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LARGE DEVIATIONS AND WANDERING EXPONENT FOR RANDOM WALK IN A DYNAMIC BETA ENVIRONMENT[1]

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Random walk in a dynamic i.i.d. beta random environment, conditioned to escape at an atypical velocity, converges to a Doob transform of the original walk. The Doob-transformed environment is correlated in time, i.i.d. in space and its marginal density function is a product of a beta density and a hypergeometric function. Under its averaged distribution, the transformed walk obeys the wandering exponent $2/3$ that agrees with Kardar–Parisi–Zhang universality. The harmonic function in the Doob transform comes from a Busemann-type limit and appears as an extremal in a variational problem for the quenched large deviation rate function.

1. Introduction. We study an exactly solvable random walk in a random environment (RWRE) in one space dimension. The walk is nearest-neighbor and the environment dynamical and product-form. Our main results (i) construct a Doob transform of the RWRE that conditions the walk on an atypical velocity, (ii) establish that the transformed walk has path fluctuation exponent $2/3$ of the Kardar–Parisi–Zhang (KPZ) class instead of the diffusive $1/2$ and (iii) describe the quenched large deviation rate function of the walk.

The three points above are tied together. The harmonic functions in the Doob transform furnish extremals of a variational formula for the quenched large deviation rate function. Explicit distributional properties of these harmonic functions enable the derivation of the path exponent. The logarithm of the harmonic function obeys the KPZ longitudinal exponent $1/3$.

This work rests on the development of analogues of percolation and polymer ideas for RWRE. The harmonic functions in the Doob transform arise through limits that correspond to Busemann functions of percolation and polymers. The

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quenched large deviation rate function is strictly above the averaged one except at their common minimum. For a deviation of small order h, the difference of these rate functions is of order h^4. These properties are exactly as for the quenched and averaged free energy of $1+1$ dimensional directed polymers [11, 18]. As for the entire KPZ class, proofs of fluctuation exponents are restricted to models with special features. A natural expectation is that the picture that emerges here should be universal for $1+1$ dimensional directed RWRE under some assumptions.

We turn to a detailed introduction of the model.

A dynamical environment is refreshed at each time step. On the two-dimensional space-time lattice $\mathbb{Z}^2$, we run time in the diagonal direction $(\frac{1}{2}, \frac{1}{2})$, and the admissible steps of the walk are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The jump probabilities are independent and identically distributed at each lattice point of $\mathbb{Z}^2$. When the walk starts at the origin, after n time steps its location is among the points (i, j) in the first quadrant (i.e., $i, j \geq 0$) with $i + j = n$.

The *environment* $\omega = (\omega_{x,x+e_1} : x \in \mathbb{Z}^2)$ is a collection of i.i.d. $[0, 1]$-valued random variables $\omega_{x,x+e_1}$ indexed by lattice points x. Set $\omega_{x,x+e_2} = 1 - \omega_{x,x+e_1}$. $(\omega_{x,x+e_1}, \omega_{x,x+e_2})$ are the jump probabilities from point $x \in \mathbb{Z}^2$ to one of the neighbors $\{x + e_1, x + e_2\}$. Transitions ω do not allow backward jumps. The distribution of the environment ω is $\mathbb{P}$ with expectation operator $\mathbb{E}$. Given a realization ω and a point $x \in \mathbb{Z}^2$, P_x^ω denotes the *quenched* path measure of the Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}^2$ that starts at x and uses transition probabilities ω:

$$\begin{aligned} &P_x^\omega(X_0 = x) = 1 \quad \text{and,} \quad \text{for } y \in \mathbb{Z}^2, n \geq 0, \text{and } i \in \{1, 2\}, \\ &P_x^\omega(X_{n+1} = y + e_i | X_n = y) = \omega_{y,y+e_i}. \end{aligned} \tag{1.1}$$

P_x^ω is a probability measure on the path space $(\mathbb{Z}^2)^{\mathbb{Z}_+}$ and $X_{\bullet}$ is the coordinate process. This is a special case of *random walk in a space-time random environment*.

This paper focuses on the *beta RWRE* where $\omega_{x,x+e_1}$ is beta-distributed. Barraquand and Corwin [6] discovered that this case is *exactly solvable*. This means that fortuitous coincidences of combinatorics and probability permit derivation of explicit formulas and precise results far deeper than anything presently available for the general case. Some limit results uncovered in an exactly solvable case are expected to be universal. These form natural conjectures for the general case.

An earlier case of exact calculations for RWRE in a static environment appeared in a series of papers by Sabot and coauthors (see [27] and references therein). They discovered and utilized special features of the multidimensional Dirichlet RWRE to prove results currently not accessible for the general multidimensional RWRE. Section 8 of [27] discusses one-dimensional RWRE in a static beta environment.

Before specializing to the dynamic beta environment, we review results for the general $1+1$ dimensional RWRE (1.1) in an i.i.d. environment.

1.1. *Nearest-neighbor space-time RWRE.* Under an i.i.d. environment for the quenched model in (1.1), the *averaged* path measure given by $P_0(\cdot) =$

$\int P_0^\omega(\cdot)\mathbb{P}(d\omega)$ is a classical random walk with admissible steps $\{e_1, e_2\}$ and transition kernel $p(e_i) = \mathbb{E}(\omega_{0,e_i})$, $i = 1, 2$. Hence there is a law of large numbers $P_0\{X_N/N \to \xi^*\} = 1$ with limiting velocity $\xi^* = (\xi_1^*, \xi_2^*) = (p(e_1), p(e_2))$. Fubini's theorem then gives the quenched law of large numbers

$$P_0^\omega\{N^{-1}X_N \to \xi^*\} = 1 \qquad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{1.2}$$

By Donsker's theorem, under P_0 the centered and diffusively rescaled walk

$$\left\{W_N(t) = \frac{X_{\lfloor Nt\rfloor} - Nt\xi^*}{\sqrt{\xi_1^*\xi_2^* N}} : t \ge 0\right\}$$

converges weakly to the process $\{(W(t), -W(t)) : t \ge 0\}$, where $W(\cdot)$ is standard one-dimensional Brownian motion.

The same functional central limit theorem (CLT) holds for the *quenched* RWRE if and only if $\mathbb{P}(\omega_{0,e_1} \in \{0, 1\}) < 1$. That is, for $\mathbb{P}$-almost every ω the distribution of $\{W_N(t) : t \ge 0\}$ under P_0^ω converges weakly to that of $\{(W(t), -W(t)) : t \ge 0\}$ (Theorem 1 of [21]). A functional CLT holds also for the quenched mean $E_0^\omega[X_N]$ with scaling $N^{1/4}$ (Corollary 3.5 of [4]). In summary, as $N \to \infty$, the quenched mean of the walk has Gaussian fluctuations on a small scale of order $N^{1/4}$, while under a typical environment the walk itself has Gaussian fluctuations on the larger scale of order $N^{1/2}$. The fluctuations of the quenched walk dominate, and hence the averaged process has Gaussian fluctuations of order $N^{1/2}$.

Let $\mathcal{U} = \{te_1 + (1-t)e_2 : 0 \le t \le 1\}$ denote the simplex of possible limiting velocities. For $\xi \in \mathcal{U}$ and $N \in \mathbb{N}$, let $[N\xi]$ denote a point closest to $N\xi$ on the antidiagonal $\{(x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 = N\}$. The averaged large deviation principle (LDP) is the standard Cramér theorem and tells us that for $\xi \in \mathcal{U}$

$$\lim_{N\to\infty} N^{-1} \log P_0\{X_N = [N\xi]\} = -I_a(\xi)$$

with rate function

$$I_a(\xi) = \xi_1 \log \frac{\xi_1}{\xi_1^*} + \xi_2 \log \frac{\xi_2}{\xi_2^*} \qquad \text{for } \xi = (\xi_1, \xi_2) \in \mathcal{U}. \tag{1.3}$$

A quenched LDP holds under the assumption

$$\mathbb{E}\big[|\log \omega_{0,e_i}|^{2+\varepsilon}\big] < \infty \qquad \text{for } i \in \{1, 2\} \text{ and some } \varepsilon > 0. \tag{1.4}$$

By Theorems 2.2, 4.1, 2.6(b) and 3.2(a) of [22], for all $\xi \in \mathcal{U}$,

$$\lim_{N\to\infty} N^{-1} \log P_0^\omega\{X_N = [N\xi]\} = -I_q(\xi) \tag{1.5}$$

exists $\mathbb{P}$-almost surely. The rate function I_q does not depend on ω. It is a nonnegative convex continuous function on $\mathcal{U}$ with a unique zero at ξ^*. By Fatou's lemma and Jensen's inequality, $I_q(\xi) \ge I_a(\xi)$ for all $\xi \in \mathcal{U}$. It is shown in [31] that $I_q(\xi) > I_a(\xi)$ for all $\xi \in \mathcal{U} \setminus \{\xi^*\}$. The proof in [31] utilizes uniform ellipticity, namely that $\mathbb{P}(\delta \le \omega_{0,e_1} \le 1-\delta) = 1$ for some $\delta > 0$, but their proof works more generally. Theorem 2.7 below states the strict inequality in the beta case.

General closed formulas for I_q have not been found. Variational representations exist, for example, in [10, 17, 22, 26, 30]. We state below one particular formula for the RWRE (1.1) on $\mathbb{Z}^2$. In the beta case, extremals for this formula are identified in Section 2.3 below, in terms of harmonic functions.

Let $\mathcal{K}$ denote the space of *integrable stationary cocycles* defined on the probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ of the environments. Elements of $\mathcal{K}$ are stochastic processes $\{B_{x,y}(\omega) : x, y \in \mathbb{Z}^2\}$ such that, for all $x, y, z \in \mathbb{Z}^2$ and $\mathbb{P}$-a.e. ω, $\mathbb{E}|B_{x,y}| < \infty$, $B_{x,y}(\omega) + B_{y,z}(\omega) = B_{x,z}(\omega)$, and $B_{x,y}(T_z\omega) = B_{x+z,y+z}(\omega)$ where T_z is the shift $(T_z\omega)_{x,x+e_i} = \omega_{x+z,x+z+e_i}$. The rate function in (1.5) is then characterized as

$$
\begin{aligned}
I_q(\xi) = -\inf_{B\in\mathcal{K}} \bigl\{ & \mathbb{E}[B_{0,e_1}]\xi_1 + \mathbb{E}[B_{0,e_2}]\xi_2 \\
& + \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \log\bigl(\omega_{0,e_1} e^{-B_{0,e_1}(\omega)} + \omega_{0,e_2} e^{-B_{0,e_2}(\omega)}\bigr)\bigr\} \\
& \text{for } \xi \in \operatorname{ri}\mathcal{U}.
\end{aligned} \tag{1.6}
$$

This formula for I_q is valid for an i.i.d. environment ω under the same moment assumption (1.4) as the LDP.

For a nearest-neighbor RWRE on $\mathbb{Z}^d$ for which all directions $\pm e_i$ satisfy (1.4) formula (1.6) appeared in Theorem 2 on page 6 of [26]. In the directed case, (1.6) is a special case of variational formula (4.7) in [14] for the point-to-point limiting free energy of a directed polymer.

When the transition probabilities in (1.1) are a small perturbation of simple symmetric random walk, under suitable space-time scaling the transition probabilities converge to the solution of the stochastic heat equation (SHE) with multiplicative noise [12]. This is also a KPZ result, for the logarithm of the SHE is a solution of the KPZ equation. The result is based on the convergence of chaos expansions, following the work [2] in the so-called intermediate disorder regime of directed polymers.

Averaged and quenched central limit theorems and large deviation estimates have also been proved for random walk in correlated dynamic environments. See, for example, [3, 7, 8, 25] and their references. However, these results do not apply to the RWRE with the correlated transition probabilities (2.5) we introduce in Section 2.1. Indeed, Theorem 2.4 below shows that the fluctuation exponent of the averaged RWRE is 2/3 instead of 1/2. See also (1.9).

1.2. *Beta RWRE.* Let $\alpha, \beta > 0$ be positive real parameter values. The standard gamma and beta functions are given by

$$
\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty s^{\alpha-1} e^{-s}\, ds \quad \text{and} \\
B(\alpha,\beta) &= \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\, ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\end{aligned} \tag{1.7}
$$

The c.d.f. of the Beta(α, β) distribution is

$$F(t; \alpha, \beta) = B(\alpha, \beta)^{-1} \int_0^t s^{\alpha-1}(1-s)^{\beta-1}\, ds \qquad \text{for } 0 < t < 1. \tag{1.8}$$

The case $\alpha = \beta = 1$ is the uniform distribution on $(0, 1)$.

For the remainder of this paper, the variables $\{\omega_{x,x+e_1} : x \in \mathbb{Z}^2\}$ in the RWRE (1.1) are i.i.d. Beta(α, β) distributed.

When $\alpha = \beta = 1$ and $\xi_1 - \xi_2 > 4/5$, Barraquand and Corwin [6] showed

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}\biggl\{ \frac{\log P_0^\omega\{X_N \cdot (e_1 - e_2) \ge N(\xi_1 - \xi_2)\} + N I_q(\xi)}{c(\xi) N^{1/3}} \le y \biggr\} \\ = F_{\mathrm{GUE}}(y),\end{aligned} \tag{1.9}$$

where the limit is the Tracy–Widom GUE distribution. Later, in a less rigorous paper, Thiery and Le Doussal [29] did the same for $\log P_0^\omega\{X_N = [N\xi]\} + N I_q(\xi)$ and all $\alpha, \beta > 0$ and $\xi \ne \xi^*$.

These results revealed that this RWRE possesses features of the $1 + 1$ dimensional Kardar–Parisi–Zhang (KPZ) universality class. Where do we find the KPZ wandering exponent $2/3$? Not in the walk (1.1), because the walk in an i.i.d. environment satisfies a standard CLT under both its quenched and averaged distributions.

We answer the question by conditioning the walk on an atypical velocity. Then the quenched process $X_\bullet$ converges to a random walk in a correlated environment which is a Doob transform of the original walk. When the environment is averaged out, at time N this walk has fluctuations of the order $N^{2/3}$, and thus has the KPZ wandering exponent. This behavior deviates from that of classical random walk: standard random walk conditioned on an atypical velocity converges to a random walk with altered transitions.

Conditioning on an atypical velocity is intimately tied to large deviations. The logarithm of the harmonic function in the Doob transform turns out to be an extremal in (1.6) and its expectation is the gradient of I_q.

Notation and conventions. $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rationals, $\mathbb{R}$ the reals and $\mathbb{C}$ the complex numbers. $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$, and $\mathbb{R}_+ = [0, \infty)$. For real a, $\lfloor a \rfloor$ is the largest integer $\le a$.

Vector notation on $\mathbb{R}^2$ is $x = (x_1, x_2) = x_1 e_1 + x_2 e_2$, with canonical basis $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The scalar product is $x \cdot y$ and the ℓ^1 norm $|x|_1 = |x_1| + |x_2|$. Coordinatewise integer parts: $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor)$. For $x \cdot (e_1 + e_2) \in \mathbb{Z}_+$, $[x]$ is a closest point to x in $\{y \in \mathbb{Z}^2 : y_1 + y_2 = x_1 + x_2\}$. Inequality $y \ge x$ is interpreted coordinatewise: $y_1 \ge x_1$ and $y_2 \ge x_2$.

Shifts T_z act on environments ω by $(T_z\omega)_{x,x+e_i} = \omega_{x+z,x+z+e_i}$ for $x, y \in \mathbb{Z}^2$. When subscripts are bulky $\omega_{x,y}$ becomes $\omega(x, y)$, with the same convention for $\pi_{x,y}$, $B_{x,y}$ and $\rho_{x,y}$. A finite or infinite sequence is denoted by $x_{i,j} = (x_i, \dots, x_j)$, for $-\infty \le i < j \le \infty$. The simplex of asymptotic velocities is $\mathcal{U} = \{te_1 + (1 - t)e_2 : 0 \le t \le 1\}$, with relative interior $\mathrm{ri}\,\mathcal{U} = \{te_1 + (1 - t)e_2 : 0 < t < 1\}$.

2. Results for beta RWRE. In Section 2.1, we construct the Doob-transformed RWRE that is the limiting process of the quenched walk conditioned on an atypical velocity. Section 2.2 states the KPZ fluctuation exponent of the averaged Doob-transformed walk. In Section 2.3, we describe the quenched large deviation rate function and its connection with the harmonic functions of the Doob transform.

Parameters $\alpha, \beta > 0$ are fixed, and the environment $\omega = (\omega_{x,x+e_1})_{x\in\mathbb{Z}^2}$ has the i.i.d. Beta(α, β) distribution. The probability space of the environment is $(\Omega, \mathfrak{S}, \mathbb{P})$ where $\mathfrak{S}$ is the Borel σ-field on the product space $\Omega = [0, 1]^{\mathbb{Z}^2}$.

2.1. *Doob transform of the quenched walk.* The first main result is the existence of a family of increment-stationary harmonic functions, indexed by directions in $\mathrm{ri}\,\mathcal{U} = \{te_1 + (1-t)e_2 : 0 < t < 1\}$.

THEOREM 2.1. *On $(\Omega, \mathfrak{S}, \mathbb{P})$, there exists a stochastic process $\{B^{\xi}_{x,y}(\omega) : x, y \in \mathbb{Z}^2, \xi \in \mathrm{ri}\,\mathcal{U}\}$ with the following properties.*

For each $\xi \in \mathrm{ri}\,\mathcal{U}$, $e^{-B^{\xi}_{0,x}}$ is a harmonic function: for all $x \in \mathbb{Z}^2$,

$$\omega_{x,x+e_1} e^{-B^{\xi}_{0,x+e_1}(\omega)} + \omega_{x,x+e_2} e^{-B^{\xi}_{0,x+e_2}(\omega)} = e^{-B^{\xi}_{0,x}(\omega)} \qquad \mathbb{P}\text{-a.s.} \tag{2.1}$$

For each $\xi \in \mathrm{ri}\,\mathcal{U}$, there is an event $\Omega^{(\xi)}$ such that $\mathbb{P}(\Omega^{(\xi)}) = 1$ and for every $\omega \in \Omega^{(\xi)}$,

$$B^{\xi}_{x,y}(\omega) = \lim_{N\to\infty} \bigl(\log P^{\omega}_x\{X_{|z_N - x|_1} = z_N\} - \log P^{\omega}_y\{X_{|z_N - y|_1} = z_N\}\bigr) \tag{2.2}$$

for all $x, y \in \mathbb{Z}^2$, and any sequence $z_N \in \mathbb{Z}^2$ such that $|z_N|_1 \to \infty$ and $z_N/|z_N|_1 \to \xi$.

