vartheta_{m,\eta}(c) + \vartheta_{m',\eta}(d) + \eta^{a/(k_*\overline{m})} + \big(1+|c|+|d|\big)^{5q}\frac{\delta^q}{\eta^{2q}} + \delta^{-k}d_k\bigg),
\end{aligned}$$

where, for a set of coefficients c and for fixed m, η,

$$\vartheta_{m,\eta}(c) = \frac{|c|_{m+1,N}^{2ha}}{\eta^{ha}} + e_{m,N}^a(c).$$

We optimize first on δ: we take $\delta = \mathfrak{a}_k^{1/(q+k)} \eta^{2q/(q+k)}(1+|c|+|d|)^{-5q/(q+k)}$ and we obtain (recall that $\lambda = \frac{q}{q+k} \in (0,1)$),

$$\begin{aligned}(1+|c|+|d|)^{5q}\frac{\delta^q}{\eta^{2q}} &= \delta^{-k} d_k = \eta^{-2k\lambda} d_k^\lambda (1+|c|+|d|)^{5k\lambda} \\ &\le \eta^{-2k\lambda} d_k^\lambda (1+|c|+|d|)^{5k}.\end{aligned}$$

It follows that

$$\begin{aligned}&\bigl|\mathbb{E}\bigl(f\bigl(Q_{N,k_*}(c,X)\bigr)\bigr) - \mathbb{E}\bigl(f\bigl(Q_{N,k_*}(d,Y)\bigr)\bigr|\\ &\quad\le C\max\bigl(1, \bigl(|c|_m^{-\frac{2}{k_* m}} + |d|_{m'}^{-\frac{2}{k_* m'}}\bigr)^a\bigr)\|f\|_\infty (1+|c|+|d|)^{5k}\\ &\qquad\times\biggl(\frac{|c|_{m+1,N}^{2ha}}{\eta^{ha}} + \frac{|d|_{m'+1,N}^{2ha}}{\eta^{ha}} + e_{m,N}^a(c) + e_{m',N}^a(d) + \eta^{a/(k_*\overline{m})} + \eta^{-2k\lambda} d_k^\lambda\biggr).\end{aligned}$$

We optimize now on η: we take $\eta = \mathfrak{a}_k^{\lambda k_*\overline{m}/(a+2\lambda k k_*\overline{m})}$, so that

$$\eta^{-2k\lambda} d_k^\lambda = \eta^{a/(k_*\overline{m})} = \mathfrak{a}_k^{\lambda a/(a+2\lambda k k_*\overline{m})} \le \mathfrak{a}_k^{\lambda a/(1+2kk_*\overline{m})},$$

the latter inequality follows from $\mathfrak{a}_k \le 1$ and, since $a, \lambda \in (0,1)$, $a + 2\lambda k k_*\overline{m} \le 1 + 2kk_*\overline{m}$. By inserting,

$$\begin{aligned}&\bigl|\mathbb{E}\bigl(f\bigl(Q_{N,k_*}(c,X)\bigr)\bigr) - \mathbb{E}\bigl(f\bigl(Q_{N,k_*}(d,Y)\bigr)\bigr|\\ &\quad\le C\max\bigl(1, \bigl(|c|_m^{-\frac{2}{k_* m}} + |d|_{m'}^{-\frac{2}{k_* m'}}\bigr)^a\bigr)\|f\|_\infty (1+|c|+|d|)^{5k}\\ &\qquad\times\biggl(\frac{|c|_{m+1,N}^{2ha}}{\eta^{ha}} + \frac{|d|_{m'+1,N}^{2ha}}{\eta^{ha}} + e_{m,N}^a(c) + e_{m',N}^a(d) + \mathfrak{a}_k^{\lambda a/(1+2kk_*\overline{m})}\biggr).\end{aligned}$$

Since $|c|_{m+1,N}^2 \le d_k^{\frac{k_*\overline{m}}{2kk_*\overline{m}+1}}$,

$$\frac{|c|_{m+1,N}^{2ha}}{\eta^{ha}} \le \mathfrak{a}_k^{ah(\frac{k_*\overline{m}}{1+2kk_*\overline{m}} - \frac{\lambda k_*\overline{m}}{a+2\lambda k k_*\overline{m}})}.$$

