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EXISTENCE CONDITIONS OF PERMANENTAL AND MULTIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS

BY NATHALIE EISENBAUM* AND FRANCK MAUNOURY†,‡

CNRS, Université Pierre et Marie Curie, Université Denis Diderot† and Télécom ParisTech‡*

Existence conditions of permanental distributions are deeply connected to existence conditions of multivariate negative binomial distributions. The aim of this paper is twofold. It answers several questions generated by recent works on this subject, but it also goes back to the roots of this field and fixes existing gaps in older papers concerning conditions of infinite divisibility for these distributions.

1. Introduction. Permanental distributions and the class of multivariate negative binomial distributions that we are interested in, have been originally considered by Griffiths (1984) [13], Griffiths and Milne (1987) [14] and Vere-Jones (1997) [25]. The recent renew of interest for these distributions mainly comes from their connections with the distribution of the local time process of Markov processes. These connections are known under the generic name of "isomorphism theorems." The first one is due to Dynkin (1983). To exploit the more recent isomorphism theorem of Eisenbaum and Kaspi (2009) [9], it was necessary to have a better understanding of the family of permanental distributions. Several authors have since made progress in that direction: Marcus and Rosen [20], Kogan and Marcus [18], Eisenbaum [6–8].

The aim of this paper is double. It answers several questions generated by [18] and [8] but it also goes back to the roots of the subject and fixes an existing gap in [14]. To briefly describe our main results, we first remind the reader of the following basic definitions. All the considered matrices are real.

A *permanental distribution* is the law of a nonnegative random vector $(X_1, X_2, \ldots, X_d)$ with Laplace transform

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{d} z_i X_i\right\}\right] = \det(I + ZA)^{-\beta}, \tag{1.1}$$

where I is the $d \times d$-identity matrix, Z is the diagonal matrix $\mathrm{Diag}((z_i)_{1\le i\le d})$, $A = (a_{ij})_{1\le i,j\le d}$ and β is a fixed positive number.

A matrix A is said to be *β-permanental* if such a random vector exists.

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A matrix A is said to be *β-positive definite* (in short β-positive) if the multivariate Taylor series expansion in $z_1^{n_1} \cdots z_d^{n_d}$ of $\det(I - ZA)^{-\beta}$ has only nonnegative coefficients [see Section 2, (2.3) for an equivalent formulation].

If the spectral radius of a β-positive definite $d \times d$-matrix A is strictly smaller than 1, there exists a nonnegative random vector $(X_1, X_2, \dots, X_d)$ with a multivariate negative binomial distribution such that its probability generating function satisfies

$$\mathbb{E}[z_1^{X_1} \cdots z_d^{X_d}] = \det(I - A)^{\beta} \det(I - ZA)^{-\beta}. \tag{1.2}$$

We mention that this multivariate negative binomial distribution corresponds to an α-permanental point process (see [24]) ζ with index $\alpha = \frac{1}{\beta}$ and kernel $\beta A(I - A)^{-1}$ with respect to the measure $\sum_{k=1}^d \delta_k$ (ζ has the same law as $\sum_{k=1}^d X_k \delta_k$).

Note that distinct matrices A and B may define the same permanental distributions (or the same multivariate negative binomial distribution). More generally, one says that A and B are *effectively equivalent* if, for every Z in $\mathbb{R}^d$,

$$\det(I + ZA) = \det(I + ZB).$$

For example, A and DAD^{-1}, for D nonsingular diagonal matrix, are effectively equivalent. They are said to be *diagonally similar*.

A necessary and sufficient condition for a matrix A to be β-positive for all $\beta > 0$ has been established by Griffiths and Milne [14]. In Section 3, we give a counterexample and correct their criteria. The gap in their proof comes from the negligence of the occurrence of zero entries in the considered matrices. Actually, this negligence has no consequence in the case when the matrices are symmetric but becomes problematic when they are not. It neither has consequences when the matrices are irreducible, but this claim requires a proof that is also given in Section 3.

Once Griffiths and Milne's criterion was fixed, we checked whether the existing results based on their initial criterion were still true. In particular, Vere-Jones NSC for a matrix to be β-permanental for all $\beta > 0$, which is formulated, thanks to Griffiths and Milne's criterion, can easily be fixed. In Section 5, we also fix the argument in [9] which shows that (up to effective equivalence) the nonsingular, β-permanental for every β, matrices are inverse M-matrices (a nonsingular matrix A is an M-matrix if A has no positive off-diagonal entry and A^{-1} has no negative entry). In the symmetric case, this characterization has been established previously by Bapat [1]. Moreover, we extend this characterization to singular matrices.

