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STRONG DIFFERENTIAL SUBORDINATES FOR NONCOMMUTATIVE SUBMARTINGALES

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We introduce a notion of strong differential subordination of noncommutative semimartingales, extending Burkholder's definition from the classical case (Ann. Probab. **22** (1994) 995–1025). Then we establish the maximal weak-type $(1, 1)$ inequality under the additional assumption that the dominating process is a submartingale. The proof rests on a significant extension of the maximal weak-type estimate of Cuculescu and a Gundy-type decomposition of an arbitrary noncommutative submartingale. We also show the corresponding strong-type (p, p) estimate for $1 < p < \infty$ under the assumption that the dominating process is a nonnegative submartingale. This is accomplished by combining several techniques, including interpolation-flavor method, Doob–Meyer decomposition and noncommutative analogue of good-λ inequalities.

1. Introduction. In the classical probability theory estimates for semimartingales and their strong differential subordinates have not only been of interest in their own right but have also found crucial applications in other fields, such as stochastic analysis, harmonic analysis and the theory of quasiconformal mappings. To explain the motivation and the main results of this paper, we first recall several estimates obtained by Burkholder [9, 10] and Hammack [13]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\geq 0}$ be a nondecreasing sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F} = \bigvee_n \mathcal{F}_n$. Assume that $f = (f_n)_{n\geq 0}$ and $g = (g_n)_{n\geq 0}$ are two adapted sequences of integrable random variables with the corresponding differences $df = (df_n)_{n\geq 0}$, $dg = (dg_n)_{n\geq 0}$ given by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geq 1$ (with an analogous formula for dg). Consider the following two conditions:

(DS) for any $n \geq 0$ we have $|dg_n| \leq |df_n|$ almost surely;
(CDS) for any $n \geq 1$ we have $|\mathbb{E}_{n-1}(dg_n)| \leq |\mathbb{E}_{n-1}(df_n)|$ almost surely,

where for any nonnegative integer n, the symbol $\mathbb{E}_n$ stands for the conditional expectation with respect to the σ-field $\mathcal{F}_n$. If the requirement (DS) is satisfied,

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then g is said to be differentially subordinate to f. If (CDS) holds true, then g is conditionally differentially subordinate to f. If both (DS) and (CDS) are satisfied, then g is strongly differentially subordinate to f.

The strong differential subordination implies many interesting estimates if we impose some additional structure on the dominating process. Let us consider a very typical case where the dominating process f is a martingale. In such a case the condition (CDS) enforces g to be a martingale as well, and the strong differential subordination reduces to the requirement (DS). Working under this assumption, Burkholder [9] proved the following sharp weak-type $(1, 1)$ and strong-type (p, p) estimates:

$$\mathbb{P}\Big(\sup_{n\geq 0} |g_n| \geq 1\Big) \leq 2\|f\|_1 \tag{1.1}$$

and

$$\|g\|_p \leq \big(p^* - 1\big)\|f\|_p, \quad 1 < p < \infty, \tag{1.2}$$

where $p^* = \max\{p, p/(p-1)\}$. These celebrated estimates have been extended in numerous directions; see the monograph [30] for an up-to-date exposition on this subject. Such estimates have been also found applications in harmonic analysis, functional analysis and the theory of quasiconformal mappings; we refer the reader to [1–4, 7, 31, 32] and the references therein.

The strong differential subordination can also be exploited under slightly weaker assumptions on the dominated process. As is shown by Burkholder in [10], if f is assumed to be a nonnegative submartingale and g is strongly differentially subordinate to f, then the weak-type $(1, 1)$ and strong-type (p, p) estimates also hold true. More precisely, we have

$$\mathbb{P}\Big(\sup_{n\geq 0} |g_n| \geq 1\Big) \leq 3\|f\|_1 \tag{1.3}$$

and

$$\|g\|_p \leq \big(p^{**} - 1\big)\|f\|_p, \quad 1 < p < \infty, \tag{1.4}$$

where $p^{**} = \max\{2p, p/(p-1)\}$. Again, the constants in the estimates above are optimal. A few years later Hammack [13] generalized the weak-type inequality (1.3) to the setting of arbitrary submartingales (i.e., with no assumption on the sign of the dominating process) and then proved that the optimal constant increases to 6. Furthermore, by constructing appropriate examples he showed that there is no version of (1.4) in this more general context.

In this paper our main interest is to investigate the strong differential subordinations in the context of noncommutative (or quantum) probability. Motivated by quantum physics, the theory of noncommutative probability has enjoyed considerable progress in recent years. As a branch of this theory, the study of noncommutative semimartingale inequalities has gained a lot of interest in the last 20 years. Starting with the seminal paper of Pisier and Xu [34], where the appropriate counterparts of Burkholder–Gundy inequalities were proposed, many classical estimates have been successfully extended to the noncommutative realm. These include the noncommutative analogue of Doob's maximal L^p estimate obtained by Junge [22], noncommutative Burkholder/Rosenthal inequalities investigated by Junge and Xu [23, 25] as well as appropriate weak-type versions of the above results due to Randrianantoanina [36–38]. We also refer to [5, 6, 11, 14–16, 20, 21, 39–41] and references therein for recent progress on this topic. In particular recently, the notion of differential subordination of martingales was generalized to the noncommutative case by the authors in [18, 19]. Via some new ideas and novel approaches, they established the weak-type $(1, 1)$ estimate and the strong-type (p, p) inequalities with constants of optimal orders as $p \to 1$ and $p \to \infty$. These results actually are noncommutative extensions of (1.1) and (1.2). As we have mentioned before, there are submartingale versions of (1.1) and (1.2)—namely, (1.3) for general submartingales (with constant 6) and (1.4) for nonnegative submartingales. This leads us to the question whether these results hold true in the noncommutative setting under some appropriate assumptions. This is exactly the main problem we plan to solve in this paper.

As we will see, the study of noncommutative submartingales and their strong differential subordinations requires the development of new methods and techniques. In the classical setting the so-called Bellman function method plays an important role. However, such method is no longer effective in our case. We should also mention that most of the arguments, which are typically used in the noncommutative setting (e.g., standard interpolation, duality), cannot be successfully applied here, or their efficiency is limited. We believe that the approach we present considerably extends the machinery which can be used in the theory of noncommutative (semi)martingales, and its appropriate modifications might play an important role in the further exploration of the subject.

Let us briefly describe the structure of the paper.

The background on noncommutative semimartingale theory, which is necessary for the treatment of the above problems, is presented in Section 2. We also provide there the noncommutative version of the strong differential subordination and discuss some of its properties.

In Section 3 we establish the maximal weak-type $(1, 1)$ estimate for arbitrary noncommutative submartingales and their strong differential subordinates which provides a noncommutative analogue of Hammack's result. This is accomplished by combining two novel ingredients. First, we construct families of certain projections (see (3.11) below) based on a significant extension of Cuculescu's weak-type estimate (see [12]). Second, we provide an appropriate modification of noncommutative Gundy-type decomposition due to Parcet and Randrianantoanina [33]. Both of these topics are of independent interest and seem to deserve further research.

Section 4 is devoted to the noncommutative extension of (1.4). Quite interestingly, we will employ the Doob–Meryer decomposition and have to split the reasoning into two parts corresponding to $1 < p \leq 2$ and $p > 2$, in which our methods will be quite different. In the case $1 < p \leq 2$, we use a certain adaptation of Gundy-type decomposition and exploit arguments which can be interpreted as a version of real interpolation. In a sense we will study a behavior of the K-functional associated with the subordinates. For $p > 2$, our approach depends heavily on the recent advance on noncommutative analogue of good-λ inequalities. In the classical case this extrapolation technique was introduced by Burkholder in [8], and it has turned out to be very powerful in a number of problems arising in harmonic analysis and probability. The noncommutative counterpart of this method, recently obtained by the authors in [17], allows us to obtain a submartingale version of the noncommutative Doob's inequality (see Lemma 4.3 and Theorem 4.15, or Remark 4.17(i)). This estimate, combined with a certain novel L^p bound for submartingale differences (see Theorem 4.12), yields the moment inequality for the strong subordinates in the range $p > 2$. Finally, we conclude the paper with two interesting byproducts—the Doob's maximal inequality and Burkholder/Rosenthal inequality for nonnegative submartingales.

2. Preliminaries. Throughout the paper we use standard notation from the theory of operator algebras; we refer the reader to [26, 27, 42] for the detailed exposition. Let H be a given Hilbert space and denote by $B(H)$ the algebra of all bounded operators acting on H. Let $\mathcal{M}$ be a von Neumann subalgebra of $B(H)$ equipped with a semifinite normal faithful trace τ. A closed densely defined operator a on H is said to be affiliated with $\mathcal{M}$ if $u^*au = a$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. Such an operator is said to be τ-measurable, if for any $\varepsilon > 0$ there exists a projection e contained in its domain, satisfying $\tau(I - e) < \varepsilon$ (here and in what follows, the letter I stands for the identity operator). The set of all τ-measurable operators will be denoted by $L^0(\mathcal{M}, \tau)$. The trace τ can be extended to a positive tracial functional on the positive part $L^0_+(\mathcal{M}, \tau)$ of $L^0(\mathcal{M}, \tau)$, and this extension is still denoted by τ. Suppose that a is a self-adjoint τ-measurable operator, and let $a = \int_{-\infty}^{\infty} \lambda \, de_\lambda$ stand for its spectral decomposition. For any Borel subset B of $\mathbb{R}$, the spectral projection of a corresponding to the set B is defined by $I_B(a) = \int_{-\infty}^{\infty} \chi_B(\lambda) \, de_\lambda$. Sometimes, with no risk of confusion, we will write $\tau(a \in B)$ instead of $\tau(I_B(a))$.

For $0 < p < \infty$, we recall that the noncommutative L^p-space associated with $(\mathcal{M}, \tau)$ is defined by $L^p(\mathcal{M}, \tau) = \{x \in L^0(\mathcal{M}, \tau) : \tau(|x|^p) < \infty\}$ equipped with the (quasi-)norm $\|x\|_p = (\tau(|x|^p))^{1/p}$, where $|x| = (x^*x)^{1/2}$ is the modulus of x. For $p = \infty$, the space $L^p(\mathcal{M}, \tau)$ coincides with $\mathcal{M}$ with its usual operator norm. We refer to the survey [35] and the references therein for more details.

The main subject of this paper is the theory of noncommutative semimartingales. Let us now present the general setup. Assume that $(\mathcal{M}_n)_{n \geq 0}$ is a filtration, that is, a nondecreasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union is weak*-dense in $\mathcal{M}$. Then for any $n \geq 0$, there exists a normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ which satisfies the following two conditions:

(i) $\mathcal{E}_n(axb) = a\mathcal{E}_n(x)b$ for all $a, b \in \mathcal{M}_n$ and $x \in \mathcal{M}$;
(ii) $\tau \circ \mathcal{E}_n = \tau$.

It can be verified readily that the conditional expectations enjoy the property $\mathcal{E}_m\mathcal{E}_n = \mathcal{E}_n\mathcal{E}_m = \mathcal{E}_{\min(m,n)}$ for all nonnegative integers m and n. Furthermore, the operator $\mathcal{E}_n$ is trace preserving, and hence it can be extended to a contractive projection from $L^p(\mathcal{M}, \tau)$ onto $L^p(\mathcal{M}_n, \tau_n)$ for all $1 \le p \le \infty$; here τ_n denotes the restriction of τ to $\mathcal{M}_n$.