We note that the above exponent is positive because $a > \lambda$. So, we choose $h \ge 1$ and such that

$$ah\biggl(\frac{k_*\overline{m}}{1+2kk_*\overline{m}} - \frac{\lambda k_*\overline{m}}{a+2\lambda k k_*\overline{m}}\biggr) \ge \frac{\lambda a}{(1+2kk_*\overline{m})},$$

so that

$$\frac{|c|_{m+1,N}^{2ha}}{\eta^{ha}} \leq \mathfrak{a}_k^{\frac{\lambda a}{(1+2kk_*\overline{m})}}.$$

A similar estimate holds with $|c|_{m+1,N}^{2ha}$ replaced by $|d|_{m'+1,N}^{2ha}$. We then obtain

$$\begin{aligned}
&\big|\mathbb{E}\big(f\big(Q_{N,k_*}(c,X)\big)\big) - \mathbb{E}\big(f\big(Q_{N,k_*}(d,Y)\big)\big|\\
&\quad \leq C \max\big(1, \big(|c|_m^{-\frac{2}{k_*m}} + |d|_{m'}^{-\frac{2}{k_*m'}}\big)^a\big)\|f\|_\infty\big(1+|c|+|d|\big)^{5k}\\
&\qquad \times \big(e_{m,N}^a(c) + e_{m',N}^a(d) + \mathfrak{a}_k^{\lambda a/(1+2kk_*\overline{m})}\big).
\end{aligned}$$

The statement now follows by recalling that $\lambda a = \theta$ and, from (3.18), $\mathfrak{d}_k \leq C(d_k(Q_{N,k_*}(c,X), Q_{N,k_*}(d,Y)) + |c|_{m+1,N}^{\frac{2(2kk_*\overline{m}+1)}{k_*\overline{m}}} + |d|_{m'+1,N}^{\frac{2(2kk_*\overline{m}+1)}{k_*\overline{m}}})$.

APPENDIX: AN ITERATED HOEFFDING'S INEQUALITY

In this section, we work with multi-indexes $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ with $1 \leq \alpha_1 < \cdots < \alpha_m$ and we look to

$$\Psi_m(c^2) = \sum_{|\alpha|=m} c^2(\alpha)\chi^\alpha,$$

where χ_n, $n \in \mathbb{N}$, denote independent Bernoulli random variables and $\chi^\alpha = \prod_{i=1}^m \chi_{\alpha_i}$. We denote

$$|c|_m^2 = \sum_{|\alpha|=m} c^2(\alpha) \quad \text{and} \quad \delta_m^2(c) = \max_n \sum_{|\alpha|=m, n\in\alpha} c^2(\alpha).$$

LEMMA A.1. *Let $p = \mathbb{P}(\chi_j = 1) \in (0,1)$. If*

$$x < \left(\frac{p}{2}\right)^N |c|_N^2 \tag{A.1}$$

then

$$\mathbb{P}\big(\Psi_N(c^2) \leq x\big) \leq \frac{2e^3}{9} N \exp\left(-\frac{x^2}{\delta_N^2(c)|c|_N^2}\right). \tag{A.2}$$

PROOF. We proceed by recurrence on N. If $N = 1$, we have

$$\begin{aligned}
\mathbb{P}\big(\Psi_N(c^2) \leq x\big) &= \mathbb{P}\left(\sum_n c^2(n)\chi_n \leq x\right)\\
&\leq \mathbb{P}\left(p\sum_n c^2(n) \leq 2x\right) + \mathbb{P}\left(\sum_n c^2(n)(p-\chi_j) \geq x\right)\\
&= \mathbb{P}\left(\sum_n c^2(n)(p-\chi_j) \geq x\right),
\end{aligned}$$

the latter inequality following from (A.1). By Hoeffding's inequality,

$$\mathbb{P}\Big(\sum_j c^2(j)(p-\chi_j) \geq x\Big) \leq \exp\Big(-\frac{2x^2}{\sum_j c^4(j)}\Big).$$

Since

$$\sum_j c^4(j) \leq \max_j c^2(j) \times \sum_j c^2(j) = \delta_1^2(c)|c|_1^2$$

(A.2) follows for $N=1$. We suppose now that (A.2) holds for $N-1$ and we prove it for N. For β with $|\beta| = N-1$, we define $c_n(\beta) = c(\beta, n)1_{\{\beta_{N-1}<n\}}$ and we write

$$\begin{aligned}\Psi_N(c^2) &= \sum_{|\alpha|=N} c^2(\alpha)\chi^\alpha = \sum_{n=N}^{\infty} \chi_n \sum_{|\beta|=N-1,\beta_{N-1}<n} c^2(\beta,n)\chi^\beta \\ &= \sum_{n=N}^{\infty} \chi_n \Psi_{N-1}(c_n^2).\end{aligned}$$