Section 5 relies on Section 4 which establishes various relations between the properties of β-permanentality and β-positivity. Indeed, they are deeply connected, for example, one can easily see that β-permanentality implies β-positivity.

Hence, the question of the description of the class of matrices that are β-permanental for all β is completely solved. The class of matrices that are β-positive for all β, is well described as well. Note that elements of these two classes correspond to infinitely divisible distributions; the question remains of the description for a fixed β of the β-permanental matrices and the β-positive matrices.

We mention a consequence of Vere-Jones results [25]: a β-permanental symmetric matrix is necessarily positive semidefinite. Conversely, for A symmetric positive semidefinite and $\beta = 1/2$, (1.1) corresponds to the distribution of $(\eta_1^2, \dots, \eta_d^2)$ with $(\eta_1, \dots, \eta_d)$ centered Gaussian vector with covariance A. Consequently, A must be $1/2$-permanental and more generally $n/2$-permanental for every positive integer n. However, for every $\beta > 0$ such that 2β is not an integer, there exist symmetric positive semidefinite matrices that are not β-positive and, therefore, not β-permanental (see the work of Bränden [3] based on Scott and Sokal [22]). This result solves a conjecture set by Shirai and Takahashi [23, 24].

The only known permanental matrices (up to effective equivalence) are symmetric positive semidefinite matrices or inverse M-matrices. Kogan and Marcus [18] have shown that if a nonsingular 3-dimensional permanental matrix is not effectively equivalent to a symmetric matrix then it is diagonally similar to an inverse M-matrix. In Section 6, we establish an analogous result for β-positive matrices: in dimension 3, an irreducible β-positive matrix which is not diagonally similar to a symmetric matrix, is necessarily diagonally similar to a matrix with no negative entry. In Section 7, we answer the question raised by Kogan and Marcus in the case of dimension greater than 3: Do there exist (up to effective equivalence) nonsingular irreducible permanental matrices that are not symmetric positive definite, nor inverse M-matrices? Thanks to the results of Section 4, we reduce the question to the search of 1-positive matrices that are not effectively equivalent to a symmetric matrix nor to a matrix with no negative entry. We actually exhibit families of such matrices and can hence give a positive answer to the question of Kogan and Marcus. This result seems quite surprising in view of [8] according to which, a permanental matrix whose 3×3-principal submatrices are not effectively equivalent to symmetric matrices, is necessarily an inverse M-matrix.

So far, one is not able to give a precise description of β-permanental matrices nor of β-positive matrices. However, we establish in Section 7, some necessary conditions for a matrix to be β-positive for a given β (see Section 7.2), that might help to find the general form of these matrices. We also establish that irreducible β-permanental matrices must satisfy a restrictive condition: their zero entries are symmetric (Theorem 5.3).

All the sections rely on a preliminary section (Section 2) where the needed notation are introduced and preliminary results on cycles of matrices are established, together with a general formula on permanents of matrices with rows and columns repetition.

2. Notation, cycles and permanents. For I, J finite sets having the same cardinality, $\Sigma(I, J)$ denotes the set of the bijections from I to J, $\Sigma(I)$ denotes the set of the permutations of I [i.e., $\Sigma(I) = \Sigma(I, I)$] and Σ_d denotes $\Sigma(\llbracket d \rrbracket)$, where $\llbracket d \rrbracket = \{1, 2, \dots, d\}$.

The β-permanent of a $d \times d$-matrix $A = (a_{ij})_{1 \le i,j \le d}$ is defined by

$$\operatorname{per}_\beta A = \sum_{\sigma \in \Sigma_d} \beta^{\nu(\sigma)} \prod_{i=1}^{d} a_{i\sigma(i)}, \tag{2.1}$$

where $\nu(\sigma)$ is the number of cycles of the permutation σ.

In particular, $\operatorname{per}_1 A$ is the permanent of A and $\operatorname{per}_{-1} A = (-1)^d \det(A)$.

To a given $d \times d$-matrix $A = (a_{ij})_{1 \le i,j \le d}$, one associates square matrices with rows and columns repetition by setting for $n_1, \dots, n_d, n'_1, \dots, n'_d$ $2d$ nonnegative integers such that $\sum_{i=1}^d n_i = \sum_{i=1}^d n'_i$:

$$A[n_1, \dots, n_d | n'_1, \dots, n'_d] = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1d} \\ A_{21} & A_{22} & \dots & A_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ A_{d1} & A_{d2} & \dots & A_{dd} \end{pmatrix},$$

where for $1 \le i, j \le d$, A_{ij} is the $n_i \times n'_j$ matrix whose elements are all equal to a_{ij}.