In the law of large numbers direction $\xi^* = (\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$, we have

$$B^{\xi^*}_{x,y}(\omega) = 0. \tag{2.3}$$

By analogy with limits of increments in percolation and polymers, we could call B^{ξ} the *Busemann function* in direction ξ. For $\xi \neq \xi^*$, the variables $B^{\xi}_{x,x+e_i}$ are marginally logarithms of beta-variables. From limit (2.2), we get

$$B^{\xi}_{x,y}(T_z\omega) = B^{\xi}_{x+z,y+z}(\omega) \quad \text{and} \quad B^{\xi}_{x,y}(\omega) + B^{\xi}_{y,z}(\omega) = B^{\xi}_{x,z}(\omega) \tag{2.4}$$

for all $x, y, z \in \mathbb{Z}^2$ and $\mathbb{P}$-a.e. ω. In other words, B^{ξ} is a member of the space $\mathcal{K}$ of integrable stationary cocycles defined above (1.6). Harmonicity (2.1) comes from limit (2.2) and the Markov property

$$\begin{aligned} P^{\omega}_x\{X_{|z_N-x|_1} = z_N\} &= \omega_{x,x+e_1} P^{\omega}_{x+e_1}\{X_{|z_N-x|_1-1} = z_N\} \\ &\quad + \omega_{x,x+e_2} P^{\omega}_{x+e_2}\{X_{|z_N-x|_1-1} = z_N\}. \end{aligned}$$

Further continuity, monotonicity and explicit distributional properties of the process B^ξ are given in Theorem 3.6.

Theorem 2.1 is proved by constructing a family of harmonic functions on quadrants to control the convergence on the right of (2.2). This approach is the RWRE counterpart of the arguments used for an exactly solvable polymer model in [16] and for the corner growth model with general i.i.d. weights in [15].

By (2.1) and (2.4),

$$\kappa^\xi_{x,x+e_i}(\omega) = \omega_{x,x+e_i} \frac{e^{-B^\xi_{0,x+e_i}(\omega)}}{e^{-B^\xi_{0,x}(\omega)}} = \omega_{x,x+e_i} e^{-B^\xi_{x,x+e_i}(\omega)}, \qquad i \in \{1,2\}, \tag{2.5}$$

defines a new transition probability on $\mathbb{Z}^2$, as a Doob-transform of the original transition ω. It is an RWRE transition as a function on Ω because, by (2.4), it obeys shifts: $\kappa^\xi_{x,x+e_i}(T_z\omega) = \kappa^\xi_{x+z,x+z+e_i}(\omega)$. The environment $\kappa^\xi(\omega) = (\kappa^\xi_{x,x+e_1}(\omega))_{x\in\mathbb{Z}^2}$ is in general correlated over locations x, except that its restriction on antidiagonals is i.i.d. as stated in the next theorem.

Let $P^{\kappa^\xi}_x$ be the quenched path measure of the Markov chain with transition probability κ^ξ. In other words, $P^{\kappa^\xi}_x$ satisfies (1.1) with $\kappa^\xi_{y,y+e_i}$ instead of $\omega_{y,y+e_i}$. $P^{\kappa^\xi}_x = P^{\kappa^\xi(\omega)}_x$ is a function of ω through its transition probability.

THEOREM 2.2. *Fix $\xi \in \mathrm{ri}\,\mathcal{U}$. Then for any $n \in \mathbb{Z}$, the random variables $\{\kappa^\xi_{x,x+e_1}(\omega) : x_1 + x_2 = n\}$ are i.i.d. We have the law of large numbers*:

$$P^{\kappa^\xi(\omega)}_0\{N^{-1}X_N \to \xi\} = 1 \qquad \textit{for } \mathbb{P}\textit{-a.e. } \omega. \tag{2.6}$$

In [16], a RWRE in a correlated environment arose as a limit of the quenched log-gamma polymer. Its transition, probability is marginally beta-distributed. Transition κ^ξ above is different: the marginal distribution of $\kappa^\xi_{x,x+e_i}$ is *not* beta. Its density function is given in Theorem 3.7.

The next theorem records the limits of conditioned quenched walks.

THEOREM 2.3. *For each fixed $\xi \in \mathrm{ri}\,\mathcal{U}$, there is an event $\Omega^{(\xi)}$ such that $\mathbb{P}(\Omega^{(\xi)}) = 1$ and the following holds for every $\omega \in \Omega^{(\xi)}$: if $z_N \in \mathbb{Z}^2$ is any sequence such that $|z_N|_1 = N$ and $z_N/N \to \xi$, then the conditioned quenched path distribution $P^\omega_0(\cdot\,|X_N = z_N)$ converges weakly on the path space $(\mathbb{Z}^2)^{\mathbb{Z}_+}$ to the Doob transformed path measure $P^{\kappa^\xi(\omega)}_0$.*

The weak convergence claim in the theorem amounts to checking that

$$\lim_{N\to\infty} P^\omega_0(X_{0,m} = x_{0,m}|X_N = z_N) = \prod_{k=0}^{m-1} \kappa^\xi_{x_k,x_{k+1}}(\omega) \qquad \text{for } \omega \in \Omega^{(\xi)}$$

for any finite path $x_{0,m}$ with $x_0 = 0$. This is an immediate consequence of limit (2.2). Combining (2.3) with the theorem above tells us that if $z_N/N \to \xi^*$, then $P_0^\omega(\cdot\,|X_N = z_N) \to P_0^\omega$. In other words, conditioning on the typical velocity ξ^* introduces no new correlations in the limit and leads back to the original path measure. This is consistent with classical random walk.

Observe that $P_0^{\kappa^\xi(\omega)}(X_{0,m} = x_{0,m}|X_N = z_N) = P_0^\omega(X_{0,m} = x_{0,m}|X_N = z_N)$ for $0 \le m \le N$. Consequently, the family $\{P_0^{\kappa^\xi}\}$ is closed under taking limits of path distributions conditioned on velocities.

Theorems 2.1, 2.2 and 2.3 are proved after the statement of Theorem 3.6.

2.2. *Fluctuation bounds.* In $1+1$ dimensional KPZ models, exponent $\frac{1}{3}$ appears in fluctuations of heights of growing interfaces and free energies of polymer models, while $\frac{2}{3}$ appears in spatial correlations and path fluctuations. The Barraquand–Corwin limit (1.9) indicated that logarithms of quenched probabilities have $N^{1/3}$ fluctuations. The theorem below shows the same exponent for process B^ξ, though only in the direction ξ, as quantified by (2.7) below. If the endpoint (m,n) deviates from $N\xi$ by N^ν for $\nu > \frac{2}{3}$, the fluctuations of $B^\xi_{0,(m,n)}$ become Gaussian. (This follows similar observations for directed polymers in [9], Corollary 1.4, and [28], Corollary 2.2.)

THEOREM 2.4. *Fix $\alpha, \beta > 0$. Fix $\xi = (\xi_1, \xi_2) \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$. Given a constant $0 < \gamma < \infty$, there exist positive finite constants c, C, and N_0, depending only on α, β, γ and ξ, such that*

$$cN^{2/3} \le \mathbb{V}\mathrm{ar}\big[B^\xi_{0,(m,n)}\big] \le CN^{2/3}$$

for all $N \ge N_0$ and $(m,n) \in \mathbb{N}^2$ such that

$$|m - N\xi_1| \vee |n - N\xi_2| \le \gamma N^{2/3}. \tag{2.7}$$

The same constants can be taken for $(\alpha, \beta, \gamma, \xi)$ varying in a compact subset of $(0,\infty)^3 \times (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$.

Theorem 2.4 was proved independently and concurrently in the present work and as part of a more general result for exactly solvable directed polymers by Chaumont and Noack (Theorem 1.2 of [9]). A proof appears in Section 4.1 of the first preprint version [5] of this paper. In the present version, we omit the proof and cite [9] for details. The translation between the Doob-transformed walk and the beta polymer is explained in Section 4.

The second fluctuation result quantifies the deviations of the walk from its limiting velocity, under the averaged measure $\mathbf{P}^\xi(\cdot) = \int P^{\kappa^\xi(\omega)}(\cdot)\mathbb{P}(d\omega)$ of the Doob-transformed RWRE. This walk is superdiffusive with the KPZ wandering exponent $\frac{2}{3}$ instead of the diffusive $\frac{1}{2}$ of classical random walk.

THEOREM 2.5. *Fix* $\alpha, \beta > 0$. *Fix* $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$. *There exist finite positive constants* C, c, r_0 *and* δ_0, *depending only on* α, β *and* ξ, *such that for* $r \ge r_0$, $\delta \in (0, \delta_0)$, *and any* $N \ge 1$ *we have*

$$\mathbf{P}_0^\xi\{|X_N - N\xi|_1 \ge rN^{2/3}\} \le Cr^{-3} \quad \text{and} \tag{2.8}$$

$$\mathbf{P}_0^\xi\{|X_N - N\xi|_1 \ge \delta N^{2/3}\} \ge c. \tag{2.9}$$

The same constants can be used for (α, β, ξ) *varying in a compact subset of* $(0, \infty)^2 \times (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$.

Theorem 2.5 is proved in Section 6. The bounds come from using harmonic functions to control the exit point of the walk from rectangles.

2.3. *Large deviations*. This section records large deviation rate functions and their relation to the process B^ξ of Theorem 2.1.

We begin with a point needed for the remainder of the paper. The next lemma links three parameters: $\xi \in \mathcal{U}$ is an asymptotic velocity of the walk, $t \in \mathbb{R}$ is dual to ξ and $0 < \lambda < \infty$ parametrizes harmonic functions constructed in Section 3.2. As ξ ranges across $\mathcal{U}$ from left to right (in the direction of ξ_1), λ goes from 0 to ∞ and back, with $\lambda = \infty$ at $\xi = \xi^*$ (see Figure 1).

The polygamma functions are $\psi_0(s) = \Gamma'(s)/\Gamma(s)$ and $\psi_n(s) = \psi'_{n-1}(s)$ for $s > 0$ and $n \in \mathbb{N}$. Properties of these functions are given in Appendix A of the first preprint [5] of this paper. Qualitatively speaking, ψ_0 is strictly concave and increasing from $\psi_0(0+) = -\infty$ to $\psi_0(\infty-) = \infty$, while ψ_1 is strictly convex and decreasing from $\psi_1(0+) = \infty$ to $\psi_1(\infty-) = 0$.

LEMMA 2.6. *Fix* $\alpha, \beta > 0$.

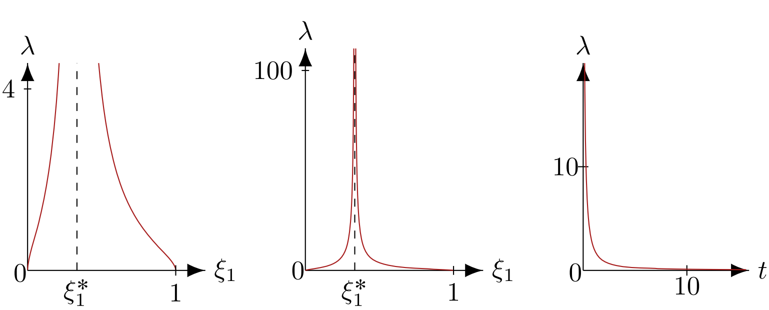

FIG. 1. *Leftmost and middle plots are of* λ *as a function of* ξ_1. *The left plot stretches the* λ*-axis to reveal the behavior away from* ξ_1^*. *The rightmost plot is of* λ *as a function of* t. *These graphs are for* $(\alpha, \beta) = (1, 2)$.

(a) *Given* $\xi = (\xi_1, 1-\xi_1) \in \mathcal{U}$ *there is a unique* $\lambda = \lambda(\xi) \in [0, \infty]$ *such that*

$$\xi_1 = \frac{\psi_1(\lambda) - \psi_1(\alpha+\lambda)}{\psi_1(\lambda) - \psi_1(\alpha+\beta+\lambda)} \qquad \text{for } \xi_1 \in [\xi_1^*, 1] \quad \text{and} \tag{2.10}$$

$$\xi_1 = 1 - \frac{\psi_1(\lambda) - \psi_1(\beta+\lambda)}{\psi_1(\lambda) - \psi_1(\alpha+\beta+\lambda)} \qquad \text{for } \xi_1 \in [0, \xi_1^*], \tag{2.11}$$

with $\lambda = 0 \Longleftrightarrow \xi \in \{e_1, e_2\}$ *and* $\lambda = \infty \Longleftrightarrow \xi = \xi^* = (\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$.

Furthermore, λ *is strictly increasing on* $\xi_1 \in [0, \xi_1^*)$ *and strictly decreasing on* $\xi_1 \in (\xi_1^*, 1]$.

(b) *Given* $t \in [0, \infty]$, *there is a unique* $\lambda = \lambda(t) \in [0, \infty]$ *such that*

$$t = \psi_0(\alpha+\beta+\lambda) - \psi_0(\lambda), \tag{2.12}$$

where $\lambda = 0 \Longleftrightarrow t = \infty$ *and* $\lambda = \infty \Longleftrightarrow t = 0$.

The proof of Lemma 2.6 is in Section 7. The formula for the quenched rate I_q in (1.5) in the beta environment can now be given; see Figure 2.

THEOREM 2.7. *Fix* $\alpha, \beta > 0$ *and let* ω *have i.i.d.* Beta(α, β) *distribution. Then for* $\xi = (\xi_1, \xi_2) \in \mathcal{U}$ *we have* $I_q(\xi^*) = 0$ *and*

$$I_q(\xi) = \begin{cases} \xi_1 \psi_0(\alpha+\beta+\lambda(\xi)) + \xi_2 \psi_0(\lambda(\xi)) & \\ \qquad - \psi_0(\alpha+\lambda(\xi)), & \xi_1 \in (\xi_1^*, 1], \\ \xi_2 \psi_0(\alpha+\beta+\lambda(\xi)) + \xi_1 \psi_0(\lambda(\xi)) & \\ \qquad - \psi_0(\beta+\lambda(\xi)), & \xi_1 \in [0, \xi_1^*), \end{cases} \tag{2.13}$$

where in both cases λ *and* ξ *determine each other uniquely via* (2.10) *and* (2.11). I_q *is a strictly convex function on* $[0, 1]$ *and satisfies* $I_q(\xi) > I_a(\xi)$ *for all* $\xi \in \mathcal{U} \setminus \{\xi^*\}$.

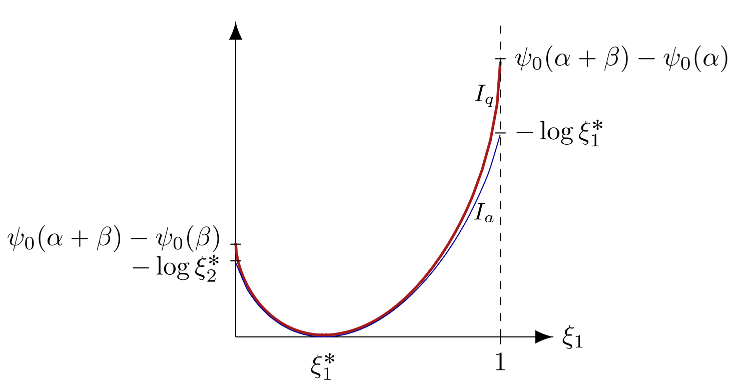

FIG. 2. *A plot showing* I_q (*higher, thicker graph*) *and* I_a (*lower, thinner graph*) *as functions of* ξ_1. *Here*, $(\alpha, \beta) = (1, 2)$.

REMARK 2.8 (Regularity of I_q). One can show that I_q is analytic away from ξ^*. We verified that I_q has at least four continuous derivatives across ξ^* by explicitly computing derivatives, and obtained the following expansion around ξ^*:

$$\begin{aligned} I_q(\xi) = {} & \frac{(\alpha+\beta)^2}{2\alpha\beta}(\xi_1-\xi_1^*)^2 + \frac{(\alpha+\beta)^3(\alpha-\beta)}{6\alpha^2\beta^2}(\xi_1-\xi_1^*)^3 \\ & + \frac{(\alpha+\beta)^4(2\alpha^2-2\alpha\beta+2\beta^2+1)}{24\alpha^3\beta^3}(\xi_1-\xi_1^*)^4 + o(|\xi_1-\xi_1^*|^4). \end{aligned} \tag{2.14}$$

These details are in Appendix B of the first preprint [5] of this paper.

For the sake of comparison, here is the expansion around ξ^* of the averaged rate function I_a from (1.3):

$$\begin{aligned} I_a(\xi) = {} & \frac{(\alpha+\beta)^2}{2\alpha\beta}(\xi_1-\xi_1^*)^2 + \frac{(\alpha+\beta)^3(\alpha-\beta)}{6\alpha^2\beta^2}(\xi_1-\xi_1^*)^3 \\ & + \frac{(\alpha+\beta)^3(\alpha^3+\beta^3)}{12\alpha^3\beta^3}(\xi_1-\xi_1^*)^4 + \mathcal{O}(|\xi_1-\xi_1^*|^5). \end{aligned} \tag{2.15}$$

The expansions of I_q and I_a agree to third order. This explains the minute difference between the two graphs in Figure 2. One can check that

$$\frac{d^4}{d\xi_1^4}\big[I_q(\xi_1, 1-\xi_1) - I_a(\xi_1, 1-\xi_1)\big]_{\xi=\xi^*} = \frac{(\alpha+\beta)^4}{\alpha^3\beta^3} > 0.$$

Thus the fourth-order terms differ in the two expansions.

EXAMPLE 2.9 (Case $\alpha=\beta=1$). In the i.i.d. uniform environment λ and I_q can be found in closed form:

$$I_q(\xi) = 1 - 2\sqrt{\xi_1\xi_2} = \sum_{n=1}^{\infty}\binom{\frac{1}{2}}{n}(-1)^{n+1}4^n\left(\xi_1-\frac{1}{2}\right)^{2n} \qquad \text{for } \xi \in \mathcal{U}. \tag{2.16}$$

The series illustrates that this rate function is analytic on the entire open segment ri$\mathcal{U}$, a property which is open for general (α, β).

We also record the convex conjugate

$$I_q^*(h) = \sup_{\xi\in\mathcal{U}}\{h\cdot\xi - I_q(\xi)\} = \lim_{n\to\infty} n^{-1}\log E_0^{\omega}[e^{h\cdot X_n}], \qquad h \in \mathbb{R}^2.$$

The second equality above is an instance of Varadhan's theorem [23], page 28. Since $(X_n - X_0)\cdot(e_1+e_2) = n$, we have $I_q^*(te_1+se_2) = s + I_q^*((t-s)e_1)$ and it suffices to consider $h = te_1$ for real t.

THEOREM 2.10. *Fix $\alpha, \beta > 0$ and let ω have i.i.d.* Beta(α, β) *distribution. For $t \geq 0$,*

$$I_q^*(te_1) = \psi_0\big(\alpha + \lambda(t)\big) - \psi_0\big(\lambda(t)\big) \quad \textit{and} \tag{2.17}$$

$$I_q^*(-te_1) = -t + \psi_0\big(\beta + \lambda(t)\big) - \psi_0\big(\lambda(t)\big), \tag{2.18}$$

where λ and t determine each other via (2.12).

Formula (2.13) for I_q appeared earlier in equations (8)–(9) of [6] where it was derived by nontrivial asymptotic analysis. We derive I_q through I_q^*, which is calculated with the help of harmonic functions we construct below.

Next, we state the connections between I_q and the processes B^ξ.