A sequence $x = (x_n)_{n\ge 0}$ in $L^1(\mathcal{M})$ is called a noncommutative martingale (respectively, submartingale or supermartingale) adapted to $(\mathcal{M}_n)_{n\ge 0}$ if for any $n \ge 0$ we have

$$\mathcal{E}_n(x_{n+1}) = x_n$$

(respectively, $\mathcal{E}_n(x_{n+1}) \ge x_n$ or $\mathcal{E}_n(x_{n+1}) \le x_n$). Note that the sub- and supermartingales need to consist of self-adjoint operators, so that the inequalities make sense. The associated difference sequence is defined, as in the commutative case, with the use of the formulae $dx_0 = x_0$ and $dx_n = x_n - x_{n-1}$ for $n \ge 1$. Sometimes, we will exploit the notation

$$\|x\|_p = \sup_{n\ge 0} \|x_n\|_p, \quad 0 < p < \infty,$$

for the pth norm of the sequence x. In the paper we will mostly deal with finite martingales $x = (x_n)_{n=0}^N$ (i.e., consisting of a finite number of operators).

In what follows we will need the so-called Burkholder/Rosenthal inequalities for finite, self-adjoint martingales (see [23] for the general statement). Suppose that $x = (x_n)_{n=0}^N$ is such a sequence with terms belonging to $L^2(\mathcal{M})$. We define the associated conditional square function $s_N(x)$ by the formula

$$s_N(x) = \left(\sum_{n=0}^N \mathcal{E}_{n-1}(dx_n^2)\right)^{1/2}.$$

Then for any $p \ge 2$ there exists a constant c_p depending only on p such that

$$c_p^{-1}\|x_N\|_p \le \|s_N(x)\|_p + \left(\sum_{n=0}^N \|dx_n\|_p^p\right)^{1/p} \le c_p\|x_N\|_p.$$

Furthermore, c_p can be taken to be of order $O(p)$ as $p \to \infty$ (see [38]).

We are ready to introduce the domination principle under which we will work in this paper.

DEFINITION 2.1. Suppose that $x = (x_n)_{n\ge 0}$, $y = (y_n)_{n\ge 0}$ are self-adjoint adapted sequences in $L^1(\mathcal{M})$. We say that y is strongly differentially subordinate to x, if the following two conditions are satisfied:

(DS) for any $n \geq 0$ and any projection $R \in \mathcal{M}_{n-1}$ we have

$$Rdy_n Rdy_n R \leq Rdx_n Rdx_n R;$$

(CDS) for any $n \geq 1$ we have

$$-\left|\mathcal{E}_{n-1}(dx_n)\right| \leq \mathcal{E}_{n-1}(dy_n) \leq \left|\mathcal{E}_{n-1}(dx_n)\right|.$$

Observe that in the commutative case this reduces to the usual definition of strong differential subordination formulated in the introductory section. Furthermore, note that the condition (CDS) is slightly weaker than the requirement

$$\left|\mathcal{E}_{n-1}(dy_n)\right| \leq \left|\mathcal{E}_{n-1}(dx_n)\right| \quad \text{for each } n \geq 1.$$

There is a weaker version of the differential subordination (i.e., the condition (DS)) which will also be of importance:

(WDS) for any $n \geq 0$ we have $dy_n^2 \leq dx_n^2$.

It is obvious that the condition (DS) implies that (WDS), and the converse is false. We refer the reader to [18], Lemma 3.3, for a detailed comparison on the conditions (DS) and (WDS).

We will show that L^p-estimates in the range $p \geq 2$ hold true under the weaker assumption (WDS) + (CDS). On the other hand, as the authors exhibited in [18], this weaker set of conditions is not sufficient for the validity of L^p estimates in the range $1 < p < 2$ even in the martingale setting. This justifies the use of the more complicated requirement (DS) in the case of $1 < p < 2$.

3. A maximal weak-type estimate. We will now handle a maximal weak-type $(1, 1)$ estimate for strong differential subordinates of arbitrary (not necessarily nonnegative) noncommutative submartingales which provides a noncommutative version of (1.3). Recall that (1.3) was firstly proved by Burkholder in [10] for nonnegative submartingales and then generalized to the general case by Hammack in [13]. The following is the precise statement.

THEOREM 3.1. *Let $x = (x_n)_{n\geq 0}$ be an arbitrary submartingale and suppose that y is strongly differentially subordinate to x. Then there exists a projection q satisfying*

$$-q \leq qy_nq \leq q \quad \textit{for all } n \tag{3.1}$$

and such that

$$\tau(I - q) \leq 327\|x\|_1. \tag{3.2}$$

Two important observations are in order. In the commutative case it is easy to see that the largest projection q satisfying (3.1) is precisely the indicator function of the set $\{\sup_{n\geq 0}|y_n|\leq 1\}$, and then (3.2) becomes

$$\mathbb{P}\Big(\sup_{n\geq 0}|y_n|>1\Big)\leq 327\|x\|_1.$$

This explains why we refer to (3.2) as to a maximal weak-type bound. The second comment is that the above result holds true in the particular case when x is a martingale. Thus, Theorem 3.1 generalizes the main result of [29], as it provides an estimate for a wider class of processes and under a weaker domination requirement.

We start with introducing certain families $(R_n)_{n\geq -1}$, $(D_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$ of projections which will play a key role in our considerations below. For an arbitrary submartingale $x=(x_n)_{n\geq 0}$, define $R_{-1}=I$ and for $n\geq 0$ inductively,

$$\begin{aligned}
R_n &= R_{n-1}I_{(-1,1)}(R_{n-1}x_nR_{n-1}),\\
D_n &= I_{[1,\infty)}(R_{n-1}x_nR_{n-1}),\\
U_n &= I_{(-\infty,-1]}(R_{n-1}x_nR_{n-1}).
\end{aligned}$$

Crucial properties of these objects, to be needed later, are gathered in the next lemma.

LEMMA 3.2. *Let $x=(x_n)_{n\geq 0}$ be an L^1-bounded submartingale. Then the following statements hold true*:

(i) *for each $n\geq 0$ the projections R_n, U_n and D_n belong to $\mathcal{M}_n$ and $R_n+U_n+D_n=R_{n-1}$*;

(ii) *for each $n\geq 0$, the projections R_n, U_n and D_n commute with $R_{n-1}x_nR_{n-1}$*;

(iii) *for each $n\geq 0$ we have*

$$-R_n\leq R_nx_nR_n\leq R_n,\qquad U_nx_nU_n\leq -U_n,\qquad D_nx_nD_n\geq D_n;$$

(iv) *for any $N\geq 0$ we have*

$$\tau(I-R_N)\leq 2\tau\big(x_N^+\big)-\tau(x_0).$$

REMARK 3.3. The expression on the right-hand side of (iv) can be bounded from above by a simple $3\|x\|_1$. However, we have decided to keep the above formulation. It should be stressed that both terms $\tau(x_0)$ and $\tau(x_N^+)$ (i.e., the measurements of the size of the starting and the terminating operator of x) are necessary due to the submartingale structure of x (cf. [13] for a similar phenomenon in the classical case). This should be contrasted with the martingale setting, where both terms could be replaced by the expression $\tau(|x_N|)$ involving just the terminating operator. A similar remark applies to three lemmas below.

PROOF OF LEMMA 3.2. The first three properties are evident, and the main difficulty lies in proving (iv). Note that for any $n \geq 1$ we have, by the submartingale property of x, the tracial property of τ and part (i) above,

$$\begin{aligned} \tau(R_{n-1}x_{n-1}R_{n-1}) &\leq \tau(R_{n-1}x_nR_{n-1}) \\ &= \tau(R_{n-1}x_n) \\ &= \tau(R_nx_n) + \tau(D_nx_n) + \tau(U_nx_n) \\ &= \tau(R_nx_nR_n) + \tau(D_nx_nD_n) + \tau(U_nx_nU_n). \end{aligned} \tag{3.3}$$

Now by Lemma 3.2(iii) we have $\tau(U_nx_nU_n) \leq -\tau(U_n)$ and

$$\tau(D_nx_nD_n) \leq 2\tau(D_nx_nD_n) - \tau(D_n) \leq 2\tau(D_nx_ND_n) - \tau(D_n),$$

where in the last passage we have exploited the submartingale property. Putting all the above facts together, we see that we have proved that

$$\begin{aligned} \tau(R_{n-1}x_{n-1}R_{n-1}) - \tau(R_nx_nR_n) &\leq 2\tau(D_nx_ND_n) - \tau(D_n) - \tau(U_n) \\ &= 2\tau(D_nx_N) - \tau(R_{n-1} - R_n). \end{aligned}$$

Summing over all $1 \leq n \leq N$, we get

$$\tau(R_0x_0R_0) - \tau(R_Nx_NR_N) \leq 2\tau\left(\sum_{n=1}^N D_nx_N\right) - \tau(R_0 - R_N).$$

Adding to this estimate the trivial bounds $\tau(U_0x_0U_0) \leq -\tau(U_0)$, $\tau(D_0x_0D_0) \geq \tau(D_0)$, we obtain that

$$\begin{aligned} \tau(x_0) &= \tau(R_0x_0R_0) + \tau(U_0x_0U_0) + \tau(D_0x_0D_0) \\ &\leq \tau\left(\left(R_N + 2\sum_{n=0}^N D_n\right)x_N\right) - \tau(R_0 - R_N) - \tau(U_0) - \tau(D_0x_0D_0) \\ &\leq \tau\left(\left(R_N + 2\sum_{n=0}^N D_n\right)x_N^+\right) - \tau(I - R_N) \leq 2\tau(x_N^+) - \tau(I - R_N), \end{aligned}$$

where in the second passage we used the fact $\tau(D_0x_0D_0) \leq \tau(D_0x_ND_0)$ (which is due to the submartingale property). □

We will also need the following further properties of the projections $(R_n)_{n\geq -1}$.