We write $A[n_1, \dots, n_d]$ for $A[n_1, \dots, n_d | n_1, \dots, n_d]$.

With this notation, one can reformulate the definition of β-positivity as it has first been enunciated by Vere-Jones [25]. Indeed, Vere-Jones [25] has established that for $\beta > 0$

$$\det(I - ZA)^{-\beta} = \sum_{n_1,\dots,n_d=0}^{\infty} \prod_{i=1}^{d} \frac{z_i^{n_i}}{n_i!} \operatorname{per}_\beta A[n_1, \dots, n_d], \tag{2.2}$$

which allows us to see that A is β-positive iff

$$\operatorname{per}_\beta A[n_1, \dots, n_d] \ge 0 \qquad \text{for every } n_1, \dots, n_d. \tag{2.3}$$

For $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_k\}$, with $1 \le i_1 < \dots < i_k \le d$ and $1 \le j_1 < \dots < j_k \le d$, we denote by $A[I \times J]$ the $k \times k$ submatrix of A such that its (r, s) entry is the (i_r, j_s) entry of A. If $I = J$, $A[I]$ denotes $A[I \times I]$.

For k in $\{1, 2, \dots, d\}$, $A^{(k)}$ denotes the $(d-1) \times (d-1)$ principal submatrix obtained by removing the kth row and kth column from A.

We also need to define $\bar{A}^{(k)}$ the $(d-1) \times (d-1)$ matrix $(a_{ik}a_{kj})_{i,j \in [\![d]\!] \setminus \{k\}}$.

A *nonnegative matrix* is a matrix such that all its entries are nonnegative.

The cardinal of a finite set I, is denoted by $|I|$ or $\#I$.

We denote by sgn the sign function:

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For a $d \times d$ real matrix A, $G(A)$ is the directed graph with vertex-set $[\![d]\!]$ and edge-set $\{(i,j) \in [\![d]\!]^2 : a_{ij} \neq 0\}$. A bidirectional edge between two vertices is a couple of edges joining theses two vertices in both ways. A cycle of A is a finite sequence $(i_1, i_2, \dots, i_n)$ of $[\![d]\!]$ such that $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{n-1} i_n} a_{i_n i_1} \neq 0$. For $a_{i_1 i_1} \neq 0$, (i_1) is a cycle of A.

In a cycle $(i_1, \dots, i_n)$, the index k in i_k has to be understood modulo n (e.g., $i_{n+1} = i_1$). Similarly, for a permutation σ in Σ_n, the integers are written modulo n [e.g., $\sigma(n+i) = \sigma(i)$ and $\sigma(i) + n = \sigma(i)$].

A cycle $(i_1, \dots, i_n)$ is said to be *semi-elementary* if:

- it is simple ($i_1, \dots, i_n$ are distinct vertices),
- two vertices i_k, i_l that are not neighbours in the cycle (i.e., $k \neq l+1$ and $l \neq k+1$) are not linked through a bidirectional edge (i.e., either $a_{i_k i_l} = 0$ or $a_{i_l i_k} = 0$).

A cycle $(i_1, \dots, i_n)$ is *elementary* if:

- it is simple,
- two vertices i_k, i_l that are not neighbours in the cycle are not linked ($a_{i_k i_l} = a_{i_l i_k} = 0$).

For A symmetric matrix, semi-elementary cycles are elementary.

2.1. *Positive cycles and symmetric cycles.* Two square matrices A and B are *signature similar* if $A = DBD^{-1}$ with D diagonal matrix with all its diagonal entries in $\{-1, +1\}$. In this section, we give a NSC for an irreducible matrix A to be signature similar to a nonnegative matrix. We also give a NSC for A to be diagonally similar to a symmetric matrix.

A cycle $(i_1, \dots, i_n)$ of a matrix $A = (a_{ij})_{1 \le i,j \le d}$ is said to be *positive* if

$$\prod_{k=1}^{n} a_{i_k i_{k+1}} > 0,$$

and *negative* if $\prod_{k=1}^{n} a_{i_k i_{k+1}} < 0$.

A cycle $(i_1, \dots, i_n)$ of A is said to be *symmetric* if

$$\prod_{k=1}^{n} a_{i_k i_{k+1}} = \prod_{k=1}^{n} a_{i_{k+1} i_k}.$$

The following lemma is due to Maybee (Theorem 4.1 in [21]).