THEOREM 2.11. (a) *Fix $\xi \in \operatorname{ri}\mathcal{U}$. Then the process B^ξ is an extremal for variational formula* (1.6). *In particular, we have*

$$\begin{aligned} I_q(\xi) &= -\mathbb{E}\big[B^\xi_{0,e_1}\big]\xi_1 - \mathbb{E}\big[B^\xi_{0,e_2}\big]\xi_2 \\ &= -\inf_{\zeta \in \operatorname{ri}\mathcal{U}} \big\{\mathbb{E}\big[B^\zeta_{0,e_1}\big]\xi_1 + \mathbb{E}\big[B^\zeta_{0,e_2}\big]\xi_2\big\}, \end{aligned} \tag{2.19}$$

where the last infimum is uniquely attained at $\zeta = \xi$.

(b) *Extend I_q homogeneously to all of $\mathbb{R}^2_+$, that is, by $I_q(c\xi) = cI_q(\xi)$ for $c > 0$ and $\xi \in \mathcal{U}$. Then the gradient of I_q satisfies*

$$\nabla I_q(\xi) = -\mathbb{E}\big[B^\xi_{0,e_1}\big]e_1 - \mathbb{E}\big[B^\xi_{0,e_2}\big]e_2, \qquad \xi \in \operatorname{ri}\mathcal{U}. \tag{2.20}$$

Corollary 4.5 and Remark 5.7 in [14] put equations (2.19)–(2.20) in the context of a general theory for directed walks in random potentials. Theorems 2.7, 2.10 and 2.11 are proved in Section 7.

Lastly, we record the LDP for the Doob-transformed RWRE. Definition (2.5) and the cocycle property in (2.4) imply that

$$P_0^{\kappa^\xi(\omega)}(X_N = x) = P_0^\omega(X_N = x)e^{-B^\xi_{0,x}(\omega)}.$$

B^ξ has i.i.d. increments along horizontal and vertical lines [Theorem 3.6(c)], and hence the law of large numbers applies: $\mathbb{P}$-almost surely

$$\lim_{N\to\infty} N^{-1} B^\xi_{0,[N\zeta]} = \mathbb{E}\big[B^\xi_{0,e_1}\big]\zeta_1 + \mathbb{E}\big[B^\xi_{0,e_2}\big]\zeta_2 = -\zeta \cdot \nabla I_q(\xi) \qquad \forall \zeta \in \operatorname{ri}\mathcal{U}.$$

The quenched LDP (1.5) of the beta walk then gives this theorem.

THEOREM 2.12. *For any fixed $\xi \in \operatorname{ri}\mathcal{U}$, the following holds $\mathbb{P}$-almost surely, simultaneously for all $\zeta \in \operatorname{ri}\mathcal{U}$:*

$$\lim_{N\to\infty} N^{-1} \log P_0^{\kappa^\xi}\big\{X_N = [N\zeta]\big\} = -I_q(\zeta) + \zeta \cdot \nabla I_q(\xi).$$

Rate function $I_q(\zeta) - \zeta \cdot \nabla I_q(\xi)$ is uniquely minimized at $\zeta = \xi$, by convexity and homogeneity of I_q. The main results have been stated and we turn to proofs.

3. Increment-stationary harmonic functions. In this section, we construct quenched harmonic functions whose probability distributions are suitably invariant under lattice translations. This is done first on restricted subsets of the lattice by solving a boundary value problem, then extended to the entire lattice by taking limits. That this is possible with explicit distributions and useful independence properties is a feature of exact solvability.

The boundaries of the positive and negative quadrants $v + \mathbb{Z}_+^2$ and $v - \mathbb{Z}_+^2$ with a corner at $v \in \mathbb{Z}^2$ are denoted by

$$\begin{aligned} \mathbb{B}_v^+ &= \{v + (i, 0), v + (0, j) : i, j \geq 0\} \quad \text{and} \\ \mathbb{B}_v^- &= \{v - (i, 0), v - (0, j) : i, j \geq 0\}. \end{aligned} \tag{3.1}$$

Hitting times of the boundaries follow analogous notation:

$$\tau_v^{\pm} = \inf\{n \geq 0 : X_n \in \mathbb{B}_v^{\pm}\}. \tag{3.2}$$

The separate axes of these boundaries are distinguished by the notation

$$\mathbb{B}_v^{(\pm 1)} = \{v \pm (i, 0) : i \geq 0\} \quad \text{and} \quad \mathbb{B}_v^{(\pm 2)} = \{v \pm (0, j) : j \geq 0\}. \tag{3.3}$$

In particular, $\mathbb{B}_v^{\pm} = \mathbb{B}_v^{(\pm 1)} \cup \mathbb{B}_v^{(\pm 2)}$.

3.1. *An involution for beta variables.* This section is technical preparation for the construction of harmonic functions. A distribution-preserving involution of triples of beta variables is defined and its properties recorded. We motivate this construction through a Dirichlet problem.

Consider backward nearest-neighbor transition probabilities $\breve{\omega}_{x,x-e_i}$, $i \in \{1, 2\}$, on the lattice $\mathbb{Z}^2$. These transition probabilities allow two steps $-e_1$ and $-e_2$ and satisfy $\breve{\omega}_{x,x-e_1} + \breve{\omega}_{x,x-e_2} = 1$ at each $x \in \mathbb{Z}^2$. Suppose a function f is given on the boundary $\mathbb{B}_0^+$ of the first quadrant $\mathbb{Z}_+^2$. When the backward walk starts in the first quadrant, the hitting time τ_0^+ is obviously finite. Then

$$H(x) = E_x^{\breve{\omega}}[f(X(\tau_0^+))] \tag{3.4}$$

defines an $\breve{\omega}$-harmonic function on the positive first quadrant. That is,

$$H(x) = \breve{\omega}_{x,x-e_1} H(x - e_1) + \breve{\omega}_{x,x-e_2} H(x - e_2) \qquad \text{for } x \in \mathbb{N}^2. \tag{3.5}$$

We solve (3.5) inductively, by beginning from the boundary values and then defining $H(x)$ once $H(x - e_1)$ and $H(x - e_2)$ have been defined. We formulate this induction in terms of ratios

$$\rho_{x,y} = H(y)/H(x).$$

The induction assumption is that the nearest-neighbor ratios $\rho_{x-e_2,x-e_1-e_2}$ and $\rho_{x-e_1,x-e_1-e_2}$ have been defined on the south and west sides of a unit square with northeast corner at x. Then, by (3.5), the ratios on the north and east sides are obtained from the equations

$$\rho_{x-e_1,x} = \frac{\breve{\omega}_{x,x-e_1}\rho_{x-e_1-e_2,x-e_1} + (1-\breve{\omega}_{x,x-e_1})\rho_{x-e_1-e_2,x-e_2}}{\rho_{x-e_1-e_2,x-e_1}}, \tag{3.6}$$

$$\rho_{x-e_2,x} = \frac{\breve{\omega}_{x,x-e_1}\rho_{x-e_1-e_2,x-e_1} + (1-\breve{\omega}_{x,x-e_1})\rho_{x-e_1-e_2,x-e_2}}{\rho_{x-e_1-e_2,x-e_2}}. \tag{3.7}$$

It is useful to augment this pair of equations with a third equation

$$\omega_{x-e_1-e_2,x-e_2} = \frac{\rho_{x-e_1-e_2,x-e_2}(\rho_{x-e_1-e_2,x-e_1}-1)}{\rho_{x-e_1-e_2,x-e_1}-\rho_{x-e_1-e_2,x-e_2}} \tag{3.8}$$

provided the denominator never vanishes. Together the three equations define an involution. In the case we specialize to below, $\omega_{x-e_1-e_2,x-e_2}$ is a forward transition probability from $x-e_1-e_2$ to $x-e_2$. The complementary transition probability from $x-e_1-e_2$ to $x-e_1$ is then

$$\omega_{x-e_1-e_2,x-e_1} = 1-\omega_{x-e_1-e_2,x-e_2}. \tag{3.9}$$

Equations (3.6)–(3.9) are illustrated by Figure 3, with x in the upper right corner of the unit square and with

$$(U,V,W) = (\rho_{x-e_1-e_2,x-e_2}, \rho_{x-e_1-e_2,x-e_1}, \breve{\omega}_{x,x-e_1}) \quad \text{and}$$
$$(U',V',W') = (\rho_{x-e_1,x}, \rho_{x-e_2,x}, \omega_{x-e_1-e_2,x-e_2}).$$

Now assume that the transition probabilities $\breve{\omega}$ come from a beta RWRE; in other words, that the variables $\{\breve{\omega}_{x,x-e_1}\}_{x\in\mathbb{Z}^2}$ are i.i.d. Beta(α,β). The next lemma indicates how to choose the distributions of the ratios of the boundary values in order to get tractable harmonic functions. We regard the parameters α,β of the environment fixed, while $0<\lambda<\infty$ parametrizes two different boundary conditions in cases (a) and (b) in the lemma.

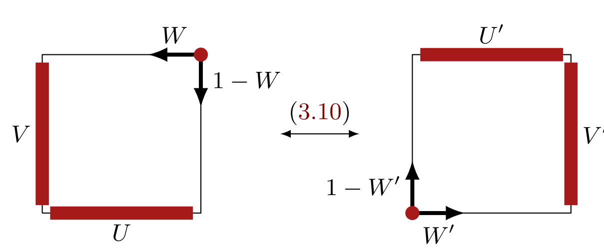

FIG. 3. *Involution* (3.10): *Respectively, weights U and V on the south and west edges and west/south transition $(W,1-W)$ become weights U' and V' on the north and east edges and east/north transition $(W',1-W')$, and vice versa.*

LEMMA 3.1. *The equations*

$$U' = \frac{WV + (1-W)U}{V}, \qquad V' = \frac{WV + (1-W)U}{U}, \quad \textit{and}$$
$$W' = \frac{U(V-1)}{V-U} \tag{3.10}$$

define an involution $(U, V, W) \mapsto (U', V', W')$ *on* $(0,1) \times (1,\infty) \times (0,1)$.

Let $0 < \alpha, \beta, \lambda < \infty$.

(a) *Suppose that* (U, V, W) *are independent variables with distributions*

$$U \sim \mathrm{Beta}(\alpha+\lambda, \beta), \qquad V^{-1} \sim \mathrm{Beta}(\lambda, \alpha), \quad \textit{and} \quad W \sim \mathrm{Beta}(\alpha, \beta). \tag{3.11}$$

Then the triples (U', V', W') *and* (U, V, W) *have the same distribution.*

(b) *Suppose that* (U, V, W) *are independent variables with distributions*

$$U^{-1} \sim \mathrm{Beta}(\lambda, \beta), \qquad V \sim \mathrm{Beta}(\beta+\lambda, \alpha), \quad \textit{and} \quad W \sim \mathrm{Beta}(\alpha, \beta). \tag{3.12}$$

Then again the triples (U', V', W') *and* (U, V, W) *have the same distribution.*

PROOF. Algebra checks the involution property. We prove part (a). Part (b) follows by switching around α and β and by switching around the axes.

Let $(W, \Gamma_\alpha, \Gamma_\beta, \Gamma_\lambda)$ be jointly independent with $W \sim \mathrm{Beta}(\alpha, \beta)$ and $\Gamma_\nu \sim \mathrm{Gamma}(\nu, 1)$. Set

$$U = \frac{\Gamma_\alpha + \Gamma_\lambda}{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda} \quad \text{and} \quad V = \frac{\Gamma_\alpha + \Gamma_\lambda}{\Gamma_\lambda}. \tag{3.13}$$

Then (U, V, W) have the desired distribution because V is independent of $\Gamma_\alpha + \Gamma_\lambda$.

Compute

$$\begin{aligned} U' &= W + (1-W)\frac{U}{V} = W + (1-W)\frac{\Gamma_\lambda}{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda}, \\ V' &= W\frac{V}{U} + 1 - W = W\frac{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda}{\Gamma_\lambda} + 1 - W, \\ W' &= \frac{U(V-1)}{(V-U)} = \frac{\Gamma_\alpha}{\Gamma_\alpha + \Gamma_\beta}. \end{aligned} \tag{3.14}$$

W' is independent of the pair (U', V') because it is independent of $\Gamma_\alpha + \Gamma_\beta$. It also clearly has the same distribution as W.

It remains to show that (U', V') has the same distribution as (U, V). Set

$$Y = \frac{\Gamma_\lambda}{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda}.$$

Observe that $U' = W + (1-W)Y$ and $V' = WY^{-1} + 1 - W$. Also

$$W' + \big(1 - W'\big)Y = Y + W'(1-Y) = \frac{\Gamma_\alpha + \Gamma_\lambda}{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda} = U$$

and similarly, $W'Y^{-1} + 1 - W' = V$. Furthermore, (Y, W') are independent and so are (Y, W). Consequently, the two pairs have the same distribution and (U', V') has the same distribution as (U, V). The lemma is proved. $\square$

Observe from (3.10) that

$$\frac{W'}{U} + \frac{1-W'}{V} = 1 \quad \text{and} \quad \frac{W}{U'} + \frac{1-W}{V'} = 1. \tag{3.15}$$

This is how the Doob transformed transition probabilities arise from a given forward transition $(W', 1 - W')$ or backward transition $(W, 1 - W)$. We derive the probability distribution of W'/U (which is the same as that of W/U'). ${}_2F_1$ below is the standard Gauss hypergeometric function

$$ {}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \tag{3.16}$$

where $(c)_k = c(c+1)\cdots(c+k-1)$ is the ascending factorial. Other rational functions of beta variables whose densities involve hypergeometric functions appear in [13, 20].

PROPOSITION 3.2. *The random variables W'/U and W/U' of Lemma 3.1 have the following density functions g_λ and $\widetilde{g}_\lambda$ on the interval* $(0, 1)$.

In case (a) *under assumption* (3.11),

$$\begin{aligned} g_\lambda(x) = {} & \frac{B(\alpha+\lambda, \alpha+\beta)}{B(\alpha+\lambda, \beta)} \cdot \frac{x^{\alpha-1}(1-x)^{\lambda-1}}{B(\lambda, \alpha)} \\ & \times {}_2F_1(\alpha+\lambda, \alpha+\lambda, 2\alpha+\beta+\lambda; x). \end{aligned} \tag{3.17}$$

In case (b) *under assumption* (3.12),

$$\begin{aligned} \widetilde{g}_\lambda(x) = {} & \frac{B(\beta+\lambda, \alpha+\beta)}{B(\beta+\lambda, \alpha)} \cdot \frac{x^{\lambda-1}(1-x)^{\beta-1}}{B(\lambda, \beta)} \\ & \times {}_2F_1(\beta+\lambda, \beta+\lambda, \alpha+2\beta+\lambda; 1-x). \end{aligned} \tag{3.18}$$

Neither g_λ nor $\widetilde{g}_\lambda$ is the density function of any beta distribution.

PROOF. Consider case (a). Let $F_{V^{-1}}$ denote the Beta(λ, α) c.d.f. of V^{-1}. Fix $0 < x < 1$. From $W'/U = (1 - V^{-1})/(1 - UV^{-1})$,

$$\begin{aligned} g_\lambda(x) &= \frac{d}{dx}\mathbb{P}\left(\frac{W'}{U} \le x\right) = \frac{d}{dx}\mathbb{P}\left(V^{-1} \ge \frac{1-x}{1-xU}\right) \\ &= \frac{1}{B(\alpha+\lambda, \beta)} \int_0^1 \frac{\partial}{\partial x}\left(1 - F_{V^{-1}}\left(\frac{1-x}{1-xu}\right)\right) \\ & \quad \times u^{\alpha+\lambda-1}(1-u)^{\beta-1}\, du \end{aligned} \tag{3.19}$$

$$= \frac{x^{\alpha-1}(1-x)^{\lambda-1}}{B(\lambda,\alpha)B(\alpha+\lambda,\beta)} \int_0^1 (1-xu)^{-\alpha-\lambda} \times u^{\alpha+\lambda-1}(1-u)^{\alpha+\beta-1}\,du.$$

The last integral equals $B(\alpha+\lambda,\alpha+\beta)\,{}_2F_1(\alpha+\lambda,\alpha+\lambda,2\alpha+\beta+\lambda;x)$ (equation (9.09) on page 161 in [19]). This verifies (3.17).

In case (b), write $W'/U = 1-(1-U^{-1})/(1-U^{-1}V)$ where the last fraction has the distribution of case (a) but with α and β interchanged. Hence we have (3.18).

Formulas (3.17) and (3.18) can be used to show that g_λ and $\widetilde{g}_\lambda$ are not beta densities. Details can be found in Proposition 3.2 in [5]. $\square$

3.2. *Harmonic functions on quadrants*. Lemma 3.1 is applied to construct two processes: $(\omega^\lambda,\rho^\lambda)$ using case (a) of the lemma and $(\widetilde{\omega}^\lambda,\widetilde{\rho}^\lambda)$ using case (b). Parameters (α,β) are fixed while in both cases $0<\lambda<\infty$. ω^λ and $\widetilde{\omega}^\lambda$ are new i.i.d. Beta(α,β) environments. ρ^λ and $\widetilde{\rho}^\lambda$ are harmonic functions on $\mathbb{Z}_+^2$ that give rise to Doob transformed transition probabilities π^λ and $\widetilde{\pi}^\lambda$, respectively.

The need for two cases (a) and (b) arises from the two-to-one connection between parameters $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$ and $0<\lambda<\infty$, given in Lemma 2.6(a). To parametrize in terms of ξ, let $\lambda(\xi)$ be given by Lemma 2.6(a) and define

$$(\bar{\omega}^\xi,\bar{\rho}^\xi,\bar{\pi}^\xi) = \begin{cases} (\omega^{\lambda(\xi)},\rho^{\lambda(\xi)},\pi^{\lambda(\xi)}), & \xi_1 \in (\xi_1^*,1) = \left(\dfrac{\alpha}{\alpha+\beta},1\right), \\ (\widetilde{\omega}^{\lambda(\xi)},\widetilde{\rho}^{\lambda(\xi)},\widetilde{\pi}^{\lambda(\xi)}), & \xi_1 \in (0,\xi_1^*) = \left(0,\dfrac{\alpha}{\alpha+\beta}\right). \end{cases} \tag{3.20}$$

This way we establish in Theorem 3.5 that $\xi \in (\mathrm{ri}\,\mathcal{U})\setminus\{\xi^*\}$ is the limiting velocity of the Doob transformed RWRE with transition $\bar{\pi}^\xi$. The law of large numbers velocity $\xi^* = (\frac{\alpha}{\alpha+\beta},\frac{\beta}{\alpha+\beta})$ does not arise from any transition π^λ or $\widetilde{\pi}^\lambda$ for a finite λ.

We now perform construction (3.6)–(3.9) of harmonic functions ρ^λ and forward transition probabilities ω^λ. Their distributional properties come from part (a) of Lemma 3.1. The inputs of the construction are boundary variables and backward transition probabilities in the bulk. We create infinitely many coupled systems indexed by the parameter $0<\lambda<\infty$. Remark 3.4 below comments on the similar construction of $(\widetilde{\omega}^\lambda,\widetilde{\rho}^\lambda)$ based on case (b) of Lemma 3.1.