LEMMA 3.4. *Let $x = (x_n)_{n\geq 0}$ be an L^1-bounded submartingale. Then, for any nonnegative integer N, we have*

$$\sum_{n=0}^N \tau(R_ndx_nR_{n-1}dx_n) \leq 4\tau(x_N^+) - 2\tau(x_0).$$

PROOF. Let us first study a single summand of the above sum corresponding to some $n \geq 1$. We have

$$\begin{aligned}\tau(R_n dx_n R_{n-1} dx_n) &= \tau\big(R_n(x_n - x_{n-1})R_{n-1}(x_n - x_{n-1})\big)\\ &= \tau(R_n x_n R_{n-1} x_n) + \tau(R_n x_{n-1} R_{n-1} x_{n-1})\\ &\quad - \tau(R_n x_n R_{n-1} x_{n-1}) - \tau(R_n x_{n-1} R_{n-1} x_n).\end{aligned}$$

The last two summands are equal by the tracial property and the fact that R_n commutes with $R_{n-1} x_n R_{n-1}$ (which implies $R_n x_n R_{n-1} = R_{n-1} x_n R_n$). This commuting property of R_n implies also that $\tau(R_n x_n R_{n-1} x_n) = \tau(R_n x_n R_n x_n)$. Furthermore, because of the traciality of τ and the inequality $R_{n-1} \geq R_n$, we see that

$$\begin{aligned}\tau(R_n x_{n-1} R_{n-1} x_{n-1}) &= \tau(R_{n-1} x_{n-1} R_n x_{n-1} R_{n-1})\\ &\leq \tau(R_{n-1} x_{n-1} R_{n-1} x_{n-1} R_{n-1})\end{aligned}$$

and hence we may write

$$\begin{aligned}&\tau(R_n dx_n R_{n-1} dx_n)\\ &\quad \leq \tau(R_n x_n R_n x_n) - \tau(R_{n-1} x_{n-1} R_{n-1} x_{n-1})\\ &\qquad + 2\tau\big(R_{n-1} x_{n-1} R_{n-1}(R_{n-1} x_{n-1} R_{n-1} - R_n x_n R_n)\big).\end{aligned} \tag{3.4}$$

Let us handle the latter expression. Since $R_n x_n R_n = R_{n-1} x_n R_n$, we have the splitting

$$\begin{aligned}&\tau\big(R_{n-1} x_{n-1} R_{n-1}(R_{n-1} x_{n-1} R_{n-1} - R_n x_n R_n)\big)\\ &\quad = \tau\big(R_{n-1} x_{n-1} R_{n-1}(x_{n-1} - x_n) R_{n-1}\big)\\ &\qquad + \tau\big(R_{n-1} x_{n-1} R_{n-1} x_n (R_{n-1} - R_n)\big).\end{aligned} \tag{3.5}$$

We have $x_{n-1} - x_n = -dx_n$, so by the properties of conditional expectation

$$\tau\big(R_{n-1} x_{n-1} R_{n-1}(x_{n-1} - x_n) R_{n-1}\big) = \tau\big(R_{n-1} x_{n-1} R_{n-1}\big(-\mathcal{E}_{n-1}(dx_n)\big)\big).$$

Lemma 3.2(iii) gives $R_{n-1} x_{n-1} R_{n-1} \geq -R_{n-1}$; furthermore, x is a submartingale, so $\mathcal{E}_{n-1}(dx_n) \geq 0$. These two observations imply that the above expression does not exceed

$$\begin{aligned}\tau\big(R_{n-1}\mathcal{E}_{n-1}(dx_n)\big) &= \tau(R_{n-1} dx_n R_{n-1})\\ &= \tau(R_{n-1} x_n R_{n-1}) - \tau(R_{n-1} x_{n-1} R_{n-1})\\ &= \tau(R_n x_n) + \tau(U_n x_n) + \tau(D_n x_n) - \tau(R_{n-1} x_{n-1})\\ &\leq \tau(R_n x_n) + \tau(U_n x_n) + \tau(D_n x_N) - \tau(R_{n-1} x_{n-1}),\end{aligned} \tag{3.6}$$

where in the last line we have exploited the submartingale property of x. Let us now analyze the second term on the right-hand side of (3.5). We have $R_{n-1} - R_n =$

$U_n + D_n$, so, by the commuting properties of U and D described in Lemma 3.2, we obtain

$$\begin{aligned}&\tau\bigl(R_{n-1}x_{n-1}R_{n-1}x_n(R_{n-1}-R_n)\bigr)\\&\quad=\tau(R_{n-1}x_{n-1}R_{n-1}D_nx_nD_n))+\tau(R_{n-1}x_{n-1}R_{n-1}U_nx_nU_n)).\end{aligned}$$

However, the operator $D_nx_nD_n$ is nonnegative, while $U_nx_nU_n$ is nonpositive; furthermore, we have $-R_{n-1}\le R_{n-1}x_{n-1}R_{n-1}\le R_{n-1}$. Consequently, the above expression does not exceed

$$\tau(D_nx_nD_n)-\tau(U_nx_nU_n)\le\tau(D_nx_N)-\tau(U_nx_n),$$

where the last passage is due to the fact that x is a submartingale. Plugging the above observations into (3.5), we get

$$\begin{aligned}&\tau\bigl(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1}-R_nx_nR_n)\bigr)\\&\quad\le\tau(R_nx_n)-\tau(R_{n-1}x_{n-1})+2\tau(D_nx_N)\end{aligned}$$

and hence, returning to (3.4), we have shown that

$$\begin{aligned}\tau(R_ndx_nR_{n-1}dx_n)\le{}&\tau(R_nx_nR_nx_n)-\tau(R_{n-1}x_{n-1}R_{n-1}x_{n-1})\\&+2\bigl(\tau(R_nx_n)-\tau(R_{n-1}x_{n-1})\bigr)+4\tau(D_nx_N).\end{aligned}$$

Consequently, using the equality $\tau(R_0dx_0R_{-1}dx_0)=\tau(R_0x_0R_0x_0)$, we get

$$\begin{aligned}&\sum_{n=0}^N\tau(R_ndx_nR_{n-1}dx_n)\\&\quad=\tau(R_0x_0R_{-1}x_0)+\sum_{n=1}^N\tau(R_ndx_nR_{n-1}dx_n)\\&\quad\le\tau(R_Nx_NR_Nx_N)+2\tau(R_Nx_N)-2\tau(R_0x_0)+4\tau\Biggl(\sum_{n=1}^ND_nx_N\Biggr).\end{aligned}\tag{3.7}$$

It remains to apply some final estimates. The operator $R_N(x_N+I)R_N$ is nonnegative and does not exceed $2R_N$ (see Lemma 3.2(iii)), so

$$\begin{aligned}\tau(R_Nx_NR_Nx_N)+\tau(R_Nx_N)&=\tau\bigl(R_Nx_NR_N(x_N+I)R_N\bigr)\\&\le\tau\bigl(R_Nx_N^+R_N(x_N+I)R_N\bigr)\\&\le2\tau\bigl(R_Nx_N^+R_N\bigr).\end{aligned}\tag{3.8}$$

Furthermore, $U_0x_0U_0$ is nonpositive, so, using the submartingale property,

$$\tau(R_Nx_N) - 2\tau(R_0x_0) + 4\tau\left(\sum_{n=1}^{N} D_nx_N\right)$$

$$\begin{aligned}
&= -2\tau(x_0) + 2\tau(D_0x_0) + 2\tau(U_0x_0) + \tau\left(\left(R_N + 4\sum_{n=1}^{N} D_n\right)x_N\right) \\
&\le -2\tau(x_0) + \tau\left(\left(R_N + 2D_0 + 4\sum_{n=1}^{N} D_n\right)x_N\right) \qquad (3.9) \\
&\le -2\tau(x_0) + \tau\left(\left(R_N + 4\sum_{n=0}^{N} D_n\right)x_N^+\right).
\end{aligned}$$

Combining the estimates (3.7), (3.8) and (3.9), we obtain the desired result. □

We conclude the analysis of $(R_n)_{n\ge 0}$ with the following statement.

LEMMA 3.5. *Let $x = (x_n)_{n\ge 0}$ be an L^1-bounded submartingale. Then, for any nonnegative integer N, we have*

$$\left\|\sum_{n=0}^{N}\left(\left|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)\right| + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\right)\right\|_1 \tag{3.10}$$

$$\le 4\tau\left(x_N^+\right) - 2\tau(x_0)$$

(*we interpret $\mathcal{E}_{-1}a = 0$ for each $a \in L^1$*).

PROOF. Let us analyze a single summand of the above sum (note that each such summand is nonnegative). First, we take a look at summands corresponding to $n \ge 1$. The trace of the term $R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}$ can be handled as in (3.6). To deal with the remaining part, we apply the triangle inequality to get

$$\begin{aligned}
&\tau\left(\left|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)\right|\right) \\
&\quad \le \tau\left(\left|(R_{n-1} - R_n)x_n(R_{n-1} - R_n)\right| + \left|(R_{n-1} - R_n)x_{n-1}(R_{n-1} - R_n)\right|\right).
\end{aligned}$$

Now, by the commuting properties of U and D, for any $n \ge 1$ we have

$$(R_{n-1} - R_n)x_n(R_{n-1} - R_n) = D_nx_nD_n + U_nx_nU_n.$$

The first term on the right is nonnegative while the second is nonpositive, so

$$\begin{aligned}
\tau\left(\left|(R_{n-1} - R_n)x_n(R_{n-1} - R_n)\right|\right) &\le \tau(D_nx_n) - \tau(U_nx_n) \\
&\le \tau(D_nx_N) - \tau(U_nx_n),
\end{aligned}$$

where in the last passage we have exploited the submartingale property. Next, we have the estimate

$$\big|(R_{n-1}-R_n)x_{n-1}(R_{n-1}-R_n)\big| \leq R_{n-1}-R_n,$$

directly from Lemma 3.2(iii). Thus, combining the above observations, we have shown that if $n \geq 1$, then

$$\begin{aligned}&\tau\big(\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big| + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\big)\\&\quad\leq \tau(R_nx_n)-\tau(R_{n-1}x_{n-1})+2\tau(D_nx_N)+\tau(R_{n-1}-R_n).\end{aligned}$$

A similar reasoning to that above yields also the appropriate upper bound for the first term in (3.10):

$$\begin{aligned}\tau\big(\big|(I-R_0)dx_0(I-R_0)\big|\big) &= \tau\big(\big|(I-R_0)x_0(I-R_0)\big|\big)\\&\leq \tau(D_0x_N)-\tau(U_0x_0).\end{aligned}$$

Summing over n, it follows from the fact $\tau(D_0x_0D_0) \leq \tau(D_0x_ND_0)$ (which is due to the submartingale property) that

$$\begin{aligned}&\left\|\sum_{n=0}^{N}\big(\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big| + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\big)\right\|_1\\&\quad= \sum_{n=0}^{N}\tau\big(\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big| + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}\big)\\&\quad= \tau(D_0x_N)-\tau(U_0x_0)\\&\qquad+\tau(R_Nx_N)-\tau(R_0x_0)+2\tau\left(\left(\sum_{n=1}^{N}D_n\right)x_N\right)+\tau(R_0-R_N)\\&\quad= \tau\left(\left(R_N+D_0+2\sum_{n=1}^{N}D_n\right)x_N\right)-\tau(x_0)+\tau(D_0x_0)+\tau(R_0-R_N)\\&\quad\leq \tau\left(\left(R_N+2\sum_{n=0}^{N}D_n\right)x_N\right)-\tau(x_0)+\tau(R_0-R_N)\\&\quad\leq 2\tau\big(x_N^+\big)-\tau(x_0)+\tau(I-R_N).\end{aligned}$$

It remains to apply Lemma 3.2(iv) to get the claim. $\square$

The next step of our analysis is to introduce yet another families of projections $(S_n)_{n\geq -1}$, $(Q_n)_{n\geq 0}$, $(T_n)_{n\geq 0}$; this time associated with the dominated process y.