LEMMA 2.1. *Let $A = (a_{ij})_{1 \le i,j \le d}$ and $B = (b_{ij})_{1 \le i,j \le d}$ be two irreducible matrices. Then A and B are diagonally similar iff $G(A) = G(B)$, and for any cycle $(i_1, \dots, i_n)$, $\prod_{k=1}^{n} a_{i_k i_{k+1}} = \prod_{k=1}^{n} b_{i_k i_{k+1}}$.*

Since a matrix $A = (a_{ij})_{1 \le i,j \le d}$ which is diagonally similar to a nonnegative matrix is also signature similar to the matrix $(|a_{ij}|)_{1 \le i,j \le d}$, one obtains the following lemma.

LEMMA 2.2. *An irreducible matrix is signature similar to a nonnegative matrix iff all its cycles are positive.*

If the zero entries of the matrix are symmetric, one can remove the irreducibility condition from Lemma 2.2. This can be seen by decomposing the matrix into a direct sum of irreducible matrices. Hence, one obtains the following lemma.

LEMMA 2.3. *A matrix with all its zero entries symmetric is signature similar to a nonnegative matrix iff all its cycles are positive.*

Assume that a matrix $A = (a_{ij})_{1\le i,j\le d}$ is diagonally similar to a symmetric matrix, then A is also diagonally similar to the matrix $(\sqrt{a_{ij}a_{ji}})_{1\le i,j\le d}$ (by assumption: $a_{ij}a_{ji} \ge 0$ for every i, j). This remark leads to the following lemma.

LEMMA 2.4. *An irreducible matrix is diagonally similar to a symmetric matrix iff all its cycles are symmetric.*

2.2. *Permanent of matrices with rows and columns repetition.* We need to establish the following formulas for the arguments of Section 7.

LEMMA 2.5. *Let B be a $n \times n$ matrix written as the following block matrix*:

$$B = \big(B_{ij}\big)_{1\le i,j\le d},$$

where for every (i, j), B_{ij} is an $n_i \times n'_j$ matrix and $n_1, \dots, n_d, n'_1, \dots, n'_d$ nonnegative integers such that $n_1 + \cdots + n_d = n'_1 + \cdots + n'_d = n$. Then we have

$$\operatorname{per} B = \sum_{\substack{\sum_i k_{ij}=n'_j \\ \sum_j k_{ij}=n_i}} \sum_{\substack{|I_{ij}|=|J_{ij}|=k_{ij} \\ \bigcup_i J_{ij}=[\![n'_i]\!] \\ \bigcup_j I_{ij}=[\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} B_{ij}[I_{ij} \times J_{ij}]\right). \tag{2.4}$$

PROOF. Denote by b_{ij} the (i, j)-entry of B. One has

$$\operatorname{per} B = \sum_{\sigma\in\Sigma_n} \prod_{i=1}^{n} b_{i\sigma(i)}.$$

For a subset I of $\mathbb{R}$ and a real number k, we define: $I - k = \{i - k : i \in I\}$ and $I + k = \{i + k : i \in I\}$. For each σ in Σ_n, we define $I_{ij} = [\![n_i]\!] \cap (\sigma^{-1}([\![n'_i]\!] +$

$\sum_{q=1}^{j-1} n'_q) - \sum_{q=1}^{j-1} n_q)$ and $J_{ij} = (\sigma([\![n_i]\!] + \sum_{q=1}^{i-1} n_q) - \sum_{q=1}^{i-1} n_q) \cap [\![n'_j]\!]$. Then we have

$$\operatorname{per} B = \sum_{\substack{\bigsqcup_i J_{ij} = [\![n'_j]\!] \\ \bigsqcup_j I_{ij} = [\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} B_{ij}[I_{ij} \times J_{ij}] \right)$$

$$= \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \sum_{\substack{|I_{ij}| = |J_{ij}| = k_{ij} \\ \bigcup_i J_{ij} = [\![n'_j]\!] \\ \bigcup_j I_{ij} = [\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} B_{ij}[I_{ij} \times J_{ij}] \right),$$

where $\bigsqcup$ means disjoint union. □

COROLLARY 2.6. *Let $n_1, \dots, n_d$, $n'_1, \dots, n'_d$ be nonnegative integers, such that $n_1 + \cdots + n_d = n'_1 + \cdots + n'_d \geq 1$. We have the following formula for a matrix with repetition of rows and columns*:

$$\operatorname{per} A[n_1, \dots, n_d | n'_1, \dots, n'_d]$$

$$= \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \left(\prod_{i,j=1}^{d} a_{ij}^{k_{ij}} \frac{\prod_{i=1}^{d} n_i! n'_i!}{\prod_{i,j=1}^{d} k_{ij}!} \right). \tag{2.5}$$

PROOF. We set $B = A[n_1, \dots, n_d | n'_1, \dots, n'_d]$. B_{ij} denotes the $n_i \times n'_j$ matrix whose elements are all equal to a_{ij} $(1 \leq i, j \leq d)$.