Let $\overline{\mathbb{P}}$ be the joint distribution of mutually independent random variables

$$\{\Delta_{(i,0)},\Delta_{(0,j)},\breve{\omega}_{x,x-e_1} : i,j\in\mathbb{N}, x\in\mathbb{N}^2\} \tag{3.21}$$

with marginals $\Delta_{(i,0)},\Delta_{(0,j)} \sim \text{Unif}(0,1)$ and $\breve{\omega}_{x,x-e_1} \sim \text{Beta}(\alpha,\beta)$. Set

$$\breve{\omega}_{x,x-e_2} = 1-\breve{\omega}_{x,x-e_1}. \tag{3.22}$$

For fixed positive a and b, let $F^{-1}(\cdot;a,b):[0,1]\to[0,1]$ denote the inverse function of the Beta(a,b) c.d.f. (1.8). For $0<\lambda<\infty$, define coupled boundary variables on the coordinate axes:

$$\rho^{\lambda}_{(i-1,0),(i,0)} = F^{-1}(\Delta_{(i,0)}; \alpha+\lambda, \beta) \qquad \text{for } i \geq 1 \quad \text{and}$$
$$\rho^{\lambda}_{(0,j-1),(0,j)} = \frac{1}{F^{-1}(\Delta_{(0,j)}; \lambda, \alpha)} \qquad \text{for } j \geq 1. \tag{3.23}$$

$\{\rho^{\lambda}_{(i-1,0),(i,0)} : i \geq 1\}$ are i.i.d. Beta$(\alpha+\lambda, \beta)$, $\{(\rho^{\lambda}_{(0,j-1),(0,j)})^{-1} : j \geq 1\}$ are i.i.d. Beta(λ, α), and the two collections are independent of each other and of $\{\breve{\omega}_{x,x-e_1} : x \in \mathbb{N}^2\}$.

For each $\lambda > 0$, apply (3.6)–(3.8) inductively to define random variables

$$\{\rho^{\lambda}_{x,x+e_1}, \rho^{\lambda}_{x,x+e_2}, \omega^{\lambda}_{x,x+e_1} : x \in \mathbb{Z}^2_+\} \tag{3.24}$$

indexed by the full quadrant. For $x \in \mathbb{Z}^2_+$, define additionally

$$\omega^{\lambda}_{x,x+e_2} = 1 - \omega^{\lambda}_{x,x+e_1}.$$

Conservation equations

$$\rho^{\lambda}_{x,x+e_1}\rho^{\lambda}_{x+e_1,x+e_1+e_2} = \rho^{\lambda}_{x,x+e_2}\rho^{\lambda}_{x+e_2,x+e_1+e_2} \tag{3.25}$$

are satisfied around all unit squares. Extend the definition of $\rho^{\lambda}_{x,x+e_i}$ from directed nearest-neighbor edges to $\rho^{\lambda}_{x,y}$ for all $x, y \in \mathbb{Z}^2_+$ so that $\rho^{\lambda}_{x,x} = 1$ and

$$\rho^{\lambda}_{x,y}\rho^{\lambda}_{y,z} = \rho^{\lambda}_{x,z} \qquad \text{for all } x, y, z \in \mathbb{Z}^2_+. \tag{3.26}$$

In the sequel, we write $\rho^{\lambda}(x, y)$ for $\rho^{\lambda}_{x,y}$ when subscripts are not convenient.

A *down-right* lattice path $\{x_j\}_{j\in\mathbb{Z}}$ is a nearest-neighbor path with increments $x_j - x_{j-1} \in \{e_1, -e_2\}$. Any bounded portion of a down-right path in $\mathbb{Z}^2_+$ can be obtained by finitely many corner flips starting from the path $x_j = (j^+, j^-)$ that lies on the coordinate axes. A single corner flip is the transformation $(U, V, W) \mapsto (U', V', W')$ in Figure 3. Figure 4 illustrates successive corners flips. By Lemma 3.1(a), each iteration of (3.6)–(3.8) preserves the properties in the next proposition. Inequalities on $\mathbb{Z}^2$ are interpreted coordinatewise.

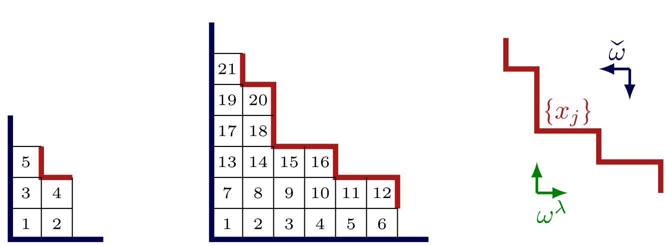

FIG. 4. *Illustration of the corner-flipping procedure. Left and center: To obtain the ρ^{λ} values on the thick edges of the down-right path inside the quadrant start with the known values on the boundary edges and consecutively flip the corners of the squares, for example, in the indicated order. Right: ratios ρ^{λ} along the down-right path, transitions $\breve{\omega}$ out of sites northeast of the path, and transitions ω^{λ} southwest of it are jointly independent.*

PROPOSITION 3.3. *Let random variables* (3.21) *and* (3.23) *be given, and define the process* (3.24) *inductively through* (3.6)–(3.8). *Then for each* $0 < \lambda < \infty$ *we have the following distributional properties.*

Random variables $\{\omega^{\lambda}_{x,x+e_1} : x \in \mathbb{Z}^2_+\}$ *are i.i.d.* $\mathrm{Beta}(\alpha, \beta)$. *For each* $x \in \mathbb{Z}^2_+$, *we have the marginal distributions*

$$\rho^{\lambda}_{x,x+e_1} \sim \mathrm{Beta}(\alpha+\lambda, \beta) \quad \textit{and} \quad \frac{1}{\rho^{\lambda}_{x,x+e_2}} \sim \mathrm{Beta}(\lambda, \alpha). \tag{3.27}$$

For any down-right path $\{x_j\}_{j\in\mathbb{Z}}$ *in* $\mathbb{Z}^2_+$, *the following random variables are all mutually independent*:

$$\{\rho^{\lambda}_{x_j,x_{j+1}} : j \in \mathbb{Z}\}, \qquad \bigcup_{j\in\mathbb{Z}} \{\breve{\omega}_{z,z-e_1} : z \geq x_j + (1,1)\}, \quad \textit{and}$$
$$\bigcup_{j\in\mathbb{Z}} \{\omega^{\lambda}_{x,x+e_1} : 0 \leq x \leq x_j - (1,1)\}.$$

In particular, we have the translation invariance of the joint distribution: *for any* $a \in \mathbb{Z}^2_+$,

$$\begin{aligned}(3.28)\qquad &\big(\omega^{\lambda}_{x,x+e_1}, \rho^{\lambda}_{u,v}, \breve{\omega}_{z,z-e_1}\big)_{x,u,v\in\mathbb{Z}^2_+, z\in\mathbb{N}^2} \\ &\overset{d}{=} \big(\omega^{\lambda}_{a+x,a+x+e_1}, \rho^{\lambda}_{a+u,a+v}, \breve{\omega}_{a+z,a+z-e_1}\big)_{x,u,v\in\mathbb{Z}^2_+, z\in\mathbb{N}^2}.\end{aligned}$$

Translation invariance (3.28) is a consequence of the down-right path statement: with a new origin at a, the edge variables $\rho^{\lambda}_{a+(i-1)e_k,a+ie_k}$ for $i \in \mathbb{N}$ and $k \in \{1,2\}$ and the bulk variables $(\breve{\omega}_{z,z-e_1})_{z\in a+\mathbb{N}^2}$ have the same joint distribution as the original ones given in (3.21) and (3.23).

Equations (3.15) give the identities

$$\frac{\omega^{\lambda}_{x,x+e_1}}{\rho^{\lambda}_{x,x+e_1}} + \frac{\omega^{\lambda}_{x,x+e_2}}{\rho^{\lambda}_{x,x+e_2}} = 1 \qquad \text{for } x \in \mathbb{Z}^2_+ \quad \text{and} \tag{3.29}$$

$$\frac{\breve{\omega}_{x,x-e_1}}{\rho^{\lambda}_{x-e_1,x}} + \frac{\breve{\omega}_{x,x-e_2}}{\rho^{\lambda}_{x-e_2,x}} = 1 \qquad \text{for } x \in \mathbb{N}^2. \tag{3.30}$$

Consider the RWRE $P^{\omega^{\lambda}}$ that uses forward transitions ω^{λ}. Combining (3.29) with (3.26) gives the following for any fixed $y \in \mathbb{Z}^2_+$:

$$\omega^{\lambda}_{x,x+e_1}\rho^{\lambda}_{x+e_1,y} + \omega^{\lambda}_{x,x+e_2}\rho^{\lambda}_{x+e_2,y} = \rho^{\lambda}_{x,y} \qquad \text{for } x \in \mathbb{Z}^2_+. \tag{3.31}$$

In other words, for any fixed y, $\rho^{\lambda}_{x,y}$ is a harmonic function of x for transition probabilities $\omega^{\lambda}_{x,x+e_i}$ on $\mathbb{Z}^2_+$. In particular, for two points $u \leq y$ in $\mathbb{Z}^2_+$,

$$\rho^{\lambda}_{u,y} = E^{\omega^{\lambda}}_u\big[\rho^{\lambda}(X_{\tau^-_y}, y)\big]. \tag{3.32}$$

By (3.30), the same function ρ^λ works for backward transitions $\breve{\omega}$ and gives

$$\rho^\lambda_{u,y} = E^{\breve{\omega}}_y\big[\rho^\lambda(u, X_{\tau_u^+})\big]. \tag{3.33}$$

Perform a Doob transform on P^{ω^λ} by introducing transition probabilities

$$\pi^\lambda_{x,x+e_i} = \frac{\omega^\lambda_{x,x+e_i}}{\rho^\lambda_{x,x+e_i}}, \qquad i \in \{1,2\}. \tag{3.34}$$

The RWRE that uses transitions π^λ is the ρ^λ-*tilted* RWRE and its quenched path measure is denoted by P^{π^λ}. Let $x_{0,k} = (x_0, \dots, x_k)$ be an up-right path from $x_0 = u$ that first enters the boundary $\mathbb{B}^-_y$ [recall (3.1)] at the endpoint x_k. Then

$$\begin{aligned} P^{\pi^\lambda}_u\{X_{0,k} = x_{0,k}\} &= \prod_{i=0}^{k-1} \frac{\omega^\lambda_{x_i,x_{i+1}}}{\rho^\lambda_{x_i,x_{i+1}}} = \frac{P^{\omega^\lambda}_u\{X_{0,k} = x_{0,k}\}}{\rho^\lambda_{u,x_k}} \\ &= \frac{P^{\omega^\lambda}_u\{X_{0,k} = x_{0,k}\}\rho^\lambda_{x_k,y}}{\rho^\lambda_{u,y}} \\ &= \frac{E^{\omega^\lambda}_u[\rho^\lambda(X_{\tau_y^-}, y), X_{0,k} = x_{0,k}]}{E^{\omega^\lambda}_u[\rho^\lambda(X_{\tau_y^-}, y)]}. \end{aligned} \tag{3.35}$$

A useful consequence for later is the following identity for the probability of hitting one of the two parts of the boundary. For fixed $u \le y$ in $\mathbb{Z}^2_+$ and $i \in \{1,2\}$, summing (3.35) over all paths entering $\mathbb{B}^-_y$ at a point of $\mathbb{B}^{(-i)}_y$ gives

$$P^{\pi^\lambda}_u\big\{X_{\tau_y^-} \in \mathbb{B}^{(-i)}_y\big\} = \frac{E^{\omega^\lambda}_u[\rho^\lambda(X_{\tau_y^-}, y), X_{\tau_y^-} \in \mathbb{B}^{(-i)}_y]}{E^{\omega^\lambda}_u[\rho^\lambda(X_{\tau_y^-}, y)]}. \tag{3.36}$$

Analogously, we define the backwards Doob transform

$$\breve{\pi}^\lambda_{x,x-e_i} = \frac{\breve{\omega}_{x,x-e_i}}{\rho^\lambda_{x-e_i,x}}, \qquad i \in \{1,2\}. \tag{3.37}$$

Then as above for fixed $u \le y$ in $\mathbb{Z}^2_+$ and a down-left path $x_{0,k}$ started from y that first enters $\mathbb{B}^+_u$ at x_k,

$$\begin{aligned} P^{\breve{\pi}^\lambda}_y\{X_{0,k} = x_{0,k}\} &= \frac{P^{\breve{\omega}}_y\{X_{0,k} = x_{0,k}\}\rho^\lambda_{u,x_k}}{\rho^\lambda_{u,y}} \\ &= \frac{E^{\breve{\omega}}_y[\rho^\lambda(u, X(\tau_u^+)), X_{0,k} = x_{0,k}]}{E^{\breve{\omega}}_y[\rho^\lambda(u, X(\tau_u^+))]}. \end{aligned} \tag{3.38}$$

REMARK 3.4. Let us comment briefly on the version of the construction above that produces $(\widetilde{\omega}^{\lambda}, \widetilde{\rho}^{\lambda})$ based on case (b) of Lemma 3.1. Instead of (3.23), begin with

$$\widetilde{\rho}^{\lambda}_{(i-1,0),(i,0)} = \frac{1}{F^{-1}(\Delta_{(i,0)}; \lambda, \beta)} \qquad \text{for } i \geq 1 \quad \text{and} \tag{3.39}$$
$$\widetilde{\rho}^{\lambda}_{(0,j-1),(0,j)} = F^{-1}(\Delta_{(0,j)}; \beta+\lambda, \alpha) \qquad \text{for } j \geq 1.$$

Equations (3.6)–(3.9) are iterated exactly as before. Proposition (3.3) is valid word for word for $(\widetilde{\omega}^{\lambda}, \widetilde{\rho}^{\lambda}, \breve{\omega})$, except that (3.27) is replaced with

$$\frac{1}{\widetilde{\rho}^{\lambda}_{x,x+e_1}} \sim \mathrm{Beta}(\lambda, \beta) \quad \text{and} \quad \widetilde{\rho}^{\lambda}_{x,x+e_2} \sim \mathrm{Beta}(\beta+\lambda, \alpha). \tag{3.40}$$

The Doob-transformed transitions are defined again by

$$\widetilde{\pi}^{\lambda}_{x,x+e_i} = \frac{\widetilde{\omega}^{\lambda}_{x,x+e_i}}{\widetilde{\rho}^{\lambda}_{x,x+e_i}}, \qquad i \in \{1,2\}, \tag{3.41}$$

with quenched path measure $P^{\widetilde{\pi}^{\lambda}}$. Equations (3.32) and (3.35) are then also valid for $(\widetilde{\omega}^{\lambda}, \widetilde{\rho}^{\lambda}, \widetilde{\pi}^{\lambda})$.

Now let $\lambda(\xi)$ be given by Lemma 2.6(a) for $\xi = (\xi_1, 1-\xi_1) \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$. Combine the two constructions $(\omega^{\lambda}, \rho^{\lambda}, \pi^{\lambda})$ and $(\widetilde{\omega}^{\lambda}, \widetilde{\rho}^{\lambda}, \widetilde{\pi}^{\lambda})$ by defining $(\bar{\omega}^{\xi}, \bar{\rho}^{\xi}, \bar{\pi}^{\xi})$ by (3.20) for all $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$. The quenched path measure of the RWRE that uses transition $\bar{\pi}^{\xi}$ is given by

$$P^{\bar{\pi}^{\xi}}_x = \begin{cases} P^{\pi^{\lambda(\xi)}}_x, & \xi_1 \in (\xi_1^*, 1), \\ P^{\widetilde{\pi}^{\lambda(\xi)}}_x, & \xi_1 \in (0, \xi_1^*). \end{cases} \tag{3.42}$$

THEOREM 3.5. *We have this almost sure law of large numbers: for all* $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$,

$$P^{\bar{\pi}^{\xi}}_0\{n^{-1}X_n \to \xi\} = 1 \qquad \overline{\mathbb{P}}\text{-almost surely.}$$

PROOF. We give the details for the case $\xi_1 \in (\xi_1^*, 1) = (\frac{\alpha}{\alpha+\beta}, 1)$ with $\lambda = \lambda(\xi)$. By translation invariance (Proposition 3.3), we can extend $\log \rho^{\lambda}$ to a process $\{\log \rho^{\lambda}_{x,y} : x, y \in \mathbb{Z}^2\}$ indexed by the entire lattice. This process has the shift-invariance and additivity properties of (2.4); in other words, it is a stationary L^1 cocycle. Such processes satisfy a uniform ergodic theorem under certain regularity assumptions as, for example, given in Theorem A.3 in the Appendix of [16]. Variable $\log \rho^{\lambda}_{0,e_i}$ is integrable and (3.29) gives the lower bound $\log \rho^{\lambda}_{x,x+e_i} \geq \log \omega^{\lambda}_{x,x+e_i}$ in terms of an i.i.d. process with strictly more than two moments. This is sufficient for Theorem A.3 of [16] which gives the almost sure limit

$$\lim_{n\to\infty} n^{-1} \max_{|x|_1\le n} \left|\log \rho^{\lambda}_{0,x} - m(\lambda)\cdot x\right| = 0, \tag{3.43}$$

with mean vector

$$\begin{aligned} m(\lambda) &= \mathbb{E}[\log \rho^{\lambda}_{0,e_1}]e_1 + \mathbb{E}[\log \rho^{\lambda}_{0,e_2}]e_2 \\ &= \bigl(\psi_0(\alpha+\lambda) - \psi_0(\alpha+\beta+\lambda)\bigr)e_1 + \bigl(\psi_0(\alpha+\lambda) - \psi_0(\lambda)\bigr)e_2. \end{aligned}$$

Proposition 3.3 says that under $\overline{\mathbb{P}}$ transitions ω^{λ} have the same distribution as ω does under $\mathbb{P}$. Then, by (1.5) and (3.43) we have $\overline{\mathbb{P}}$-almost surely

$$\begin{aligned} &\lim_{n\to\infty} n^{-1} \log P_0^{\pi^{\lambda}}\{X_n = [n\zeta]\} \\ &\quad = \lim_{n\to\infty} \bigl(n^{-1}\log P_0^{\omega^{\lambda}}\{X_n = [n\zeta]\} - n^{-1}\log \rho^{\lambda}_{0,[n\zeta]}\bigr) \\ &\quad = -I_q(\zeta) + \zeta_1\psi_0(\alpha+\beta+\lambda) + \zeta_2\psi_0(\lambda) - \psi_0(\alpha+\lambda). \end{aligned}$$

In other words, the distribution of X_n/n under $P_0^{\pi^{\lambda}}$ satisfies a (quenched) large deviation principle with rate function

$$I_q^{\lambda}(\zeta) = I_q(\zeta) - \zeta_1\psi_0(\alpha+\beta+\lambda) - \zeta_2\psi_0(\lambda) + \psi_0(\alpha+\lambda).$$

By the strict convexity of I_q and its expression (2.13), $I_q^{\lambda}(\zeta)$ has a unique zero at $\zeta = \xi$ with ξ_1 given by the right-hand side of (2.10). This proves Theorem 3.5. □

Lastly, we record continuity and monotonicity satisfied by the boundary variables defined in (3.23) and extended by the construction to all $x, y \in \mathbb{Z}_+^2$:

$$\rho^{\gamma}_{x,y} \underset{\gamma\to\lambda}{\longrightarrow} \rho^{\lambda}_{x,y}, \qquad \omega^{\gamma}_{x} \underset{\gamma\to\lambda}{\longrightarrow} \omega^{\lambda}_{x}, \quad \text{and} \tag{3.44}$$

$$\gamma > \lambda > 0 \Longrightarrow \rho^{\gamma}_{x,x+e_1} > \rho^{\lambda}_{x,x+e_1} \quad \text{and} \quad \rho^{\gamma}_{x,x+e_2} < \rho^{\lambda}_{x,x+e_2}. \tag{3.45}$$

Limits (3.44) are valid also for $(\widetilde{\omega}, \widetilde{\rho})$, but the monotonicity is reversed:

$$\gamma > \lambda > 0 \Longrightarrow \widetilde{\rho}^{\gamma}_{x,x+e_1} < \widetilde{\rho}^{\lambda}_{x,x+e_1} \quad \text{and} \quad \widetilde{\rho}^{\gamma}_{x,x+e_2} > \widetilde{\rho}^{\lambda}_{x,x+e_2}. \tag{3.46}$$

3.3. *Global harmonic functions.* In this section, we construct the process $B^{\xi}_{x,y}$ discussed in Section 2.1. We summarize the construction in Theorem 3.6, derive the claims of Section 2.1, then prove Theorem 3.6 piece by piece.

The probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ is the product space $\Omega = [0,1]^{\mathbb{Z}^2}$ of beta environments $\omega = (\omega_{x,x+e_1} : x \in \mathbb{Z}^2)$ where the variables $\omega_{x,x+e_1}$ are i.i.d. Beta(α,β)-distributed. Shift mappings T_z act by $(T_z\omega)_{x,x+e_i} = \omega_{x+z,x+z+e_i}$ for $x, z \in \mathbb{Z}^2$. Velocities $\xi \in \operatorname{ri}\mathcal{U}$ are denoted by $\xi = (\xi_1,\xi_2) = (\xi_1, 1-\xi_1)$, and the distinguished velocity is $\xi^* = (\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$. A down-right path $\{x_i\} \subset \mathbb{Z}^2$ in part (c) below satisfies $x_{i+1} - x_i \in \{e_1, -e_2\}$. The increment distributions in part (a.1) below are those of case (a) of Lemma 3.1, while part (a.2) corresponds to case (b) of Lemma 3.1.

THEOREM 3.6. *Fix $0 < \alpha, \beta < \infty$. On the probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ there exists a stochastic process $\{B^{\xi}_{x,y}(\omega) : x, y \in \mathbb{Z}^2, \xi \in \operatorname{ri}\mathcal{U}\}$ with the following properties*:

I. Distribution and expectations.

(a) *For $\xi = \xi^*$ the process $B^{\xi^*}_{x,y}$ is identically zero. For $\xi \in (\operatorname{ri}\mathcal{U}) \setminus \{\xi^*\}$, the marginal distributions and expectations are as follows, with $\lambda(\xi)$ given by* (2.10)–(2.11).

(a.1) *For $\xi_1 \in (\xi_1^*, 1)$,*

$$e^{B^{\xi}_{x,x+e_1}} \sim \operatorname{Beta}\bigl(\alpha + \lambda(\xi), \beta\bigr) \quad \textit{and} \quad e^{-B^{\xi}_{x,x+e_2}} \sim \operatorname{Beta}\bigl(\lambda(\xi), \alpha\bigr),$$

and so

$$\begin{aligned} &\mathbb{E}\bigl[B^{\xi}_{x,x+e_1}\bigr] = \psi_0\bigl(\alpha + \lambda(\xi)\bigr) - \psi_0\bigl(\alpha + \beta + \lambda(\xi)\bigr) \quad \textit{and} \\ &\mathbb{E}\bigl[B^{\xi}_{x,x+e_2}\bigr] = \psi_0\bigl(\alpha + \lambda(\xi)\bigr) - \psi_0\bigl(\lambda(\xi)\bigr). \end{aligned} \tag{3.47}$$

(a.2) *For $\xi_1 \in (0, \xi_1^*)$,*

$$e^{-B^{\xi}_{x,x+e_1}} \sim \operatorname{Beta}\bigl(\lambda(\xi), \beta\bigr) \quad \textit{and} \quad e^{B^{\xi}_{x,x+e_2}} \sim \operatorname{Beta}\bigl(\beta + \lambda(\xi), \alpha\bigr),$$

and so

$$\begin{aligned} &\mathbb{E}\bigl[B^{\xi}_{x,x+e_1}\bigr] = \psi_0\bigl(\beta + \lambda(\xi)\bigr) - \psi_0\bigl(\lambda(\xi)\bigr) \quad \textit{and} \\ &\mathbb{E}\bigl[B^{\xi}_{x,x+e_2}\bigr] = \psi_0\bigl(\beta + \lambda(\xi)\bigr) - \psi_0\bigl(\alpha + \beta + \lambda(\xi)\bigr). \end{aligned} \tag{3.48}$$

(b) *For any $z \in \mathbb{Z}^2$, the variables $\{B^{\xi}_{x,y}(\omega) : x, y \not\leq z, \xi \in \operatorname{ri}\mathcal{U}\}$ are independent of the variables $\{\omega_{x+e_1} : x \leq z\}$.*

(c) *For a fixed $\xi \in (\operatorname{ri}\mathcal{U}) \setminus \{\xi^*\}$, the joint distribution of (ω, B^{ξ}) is the same as that of $(\bar{\omega}^{\xi}, \log \bar{\rho}^{\xi})$ defined in* (3.20). *This distribution is described in Proposition* 3.3 *and Remark* 3.4. *In particular, on any down-right path $\{x_i\}_{i\in\mathbb{Z}}$ on $\mathbb{Z}^2$ the variables $\{B^{\xi}_{x_i,x_{i+1}}\}_{i\in\mathbb{Z}}$ are independent.*

(d) *The quenched large deviation rate function of* (1.5) *satisfies*

$$I_q(\xi) = -\inf_{\zeta\in\operatorname{ri}\mathcal{U}} \bigl\{\mathbb{E}\bigl[B^{\zeta}_{0,e_1}\bigr]\xi_1 + \mathbb{E}\bigl[B^{\zeta}_{0,e_2}\bigr]\xi_2\bigr\} \textit{for all } \xi \in \mathcal{U}.$$

The infimum is uniquely attained at $\zeta = \xi$.

II. Pointwise properties. *There exists an event $\Omega_0 \subset \Omega$ such that $\mathbb{P}(\Omega_0) = 1$ and the following statements hold for all $\omega \in \Omega_0$, $\xi, \zeta \in \operatorname{ri}\mathcal{U}$, and $x, y, z \in \mathbb{Z}^2$*:

(e) *Cocycle properties*: *stationarity*

$$B^{\xi}_{x+z,y+z}(\omega) = B^{\xi}_{x,y}(T_z\omega) \tag{3.49}$$

and additivity

$$B^{\xi}_{x,y}(\omega) + B^{\xi}_{y,z}(\omega) = B^{\xi}_{x,z}(\omega). \tag{3.50}$$

In particular, $B^{\xi}_{x,x}(\omega) = 0$ *and* $B^{\xi}_{x,y}(\omega) = -B^{\xi}_{y,x}(\omega)$.

(f) *Harmonic increments*:

$$\omega_{x,x+e_1} e^{-B^{\xi}_{x,x+e_1}(\omega)} + \omega_{x,x+e_2} e^{-B^{\xi}_{x,x+e_2}(\omega)} = 1. \tag{3.51}$$

(g) *Monotonicity*: *If* $\xi \cdot e_1 < \zeta \cdot e_1$, *then*

$$B^{\xi}_{x,x+e_1} \geq B^{\zeta}_{x,x+e_1} \quad \text{and} \quad B^{\xi}_{x,x+e_2} \leq B^{\zeta}_{x,x+e_2}.$$

(h) $B^{\xi}_{x,y}(\omega)$ *is a cadlag function of* $\xi_1 \in (0, 1)$.

III. Limits. *For each fixed* $\xi \in \mathrm{ri}\,\mathcal{U}$, *there exists an event* $\Omega^{(\xi)}_0 \subset \Omega$ *that can vary with* ξ, *has* $\mathbb{P}(\Omega^{(\xi)}_0) = 1$, *and is such that the following statements hold for each* $\omega \in \Omega^{(\xi)}_0$ *and* $x, y \in \mathbb{Z}^2$:

(i) *For any sequence* $\xi^n \in \mathrm{ri}\,\mathcal{U}$ *with* $\xi^n \to \xi$, *we have* $B^{\xi^n}_{x,y}(\omega) \to B^{\xi}_{x,y}(\omega)$.

(j) *For any sequence* $z_N \in \mathbb{Z}^2$ *with* $|z_N|_1 \to \infty$ *and* $z_N/N \to \xi$ *we have*

$$\begin{aligned} B^{\xi}_{x,y}(\omega) = \lim_{N\to\infty} \big(&\log P^{\omega}_x\{X_{|z_N - x|_1} = z_N\} \\ &- \log P^{\omega}_y\{X_{|z_N - y|_1} = z_N\}\big). \end{aligned} \tag{3.52}$$

Comments about the theorem. The shift-invariant process $(\bar{\omega}^{\xi}, \log \bar{\rho}^{\xi})$ in part (c) was constructed in (3.20) in Section 3.2 on the quadrant $\mathbb{Z}^2_+$. For part (c) above to make sense, extend $(\bar{\omega}^{\xi}, \log \bar{\rho}^{\xi})$ to the full lattice $\mathbb{Z}^2$ by Kolmogorov's extension theorem. It is also important to distinguish when ξ is fixed and when it can vary. The distributional equality of B^{ξ} and $\log \bar{\rho}^{\xi}$ is not valid jointly across different ξ because the joint distribution of $\{B^{\xi}\}$ is not the one constructed in Section 3.2 through a coupling with uniform random variables. Note the distinction between (h) and (i): at fixed ξ there is continuity almost surely, but globally over ξ the path is cadlag.

We prove the results of Section 2.1. As given in (2.5), the transformed transition probability is defined by $\kappa^{\xi}_{x,x+e_i}(\omega) = \omega_{x,x+e_i} e^{-B^{\xi}_{x,x+e_i}(\omega)}$.

PROOF OF THEOREM 2.1. Theorem 2.1 is a subset of Theorem 3.6. □

PROOF OF THEOREM 2.2. Express $\kappa^{\xi}_{x,x+e_1}$ using $(B^{\xi}_{x,x+e_1}, B^{\xi}_{x,x+e_2})$:

$$\kappa^{\xi}_{x,x+e_1} = \frac{e^{B^{\xi}_{x,x+e_2}} - 1}{e^{B^{\xi}_{x,x+e_2}} - e^{B^{\xi}_{x,x+e_1}}}. \tag{3.53}$$

To prove the formula, substitute in limits (3.52) and use the Markov property. Note that this is the analogue of $W'/U = (V-1)/(V-U)$ from (3.10).

Given $n \in \mathbb{Z}$, define the down-right path $\{x^j\}$ by

$$x^{2k} = (n+k, -k) \quad \text{and} \quad x^{2k+1} = (n+k+1, -k) \qquad \text{for } k \in \mathbb{Z}.$$

The antidiagonal $\{x : x_1 + x_2 = n\}$ is the subsequence $\{x^{2k}\}$, and

$$(B^{\xi}_{x^{2k},x^{2k}+e_1}, B^{\xi}_{x^{2k},x^{2k}+e_2}) = (B^{\xi}_{x^{2k},x^{2k+1}}, -B^{\xi}_{x^{2k-1},x^{2k}}).$$

These pairs are i.i.d. by part (c) of Theorem 3.6.

For $\xi \in \mathcal{U} \setminus \{\xi^*\}$, the law of large numbers part (2.6) of Theorem 2.2 follows from Theorem 3.5 and the observation that (ω, B^{ξ}) has the same distribution as $(\bar{\omega}^{\xi}, \log \bar{\rho}^{\xi})$, as stated in part (c) of Theorem 3.6.

For $\xi = \xi^*$, $P_0^{\kappa^{\xi^*}(\omega)} = P_0^{\omega}$, the original path measure in an i.i.d. environment, and the LLN is the one in (1.2). $\square$

PROOF OF THEOREM 2.3. Immediate from the limits (3.52). $\square$

THEOREM 3.7. *For $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$, random variable κ^{ξ}_{0,e_1} is not beta distributed. Let $\lambda(\xi)$ be given by* (2.10)–(2.11) *and let g_λ and $\widetilde{g}_\lambda$ be the functions defined in* (3.17)–(3.18). *Then the density function $f^{\xi}(x)$ of κ^{ξ}_{0,e_1} for $0 < x < 1$ is given by*

$$f^{\xi}(x) = \begin{cases} g_{\lambda(\xi)}(x), & \xi_1 \in (\xi_1^*, 1), \\ \widetilde{g}_{\lambda(\xi)}(x), & \xi_1 \in (0, \xi_1^*). \end{cases} \tag{3.54}$$

PROOF. This comes from Proposition 3.2. Formula (3.53), the independence of B^{ξ}_{0,e_1} and B^{ξ}_{0,e_2}, and their distributions given in part (a) of Theorem 3.6 imply that κ^{ξ}_{0,e_1} has the distribution of W'/U in Proposition 3.2. $\square$

We begin now with some preliminaries toward the proof of Theorem 3.6. In addition to the probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ with its beta environment ω, we use the coupled processes $\{\bar{\omega}^{\xi}_{x,x+e_1}, \bar{\rho}^{\xi}_{x,y} : x, y \in \mathbb{Z}^2_+\}$ under distribution $\overline{\mathbb{P}}$, constructed in Section 3.2 with properties given in Proposition 3.3 and the subsequent discussion. Each environment $\bar{\omega}^{\xi}$ has the i.i.d. Beta(α, β) distribution of the original environment ω. The construction of $B^{\xi}_{x,y}$ is based on the limits (3.52). These limits are proved by bounding ratios of hitting probabilities with variables $\bar{\rho}^{\xi}$ from (3.20) whose distributions we control.

We begin with two lemmas that do not use the beta distributions. The setting for Lemmas 3.8 and 3.9 is the following: $a \in \mathbb{Z}^2$ and on the quadrant $\mathbb{S} = a + \mathbb{Z}^2_+$ we have a Markov transition probability p such that

$$0 < p_{x,x+e_1} = 1 - p_{x,x+e_2} < 1 \qquad \text{for all } x \in \mathbb{S}. \tag{3.55}$$

Let P_x with expectation E_x denote the Markov chain with transition p starting at $x \in \mathbb{S}$. Use the standard notation for hitting probabilities:

$$F(x, y) = P_x(\exists n \geq 0 : X_n = y).$$

LEMMA 3.8. *The following inequalities hold for all $y \in \mathbb{S} = a + \mathbb{Z}_+^2$:*

$$\frac{F(a+e_1, y+e_2)}{F(a, y+e_2)} \leq \frac{F(a+e_1, y)}{F(a, y)} \leq \frac{F(a+e_1, y+e_1)}{F(a, y+e_1)}. \tag{3.56}$$

The first two numerators can vanish but the denominators are all positive. The same inequalities hold with e_1 and e_2 switched around.

PROOF. The second statement follows by applying (3.56) to the transition probability $\widetilde{p}$ obtained by reflecting p across the diagonal passing through a: for $x = (x_1, x_2) \in \mathbb{Z}_+^2$ set $\widetilde{p}_{a+x,a+x+e_i} = p_{a+\widetilde{x},a+\widetilde{x}+e_{3-i}}$, where $\widetilde{x} = (x_2, x_1)$.

We prove claim (3.56) by induction on y. It is convenient to use the ratios

$$U_{x,y} = \frac{F(x, y)}{F(x, y-e_1)} \quad \text{and} \quad V_{x,y} = \frac{F(x, y)}{F(x, y-e_2)} \qquad \text{for } x \leq y \text{ in } \mathbb{S}.$$

The numerator does not vanish but the denominator can vanish and then the ratio has value ∞.

Equation (3.56) holds trivially for $y = a + \ell e_2$ with $\ell \geq 0$ because the first two numerators vanish while the other probabilities are positive. Hence we may assume $y \geq a + e_1$. By a shift of y, (3.56) is equivalent to having

$$\begin{aligned} &U_{a,y} \leq U_{a+e_1,y} &&\text{for } y \geq a + 2e_1 \quad \text{and} \\ &V_{a,y} \geq V_{a+e_1,y} &&\text{for } y \geq a + e_1 + e_2. \end{aligned} \tag{3.57}$$

We check the boundaries first. For $y = a + ke_1$ for $k \geq 2$, $U_{a,y} = U_{a+e_1,y} = p_{y-e_1,y}$. For $y = a + e_1 + \ell e_2$ for $\ell \geq 1$,

$$V_{a,y} = \frac{F(a, y-e_2)p_{y-e_2,y} + F(a, y-e_1)p_{y-e_1,y}}{F(a, y-e_2)} > p_{y-e_2,y} = V_{a+e_1,y}.$$

It remains the check (3.57) for $y = a + ke_1 + \ell e_2$ for $k \geq 2$ and $\ell \geq 1$. For $y \geq x + e_1 + e_2$, the Markov property and assumption (3.55) give

$$F(x, y) = F(x, y-e_1)p_{y-e_1,y} + F(x, y-e_2)p_{y-e_2,y}$$

from which we derive the identities

$$U_{x,y} = p_{y-e_1,y} + p_{y-e_2,y}\frac{U_{x,y-e_2}}{V_{x,y-e_1}} \quad \text{and} \quad V_{x,y} = p_{y-e_1,y}\frac{V_{x,y-e_1}}{U_{x,y-e_2}} + p_{y-e_2,y}$$

also for $y \geq x + e_1 + e_2$.