Set $S_{-1} = I$ and for $n \geq 0$, by induction,

$$\begin{aligned} S_n &= S_{n-1} I_{(-1,1)}\left(S_{n-1}\left(\sum_{k=0}^{n} R_{k-1} dy_k R_{k-1}\right) S_{n-1}\right), \\ Q_n &= I_{[1,\infty)}\left(S_{n-1}\left(\sum_{k=0}^{n} R_{k-1} dy_k R_{k-1}\right) S_{n-1}\right), \\ T_n &= I_{(-\infty,-1]}\left(S_{n-1}\left(\sum_{k=0}^{n} R_{k-1} dy_k R_{k-1}\right) S_{n-1}\right). \end{aligned} \tag{3.11}$$

The crucial property of $(S_n)_{n \geq -1}$ is described in the next lemma. Before we proceed, let us introduce a Gundy-type decomposition for y:

$$dy_n = d\alpha_n + d\beta_n + d\gamma_n + d\delta_n,$$

where

$$\begin{aligned} d\alpha_n &= R_{n-1} dy_n R_n + R_n dy_n R_{n-1} - R_n dy_n R_n \\ &\quad - \mathcal{E}_{n-1}(R_{n-1} dy_n R_n + R_n dy_n R_{n-1} - R_n dy_n R_n), \\ d\beta_n &= \mathcal{E}_{n-1}(R_{n-1} dy_n R_n + R_n dy_n R_{n-1} - R_n dy_n R_n), \\ d\gamma_n &= R_n dy_n (I - R_{n-1}), \\ d\delta_n &= (I - R_n) dy_n - (R_{n-1} - R_n) dy_n R_n. \end{aligned}$$

LEMMA 3.6. *For any integer $N \geq -1$, we have*

$$\tau(I - S_N) \leq 216\tau(x_N^+) - 108\tau(x_0).$$

PROOF. For the sake of clarity, it is convenient to split the quite lengthy reasoning into a few intermediate parts.

Step 1 (Preliminary observations). We start with the identity

$$\tau(I - S_N) = \sum_{n=0}^{N} \tau(S_{n-1} - S_n) = \sum_{n=0}^{N} \big(\tau(Q_n) + \tau(T_n)\big). \tag{3.12}$$

Now, by the very definition of Q_n, we have

$$\begin{aligned} \tau(Q_n) &= \tau\left(Q_n\left(\sum_{k=0}^{n} R_{k-1} dy_k R_{k-1}\right) Q_n \geq 1\right) \\ &= \tau\left(Q_n\left(\sum_{k=0}^{n} R_{k-1}(d\alpha_k + d\beta_k + d\gamma_k + d\delta_k) R_{k-1}\right) Q_n \geq 1\right). \end{aligned} \tag{3.13}$$

Note that $R_{k-1}d\gamma_k R_{k-1} = 0$, $R_{k-1}d\alpha_k R_{k-1} = d\alpha_k$, $R_{k-1}d\beta_k R_{k-1} = d\beta_k$ and $R_{k-1}d\delta_k R_{k-1} = (R_{k-1} - R_k)dy_k(R_{k-1} - R_k)$. Consequently, the above expression is not bigger than

$$\begin{aligned}\tag{3.14} &\tau(Q_n\alpha_n Q_n \geq 1/3) + \tau(Q_n\beta_n Q_n \geq 1/3)\\ &\quad + \tau\left(Q_n\left(\sum_{k=0}^{n}(R_{k-1} - R_k)dy_k(R_{k-1} - R_k)\right)Q_n \geq 1/3\right).\end{aligned}$$

We will treat each of these three summands separately.

Step 2 (Bound for the summand involving α). By Chebyshev's inequality and the fact that α is an L^2-bounded martingale, we obtain

$$\tau(Q_n\alpha_n Q_n \geq 1/3) \leq 9\tau\big((Q_n\alpha_n Q_n)^2\big) \leq 9\tau\big(Q_n\alpha_n^2 Q_n\big) \leq 9\tau\big(Q_n\alpha_N^2\big).$$

Hence, summing over n and using the fact that the sum of Q_n's is not bigger than I, we get

$$\tag{3.15} \sum_{n=0}^{N}\tau(Q_n\alpha_n Q_n \geq 1/3) \leq 9\sum_{n=0}^{N}\tau\big(Q_n\alpha_N^2\big) \leq 9\tau\big(\alpha_N^2\big) = 9\sum_{n=0}^{N}\tau\big(d\alpha_n^2\big).$$

Directly from the definition of $d\alpha_n$ we infer that

$$\begin{aligned}\tag{3.16} \tau\big(d\alpha_n^2\big) &\leq \tau\big((R_{n-1}dy_n R_n + R_n dy_n R_{n-1} - R_n dy_n R_n)^2\big)\\ &= 2\tau(R_n dy_n R_{n-1} dy_n) - \tau(R_n dy_n R_n dy_n)\\ &\leq 2\tau(R_{n-1}dy_n R_n dy_n).\end{aligned}$$

Applying the differential subordination of y to x, we obtain

$$R_{n-1}dy_n R_{n-1}dy_n R_{n-1} \leq R_{n-1}dx_n R_{n-1}dx_n R_{n-1},$$

and hence also $R_n dy_n R_{n-1} dy_n R_n \leq R_n dx_n R_{n-1} dx_n R_n$, since $R_n \leq R_{n-1}$. Passing to the trace, this implies

$$\tau(R_{n-1}dy_n R_n dy_n) = \tau(R_n dy_n R_{n-1} dy_n R_n) \leq \tau(R_n dx_n R_{n-1} dx_n R_n).$$

Plugging this into (3.16) and then returning to (3.15), we obtain

$$\begin{aligned}\tag{3.17} \sum_{n=0}^{N}\tau(Q_n\alpha_n Q_n \geq 1/3) &\leq 18\tau\left(\sum_{n=0}^{N} R_n dx_n R_{n-1} dx_n R_n\right)\\ &\leq 72\tau\big(x_N^+\big) - 36\tau(x_0),\end{aligned}$$

where in the last line we exploited the estimate of Lemma 3.4.

Step 3 (Bound for the term involving β). Let us first find an appropriate upper bound for $d\beta_n$. We have

$$\begin{aligned}
d\beta_n &= \mathcal{E}_{n-1}(-R_{n-1}dy_nR_{n-1} + R_{n-1}dy_nR_n + R_ndy_nR_{n-1} - R_ndy_nR_n) \\
&\quad + R_{n-1}\mathcal{E}_{n-1}(dy_n)R_{n-1} \\
&= -\mathcal{E}_{n-1}\big((R_{n-1} - R_n)dy_n(R_{n-1} - R_n)\big) + R_{n-1}\mathcal{E}_{n-1}(dy_n)R_{n-1} \\
&\le -\mathcal{E}_{n-1}\big((R_{n-1} - R_n)dy_n(R_{n-1} - R_n)\big) + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1},
\end{aligned}$$

where in the last line we have exploited the conditional differential subordination of y to x. Next, by the differential subordination of y to x, we have

$$R_{n-1}dy_nR_{n-1}dy_nR_{n-1} \le R_{n-1}dx_nR_{n-1}dx_nR_{n-1},$$

so

$$\begin{aligned}
&(R_{n-1} - R_n)dy_n(R_{n-1} - R_n)dy_n(R_{n-1} - R_n) \\
&\quad \le (R_{n-1} - R_n)dy_nR_{n-1}dy_n(R_{n-1} - R_n) \\
&\quad \le (R_{n-1} - R_n)dx_nR_{n-1}dx_n(R_{n-1} - R_n) \\
&\quad = (R_{n-1} - R_n)(x_n - x_{n-1})R_{n-1}(x_n - x_{n-1})(R_{n-1} - R_n) \\
&\quad = (R_{n-1} - R_n)x_nR_{n-1}x_n(R_{n-1} - R_n) \\
&\qquad + (R_{n-1} - R_n)x_{n-1}R_{n-1}x_{n-1}(R_{n-1} - R_n) \\
&\qquad - (R_{n-1} - R_n)x_{n-1}R_{n-1}x_n(R_{n-1} - R_n) \\
&\qquad - (R_{n-1} - R_n)x_nR_{n-1}x_{n-1}(R_{n-1} - R_n).
\end{aligned}$$

Since R_n commutes with $R_{n-1}x_nR_{n-1}$, the above sum is equal to

$$\begin{aligned}
&(R_{n-1} - R_n)x_n(R_{n-1} - R_n)x_n(R_{n-1} - R_n) \\
&\quad + (R_{n-1} - R_n)x_{n-1}R_{n-1}x_{n-1}(R_{n-1} - R_n) \\
&\quad - (R_{n-1} - R_n)x_{n-1}(R_{n-1} - R_n)x_n(R_{n-1} - R_n) \\
&\quad - (R_{n-1} - R_n)x_n(R_{n-1} - R_n)x_{n-1}(R_{n-1} - R_n)
\end{aligned}$$

(note that the second summand has not changed) which can be further transformed into

$$\begin{aligned}
&(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)dx_n(R_{n-1} - R_n) \\
&\quad + (R_{n-1} - R_n)x_{n-1}R_nx_{n-1}(R_{n-1} - R_n).
\end{aligned} \tag{3.18}$$

Let us handle the second term in the latter expression. Since $R_n \le R_{n-1}$, we have

$$\begin{aligned}
&(R_{n-1} - R_n)x_{n-1}R_nx_{n-1}(R_{n-1} - R_n) \\
&\quad \le (R_{n-1} - R_n)x_{n-1}R_{n-1}x_{n-1}(R_{n-1} - R_n).
\end{aligned}$$

This is not bigger than $R_{n-1} - R_n$. Indeed, by Lemma 3.2(iii), we have the estimate $R_{n-1}x_{n-1}R_{n-1} \leq R_{n-1}$ which yields $R_{n-1}x_{n-1}R_{n-1}x_{n-1}R_{n-1} \leq R_{n-1}$ and hence also the desired inequality. This enables us to bound the expression in (3.18) from above by a convenient square,

$$\big(\big|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)\big| + R_{n-1} - R_n\big)^2.$$

Putting all the above facts together, we conclude that

$$\begin{aligned}&(R_{n-1} - R_n)dy_n(R_{n-1} - R_n)dy_n(R_{n-1} - R_n)\\ &\quad \leq \big(\big|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)\big| + R_{n-1} - R_n\big)^2,\end{aligned}$$

which implies

$$\begin{aligned}&\big|(R_{n-1} - R_n)dy_n(R_{n-1} - R_n)\big|\\ &\quad \leq \big|(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)\big| + R_{n-1} - R_n,\end{aligned} \tag{3.19}$$

and hence

$$\begin{aligned}d\beta_n \leq \mathcal{E}_{n-1}\big(\big|&(R_{n-1} - R_n)dx_n(R_{n-1} - R_n)\big|\\ &+ R_{n-1} - R_n\big) + R_{n-1}\mathcal{E}_{n-1}(dx_n)R_{n-1}.\end{aligned}$$

Denoting the right-hand side by $d\tilde{\beta}_n$, we see that $\beta_n \leq \tilde{\beta}_n$ and $\tilde{\beta}_n$ is nonnegative. Therefore, by Chebyshev's inequality, we get

$$\tau(Q_n\beta_nQ_n \geq 1/3) \leq \tau(Q_n\tilde{\beta}_nQ_n \geq 1/3) \leq 3\tau(Q_n\tilde{\beta}_nQ_n) \leq 3\tau(Q_n\tilde{\beta}_N),$$

and hence, summing over n, we arrive at

$$\begin{aligned}\sum_{n=0}^{N}\tau(Q_n\beta_nQ_n \geq 1/3) &\leq 3\tau(\tilde{\beta}_N)\\ &= 3\sum_{n=0}^{N}\tau(d\tilde{\beta}_n)\\ &\leq 12\tau\big(x_N^+\big) - 6\tau(x_0) + 3\tau(I - R_N)\\ &\leq 18\tau\big(x_N^+\big) - 9\tau(x_0),\end{aligned}$$

where the last two estimates follow from Lemma 3.2(iv) and Lemma 3.5.