Then

$$\begin{aligned}&\mathbb{P}(\Psi_N(c^2) \leq x) \\ &\quad \leq \mathbb{P}\Big(\sum_{n=N}^{\infty} \Psi_{N-1}(c_n^2) \leq \frac{2x}{p}\Big) + \mathbb{P}\Big(\sum_{n=N}^{\infty} (p-\chi_n)\Psi_{N-1}(c_n^2) \geq x\Big) =: a+b.\end{aligned}$$

We estimate first b. We write

$$\sum_{n=N}^{\infty} \Psi_{N-1}(c_n^2) = \sum_{|\beta|=N-1} d_n^2(\beta)\chi^\beta \quad \text{with } d^2(\beta) = \sum_{n>\beta_{N-1}}^{\infty} c^2(\beta,n).$$

Notice that

$$|d|_{N-1}^2 = \sum_{|\beta|=N-1} \sum_{n>\beta_{N-1}}^{\infty} c^2(\beta,n) = \sum_{|\alpha|=N} c^2(\alpha) = |c|_N^2$$

and

$$\begin{aligned}\delta_{N-1}^2(d) &= \max_k \sum_{|\alpha|=N-1,k\in\alpha} d^2(\alpha) = \max_k \sum_{|\alpha|=N-1,k\in\alpha} \sum_{n>\alpha_{N-1}}^{\infty} c^2(\alpha,n) \\ &\leq \max_k \sum_{|\beta|=N,k\in\beta} c^2(\beta) = \delta_N^2(c).\end{aligned}$$

We also have

$$\frac{2x}{p} < \frac{2}{p}\Big(\frac{p}{2}\Big)^N |c|_N^2 = \Big(\frac{p}{2}\Big)^{N-1} |d|_N^2$$

so we can use the recurrence hypothesis and we get

$$\begin{aligned} b &= \mathbb{P}\left(\Psi_{N-1}(d^2) \le \frac{2x}{p}\right) \le \frac{2e^3}{9}(N-1)\exp\left(-\frac{(2x/p)^2}{\delta^2_{N-1}(d)|d|^2_{N-1}}\right) \\ &\le \frac{2e^3}{9}(N-1)\exp\left(-\frac{x^2}{\delta^2_N(c)|c|^2_N}\right). \end{aligned} \tag{A.3}$$

We estimate now a. We use Corollary 1.4, page 1654 in Bentkus [8], which asserts the following: if $M_k, k \in \mathbb{N}$ is a martingale such that $|M_k - M_{k-1}| \le h_k$ almost surely, then for every $n \in \mathbb{N}$,

$$\mathbb{P}(M_n \ge x) \le \frac{2e^3}{9}\exp\left(-\frac{x^2}{\sum_{j=1}^n h_j^2}\right). \tag{A.4}$$

Since $0 \le \chi_n \le 1$, we have

$$\Psi_{N-1}(c_n^2) \le \sum_{|\beta|=n, \beta_{N-1}<n} c^2(\beta, n) =: h_n.$$

Notice that $h_n \le \delta^2_N(c)$ so that

$$\sum_{j=1}^n h_j^2 \le \delta^2_N(c)\sum_{j=1}^n h_j = \delta^2_N(c)|c|^2_N.$$

So, using (A.4),

$$a = \mathbb{P}\left(\sum_{j=1}^\infty (p-\chi_j)\Psi_{N-1}(c_n) \ge x\right) \le \frac{2e^3}{9}\exp\left(-\frac{x^2}{\delta^2_N(c)|c|^2_N}\right).$$

This, together with (A.3), gives (A.2). □

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LAMA (UMR CNRS, UPEMLV, UPEC),
MATHRISK INRIA,
UNIVERSITÉ PARIS-EST
F-77454 MARNE-LA-VALLÉE
FRANCE
E-MAIL: bally@univ-mlv.fr

DIP. MATEMATICA AND INDAM-GNAMPA
UNIVERSITÀ DI ROMA "TOR VERGATA"
VIA DELLA RICERCA SCIENTIFICA 1
I-00133 ROMA
ITALY
E-MAIL: caramell@mat.uniroma2.it