Now proceed by induction on $y \geq a + 2e_1 + e_2$, beginning with $y = a + 2e_1 + e_2$, and then taking e_1 and e_2 steps. The boundary cases checked above together with the induction assumption give $U_{a,y-e_2} \leq U_{a+e_1,y-e_2}$ and $V_{a,y-e_1} \geq V_{a+e_1,y-e_1}$. Then the identities above give $U_{a,y} \leq U_{a+e_1,y}$ and $V_{a,y} \geq V_{a+e_1,y}$. $\square$

LEMMA 3.9. *Let $v \ge a$ on $\mathbb{Z}_+^2$ and set $y = v + e_1 + e_2$. Suppose $f(x) > 0$ for x on the boundary $\mathbb{B}_y^-$. We have the following inequalities.*

For $a + e_1 \le v$:

$$\begin{aligned}\frac{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-2)}]}{E_{a+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-2)}]} &\le \frac{F(a,v)}{F(a+e_1,v)} \\ &\le \frac{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-1)}]}{E_{a+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-1)}]}.\end{aligned} \tag{3.58}$$

For $a + e_2 \le v$:

$$\begin{aligned}\frac{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-1)}]}{E_{a+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-1)}]} &\le \frac{F(a,v)}{F(a+e_2,v)} \\ &\le \frac{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-2)}]}{E_{a+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-2)}]}.\end{aligned} \tag{3.59}$$

PROOF. For the proof, fix v and do induction on $|v-a|_1 \ge 1$. Consider the case $a = v - ke_1$ for $k \ge 1$. Then

$$\frac{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-2)}]}{E_{a+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-2)}]} = p_{a,a+e_1} = \frac{F(a,v)}{F(a+e_1,v)}.$$

On the other hand, when the walk is required to hit $\mathbb{B}_y^{(-1)}$, both steps e_1 and e_2 are feasible from a, and so

$$E_a\big[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-1)}\big] \ge p_{a,a+e_1} E_{a+e_1}\big[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-1)}\big].$$

This establishes (3.58). Equation (3.59) for $a = v - ke_2$ for $k \ge 1$ follows in a symmetric manner. In particular, we have the full conclusion for $|v-a|_1 = 1$.

Suppose (3.58)–(3.59) hold for all pairs $a \le v$ with $|v-a|_1 = \ell \ge 1$. Consider $a \le v$ with $|v-a|_1 = \ell + 1$. We have the result when $a \in \{v - (\ell+1)e_1, v - (\ell+1)e_2\}$. Thus assume that $a < v$ coordinatewise. For $i \in \{1,2\}$, take the identity

$$\begin{aligned}p_{a,a+e_1} &\frac{E_{a+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} \\ &+ p_{a,a+e_2} \frac{E_{a+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} = 1\end{aligned}$$

and rearrange it to yield the two identities

$$\frac{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_{a+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} = p_{a,a+e_1} + p_{a,a+e_2} \frac{E_{a+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_{a+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} \times \left(\frac{E_{a+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_{a+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} \right)^{-1}, \tag{3.60}$$

$$\frac{E_a[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_{a+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} = p_{a,a+e_2} + p_{a,a+e_1} \frac{E_{a+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_{a+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} \times \left(\frac{E_{a+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]}{E_{a+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in \mathbb{B}_y^{(-i)}]} \right)^{-1}. \tag{3.61}$$

Derive the analogous equations for ratios of hitting probabilities from

$$p_{a,a+e_1} \frac{F(a+e_1, v)}{F(a, v)} + p_{a,a+e_2} \frac{F(a+e_2, v)}{F(a, v)} = 1.$$

Apply the induction assumption on the right-hand sides of (3.60) and (3.61) and their counterparts for the ratios of hitting probabilities. This verifies (3.58) and (3.59) for u. $\square$

The remainder of the proof relies on the beta environment. The next proposition gives control over limits of hitting probability ratios through the harmonic functions constructed for Proposition 3.3. Quenched hitting probabilities are denoted by

$$F^\omega(x, y) = F^\omega_{x,y} = P^\omega_x(\exists n \geq 0 : X_n = y). \tag{3.62}$$

When $x \leq y$, this is of course $F^\omega_{x,y} = P^\omega_x(X_{|y-x|_1} = y)$, which we also use occasionally when the notation is not too heavy.

PROPOSITION 3.10. *Fix $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \{\xi^*\}$. If $\eta, \zeta \in \mathrm{ri}\,\mathcal{U}$ are such that*

$$\eta_1 < \xi_1 < \zeta_1 \tag{3.63}$$

then for all $x \in \mathbb{Z}_+^2$ *and* $z \in \mathbb{Z}^2$ *we have almost surely*

$$\varlimsup_{N\to\infty} \frac{F^{\bar{\omega}^\xi}_{x,[N\zeta]+z}}{F^{\bar{\omega}^\xi}_{x+e_1,[N\zeta]+z}} \le \bar{\rho}^\xi_{x,x+e_1} \le \varliminf_{N\to\infty} \frac{F^{\bar{\omega}^\xi}_{x,[N\eta]+z}}{F^{\bar{\omega}^\xi}_{x+e_1,[N\eta]+z}} \quad \textit{and}$$

$$\varliminf_{N\to\infty} \frac{F^{\bar{\omega}^\xi}_{x,[N\zeta]+z}}{F^{\bar{\omega}^\xi}_{x+e_2,[N\zeta]+z}} \ge \bar{\rho}^\xi_{x,x+e_2} \ge \varlimsup_{N\to\infty} \frac{F^{\bar{\omega}^\xi}_{x,[N\eta]+z}}{F^{\bar{\omega}^\xi}_{x+e_2,[N\eta]+z}}.$$

Recall from (3.20) *that these inequalities split into two separate results*: *one for* $(\omega^{\lambda(\xi)}, \rho^{\lambda(\xi)})$ *when* $\xi_1 \in (\xi_1^*, 1)$ *and one for* $(\widetilde{\omega}^{\lambda(\xi)}, \widetilde{\rho}^{\lambda(\xi)})$ *when* $\xi_1 \in (0, \xi_1^*)$.

PROOF. The inequalities claimed are all proved the same way. We illustrate with the first one. Let $y_N = [N\zeta] + z + e_1 + e_2$. By (3.58), then (3.36), (3.32) and (3.26),

$$\begin{aligned}
\frac{F^{\bar{\omega}^\xi}_{x,[N\zeta]+z}}{F^{\bar{\omega}^\xi}_{x+e_1,[N\zeta]+z}} &\le \frac{E^{\bar{\omega}^\xi}_x[\bar{\rho}^\xi(X_{\tau^-_{y_N}}, y_N), X_{\tau^-_{y_N}} \in \mathbb{B}^{(-1)}_{y_N}]}{E^{\bar{\omega}^\xi}_{x+e_1}[\bar{\rho}^\xi(X_{\tau^-_{y_N}}, y_N), X_{\tau^-_{y_N}} \in \mathbb{B}^{(-1)}_{y_N}]} \\
&= \frac{P^{\bar{\pi}^\xi}_x\{X_{\tau^-_{y_N}} \in \mathbb{B}^{(-1)}_{y_N}\} E^{\bar{\omega}^\xi}_x[\bar{\rho}^\xi(X_{\tau^-_{y_N}}, y_N)]}{P^{\bar{\pi}^\xi}_{x+e_1}\{X_{\tau^-_{y_N}} \in \mathbb{B}^{(-1)}_{y_N}\} E^{\bar{\omega}^\xi}_{x+e_1}[\bar{\rho}^\xi(X_{\tau^-_{y_N}}, y_N)]} \\
&= \frac{P^{\bar{\pi}^\xi}_x\{X_{\tau^-_{y_N}} \in \mathbb{B}^{(-1)}_{y_N}\}}{P^{\bar{\pi}^\xi}_{x+e_1}\{X_{\tau^-_{y_N}} \in \mathbb{B}^{(-1)}_{y_N}\}} \cdot \bar{\rho}^\xi_{x,x+e_1}.
\end{aligned}$$

The probabilities in the last expression converge to one by the law of large numbers of Theorem 3.5 because by (3.63) the ξ-ray passes ζ on the left. $\square$

COROLLARY 3.11. *Fix* $\xi \in \mathrm{ri}\,\mathcal{U}$. *Let* ω *be an i.i.d.* Beta(α, β) *environment. Then almost surely, for all* $z \in \mathbb{Z}^2$, *the limits*

$$\lim_{N\to\infty} \frac{F^\omega_{0,[N\xi]+z}}{F^\omega_{e_1,[N\xi]+z}} \quad \textit{and} \quad \lim_{N\to\infty} \frac{F^\omega_{e_2,[N\xi]+z}}{F^\omega_{0,[N\xi]+z}} \tag{3.64}$$

exist and are independent of z.

(a) *When* $\xi = \xi^*$, *the limits equal* 1.
(b) *For* $\xi \ne \xi^*$, *let* $\lambda = \lambda(\xi)$ *be determined by Lemma* 2.6(a).

(b.i) *If* $\xi_1 \in (\xi_1^*, 1)$ *the two limits in* (3.64) *are*, *respectively*, Beta$(\alpha + \lambda, \beta)$ *and* Beta(λ, α) *distributed.*

(b.ii) *If* $\xi_1 \in (0, \xi_1^*)$, *the reciprocals of the two limits in* (3.64) *are*, *respectively*, Beta(λ, β) *and* Beta$(\beta + \lambda, \alpha)$ *distributed.*

PROOF. Consider the case $\xi_1 \in (\xi_1^*, 1)$ and the first limit of (3.64). Let $\gamma < \lambda < \delta$. By Lemma 2.6(a), the velocities associated with these parameters in the range $(\xi_1^*, 1)$ satisfy $\xi_1(\gamma) > \xi_1 = \xi_1(\lambda) > \xi_1(\delta)$. By Proposition 3.10,

$$\rho^{\gamma}_{0,e_1} \le \varliminf_{N\to\infty} \frac{F^{\omega^\gamma}_{0,[N\xi]+z}}{F^{\omega^\gamma}_{e_1,[N\xi]+z}} \quad \text{and} \quad \varlimsup_{N\to\infty} \frac{F^{\omega^\delta}_{0,[N\xi]+z}}{F^{\omega^\delta}_{e_1,[N\xi]+z}} \le \rho^{\delta}_{0,e_1}. \tag{3.65}$$

Since $\rho^\gamma, \rho^\delta \to \rho^\lambda$ as $\gamma, \delta \to \lambda$ by (3.44), and since ρ^λ_{0,e_1} is Beta$(\alpha+\lambda, \beta)$-distributed, we have that

$$\varliminf_{N\to\infty} \frac{F^{\omega}_{0,[N\xi]+z}}{F^{\omega}_{e_1,[N\xi]+z}} \quad \text{and} \quad \varlimsup_{N\to\infty} \frac{F^{\omega}_{0,[N\xi]+z}}{F^{\omega}_{e_1,[N\xi]+z}}$$

are both Beta$(\alpha+\lambda, \beta)$ random variables. Since $\varliminf \le \varlimsup$, their equality in distribution implies their $\mathbb{P}$-almost sure equality. Same reasoning works for the second limit of (3.64). Existence of the limits and claim (b.i) are proved.

To argue that the limit with $z = 0$ equals the limit with an arbitrary $z = (z_1, z_2)$, pick integers k_N so that $[k_N\xi]_2 = [N\xi]_2 + z_2$. Then, depending on the relative locations of $[k_N\xi]_1$ and $[N\xi]_1 + z_1$, by (3.56) either

$$\frac{F^{\omega}_{0,[k_N\xi]}}{F^{\omega}_{e_1,[k_N\xi]}} \ge \frac{F^{\omega}_{0,[N\xi]+z}}{F^{\omega}_{e_1,[N\xi]+z}} \tag{3.66}$$

or the opposite inequality is valid for infinitely many N. In the limit, we get again an almost sure inequality between two Beta$(\alpha+\lambda, \beta)$ random variables, which therefore must coincide almost surely.

For $\xi_1 \in (0, \xi_1^*)$, one can repeat the same steps but use instead the processes $(\widetilde{\omega}, \widetilde{\rho})$ that follow part (b) of Lemma 3.1.

For the case $\xi = \xi^*$, pick $\eta, \zeta \in \mathrm{ri}\,\mathcal{U}$ such that $\eta_1 < \xi_1^* < \zeta_1$. By (3.56),

$$\frac{F^{\omega}_{0,[N\zeta]+z}}{F^{\omega}_{e_1,[N\zeta]+z}} \le \frac{F^{\omega}_{0,[N\xi^*]+z}}{F^{\omega}_{e_1,[N\xi^*]+z}} \le \frac{F^{\omega}_{0,[N\eta]+z}}{F^{\omega}_{e_1,[N\eta]+z}}.$$

By the cases already proved, the left and right ratios converge to random variables with distributions Beta$(\alpha+\lambda(\zeta), \beta)$ and Beta$(\lambda(\eta), \beta)^{-1}$, respectively. These random variables converge to 1 as we let $\zeta, \eta \to \xi^*$ which sends $\lambda(\zeta), \lambda(\eta) \to \infty$ (see Lemma 2.6(a) and the middle plot of Figure 1). The second ratio in (3.64) for $\xi = \xi^*$ is handled similarly. $\square$

PROOF OF THEOREM 3.6. We begin by constructing the process for a fixed $\xi \in \mathrm{ri}\,\mathcal{U}$, then do it simultaneously for a dense countable subset of $\mathrm{ri}\,\mathcal{U}$, and finally capture all of $\mathrm{ri}\,\mathcal{U}$ with limits.

Fix $\xi \in \mathrm{ri}\,\mathcal{U}$. Using Corollary 3.11 and shifts $P^{\omega}_{x+a}(X_n = v) = P^{T_x\omega}_{a}(X_n = v - x)$, we define

$$B^{\xi}_{x,y}(\omega) = \lim_{N\to\infty}\{\log P^{\omega}_{x}(X_{|z_N-x|_1} = z_N) - \log P^{\omega}_{y}(X_{|z_N-y|_1} = z_N)\}$$
$$= \lim_{N\to\infty}(\log F^{\omega}_{x,z_N} - \log F^{\omega}_{y,z_N}), \tag{3.67}$$

as an almost sure limit, for all $x, y \in \mathbb{Z}^2$, and for any sequence $z_N = [N\xi] + z$ with an arbitrary fixed z. (The second line is the same as the first, stated to illustrate the alternative notation we use.) The limit is independent of the choice of z. The marginal distributional properties (a), stationary cocycle properties (e), and harmonicity (f) stated in Theorem 3.6 follow from Corollary 3.11 and the structure of the limits.

Proof of part (b). The independence of the weights $\{\omega_x\}$ and construction (3.67) imply directly the first independence claim of part (b).

Proof of part (c) *for fixed* ξ. We write the details for the case $\xi_1 \in (\xi_1^*, 1)$. Consider the joint distribution of m weights ω_{z_h} for $1 \le h \le m$ and $k + \ell$ nearest-neighbor increments $B^{\xi}_{x_i,x_i+e_1}$ and $B^{\xi}_{y_j,y_j+e_2}$ for $1 \le i \le k$ and $1 \le j \le \ell$. By a shift, we may assume that z_h, x_i, y_j all lie in $\mathbb{Z}^2_+$. Let $\gamma < \lambda(\xi) < \delta$ as in the proof of Corollary 3.11. Limit (3.67) works also in environments ω^{γ} and ω^{δ} since they have the same i.i.d. beta distribution as ω. Let $r_h, s_i, t_j \in \mathbb{R}$. Then inequalities (3.65) and their counterparts for e_2 give us these bounds:

$$\begin{aligned}
&\mathbb{P}\{\omega_{z_h} \le r_h, e^{B^{\xi}_{x_i,x_i+e_1}} \le s_i, e^{B^{\xi}_{y_j,y_j+e_2}} \ge t_j\ \forall h, i, j\} \\
&\quad = \overline{\mathbb{P}}\{\omega^{\gamma}_{z_h} \le r_h, e^{B^{\xi}_{x_i,x_i+e_1}(\omega^{\gamma})} \le s_i, e^{B^{\xi}_{y_j,y_j+e_2}(\omega^{\gamma})} \ge t_j\ \forall h, i, j\} \\
&\quad \le \overline{\mathbb{P}}\{\omega^{\gamma}_{z_h} \le r_h, \rho^{\gamma}_{x_i,x_i+e_1} \le s_i, \rho^{\gamma}_{y_j,y_j+e_2} \ge t_j\ \forall h, i, j\}
\end{aligned}$$

and from the other side

$$\begin{aligned}
&\mathbb{P}\{\omega_{z_h} \le r_h, e^{B^{\xi}_{x_i,x_i+e_1}} \le s_i, e^{B^{\xi}_{y_j,y_j+e_2}} \ge t_j\ \forall h, i, j\} \\
&\quad = \overline{\mathbb{P}}\{\omega^{\delta}_{z_h} \le r_h, e^{B^{\xi}_{x_i,x_i+e_1}(\omega^{\delta})} \le s_i, e^{B^{\xi}_{y_j,y_j+e_2}(\omega^{\delta})} \ge t_j\ \forall h, i, j\} \\
&\quad \ge \overline{\mathbb{P}}\{\omega^{\delta}_{z_h} \le r_h, \rho^{\delta}_{x_i,x_i+e_1} \le s_i, \rho^{\delta}_{y_j,y_j+e_2} \ge t_j\ \forall h, i, j\}.
\end{aligned}$$

Letting $\gamma, \delta \to \lambda(\xi)$ brings the bounds together by (3.44):

$$\begin{aligned}
&\mathbb{P}\{\omega_{z_h} \le r_h, e^{B^{\xi}_{x_i,x_i+e_1}} \le s_i, e^{B^{\xi}_{y_j,y_j+e_2}} \ge t_j\ \forall h, i, j\} \\
&\quad = \overline{\mathbb{P}}\{\omega^{\lambda(\xi)}_{z_h} \le r_h, \rho^{\lambda(\xi)}_{x_i,x_i+e_1} \le s_i, \rho^{\lambda(\xi)}_{y_j,y_j+e_2} \ge t_j\ \forall h, i, j\}.
\end{aligned} \tag{3.68}$$

Thus the joint distribution of (ω, B^{ξ}) is the same as that of $(\omega^{\lambda(\xi)}, \log \rho^{\lambda(\xi)})$ described in Proposition 3.3. The independence of nearest-neighbor B^{ξ}-increments along a down-right path follows.

Let Ω_0 be the event of full $\mathbb{P}$-probability on which the process $B^{\xi}_{x,y}$ is defined by (3.67) for all ξ in the countable set $\mathcal{U}_0 = (\mathrm{ri}\,\mathcal{U}) \cap \mathbb{Q}^2$.

Consider $\xi \in \mathrm{ri}\,\mathcal{U}$ and $\zeta, \eta \in \mathcal{U}_0$ with $\eta_1 < \xi_1 < \zeta_1$. Take a (possibly random) sequence z_N with $z_N/N \to \xi$. Let $M = |z_N|_1$. For large enough N, we have $[M\eta] \cdot e_1 < z_N \cdot e_1 < [M\zeta] \cdot e_1$ and $[M\eta] \cdot e_2 > z_N \cdot e_2 > [M\zeta] \cdot e_2$. By Lemma 3.8, we have for such N

$$\frac{F^{\omega}_{x,[M\zeta]}}{F^{\omega}_{x+e_1,[M\zeta]}} \le \frac{F^{\omega}_{x,z_N}}{F^{\omega}_{x+e_1,z_N}} \le \frac{F^{\omega}_{x,[M\eta]}}{F^{\omega}_{x+e_1,[M\eta]}}.$$

The already established limit (3.67) with sequences $[M\zeta]$ and $[M\eta]$ gives

$$\begin{aligned} B^{\zeta}_{x,x+e_1} &\le \varliminf_{N\to\infty} \{\log P^{\omega}_x(X_{|z_N-x|_1} = z_N) - \log P^{\omega}_{x+e_1}(X_{|z_N-x|_1-1} = z_N)\} \\ &\le \varlimsup_{N\to\infty} \{\log P^{\omega}_x(X_{|z_N-x|_1} = z_N) - \log P^{\omega}_{x+e_1}(X_{|z_N-x|_1-1} = z_N)\} \\ &\le B^{\eta}_{x,x+e_1}. \end{aligned} \tag{3.69}$$

The reverse inequalities hold when e_1 is replaced by e_2.