Step 4 (Bound for the term involving δ). All the crucial observations have been made in the previous step. First, by Chebyshev's inequality, we have

$$\begin{aligned}
&\tau\left(Q_n\left(\sum_{k=0}^{n}(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right)Q_n\geq 1/3\right)\\
&\quad\leq \tau\left(Q_n\left(\sum_{k=0}^{n}\left|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right|\right)Q_n\geq 1/3\right)\\
&\quad\leq 3\tau\left(Q_n\left(\sum_{k=0}^{n}\left|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right|\right)Q_n\right)\\
&\quad\leq 3\tau\left(Q_n\left(\sum_{k=0}^{N}\left|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right|\right)Q_n\right).
\end{aligned}\tag{3.20}$$

Summing over n, we obtain that

$$\begin{aligned}
&\sum_{n=0}^{N}\tau\left(Q_n\left(\sum_{k=0}^{N}(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right)Q_n\geq 1/3\right)\\
&\quad\leq 3\tau\left(\sum_{k=0}^{N}\left|(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right|\right).
\end{aligned}$$

In light of (3.19), this can be bounded from above by

$$3\tau(I-R_N)+3\tau\left(\sum_{k=0}^{N}\left|(R_{k-1}-R_k)dx_k(R_{k-1}-R_k)\right|\right).$$

The first trace can be handled with the use of Lemma 3.2(iv). The second trace is not bigger than the left-hand side of (3.10) by the submartingale property of x. Combining these observations with (3.20), we finally obtain

$$\tau\left(Q_n\left(\sum_{k=0}^{n}(R_{k-1}-R_k)dy_k(R_{k-1}-R_k)\right)Q_n\geq 1/3\right)\leq 18\tau\left(x_N^+\right)-9\tau(x_0).$$

Step 5 (Conclusion). Having completed the analysis of the three terms in (3.14), we combine it with (3.13) and obtain

$$\sum_{n=0}^{N}\tau(Q_n)\leq 108\tau\left(x_N^+\right)-54\tau(x_0).$$

The same analysis (or simply the replacement of y with $-y$ in all relevant places) gives the corresponding bound for the projections T:

$$\sum_{n=0}^{N}\tau(T_n)\leq 108\tau\left(x_N^+\right)-54\tau(x_0).$$

Summing the last two estimates yields the claim by virtue of (3.12). $\square$

We are ready for the proof of the weak-type estimate.

PROOF OF THEOREM 3.1. The desired projection q is given by the intersection

$$q = \bigwedge_{n=-1}^{\infty} S_n \wedge \bigwedge_{n=-1}^{\infty} R_n.$$

To show that $-q \le q y_n q \le q$ for each n, fix a vector $\xi \in q(H)$. Then, $\xi \in S_n(H)$ and $\xi \in R_{k-1}(H)$ for each $k \le n$, so

$$\begin{aligned}
\left|\langle y_n\xi, \xi\rangle\right| &= \left|\sum_{k=0}^{n} \langle dy_k R_{k-1} S_n\xi, R_{k-1} S_n\xi\rangle\right| \\
&= \left|\left\langle S_n\left(\sum_{k=0}^{n} R_{k-1} dy_k R_{k-1}\right) S_n\xi, \xi\right\rangle\right| < \|\xi\|^2,
\end{aligned}$$

where the last estimate follows from the very definition of S_n. It remains to show the upper bound for $\tau(I-q)$. This is an immediate consequence of Lemma 3.2(iv) and Lemma 3.6; indeed, we have

$$\begin{aligned}
\tau(I - S_N \wedge R_N) &\le \tau(I - S_N) + \tau(I - R_N) \\
&\le 218\tau\left(x_N^+\right) - 109\tau(x_0) \le 327\|x\|_1.
\end{aligned}$$

Letting $N \to \infty$ completes the proof. $\square$

We conclude this section by a simple, yet important, application of the above weak-type bound which serves as a motivation for our further considerations.

THEOREM 3.7. *Suppose that $\tau(I) = 1$ and $0 < p < 1$. If $x = (x_n)_{n\ge 0}$ is an arbitrary submartingale, and y is strongly differentially subordinate to x, then*

$$\|y\|_p \le \frac{654}{(1-p)^{-1}} \|x\|_1.$$

The order $(1-p)^{-1}$ is the best possible, as it is already optimal in the case when x is assumed to be a classical martingale.

PROOF. We will first show that for any nonnegative integer N and any positive number t we have

$$t\tau\left(|y_N| > t\right) \le 654\|x\|_1. \tag{3.21}$$

By homogeneity we may assume that $t = 1$. By Theorem 3.1 there is a projection q such that $-q \le q y_N q \le q$ and $\tau(I-q) \le 327\|x\|_1$. Observe that the projection $I_{(1,\infty)}(y_N)$ is equivalent to a subprojection of $I - q$. Indeed, if a vector ξ belongs to $q(H)$, then $\langle y_N\xi,\xi\rangle = \langle q y_N q\xi,\xi\rangle \le \|\xi\|^2$, and hence this vector cannot lie in $I_{(1,\infty)}(y_N)(H)$, unless $\xi = 0$. This gives the equivalence and implies the inequality

$$\tau(y_N > 1) \le \tau(I - q) \le 327\|x\|_1.$$

We prove analogously the symmetric estimate $\tau(y_N < -1) \le 327\|x\|_1$, and hence (3.21) follows. Consequently, if we set $a = 654\|x\|_1$, then

$$\begin{aligned}
\|y_N\|_p^p &= p\int_0^\infty t^{p-1}\tau\big(|y_N| > t\big)\,\mathrm{d}t \\
&= p\int_0^a t^{p-1}\tau\big(|y_N| > t\big)\,\mathrm{d}t + p\int_a^\infty t^{p-1}\tau\big(|y_N| > t\big)\,\mathrm{d}t \\
&\le p\int_0^a t^{p-1}\,\mathrm{d}t + p\int_a^\infty t^{p-2}a\,\mathrm{d}t \\
&= \frac{a^p}{1-p}.
\end{aligned}$$

Since N was arbitrary, the proof is complete. $\square$

4. Moment estimates for $1 < p < \infty$. The primary goal of this section is to prove the strong-type (p, p) inequality $(1 < p < \infty)$ for noncommutative submartingales and their strong differential subordinates. As we mentioned in the Introduction, this inequality fails in general even in the classical case, and to overcome this problem one imposes the additional sign assumption on the dominating process. We will proceed similarly and assume, throughout this section, that $x = (x_n)_{n\ge 0}$ is a nonnegative submartingale. We shall establish the following noncommutative version of (1.4).

THEOREM 4.1. *Suppose that $x = (x_n)_{n\ge 0}$ is a nonnegative submartingale. For any $1 < p < \infty$, there is a finite constant C_p depending only on p such that the following holds. If y is strongly differentially subordinate to x (i.e., y and x satisfy the conditions (DS) + (CDS)), then*

$$\|y_N\|_p \le C_p\|x_N\|_p, \quad N = 0, 1, 2, \ldots. \tag{4.1}$$

Furthermore, if $p \ge 2$, the same inequality holds true under the "weaker" strong differential subordination (WDS) + (CDS) of Section 2.

REMARK 4.2. Our proof will give C_p of orders $O((p-1)^{-1})$ as $p \to 1+$ and $O(p^4)$ as $p \to \infty$. The first order is optimal, as it is already the best possible in the commutative setting. Unfortunately, the second order does not seem to be the optimal, and our guess is that the best order is $O(p^2)$, the same as in the noncommutative Doob's inequality. The reason for this conjecture is the following. Let $a = (a_n)_{n\ge 0}$ be a sequence of positive operators adapted to some filtration $(\mathcal{M}_n)_{n\ge 0}$.

Consider the "extended" filtration $\widetilde{\mathcal{M}} = (\mathcal{M}_0, \mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_2, \ldots)$, in which each algebra $\mathcal{M}_k$ appears twice. Then the process $x = (x_n)_{n\geq 0}$ with the difference defined by $dx_{2n} = \mathcal{E}_{n-1}a_n$ and $dx_{2n+1} = a_n - \mathcal{E}_{n-1}(a_n)$, $n = 0, 1, 2, \ldots$, is a nonnegative submartingale with respect to $\widetilde{\mathcal{M}}$. Furthermore, the sequence $y = (y_n)_{n\geq 0}$, defined by $dy_{2n} = \mathcal{E}_{n-1}a_n$ and $dy_{2n+1} = 0$, is strongly differentially subordinate to x (no matter which domination—(DS) + (CDS) or (WDS) + (CDS)—we consider). Therefore, (4.1) yields

$$\left\| \sum_{n=0}^{N} \mathcal{E}_{n-1}(a_n) \right\|_p \leq C_p \left\| \sum_{n=0}^{N} a_n \right\|_p,$$

for $N = 0, 1, 2, \ldots$. If we drop the adaptedness assumption on the sequence $a = (a_n)_{n\geq 0}$, the above estimate is just the noncommutative Doob's L^p inequality established by Junge in [22]. As Junge and Xu showed in [24], the optimal order of the constant in this inequality, as $p \to \infty$, is $O(p^2)$. Thus, we believe that the optimal order of the constant in (4.1) should also be quadratic.

The proof of the L^p bound between x and y will involve the Doob–Meyer decompositions of these sequences, which we briefly recall. For any $n \geq 0$, we may write

$$dx_n = du_n + da_n, \qquad dy_n = dv_n + db_n,$$

where $du_n = dx_n - \mathcal{E}_{n-1}(dx_n)$, $da_n = \mathcal{E}_{n-1}(dx_n)$ and, similarly, $dv_n = dy_n - \mathcal{E}_{n-1}(dy_n)$, $db_n = \mathcal{E}_{n-1}(dy_n)$. Note that du and dv are martingale differences, while da, db are predictable processes and da consists of nonnegative operators. The L^p bound for y will be obtained by providing appropriate estimates for $\|v\|_p$ and $\|b\|_p$. As we have already mentioned in the introductory section, the analysis in the cases $1 < p \leq 2$ and $p > 2$ will be quite different; we have decided to split the remaining part of this section accordingly.

4.1. *The case* $1 < p \leq 2$. We start with the estimate for the finite variation term $\|b_N\|_p$. It is quite interesting to note that the lemma below provides the *best* constants, even in the commutative case (see Wang [43]).

LEMMA 4.3. *For* $1 \leq p \leq 2$, *we have* $\|b_N\|_p \leq \|a_N\|_p \leq p\|x_N\|_p$.