Equation (3.69) proves that the monotonicity in part (g) holds for $\xi, \zeta \in \mathcal{U}_0$, $\omega \in \Omega_0$ and $x \in \mathbb{Z}^2$. Consequently, for any $\xi \in (\mathrm{ri}\,\mathcal{U}) \setminus \mathcal{U}_0$ we can *define* $B^{\xi}_{x,x+e_i}(\omega)$ for $\omega \in \Omega_0$ by the monotone limit

$$B^{\xi}_{x,x+e_i}(\omega) = \lim_{\mathcal{U}_0 \ni \zeta \to \xi, \zeta_1 > \xi_1} B^{\zeta}_{x,x+e_i}(\omega) \tag{3.70}$$

as ζ_1 decreases to ξ_1. By shrinking Ω_0, we can assume that (3.70) holds also when $\xi \in \mathcal{U}_0$. (This is because the monotonicity gives an inequality in (3.70), but the two sides agree in distribution, and hence agree almost surely.) By additivity on the right-hand side, we can extend (3.70) to define

$$B^{\xi}_{x,y}(\omega) = \lim_{\mathcal{U}_0 \ni \zeta \to \xi, \zeta_1 > \xi_1} B^{\zeta}_{x,y}(\omega) \qquad \text{for all } x, y \in \mathbb{Z}^2 \text{ and } \omega \in \Omega_0.$$

This definition in terms of right limits extends the properties proved thus far to all ξ. Furthermore, cadlag paths [part (h)] have also been established.

Fix $\xi \in \mathrm{ri}\,\mathcal{U}$ and $i \in \{1, 2\}$. The almost sure continuity of $\zeta \mapsto B^{\zeta}_{0,e_i}$ at $\zeta = \xi$ follows from monotonicity (g) and from the continuity of $\zeta \mapsto \mathbb{E}[B^{\zeta}_{0,e_i}]$, which itself is a consequence of continuity of the polygamma functions in (3.47) and (3.48). Claim (i) follows from the cocycle property in part (e).

Continue with a fixed $\xi \in \mathrm{ri}\,\mathcal{U}$. Let $\zeta, \eta \to \xi$ in (3.69) and use the almost sure continuity we just proved. This shows that limit (3.52) holds $\mathbb{P}$-almost surely, simultaneously for all $x \in \mathbb{Z}^2$, $y \in \{x + e_1, x + e_2\}$, and any sequence z_N with $z_N/N \to \xi$. The case of a general $y \in \mathbb{Z}^2$ follows from additivity. Part (j) is done.

Part (d). When $\xi_1 \in [\xi_1^*, 1]$, the variational formula for I_q comes from (3.47) and the explicit calculations in (7.7) in Section 7. To minimize the formula differentiate,

$$\big(\psi_0(\alpha + \lambda(\zeta)) - \psi_0(\alpha + \beta + \lambda(\zeta))\big)\xi_1 + \big(\psi_0(\alpha + \lambda(\zeta)) - \psi_0(\lambda(\zeta))\big)\xi_2$$

in ζ_1 and set it to 0. This gives the equation

$$\xi_1 = \frac{\psi_1(\lambda(\zeta)) - \psi_1(\alpha+\lambda(\zeta))}{\psi_1(\lambda(\zeta)) - \psi_1(\alpha+\beta+\lambda(\zeta))}$$

which by Lemma 2.6(a) has a unique solution at $\zeta = \xi$, similarly for $\xi_1 \in [0, \xi_1^*]$. $\square$

4. Stationary beta polymer. By looking at the random walk paths under the Doob-transformed RWRE in reverse direction, we can view this model as a stationary directed polymer model, called the beta polymer. We establish this connection in the present section, and then use it in the next two sections to rely on recently published estimates in [9] for the technical work behind our Theorems 2.4 and 2.5. The polymer model described here is case (1.4) in [9], with their parameter triple (μ, β, θ) corresponding to our (α, β, λ). The notation $Z_{m,n}$ and $Q_{m,n}$ used below matches that of [9].

Recall the backward transition probabilities $\breve{\omega}$, introduced in (3.21) and (3.22), and random variables $(\rho^\lambda, \omega^\lambda)$ from (3.24). The quenched stationary *beta polymer* is a polymer distribution on up-right paths on the nonnegative first quadrant $\mathbb{Z}_+^2$ that start at the origin. In our notation, this model uses potential $V(x-e_j, e_j) = \log \breve{\omega}_{x,x-e_j}$ across edges $(x-e_j, x)$ for $x \in \mathbb{N}$, and potential $V(x-e_j, e_j) = \log \rho^\lambda_{x-e_j,x}$ across boundary edges $(x-e_j, x)$ for $x \in \mathbb{B}_0^{(+j)} \setminus \{0\}$, $j \in \{1,2\}$. Fix a point $v = (m,n) \in \mathbb{N}^2$. The point-to-point partition function for paths from 0 to v is

$$Z_{m,n} = \sum_{y_{0,m+n}} e^{\sum_{i=0}^{m+n-1} V(y_i, y_{i+1}-y_i)},$$

where the sum is over up-right paths $y_{0,m+n} = (y_0, \dots, y_{m+n})$ from 0 to $v = (m,n)$.

If $x_i = y_{m+n-i}$ denotes the reversed path and $\ell = \min\{i : x_i \in \mathbb{B}_0^+\}$ is the time of its first entry into the boundary $\mathbb{B}_0^+$, then

$$\begin{aligned} e^{\sum_{i=0}^{m+n-1} V(y_i, y_{i+1}-y_i)} &= \rho^\lambda_{0,x_\ell} \prod_{i=0}^{\ell-1} \breve{\omega}_{x_i,x_{i+1}} = \rho^\lambda_{0,x_\ell} P_v^{\breve{\omega}}\{X_{0,\ell} = x_{0,\ell}\} \\ &= \rho^\lambda_{0,x_\ell} P_v^{\breve{\omega}}\{X_{0,\tau_0^+} = x_{0,\ell}\}. \end{aligned}$$

Summing up over the paths gives the first equality below, and the second one comes from (3.33):

$$Z_{m,n} = E_v^{\breve{\omega}}[\rho^\lambda(0, X_{\tau_0^+})] = \rho^\lambda_{0,v} \qquad \text{for } v = (m,n). \tag{4.1}$$

The quenched polymer measure on up-right paths $y_{0,m+n} = (y_0, \dots, y_{m+n})$ from 0 to $v = (m,n)$ is

$$Q_{m,n}(y_{0,m+n}) = \frac{e^{\sum_{i=0}^{m+n-1} V(y_i, y_{i+1}-x_i)}}{Z_{m,n}}.$$

Letting again $x_i = y_{m+n-i}$ and $\ell = \min\{i : x_i \in \mathbb{B}_0^+\}$ and using (3.38),

$$
(4.2)\qquad \begin{aligned} Q_{m,n}(y_{0,n}) &= \frac{\rho^{\lambda}_{0,x_\ell} P_v^{\breve{\omega}}\{X_{0,\ell} = x_{0,\ell}\}}{\rho^{\lambda}_{0,v}} \\ &= P_v^{\breve{\pi}^{\lambda}}\{X_{0,\ell} = x_{0,\ell}\} \qquad \text{for } v = (m,n). \end{aligned}
$$

Thus (the reverse of) the polymer path under $Q_{m,n}$ is obtained by running the Doob-transformed RWRE under $P_v^{\breve{\pi}^{\lambda}}$ until it hits the boundary $\mathbb{B}_0^+$, and then following the boundary to the origin.

5. The variance of the increment-stationary harmonic functions. The method for bounding the fluctuations of the walk for Theorem 2.5 is to control the exit point of the walk from rectangles. This is achieved with the help of the harmonic functions ρ^{λ} and $\widetilde{\rho}^{\lambda}$ constructed in Section 3.2. We work exclusively with ρ^{λ} and omit the analogous statements and proofs for $\widetilde{\rho}^{\lambda}$. Equivalently, we are treating explicitly only the case $\xi_1 \in (\xi_1^*, 1)$ and omitting the details for $\xi_1 \in (0, \xi_1^*)$.

This section gives the connection between the fluctuations of $\log \rho^{\lambda}$ and the entry point on the boundary, and bounds on the variance of $\log \rho^{\lambda}$. Theorem 2.4 is proved at the end of the section.

Recall the beta integral $B(a,b)$ and the c.d.f. $F(s; a, b)$ of the Beta(a,b) distribution from (1.7) and (1.8). Define

$$
\widetilde{L}(s,a,b) = -\frac{1}{s} \cdot \frac{\frac{\partial}{\partial a} F(s;a,b)}{\frac{\partial}{\partial s} F(s;a,b)}.
$$

Note that $\frac{\partial}{\partial a} B(a,b) = (\psi_0(a) - \psi_0(a+b))B(a,b)$. A computation gives

$$
(5.1)\qquad \begin{aligned} \widetilde{L}(s,a,b) = -s^{-a}(1-s)^{1-b} \int_0^s & t^{a-1}(1-t)^{b-1} \\ & \times \bigl[\log t - \bigl(\psi_0(a) - \psi_0(a+b)\bigr)\bigr]\,dt. \end{aligned}
$$

Observe that

$$
\widetilde{L}(s,a,b) = s^{-a}(1-s)^{1-b} B(a,b)\,\mathrm{Cov}\bigl(-\log W, \mathbb{1}\{W \le s\}\bigr),
$$

where $W \sim \text{Beta}(a,b)$. Since $-\log t$ and $\mathbb{1}\{t \le s\}$ are decreasing functions of t, we see that $\widetilde{L}(s,a,b) > 0$ for all $s \in (0,1)$ and $a, b > 0$.

Recall hitting times τ_v^- and τ_0^+ defined in (3.2). Let $\overline{\mathbb{V}}\mathrm{ar}$ denote the variance under the coupling $\overline{\mathbb{P}}$ of Section 3.2. Let $\breve{\mathbf{P}}_v^{\lambda}(\cdot) = \overline{\mathbb{E}} P_v^{\breve{\pi}^{\lambda}}(\cdot)$ denote the averaged measure of the RWRE that utilizes the backward Doob-transformed transition probability $\breve{\pi}^{\lambda}$ of (3.37). Its expectation is $\breve{\mathbf{E}}_v^{\lambda}[\cdot] = \overline{\mathbb{E}} E_v^{\breve{\pi}^{\lambda}}[\cdot]$.

THEOREM 5.1. *The following holds for all* $\alpha, \beta, \lambda > 0$ *and all* $v = (m, n) \in \mathbb{N}^2$:

$$\overline{\mathbb{V}}\mathrm{ar}(\log \rho^{\lambda}_{0,v}) = n(\psi_1(\lambda) - \psi_1(\alpha+\lambda)) - m(\psi_1(\alpha+\lambda) - \psi_1(\alpha+\beta+\lambda))$$

$$+ 2\breve{\mathbf{E}}^{\lambda}_{v}\left[\sum_{i=0}^{X(\tau_0^+)\cdot e_1 - 1} \widetilde{L}(\rho^{\lambda}_{ie_1,(i+1)e_1}, \alpha+\lambda, \beta)\right] \tag{5.2}$$

$$= m(\psi_1(\alpha+\lambda) - \psi_1(\alpha+\beta+\lambda)) - n(\psi_1(\lambda) - \psi_1(\alpha+\lambda))$$

$$+ 2\breve{\mathbf{E}}^{\lambda}_{v}\left[\sum_{i=0}^{X(\tau_0^+)\cdot e_2 - 1} \widetilde{L}(1/\rho^{\lambda}_{ie_2,(i+1)e_2}, \lambda, \alpha)\right]. \tag{5.3}$$

An empty sum (e.g., $\sum_{i=0}^{-1}$) equals 0. Thus the $\breve{\mathbf{E}}^{\lambda}_{v}$ expectation on the right-hand side of (5.2) is in fact over the event $\{X(\tau_0^+) \in \mathbb{B}_0^{(+1)}\}$. When v is chosen (approximately) in the direction $\xi(\lambda)$ so that the first two terms on the right-hand side of (5.2) (approximately) cancel, the equation expresses the KPZ relation that in $1+1$ dimension the wandering exponent is twice the free energy exponent.

Theorem 5.1 is the same as Proposition 1.1 in [9], via the connections (4.1) and (4.2) between the RWRE and the polymer. Theorem 5.1 is also proved in Section 4.1 of the first preprint version [5] of this paper.

Starting from the identity in Theorem 5.1, a series of coupling arguments and estimates leads to upper and lower bounds on the fluctuations of $\log \rho^{\lambda}$. Theorem 5.2 below follows from Theorem 1.2 of [9]. It is also proved in Sections 4.2 and 4.3 of [5]. Here, $\xi(\lambda)$ is given by (2.10).

THEOREM 5.2. *Fix* $\alpha, \beta > 0$. *Fix* $\lambda > 0$. *Given a constant* $0 < \gamma < \infty$, *there exist positive finite constants* c, C *and* N_0, *depending only on* α, β, γ *and* λ, *such that*

$$cN^{2/3} \le \mathbb{V}\mathrm{ar}[\rho^{\lambda}_{0,me_1+ne_2}] \le CN^{2/3} \tag{5.4}$$

for all $N \ge N_0$ *and* $(m, n) \in \mathbb{N}^2$ *such that*

$$|m - N\xi_1(\lambda)| \vee |n - N\xi_2(\lambda)| \le \gamma N^{2/3}.$$

The same constants c, C *and* N_0 *can be taken for* $(\alpha, \beta, \gamma, \lambda)$ *varying in a compact subset of* $(0, \infty)^4$.

PROOF OF THEOREM 2.4. By virtue of Theorem 3.6(c), Theorem 5.2 implies Theorem 2.4 for the case $\xi_1 \in (\xi_1^*, 1)$. The remaining case $\xi_1 \in (0, \xi_1^*)$ follows from the (omitted) version of Theorem 5.2 for $\widetilde{\rho}^{\lambda}$. $\square$

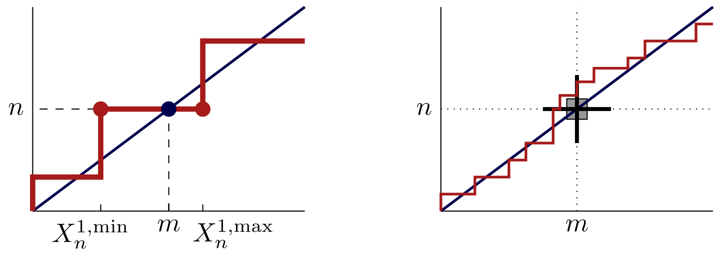

FIG. 5. *In both plots, the diagonal line points in direction $\xi(\lambda)$. Left: the definition of $X_n^{1,\min}$ and $X_n^{1,\max}$. Right: illustration of* (6.1) *and* (6.4). *The four arms of the cross centered at $(m,n) = \lfloor N\xi \rfloor$ are of length $rN^{2/3}$ each. The shaded box, also centered at (m,n), has sides of length $2\delta N^{2/3}$. For large r, the path has a high probability of entering and exiting through the cross and never touching the dotted lines. For small δ, there is a positive probability, uniformly in N, the path stays left of the top edge of the shaded box, completely avoiding the box.*

6. Path fluctuations. This section proves results about path fluctuations, from which Theorem 2.5 follows. For an up-right path $X_{0,\infty}$ started at the origin and an integer $n \geq 0$ let

$$X_n^{1,\min} = \min\{m \geq 0 : X_{m+n} \cdot e_2 = n\} \quad \text{and}$$
$$X_n^{1,\max} = \max\{m \geq 0 : X_{m+n} \cdot e_2 = n\}.$$

Then $X_n^{1,\min} e_1 + n e_2$ and $X_n^{1,\max} e_1 + n e_2$ are, respectively, the leftmost and rightmost points of the path on the horizontal line $ne_2 + \mathbb{Z}_+ e_1$. See the left panel in Figure 5. The vertical counterparts are given by

$$X_m^{2,\min} = \min\{n \geq 0 : X_{m+n} \cdot e_1 = m\} \quad \text{and}$$
$$X_m^{2,\max} = \max\{n \geq 0 : X_{m+n} \cdot e_1 = m\}.$$

Again, the next result is stated and proved for $\xi_1 \in (\xi_1^*, 1)$ only. The other case works similarly. Recall $\xi(\lambda)$ from (2.10). Let $\mathbf{P}_0^\lambda = \overline{\mathbb{E}} P_0^{\pi^\lambda}$, with expectation $\mathbf{E}_0^\lambda = \overline{\mathbb{E}} E_0^{\pi^\lambda}$. By Theorem 3.5, $\xi(\lambda)$ is the LLN direction for $\mathbf{P}_0^\lambda$.

THEOREM 6.1. *Fix $\alpha, \beta, \lambda > 0$.*

(a) Upper bound. *There exist finite positive constants r_0 and C depending on α, β, and λ, such that for all $r \geq r_0$, $N \geq 1$ and $(m,n) = \lfloor N\xi(\lambda) \rfloor$,*

$$\mathbf{P}_0^\lambda\{X_n^{1,\min} < m - rN^{2/3}\} \leq Cr^{-3} \quad \textit{and}$$
$$\mathbf{P}_0^\lambda\{X_n^{1,\max} > m + rN^{2/3}\} \leq Cr^{-3}. \tag{6.1}$$

From this, it follows that

$$\mathbf{E}_0^\lambda\bigl[\bigl|(m - X_n^{1,\min})^+\bigr|^p\bigr]^{1/p} \le \left(1 + \frac{Cp}{3-p}\right)^{1/p} N^{2/3} \quad \textit{and}$$
$$\mathbf{E}_0^\lambda\bigl[\bigl|(X_n^{1,\max} - m)^+\bigr|^p\bigr]^{1/p} \le \left(1 + \frac{Cp}{3-p}\right)^{1/p} N^{2/3}. \tag{6.2}$$

(b) Lower bound. *There exist finite positive constants δ and c depending on α, β, and λ, such that for any integer $N \ge 1$ such that $(m,n) = \lfloor N\xi(\lambda)\rfloor \in \mathbb{N}^2$ we have*

$$\mathbf{E}_0^\lambda\bigl[(m - X_n^{1,\min})^+\bigr] \ge cN^{2/3} \quad \textit{and} \tag{6.3}$$

$$\mathbf{P}_0^\lambda\bigl\{X_n^{1,\min} \le X_n^{1,\max} < m - \delta N^{2/3}\bigr\} \ge c. \tag{6.4}$$

Similar bounds hold for the vertical counterparts $X_m^{2,\min}$ and $X_m^{2,\max}$. The same constants can be used for all (α, β, λ) in a compact subset of $(0,\infty)^3$.

PROOF. Abbreviate $u = (m,n) = \lfloor N\xi(\lambda)\rfloor$. Inequality (6.1) is trivial if $rN^{1/3} \ge m$. We hence assume that $rN^{2/3} < m$.

Note that

$$(m - X_n^{1,\min})^+ = m - X(\tau_{(m,n)}^-)\cdot e_1. \tag{6.5}$$

Thus, the first probability in (6.1) equals

$$\mathbf{P}_0^\lambda\bigl\{X(\tau_u^-)\cdot e_1 < m - rN^{2/3}\bigr\} = \breve{\mathbf{P}}_u^\lambda\bigl\{X(\tau_0^+)\cdot e_1 > rN^{2/3}\bigr\}. \tag{6.6}$$

Applying Lemma 4.7 of [9] and the connection (4.2) gives

$$\breve{\mathbf{P}}_u^\lambda\bigl\{X(\tau_0^+)\cdot e_1 > rN^{2/3}\bigr\} \le Cr^{-3}.$$

(This is also (4.24) in [5].) This proves the first inequality in (6.1).

For the second inequality, set $N_0 = \lfloor \frac{m + rN^{2/3}}{\xi_1(\lambda)}\rfloor$ and $(m_0, n_0) = \lfloor N_0\xi(\lambda)\rfloor$. Then $m_0 \le m + rN^{2/3}$ and, therefore, if $X_n^{1,\max} > m + rN^{2/3}$, then $X_{m_0}^{2,\min} \le n$. But we also have

$$\begin{aligned} n_0 > N_0\xi_2(\lambda) - 1 &\ge \frac{m\xi_2(\lambda)}{\xi_1(\lambda)} + \frac{\xi_2(\lambda)}{\xi_1(\lambda)} rN^{2/3} - 1 - \xi_2(\lambda) \\ &\ge n + \frac{\xi_2(\lambda)}{\xi_1(\lambda)} rN^{2/3} - 1 - \xi_2(\lambda) - \frac{\xi_2(\lambda)}{\xi_1(\lambda)} \\ &\ge n + \frac{\xi_2(\lambda)}{2\xi_1(\lambda)} rN^{2/3} \ge n + \frac{\xi_2(\lambda_1)}{2\xi_1(\lambda_2)} rN^{2/3}, \end{aligned}$$

provided $r \ge 2(1 + \xi_1(\lambda_1) + \frac{\xi_1(\lambda_1)}{\xi_2(\lambda_2)})$. The upshot is that if $X_n^{1,\max} > m + rN^{2/3}$ then $X_{m_0}^{2,\min} < n_0 - \frac{\xi_2(\lambda_1)}{2\xi_1(\lambda_2)} rN_0^{2/3}$. The second inequality in (6.1) thus follows from the vertical version of the first inequality, but with N_0 and $r_0 = \frac{\xi_2(\lambda_1)}{2\xi_1(\lambda_2)} r$ playing the roles of N and r, respectively.

Bounds (6.2) follow from (6.1). For example, for the first bound abbreviate $Y = (m - X_n^{1,\min})^+$ and write

$$\begin{aligned}\mathbf{E}_0^\lambda[(N^{-2/3}Y)^p] &= \int_0^\infty pr^{p-1}\mathbf{P}_0^\lambda(Y > rN^{2/3})\,dr \\ &\le \int_0^1 pr^{p-1}\,dr + C\int_1^\infty pr^{p-4}\,dp = 1 + \frac{Cp}{3-p}.\end{aligned}$$

Next, applying Lemma 4.2 of [9] we have

$$\breve{\mathbf{E}}_v^\lambda\Bigg[\sum_{i=0}^{X(\tau_0^+)\cdot e_1 - 1} \widetilde{L}(\rho^\lambda_{ie_1,(i+1)e_1}, \alpha+\lambda, \beta)\Bigg] \le C(\breve{\mathbf{E}}_v^\lambda[X(\tau_0^+)\cdot e_1] + 1). \tag{6.7}$$

(This is also (4.15) in [5].) Now, bound (6.3) follows from stringing together (6.7), (5.2) and the lower bound in (5.4), then reversing the picture in (6.5). To get (6.4), first write

$$\begin{aligned}cN^{2/3} \le \mathbf{E}_0^\lambda[Y] &= \mathbf{E}_0^\lambda[Y\mathbb{1}\{Y \le \delta N^{2/3}\}] + \mathbf{E}_0^\lambda[Y\mathbb{1}\{Y > \delta N^{2/3}\}] \\ &\le \delta N^{2/3} + \mathbf{E}_0^\lambda[Y^2]^{1/2}\mathbf{P}_0^\lambda(Y > \delta N^{2/3})^{1/2}.\end{aligned}$$

Applying (6.2) with say $p = 2$ and taking $\delta \le c/2$, we get

$$\mathbf{P}_0^\lambda\{X_n^{1,\min} < m - \delta N^{2/3}\} \ge \frac{c}{2\sqrt{1+2C}}. \tag{6.8}$$

Now take $\delta_0 > 2\delta$, $N_0 = N + \lfloor \delta N^{2/3}\rfloor$, and $(m_0, n_0) = \lfloor N_0\xi(\lambda)\rfloor$. Note that $m_0 \le \lfloor N\xi_1(\lambda)\rfloor + \delta N^{2/3} = m + \delta N^{2/3}$. This forces

$$m_0 - \delta_0 N_0^{2/3} \le m + \delta N^{2/3} - 2\delta N^{2/3} = m - \delta N^{2/3}.$$

Since $n \le n_0$, we have that if $X_{n_0}^{1,\min} < m_0 - \delta_0 N_0^{2/3}$, then

$$X_n^{1,\max} \le X_{n_0}^{1,\min} < m_0 - \delta_0 N_0^{2/3} \le m - \delta N^{2/3}.$$

Bound (6.4) follows from the above and (6.8) with N_0 and δ_0 playing the roles of N and δ, respectively. $\square$

PROOF OF THEOREM 2.5. We only argue for the case $\xi_1 \in (\xi_1^*, 1)$, the other case being similar. By Theorem 3.6(c), the distribution of $P_0^{\kappa^\xi}$ under $\mathbb{P}$ is the same as that of $P_0^{\pi^\lambda}$ under $\overline{\mathbb{P}}$, provided λ and ξ are in duality via (2.10). Hence, $\mathbf{P}_0^\xi = \mathbf{P}_0^\lambda$. The claims of the theorem follow from (6.1) and (6.4); see Figure 5. $\square$

7. Proofs of the large deviation results. PROOF OF LEMMA 2.6. Define the function

$$f(\lambda) = \frac{\psi_1(\lambda) - \psi_1(\alpha + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)}.$$

We prove that f is strictly decreasing in $\lambda > 0$. Its derivative is

$$\begin{aligned} f'(\lambda) = &\frac{(\psi_2(\lambda) - \psi_2(\alpha + \lambda))}{(\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda))} \\ &- \frac{(\psi_1(\lambda) - \psi_1(\alpha + \lambda))(\psi_2(\lambda) - \psi_2(\alpha + \beta + \lambda))}{(\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda))^2}. \end{aligned}$$

Since ψ_1 is strictly decreasing, $f'(\lambda) < 0$ is equivalent to

$$\frac{\psi_2(\lambda) - \psi_2(\alpha + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \lambda)} < \frac{\psi_2(\lambda) - \psi_2(\alpha + \beta + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)}. \tag{7.1}$$

This in turn follows from $\psi_2 \circ \psi_1^{-1}$ being strictly concave (see Lemma 5.3 in [6] or Lemma A.3 in [5]).

We have so far shown that f is strictly decreasing. Since $\psi_1(\lambda) \to \infty$ as $\lambda \searrow 0$ we have $f(\lambda) \to 1$ as $\lambda \searrow 0$. A Taylor expansion of ψ_1 (see [5], Lemma A.2) gives $\lambda^2(\psi_1(\lambda) - \psi_1(a + \lambda)) \to a$ as $\lambda \to \infty$, and thus $f(\lambda) \to \frac{\alpha}{\alpha+\beta}$ as $\lambda \to \infty$. The claims in part (a) for $\xi_1 \in [\xi_1^*, 1]$ now follow. The case $\xi_1 \in [0, \xi_1^*]$ comes by interchanging the roles of α and β and those of ξ_1 and ξ_2.

Define the function

$$g(\lambda) = \psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda).$$

Since ψ_1 is strictly decreasing, we see that

$$g'(\lambda) = \psi_1(\alpha + \beta + \lambda) - \psi_1(\lambda) < 0.$$

Hence, g is strictly decreasing. As $\lambda \searrow 0$ we have $\psi_0(\lambda) \to -\infty$ and $g(\lambda) \to \infty$. Combining 6.3.5 and 6.3.16 from [1] gives

$$g(\lambda) = -\frac{1}{\alpha + \beta + \lambda} + \frac{1}{\lambda} - \sum_{k=1}^{\infty} \left(\frac{1}{\alpha + \beta + \lambda + k} - \frac{1}{\lambda + k} \right).$$

Hence $g(\lambda) \to 0$ as $\lambda \to \infty$. Part (b) follows and Lemma 2.6 is proved. □

PROOFS OF THEOREMS 2.7 AND 2.10. We utilize the ratios ρ^λ and transitions ω^λ from Section 3.2. By Proposition 3.3, ω^λ under $\overline{\mathbb{P}}$ (defined on page 2202) has the same distribution as the original environment ω under $\mathbb{P}$.

By the ergodic theorem,

$$\begin{aligned} n^{-1} \log \rho^\lambda_{0,ne_2} &= n^{-1} \sum_{i=0}^{n-1} \log \rho^\lambda_{ie_2,(i+1)e_2} \\ &\underset{n\to\infty}{\longrightarrow} \overline{\mathbb{E}}[\log \rho^\lambda_{0,e_2}] = \psi_0(\alpha + \lambda) - \psi_0(\lambda). \end{aligned} \tag{7.2}$$

(Recall that the logarithm of a Gamma$(\nu, 1)$ has expected value $\psi_0(\nu)$ and that a Beta(a, b) is a ratio of a Gamma$(a, 1)$ and a Gamma$(a+b, 1)$. By Proposition 3.3, $\rho^{-1}_{ie_2,(i+1)e_2}$ are i.i.d. Beta(λ, α).)

Harmonicity of $\rho_{\bullet,ne_2}$ implies, as for (3.32), that for $x \in \mathbb{Z}^2_+$ with $|x|_1 \le n$,

$$\rho^{\lambda}_{x,ne_2} = \sum_{j=0}^{n} P^{\omega^{\lambda}}_x \{X_{n-|x|_1} = je_1 + (n-j)e_2\} \rho^{\lambda}_{je_1+(n-j)e_2,ne_2}. \tag{7.3}$$

A term above is nonzero exactly when j is between $x \cdot e_1$ and $n - x \cdot e_2$. Abbreviate

$$R_{i,n} = \rho^{\lambda}_{(i+1)e_1+(n-i-1)e_2,ie_1+(n-i)e_2}.$$

For fixed n, under $\overline{\mathbb{P}}$, variables $R_{i,n}$ are i.i.d. and each distributed as $\rho^{\lambda}_{e_1,e_2}$. Rewrite (7.3) for $x = 0$ as

$$\rho^{\lambda}_{0,ne_2} = \sum_{j=0}^{n} P^{\omega^{\lambda}}_0 \{X_n = je_1 + (n-j)e_2\} \prod_{i=0}^{j-1} R_{i,n}. \tag{7.4}$$

By standard asymptotics (detailed justification on page 52 of [5]),

$$\begin{aligned}
\frac{1}{n} \log \rho^{\lambda}_{0,ne_2} &\approx \max_{0 \le j \le n} \left\{ \frac{1}{n} \log P^{\omega^{\lambda}}_0 \{X_n = je_1 + (n-j)e_2\} + \frac{1}{n} \sum_{i=0}^{j-1} \log R_{i,n} \right\} \\
&= \sup_{\xi \in \mathcal{U}} \left\{ n^{-1} \log P^{\omega^{\lambda}}_0 \{X_n = [n\xi]\} + n^{-1} \sum_{i=0}^{[n\xi]\cdot e_1 - 1} \log R_{i,n} \right\} \\
&\underset{n\to\infty}{\longrightarrow} \sup_{\xi \in \mathcal{U}} \{ -I_q(\xi) + \xi \cdot e_1 (\psi_0(\alpha+\beta+\lambda) - \psi_0(\lambda)) \}.
\end{aligned}$$

The above and (7.2) give the equation

$$\psi_0(\alpha+\lambda) - \psi_0(\lambda) = \sup_{\xi \in \mathcal{U}} \{ \xi_1 (\psi_0(\alpha+\beta+\lambda) - \psi_0(\lambda)) - I_q(\xi) \}. \tag{7.5}$$

For $t \in \mathbb{R}$, let

$$f(t) = I^*_q(te_1) = \sup_{s \in \mathbb{R}} \{ ts - I_q(se_1 + (1-s)e_2) \},$$

where of course $I_q(\xi) = \infty$ for $\xi \notin \mathcal{U}$ (i.e., $s \notin [0, 1]$). For $t \ge 0$ and $\lambda(t)$ defined by Lemma 2.6(b) equation (7.5) gives $f(t) = \psi_0(\alpha+\lambda(t)) - \psi_0(\lambda(t))$. This proves (2.17). Furthermore, we have

$$\begin{aligned}
f'(t) &= (\psi_1(\alpha+\lambda(t)) - \psi_1(\lambda(t)))\lambda'(t) \\
&= \frac{\psi_1(\alpha+\lambda(t)) - \psi_1(\lambda(t))}{\psi_1(\alpha+\beta+\lambda(t)) - \psi_1(\lambda(t))} \underset{t\searrow 0}{\longrightarrow} \frac{\alpha}{\alpha+\beta},
\end{aligned}$$

where the last limit has already been shown at the end of the proof of Lemma 2.6(a). Consequently, $f'(0+) = \frac{\alpha}{\alpha+\beta}$. Since f is convex, we get that

$$f'(t\pm) \le \frac{\alpha}{\alpha+\beta} \qquad \text{for } t \le 0. \tag{7.6}$$

We have $I_q(\xi^*) = 0$ because the RWRE under the averaged measure $\int P_0^{\omega^\lambda}(\cdot)\overline{\mathbb{P}}(d\bar{\omega})$ is simple random walk, thus RWRE with transitions ω^λ satisfies an almost-sure law of large numbers with velocity given by

$$\overline{\mathbb{E}}[\omega^\lambda_{0,e_1}e_1 + \omega^\lambda_{0,e_2}e_2] = \left(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right) = \xi^*.$$

Let $\xi \in \mathcal{U}$ with $\xi_1 \ge \frac{\alpha}{\alpha+\beta}$. The second equality in the next computation comes from (7.6):

$$\begin{aligned} I_q(\xi) &= \sup_{t\in\mathbb{R}}\{t\xi_1 - f(t)\} = \sup_{t>0}\{t\xi_1 - f(t)\} \\ &= \sup_{\lambda>0}\{\xi_1(\psi_0(\alpha+\beta+\lambda) - \psi_0(\lambda)) - \psi_0(\alpha+\lambda) + \psi_0(\lambda)\} \qquad (7.7) \\ &= \xi_1\psi_0(\alpha+\beta+\lambda(\xi_1)) + (1-\xi_1)\psi_0(\lambda(\xi_1)) - \psi_0(\alpha+\lambda(\xi_1)) \qquad (7.8) \end{aligned}$$

because condition (2.10) picks out the maximizer above.

To derive $I_q(\xi)$ for $\xi_1 \in [0, \frac{\alpha}{\alpha+\beta}]$, switch around α and β and the axes and then apply the first formula of (2.13) already proved.

To compute $I_q^*(te_1)$ for $t < 0$, write temporarily $f_{\alpha,\beta}(t)$ and $I_{\alpha,\beta}(\xi)$ to make the dependence on the parameters α, β explicit. Then

$$\begin{aligned} f_{\alpha,\beta}(t) &= t + \sup_{0\le s\le 1}\{(-t)(1-s) - I_{\alpha,\beta}(se_1 + (1-s)e_2)\} \\ &= t + \sup_{0\le s\le 1}\{(-t)(1-s) - I_{\beta,\alpha}((1-s)e_1 + se_2)\} = t + f_{\beta,\alpha}(-t). \end{aligned}$$

Formula (2.18) follows. In particular, $f'_{\alpha,\beta}(0-) = 1 - \frac{\beta}{\alpha+\beta} = f'_{\alpha,\beta}(0+)$ and $f_{\alpha,\beta}$ is everywhere differentiable. Thus, $I_q = I_{\alpha,\beta}$ is strictly convex on $\mathcal{U}$.

We have now verified formula (2.13) for I_q and Theorem 2.10. By Lemma 8.1 of [24], the statement $I_q(\xi) > I_a(\xi)\ \forall \xi \in \mathcal{U}\setminus\{\xi^*\}$ is equivalent to

$$I_q^*(t) < I_a^*(t) \qquad \text{for all } t \ne 0.$$

(The case $t = 0$ corresponds to $\xi = \xi^*$ and thus leads to an equality.)

Substituting the above functions, this becomes

$$\psi_0(\alpha+\lambda(t)) - \psi_0(\lambda(t)) < \log(\xi_1^* e^t + \xi_2^*) \quad \text{and}$$
$$-t + \psi_0(\beta+\lambda(t)) - \psi_0(\lambda(t)) < \log(\xi_1^* e^{-t} + \xi_2^*)$$

for all $t > 0$. Using (2.12) and rearranging, the above is equivalent to

$$e^{\psi_0(\alpha+\lambda)} < \xi_1^* e^{\psi_0(\alpha+\beta+\lambda)} + \xi_2^* e^{\psi_0(\lambda)} \quad \text{and}$$
$$e^{\psi_0(\beta+\lambda)} < \xi_1^* e^{\psi_0(\lambda)} + \xi_2^* e^{\psi_0(\alpha+\beta+\lambda)} \qquad \text{for all } \lambda \geq 0.$$

Since $\xi_1^*(\alpha + \beta + \lambda) + \xi_2^*\lambda = \alpha + \lambda$ and $\xi_1^*\lambda + \xi_2^*(\alpha + \beta + \lambda) = \beta + \lambda$, the above inequalities follow if $e^{\psi_0(x)}$ is strictly convex. Its second derivative is $e^{\psi_0(x)}(\psi_2(x) + \psi_1(x)^2)$. An exercise in calculus (see Lemma A.5 in [5]) shows this is positive. We have shown that $I_q(\xi) > I_a(\xi)$ for all $\xi \in \mathcal{U}$ with $\xi \neq \xi^*$. The proofs of Theorems 2.7 and 2.10 are complete. $\square$

PROOF OF THEOREM 2.11. Equation (2.19) was proved for part (d) of Theorem 3.6, without appeal to the general variational formula (1.6). Substitution of B^ξ on the right-hand side of (1.6) verifies that the infimum is attained at $B = B^\xi$. Formula (2.19) remains valid for I_q extended to all of $\mathbb{R}_+^2$. This and calculus verify (2.20). $\square$

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