PROOF. By the conditional differential subordination (i.e., the condition (CDS)), we have $-a_n \leq b_n \leq a_n$ and hence $\|b_N\|_p \leq \|a_N\|_p$ for each N. Therefore, it suffices to prove the bound $\|a_N\|_p \leq p\|x_N\|_p$. To this end, consider the sequence

$$w_n = a_n^p - pa_n^{p-1}x_n, \quad n = 0, 1, 2, \ldots, N.$$

We will prove that w_n is trace decreasing. Recall that $(a_n)_{n\geq 0}$ is predictable, so for any $n\geq 0$,

$$\begin{aligned}\tau\big(a_{n+1}^p - pa_{n+1}^{p-1}x_{n+1}\big) &= \tau\big(\mathcal{E}_n\big(a_{n+1}^p - pa_{n+1}^{p-1}x_{n+1}\big)\big)\\ &= \tau\big(a_{n+1}^p - pa_{n+1}^{p-1}\mathcal{E}_n(x_{n+1})\big)\\ &= \tau\big(a_{n+1}^p - pa_{n+1}^{p-1}(x_n + da_{n+1})\big).\end{aligned}$$

We have $a_n\leq a_{n+1}$ and $1\leq p\leq 2$, therefore, $a_n^{p-1}\leq a_{n+1}^{p-1}$, and

$$\tau\big(a_{n+1}^{p-1}x_n\big) = \tau\big(x_n^{1/2}a_{n+1}^{p-1}x_n^{1/2}\big) \geq \tau\big(x_n^{1/2}a_n^{p-1}x_n^{1/2}\big) = \tau\big(a_n^{p-1}x_n\big).$$

Furthermore, by Young's inequality,

$$\tau\big(a_{n+1}^{p-1}a_n\big) \leq \frac{p-1}{p}\tau\big(a_{n+1}^p\big) + \frac{1}{p}\tau\big(a_n^p\big)$$

which is equivalent to

$$\tau\big(a_{n+1}^p\big) \leq \tau\big(a_n^p\big) + p\tau\big(a_{n+1}^{p-1}(a_{n+1}-a_n)\big) = \tau\big(a_n^p\big) + p\tau\big(a_{n+1}^{p-1}da_{n+1}\big).$$

Using these observations above, we get

$$\tau(w_{n+1}) = \tau\big(a_{n+1}^p - pa_{n+1}^{p-1}x_{n+1}\big) \leq \tau\big(a_n^p - pa_n^{p-1}x_n\big) = \tau(w_n),$$

as we have claimed. Consequently, we have $\tau(w_N)\leq\tau(w_0)=0$. Using Young's inequality, we get

$$p\tau\big(a_N^{p-1}x_N\big) = \tau\big(a_N^{p-1}(px_N)\big) \leq \frac{p-1}{p}\tau\big(a_N^p\big) + \frac{1}{p}\tau\big((px_N)^p\big),$$

or, equivalently,

$$\tau(w_N) \geq \frac{1}{p}\tau\big(a_N^p - p^p x_N^p\big).$$

Combining this with $\tau(w_N)\leq 0$, we obtain that $\|a_N\|_p\leq p\|x_N\|_p$. The proof is complete. $\square$

PROOF OF THEOREM 4.1 FOR $p=2$. As we have just shown above, we have $\|b_N\|_2\leq 2\|x_N\|_2$; so, it remains to provide an L^2 bound for v. This sequence is a martingale; so, by properties of conditional expectations,

$$\begin{aligned}\|v_N\|_2^2 &= \sum_{n=0}^N \tau\big(dv_n^2\big)\\ &= \sum_{n=0}^N \tau\big(\big(dy_n - \mathcal{E}_{n-1}(dy_n)\big)^2\big)\\ &= \sum_{n=0}^N \tau\big(dy_n^2 - \big(\mathcal{E}_{n-1}(dy_n)\big)^2\big) \leq \sum_{n=0}^N \tau\big(dy_n^2\big).\end{aligned}$$

Furthermore, since x is a nonnegative submartingale,

$$\tau(x_n^2) = \tau(dx_n^2 + 2x_{n-1}dx_n + x_{n-1}^2) \geq \tau(dx_n^2 + x_{n-1}^2).$$

Hence,

$$\sum_{n=0}^{N} \tau(dx_n^2) \leq \tau(x_N^2).$$

It remains to use the "weak" differential subordination of y to x (i.e., the condition (WDS)) to obtain $\|v_N\|_2 \leq \|x_N\|_2$. This implies $\|y_N\|_2 \leq \|v_N\|_2 + \|b_N\|_2 \leq 3\|x_N\|_2$ and completes the proof. □

For $1 < p < 2$ the analysis of $\|v_N\|_p$ will be more elaborate and will exploit the projections $(R_n)_{n\geq 0}$ studied in the previous sections. Since x is nonnegative, these objects are given by the recursive formula $R_{-1} = 0$ and $R_n = R_{n-1}I_{[0,1)}(R_{n-1}x_nR_{n-1})$ for $n \geq 0$. We will require appropriate versions of lemmas studied in Section 3, taking into account that x is nonnegative. We start with the following version of property (iv) of Lemma 3.2.

LEMMA 4.4. *For any $N \geq 0$ we have $\tau(I - R_N) \leq \tau((I - R_N)x_N)$.*

PROOF. We repeat the argument of Cuculescu [12]. For any $n \geq 0$, by the very definition of R_n and the submartingale property of x,

$$\begin{aligned}\tau(R_{n-1} - R_n) &\leq \tau\big((R_{n-1} - R_n)x_n(R_{n-1} - R_n)\big)\\ &\leq \tau\big((R_{n-1} - R_n)x_N(R_{n-1} - R_n)\big) = \tau\big((R_{n-1} - R_n)x_N\big).\end{aligned}$$

Summing the estimate over $n = 0, 1, 2, \ldots, N$, we get the claim. □

We also need the following version of Lemma 3.4.

LEMMA 4.5. *For any nonnegative integer N we have*

$$\sum_{n=0}^{N} \tau(R_n dx_n R_{n-1} dx_n) \leq \tau(R_N x_N R_N x_N) + 2\tau\big((I - R_N)x_N\big).$$

PROOF. Arguing as the proof of Lemma 3.4, we show that

$$\begin{aligned}&\tau(R_n dx_n R_{n-1} dx_n)\\ &\quad\leq \tau(R_n x_n R_n x_n) - \tau(R_{n-1}x_{n-1}R_{n-1}x_{n-1})\\ &\qquad + 2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1} - R_n x_n R_n)\big)\end{aligned} \tag{4.2}$$

(this is (3.4)). We will now analyze the last trace. Since x is a nonnegative submartingale, we have

$$\begin{aligned}
&\tau(R_{n-1}x_{n-1}R_{n-1}x_{n-1}R_{n-1})\\
&\quad=\tau\big((R_{n-1}x_{n-1}R_{n-1})^{1/2}x_{n-1}(R_{n-1}x_{n-1}R_{n-1})^{1/2}\big)\\
&\quad\le\tau\big((R_{n-1}x_{n-1}R_{n-1})^{1/2}\mathcal{E}_{n-1}(x_n)(R_{n-1}x_{n-1}R_{n-1})^{1/2}\big)\\
&\quad=\tau(R_{n-1}x_{n-1}R_{n-1}x_nR_{n-1}).
\end{aligned}$$

Consequently, using the equality $R_nx_nR_n=R_nx_nR_{n-1}$, we may write

$$\begin{aligned}
&2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1}-R_nx_nR_n)\big)\\
&\quad\le 2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}-R_n)x_nR_{n-1}\big)\\
&\quad=2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}-R_n)x_n(R_{n-1}-R_n)\big)
\end{aligned}$$

(the last equality follows from the fact that R_n, and hence also $R_{n-1}-R_n$, commutes with $R_{n-1}x_nR_{n-1}$). But $R_{n-1}x_{n-1}R_{n-1}\le I$, so we obtain

$$\begin{aligned}
2\tau\big(R_{n-1}x_{n-1}R_{n-1}(R_{n-1}x_{n-1}R_{n-1}-R_nx_nR_n)\big)&\le 2\tau\big((R_{n-1}-R_n)x_n\big)\\
&\le 2\tau\big((R_{n-1}-R_n)x_N\big).
\end{aligned}$$

Plugging this into (4.2) and summing over $n=0,1,2,\dots,N$, we get

$$\sum_{n=0}^{N}\tau(R_ndx_nR_{n-1}dx_n)\le\tau(R_Nx_NR_Nx_N)+2\tau\big((I-R_N)x_N\big). \qquad\square$$

Finally, we will need the following analogue of Lemma 3.5.

LEMMA 4.6. *For any nonnegative integer N we have*

$$\Big\|\sum_{n=0}^{N}\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big|\Big\|_1\le 2\tau\big((I-R_N)x_N\big).$$

PROOF. By the triangle inequality and the fact that x is a nonnegative submartingale satisfying $R_nx_nR_n\le I$ for each n, we see that

$$\begin{aligned}
&\tau\big(\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big|\big)\\
&\quad\le\tau\big((R_{n-1}-R_n)x_n(R_{n-1}-R_n)\big)+\tau\big((R_{n-1}-R_n)x_{n-1}(R_{n-1}-R_n)\big)\\
&\quad\le\tau\big((R_{n-1}-R_n)x_N(R_{n-1}-R_n)\big)+\tau(R_{n-1}-R_n).
\end{aligned}$$

It remains to sum over $n=0,1,2,\dots,N$ and use Lemma 4.4. $\square$

The proof of the L^p bound for v will rest on the following intermediate weak-type inequality.

THEOREM 4.7. *Suppose that $x = (x_n)_{n\geq 0}$ is a nonnegative submartingale, and $y = (y_n)_{n\geq 0}$ is strongly differentially subordinate to x. Then, for any nonnegative integer N, we have*

$$\tau\big(|\upsilon_N| \geq 4\big) \leq 2\tau(R_N x_N R_N x_N) + 9\tau\big((I - R_N)x_N\big). \tag{4.3}$$

REMARK 4.8. The inequality (4.3) can be interpreted in the language of real interpolation theory. The left-hand side is a tail of υ_N which after homogenization and integration leads to the pth norm of υ_N up to some multiplicative constant. The right-hand side can be viewed as a K-functional of the operator x_N, corresponding to the interpolating spaces L^1 and L^2. Indeed, let us look at this expression in the commutative context. As we have already discussed earlier, the projection R_N corresponds to the indicator function of the set $\{\max_{0\leq n\leq N} x_n \leq 1\}$, and hence, roughly speaking, the operator

$$2R_N x_N R_N x_N R_N + 9(I - R_N)x_N$$

is equal to the quadratic term x_N^2, when x_N is small, and to the linear term $9x_N$, when x_N is large. This is precisely the intuition behind the K-functional.

PROOF OF THEOREM 4.7. We decompose $d\upsilon_n$ using a splitting similar to that introduced in Section 3:

$$d\upsilon_n = d\alpha_n + d\beta_n + d\gamma_n + d\delta_n,$$

where this time,

$$\begin{aligned}
d\alpha_n &= R_{n-1}dy_n R_n + R_n dy_n R_{n-1} - R_n dy_n R_n \\
&\qquad - \mathcal{E}_{n-1}(R_{n-1}dy_n R_n + R_n dy_n R_{n-1} - R_n dy_n R_n), \\
d\beta_n &= -\mathcal{E}_{n-1}\big((R_{n-1} - R_n)dy_n(R_{n-1} - R_n)\big), \\
d\gamma_n &= \big(R_n dy_n - \mathcal{E}_{n-1}(dy_n)\big)(I - R_{n-1}), \\
d\delta_n &= (I - R_n)dy_n - (R_{n-1} - R_n)dy_n R_n - (I - R_{n-1})\mathcal{E}_{n-1}(dy_n)R_{n-1}.
\end{aligned}$$

We write

$$\tau\big(|\upsilon_N| \geq 4\big) \leq \tau\big(|\alpha_N| \geq 1\big) + \tau\big(|\beta_N| \geq 1\big) + \tau\big(|\gamma_N| \geq 1\big) + \tau\big(|\delta_N| \geq 1\big)$$

and analyze each term on the right separately. Arguing as in the proof of Lemma 3.6, we obtain

$$\begin{aligned}
\tau\big(|\alpha_N| \geq 1\big) &\leq \tau\big(\alpha_N^2\big) \\
&= \sum_{n=0}^{N} \tau\big(d\alpha_n^2\big)
\end{aligned}$$

$$\leq 2\sum_{n=0}^{N}\tau(R_n dy_n R_{n-1} dy_n)$$
$$\leq 2\sum_{n=0}^{N}\tau(R_n dx_n R_{n-1} dx_n)$$
$$\leq 2\tau(R_N x_N R_N x_N) + 4\tau\big((I-R_N)x_N\big),$$

where in the last line we exploited Lemma 4.5. Furthermore, again by the reasoning presented in the proof of Lemma 3.6 (see (3.19)), we get

$$\begin{aligned}\tau\big(|\beta_N|\geq 1\big) &\leq \tau\big(|\beta_N|\big)\\ &\leq \left\|\sum_{n=0}^{N}|d\beta_n|\right\|_1\\ &\leq \left\|\sum_{n=0}^{N}\big(\big|(R_{n-1}-R_n)dx_n(R_{n-1}-R_n)\big| + (R_{n-1}-R_n)\big)\right\|_1\\ &\leq 3\tau\big((I-R_N)x_N\big),\end{aligned}$$

where the latter passage is due to Lemmas 4.4 and 4.6. To handle the terms $\tau(|\gamma_N|\geq 1)$ and $\tau(|\delta_N|\geq 1)$, note that the right support of $d\gamma_n$ satisfies $r(d\gamma_n)\leq I-R_{n-1}\leq I-R_N$, so

$$\bigvee_{n=0}^{N} r(d\gamma_n)\leq I-R_N.$$

Therefore, by Lemma 4.4,

$$\tau\big(|\gamma_N|\geq 1\big)\leq \tau\left(\bigvee_{n=0}^{N} r(d\gamma_n)\right)\leq \tau\big((I-R_N)x_N\big).$$

A similar analysis of the left support of $d\delta_n$ gives

$$\tau\big(|\delta_N|\geq 1\big)\leq \tau\left(\bigvee_{n=0}^{N} \ell(d\delta_n)\right)\leq \tau\big((I-R_N)x_N\big).$$

Putting all the above facts together we get the claim. □

Arguing as in Jiao et al. [18], Theorem 5.1(i), the weak-type bound (4.3) yields the following L^p estimate.

THEOREM 4.9. *For any nonnegative integer N and any $B>1$, we have*

$$\|v_N\|_p \leq \frac{4B^{p-1}}{B^{p-1}-1}\left(9B^p - 3 + \frac{4B^p(B^p-1)}{1-B^{p-2}}\right)^{1/p}\|x_N\|_p.$$

PROOF OF THEOREM 4.1 FOR $1 < p < 2$. Combining Theorem 4.9 with Lemma 4.3, we get the desired L^p estimate between x and y:

$$\|y_N\|_p \le \left(p + \frac{4B^{p-1}}{B^{p-1}-1}\left(9B^p - 3 + \frac{4B^p(B^p-1)}{1-B^{p-2}}\right)^{1/p}\right)\|x_N\|_p.$$

Note that the constant is indeed of order $O((p-1)^{-1})$ as $p \to 1+$. □

4.2. *The case $p > 2$.* In [23] Junge and Xu proved that if $z = (z_n)_{n=0}^N$ is a martingale, then we have

$$\left(\sum_{n=0}^N \|dz_n\|_p^p\right)^{1/p} \le 2^{1-2/p}\|z_N\|_p. \tag{4.4}$$

We will prove that the same inequality is true if z is assumed to be a nonnegative submartingale. To this end, we first strengthen (4.4) slightly.

LEMMA 4.10. *Let $z = (z_n)_{n=0}^N$ be a martingale. Then we have the estimate*

$$\left(2^{p-2}\|dz_0\|_p^p + \sum_{n=1}^N \|dz_n\|_p^p\right)^{1/p} \le 2^{1-2/p}\|z_N\|_p. \tag{4.5}$$

PROOF. We argue as in [23]. With no loss of generality we may assume that $\|z_N\|_p = 1$. We have

$$\left(2^{p-2}\|dz_0\|_p^p + \sum_{n=1}^N \|dz_n\|_p^p\right)^{1/p} = \tau\left(b_0 \cdot 2^{1-2/p} dz_0\right) + \sum_{1\le k\le N} \tau(b_k dz_k), \tag{4.6}$$

where $\sum_{0\le k\le N} \|b_k\|_{p'}^{p'} \le 1$. By approximation and the interpolation results of Kosaki [28], there exist continuous functions $Z, B_k : \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\} \to \mathcal{M}$ analytic in the interior of the strip such that $z_N = Z(2/p)$, $b_k = B_k(2/p)$ and

$$\begin{gathered}\sup_{t\in\mathbb{R}} \max\{\|Z(it)\|_\infty, \|Z(1+it)\|_2\} \le 1,\\ \sup_{t\in\mathbb{R}} \max\left\{\sum_{0\le k\le N} \|B_k(it)\|_1, \left(\sum_{0\le k\le N} \|B_k(1+it)\|_2^2\right)^{1/2}\right\} \le 1.\end{gathered} \tag{4.7}$$

Consider the analytic function

$$F(z) = \tau\left(2^{1-z} B_0(z)\mathcal{E}_0 Z(z) + \sum_{0<k\le N} B_k(z)(\mathcal{E}_k - \mathcal{E}_{k-1})Z(z)\right).$$

By Hölder's inequality

$$\begin{aligned}|F(it)| \le \tau\Big(\sum_{0\le k\le N}|B_k(it)|\Big) \\ \times \max\{\|2\mathcal{E}_0 Z(it)\|_\infty, \|(\mathcal{E}_1-\mathcal{E}_0)Z(it)\|_\infty, \dots, \\ \|(\mathcal{E}_N-\mathcal{E}_{N-1})Z(it)\|_\infty\}\end{aligned}$$

which is not bigger than 2 by the assumptions in (4.7). Similarly, we have

$$\begin{aligned}|F(1+it)| &\le \tau\Big(\sum_{0\le k\le N}|B_k(1+it)|^2\Big)^{1/2} \\ &\quad \times \tau\Big(|\mathcal{E}_0 Z(1+it)|^2 + \sum_{0<k\le N}|(\mathcal{E}_k-\mathcal{E}_{k-1})Z(1+it)|^2\Big)^{1/2} \le 1.\end{aligned}$$

Hence, by the three lines lemma, we get $F(2/p) \le 2^{1-2/p}$ which is the desired claim (see (4.6)). $\square$

The estimate (4.5) allows to obtain the following trace inequality which is of independent interest.

LEMMA 4.11. *For any $a, b \in \mathcal{M}$ we have*

$$\tau(|a+b|^p) \ge \tau(|a|^p) + p\tau(|a|^{p-2}ab) + 2^{2-p}\tau(|b|^p).$$

PROOF. Let s be a positive number and introduce the centered random variable ξ with the distribution

$$\mathbb{P}(\xi=-s) = \frac{1}{s+1} = 1-\mathbb{P}(\xi=1).$$

Consider the martingale given by $z_0 = 1\otimes a$, $z_1 = 1\otimes a + \xi\otimes b$ (on the von Neumann algebra $L^\infty(\Omega,\mathcal{F},\mathbb{P})\otimes\mathcal{M}$ with the natural filtration). By (4.5) we get

$$\begin{aligned}&2^{p-2}\tau(|a|^p) + \frac{s}{s+1}\tau(|b|^p) + \frac{s^p}{s+1}\tau(|b|^p) \\ &\quad \le 2^{p-2}\Big(\frac{s}{s+1}\tau(|a+b|^p) + \frac{1}{s+1}\tau(|a-sb|^p)\Big),\end{aligned}$$

or, equivalently,

$$\begin{aligned}&2^{p-2}\Big(\frac{\tau(|a+b|^p)-\tau(|a|^p)}{s+1} + \frac{\tau(|a-sb|^p)-\tau(|a|^p)}{s(s+1)}\Big) \\ &\quad \ge \frac{\tau(|b|^p)}{s+1} + \frac{s^{p-1}\tau(|b|^p)}{s+1}.\end{aligned}$$

It remains to let $s\to 0$ to obtain the claim. $\square$

Now we prove the following "submartingale" version of (4.4).

THEOREM 4.12. *Suppose that $x = (x_n)_{n=0}^N$ is a nonnegative submartingale. Then,*

$$\left(\sum_{n=0}^{N} \|dx_n\|_p^p\right)^{1/p} \leq 2^{1-2/p}\|x_N\|_p. \tag{4.8}$$

PROOF. By the previous lemma, applied to $a = x_{n-1}$ and $b = dx_n$, we have

$$\|x_n\|_p^p - \|x_{n-1}\|_p^p \geq p\tau\big(x_{n-1}^{p-1} dx_n\big) + 2^{2-p}\|dx_n\|_p^p \geq 2^{2-p}\|dx_n\|_p^p.$$

Summing over n completes the proof. □

We turn our attention to the upper bound for $\|b_N\|_p$. We will exploit the noncommutative good-λ inequalities developed by Jiao et al. in [17, 18]. Let us briefly recall the framework. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra equipped with some filtration $(\mathcal{M}_n)_{n\geq 0}$. Suppose that $q = (q_n)_{n=0}^N$ is an adapted finite self-adjoint martingale, and r, s are self-adjoint operators. For any $\lambda > 0$, consider the projections $S_{-1}^\lambda, S_0^\lambda, S_1^\lambda, \dots, S_N^\lambda$ given by $S_{-1}^\lambda = I$, and $S_n^\lambda = S_{n-1}^\lambda I_{(-\lambda,\lambda)}(S_{n-1}^\lambda q_n S_{n-1}^\lambda)$ for $n = 1, 2, \dots, N$.

DEFINITION 4.13. The triple (q, r, s) is said to satisfy the *good-λ testing condition* if the following two requirements are fulfilled:

(i) For all $\lambda > 0$, we have

$$\sum_{n=0}^{N}\sum_{k=n+1}^{N} \tau\big((S_{n-1}^\lambda - S_n^\lambda)dq_k S_{n-1}^\lambda dq_k(S_{n-1}^\lambda - S_n^\lambda)\big) \leq \tau\big((I - S_N^\lambda)r^2\big).$$

(ii) For each $0 \leq n \leq N$ and any projection $P \in \mathcal{M}_n$,

$$\tau\big(P dq_n^2 P\big) \leq \tau\big(P s^2 P\big).$$

Let us stress here that we do not assume that r, s are $\mathcal{M}_N$-measurable. One of the main results of [17] is the following.

THEOREM 4.14. *If (q, r, s) satisfies the good-λ testing condition, then for any $p > 2$,*

$$\|q_N\|_p \leq \frac{12p}{(1 - (1 + \frac{1}{p})^{2-p})^{1/2}}\big(\|r\|_p^2 + \|s\|_p^2\big)^{1/2}. \tag{4.9}$$

Equipped with the above statement, we will prove the following statement.

THEOREM 4.15. *Fix $p > 2$. Then, for any finite nonnegative submartingale $x = (x_n)_{n \geq 0}$, we have*

$$\|b_N\|_p \leq \|a_N\|_p \leq C_p \|x_N\|_p, \tag{4.10}$$

where C_p is of order $O(p^2)$ as $p \to \infty$.

PROOF. Fix $p > 2$. It suffices to show the second estimate in (4.10). We start with an appropriate complication of the underlying von Neumann algebra which will enable us to fit the assertion into the framework of good-λ inequalities. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space, and let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$ be a sequence of independent Rademacher variables. Consider the algebra $\mathcal{N} = \mathbb{M}_{N+2} \otimes L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \otimes \mathcal{M}$, where $\mathbb{M}_{N+2}$ denotes the algebra of $(N+2) \times (N+2)$ matrices with the standard trace. So, we may interpret $\mathcal{N}$ as the algebra of $(N+2) \times (N+2)$ matrices, whose entries are random elements of $\mathcal{M}$. We equip $\mathcal{N}$ with the usual tensor trace ν and the filtration $(\mathcal{N}_n)_{n=0}^N = (\mathbb{M}_{N+2} \otimes L^\infty(\Omega, \mathcal{F}_n, \mathbb{P}) \otimes \mathcal{M}_{n-1})_{n=0}^N$, where $\mathcal{F}_n$ stands for the σ-field generated by the variables $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$. Notice that we have used the algebra $\mathcal{M}_{n-1}$ on the third factor (with the convention that $\mathcal{M}_{-1} = \mathcal{M}_0$). Consider the operator

$$\begin{aligned} r = s &= e_{1,1} \otimes 1 \otimes x_N^{1/2} + \sum_{n=0}^N e_{n+2,n+2} \otimes 1 \otimes |dx_n|^{1/2} \\ &= \begin{bmatrix} x_N^{1/2} & 0 & 0 & \dots & 0 \\ 0 & |dx_0|^{1/2} & 0 & \dots & 0 \\ 0 & 0 & |dx_1|^{1/2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & |dx_N|^{1/2} \end{bmatrix}, \end{aligned}$$

and the sequence $q = (q_n)_{n=0}^N$ given by

$$\begin{aligned} q_n &= \sum_{k=0}^n (e_{1,k+2} + e_{k+2,1}) \otimes \varepsilon_k \otimes da_k^{1/2} \\ &= \begin{bmatrix} 0 & \varepsilon_0 da_0^{1/2} & \varepsilon_1 da_1^{1/2} & \dots & \varepsilon_n da_n^{1/2} & 0 & \dots & 0 \\ \varepsilon_0 da_0^{1/2} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \varepsilon_1 da_1^{1/2} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \varepsilon_n da_n^{1/2} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}. \end{aligned}$$

Of course q is a martingale adapted to $(\mathcal{N}_n)_{n=0}^N$. This is due to the fact that the sequence da is predictable. Let us verify that the triple (q, r, s) satisfies the good-λ testing condition. We have

$$dq_k^2 = \big((e_{1,k+2} + e_{k+2,1}) \otimes \varepsilon_k \otimes da_k^{1/2}\big)^2 = (e_{1,1} + e_{k+2,k+2}) \otimes 1 \otimes da_k,$$

so,

$$\begin{aligned}
&\sum_{n=0}^N \sum_{k=n+1}^N \nu\big((S_{n-1}^\lambda - S_n^\lambda) dq_k S_{n-1}^\lambda dq_k (S_{n-1}^\lambda - S_n^\lambda)\big) \\
&\quad \le \nu\left(\sum_{n=0}^N \sum_{k=n+1}^N (S_{n-1}^\lambda - S_n^\lambda) dq_k^2 (S_{n-1}^\lambda - S_n^\lambda)\right) \\
&\quad = \nu\left(\sum_{n=0}^N \sum_{k=n+1}^N (S_{n-1}^\lambda - S_n^\lambda)\big((e_{1,1} + e_{k+2,k+2}) \otimes 1 \otimes da_k\big)\right) \\
&\quad = \nu\left(\sum_{n=0}^N \sum_{k=n+1}^N (S_{n-1}^\lambda - S_n^\lambda)\big((e_{1,1} + e_{k+2,k+2}) \otimes 1 \otimes dx_k\big)\right).
\end{aligned}$$

We split the latter expression into two parts:

$$\begin{aligned}
&\nu\left(\sum_{n=0}^N \sum_{k=n+1}^N (S_{n-1}^\lambda - S_n^\lambda)(e_{1,1} \otimes 1 \otimes dx_k)\right) \\
&\qquad + \nu\left(\sum_{n=0}^N \sum_{k=n+1}^N (S_{n-1}^\lambda - S_n^\lambda)(e_{k+2,k+2} \otimes 1 \otimes dx_k)\right) \\
&\quad \le \sum_{n=0}^N \nu\big((S_{n-1}^\lambda - S_n^\lambda)\big(e_{1,1} \otimes 1 \otimes (x_N - x_n)\big)\big) \\
&\qquad + \nu\left(\sum_{k=0}^N \sum_{n=0}^{k-1} (S_{n-1}^\lambda - S_n^\lambda)\big(e_{k+2,k+2} \otimes 1 \otimes |dx_k|\big)\right) \\
&\quad \le \nu\left(\big(I - S_N^\lambda\big)\left(e_{1,1} \otimes 1 \otimes x_N + \sum_{k=0}^N e_{k+2,k+2} \otimes 1 \otimes |dx_k|\right)\right) \\
&\quad = \nu\big((I - S_N^\lambda) r^2\big)
\end{aligned}$$

which is the condition (i). Concerning the assumption (ii), we check that for any projection $P \in \mathcal{N}_n$,

$$\begin{aligned}
\nu\big(P dq_n^2 P\big) &= \tau\big(P\big((e_{1,1} + e_{n+2,n+2}) \otimes 1 \otimes da_n\big) P\big) \\
&= \tau\big(P\big((e_{1,1} + e_{n+2,n+2}) \otimes 1 \otimes dx_n\big) P\big) \le \tau\big(P s^2 P\big),
\end{aligned}$$

where in the second passage we have used the equality $da_n = \mathcal{E}_{n-1}(dx_n)$ (and the fact that the third factor in $\mathcal{N}_n = \mathbb{M}_{N+2} \otimes L^\infty(\Omega, \mathcal{F}_n, \mathbb{P}) \otimes \mathcal{M}_{n-1}$ is the algebra $\mathcal{M}_{n-1}$). So, the good-λ testing condition holds true and therefore the inequality (4.9) gives

$$\|q_N\|_{2p} \le \frac{24p}{(1-(1+\frac{1}{2p})^{2-2p})^{1/2}} \cdot 2^{1/2}\|r\|_{2p}. \tag{4.11}$$

However, we have $q_N^2 \ge e_{1,1} \otimes 1 \otimes a_N$ which implies $\|q_N\|_{2p}^2 \ge \|a_N\|_p$. Furthermore, we have

$$\|r\|_{2p}^2 = \left(\|x_N\|_p^p + \sum_{n=0}^N \|dx_n\|_p^p\right)^{1/p}.$$

By Theorem 4.12, the expression on the right does not exceed $(1+2^{p-2})^{1/p}\|x_N\|_p$. Putting all the above observations together, we get the desired estimate (4.10). □

We turn our attention to the estimate for the martingale part $\|v\|_p$. This is the only missing part of the proof of Theorem 4.1.

THEOREM 4.16. *We have $\|v_N\|_p \le \kappa_p \|x_N\|_p$, where κ_p is of order $O(p^4)$ as $p \to \infty$.*

PROOF. First apply Burkholder–Rosenthal inequality (see [23]) to obtain

$$\|v_N\|_p \le c_p\left[\big\|s_N(v)\big\|_p + \left(\sum_{n=0}^N \|dv_n\|_p^p\right)^{1/p}\right],$$

where c_p is of order $O(p)$ as $p \to \infty$. By the assumption (WDS) we have

$$\begin{aligned}
\mathcal{E}_{n-1}(dv_n^2) &= \mathcal{E}_{n-1}\big((dy_n - \mathcal{E}_{n-1}(dy_n))^2\big) \\
&\le \mathcal{E}_{n-1}(dy_n^2) \\
&\le \mathcal{E}_{n-1}(dx_n^2) = \mathcal{E}_{n-1}(du_n^2) + da_n^2,
\end{aligned}$$

so, $s_N(v)^2 \le s_N(u)^2 + s_N(a)^2$ and $\|s_N(v)\|_p \le \|s_N(u)\|_p + \|s_N(a)\|_p$. In addition the triangle inequality and (WDS) + (CDS) assumptions give that

$$\left(\sum_{n=0}^N \|dv_n\|_p^p\right)^{1/p} \le \left(\sum_{n=0}^N \|du_n\|_p^p\right)^{1/p} + \left(\sum_{n=0}^N \|da_n\|_p^p\right)^{1/p}.$$

Combining the above observations, we obtain

$$\begin{aligned}
&\big\|s_N(v)\big\|_p + \left(\sum_{n=0}^N \|dv_n\|_p^p\right)^{1/p} \\
&\quad \le \big\|s_N(u)\big\|_p + \left(\sum_{n=0}^N \|du_n\|_p^p\right)^{1/p} + \big\|s_N(a)\big\|_p + \left(\sum_{n=0}^N \|da_n\|_p^p\right)^{1/p}
\end{aligned}$$

$$\le \tilde{c}_p \|u_N\|_p + (1 + 2^{1-2/p})\|a_N\|_p$$
$$\le \tilde{c}_p \|x_N\|_p + (\tilde{c}_p + 1 + 2^{1-2/p})\|a_N\|_p,$$

with $\tilde{c}_p$ of order $O(p)$. Here, in the second passage we have used the reverse Burkholder/Rosenthal inequality (which gave the estimate for the terms involving u), the inequality

$$\|s_N(a)\|_p = \left\| \left(\sum_{n=0}^{N} da_n^2 \right)^{1/2} \right\|_p \le \|a_N\|_p \tag{4.12}$$

and Theorem 4.12 applied to the submartingale $(a_n)_{n=0}^N$. Putting all the above facts together and combining them with the estimate (4.10), we obtain the claim. □

PROOF OF THEOREM 4.1 FOR $2 < p < \infty$. Combining Theorem 4.15 with Theorem 4.16, we get the desired L^p estimate between x and y:

$$\|y_N\|_p \le c_p \|x_N\|_p,$$

where c_p is of order $O(p^4)$ as $p \to \infty$. □

We conclude with the following interesting questions:

REMARK 4.17. (i) Let us write the estimates of Lemma 4.3 and Theorem 4.15 in the language of noncommutative Doob's inequality. We have shown that if $x = (x_n)_{n\ge 0}$ is a nonnegative submartingale, then for $1 \le p < \infty$ we have

$$\left\| \sum_{n=0}^{N} \mathcal{E}_{n-1}(dx_n) \right\|_p \le C_p \left\| \sum_{n=0}^{N} dx_n \right\|_p,$$

with $C_p = O(p^2)$ as $p \to \infty$. Is this order optimal? This is not clear, since the sequence dx above needs to be adapted and hence the estimate does not generalize Doob's inequality.

(ii) We also have the following Burkholder/Rosenthal inequality of noncommutative nonegative submartingales. Suppose that $x = (x_n)_{n\ge 0}$ is a nonnegative submartingale. Then, for $2 \le p < \infty$ we have

$$\|s_N(x)\|_p + \left(\sum_{n=0}^{N} \|dx_k\|_p^p \right)^{1/p} \le C_p \|x_N\|_p,$$

where $C_p = O(p^2)$ as $p \to \infty$. Indeed, from Theorem 4.12, it suffices to prove that

$$\|s_N(x)\|_p \le C_p \|x_N\|_p.$$

This can be verified by using the Burkholder/Rosenthal inequalities for noncommutative martingales, (4.12) and (4.10). At the time of this writing, we do not know if the order $O(p^2)$ as $p \to \infty$ is optimal or not.

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