Applying formula (2.4) to B we obtain

$$\operatorname{per} A[n_1, \dots, n_d | n'_1, \dots, n'_d]$$

$$= \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \left(\prod_{i,j=1}^{d} a_{ij}^{k_{ij}} k_{ij}! \right) \# \Bigg\{ (I_{ij}, J_{ij})_{1 \leq i,j \leq d} : |I_{ij}| = |J_{ij}| = k_{ij};$$

$$\bigcup_i J_{ij} = [\![n'_j]\!]; \bigcup_j I_{ij} = [\![n_i]\!] \Bigg\}.$$

We have

$$\# \left\{ (I_{ij}, J_{ij})_{1 \leq i,j \leq d} : |I_{ij}| = |J_{ij}| = k_{ij}; \bigcup_i J_{ij} = [\![n'_j]\!]; \bigcup_j I_{ij} = [\![n_i]\!] \right\}$$

$$= \# \left\{ (I_{ij})_{1 \leq i,j \leq d} : |I_{ij}| = k_{ij}; \bigcup_j I_{ij} = [\![n_i]\!] \right\}$$

$$\times \#\left\{(J_{ij})_{1\le i,j\le d} : |J_{ij}| = k_{ij}; \bigcup_i J_{ij} = [\![n'_j]\!]\right\}$$

$$= \left(\prod_{i=1}^{d} \binom{n_i}{(k_{ij})_j}\right)\left(\prod_{j=1}^{d} \binom{n'_j}{(k_{ij})_i}\right),$$

where $\binom{n_i}{(k_{ij})_j} = \binom{n_i}{k_{i1}\cdots k_{id}}$ and $\binom{n_j}{(k_{ij})_i} = \binom{n'_j}{k_{1j}\cdots k_{dj}}$ denotes the multinomial coefficients.

Hence, one obtains

$$\operatorname{per} A[n_1, \dots, n_d | n'_1, \dots, n'_d]$$
$$= \sum_{\substack{\sum_i k_{ij} = n'_j \\ \sum_j k_{ij} = n_i}} \left(\prod_{i,j=1}^{d} a_{ij}^{k_{ij}} k_{ij}!\right)\left(\prod_{i=1}^{d} \binom{n_i}{(k_{ij})_j}\right)\left(\prod_{j=1}^{d} \binom{n'_j}{(k_{ij})_i}\right),$$

which leads to (2.5). $\square$

3. NSC to be β-positive for all $\beta > 0$. According to Griffiths and Milne [14], a $d \times d$-matrix A is β-positive for all positive β iff:

(i) For any i, j in $[\![d]\!]$: $a_{ii} \ge 0$, and if $i \ne j$: $a_{ij}a_{ji} \ge 0$.
(ii) For any elementary cycle $(i_1, \dots, i_n)$ of $A + A^t$: $\prod_{k=1}^{n}(a_{i_k i_{k+1}} + a_{i_{k+1} i_k}) \ge 0$.

Set

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and note that B is a counterexample. Indeed, B is clearly β-positive for all $\beta > 0$, but $(b_{12} + b_{21})(b_{23} + b_{32})(b_{31} + b_{13}) < 0$.

The problem with the proof of the above criterion is located in [14] page 18, line 15: a cycle of $A + A^t$ may not correspond to a cycle of A. Under Condition (i), for a given subset $i_1, \dots, i_n$ of $[\![d]\!]$, $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1} \ge 0$ does not necessarily imply that $(a_{i_1 i_2} + a_{i_2 i_1})(a_{i_2 i_3} + a_{i_3 i_2}) \cdots (a_{i_n i_1} + a_{i_1 i_n}) \ge 0$. Nevertheless, when A is symmetric or when all its entries are nonzero, this implication is correct.

Hence, under the additional assumption that A is symmetric or A has no zero entry, Griffiths and Milne's criterion is correct. For the remaining cases, we present below two ways to fix the argument of [14]. Either we extend Condition (ii) to semi-elementary cycles (Theorem 3.1), either we assume that the matrix A is irreducible (Corollary 3.3). This enables us to give a complete answer to the question of β-positivity for all $\beta > 0$.

THEOREM 3.1. *A matrix A is β-positive for all $\beta > 0$ iff the semi-elementary cycles of A are positive.*

PROOF. *Sufficiency*. Assume that for any semi-elementary cycle $(i_1, \dots, i_n)$ of A

$$\prod_{k=1}^{n} a_{i_k i_{k+1}} > 0. \tag{3.1}$$

Then we have (3.1) for any simple cycle $(i_1, \dots, i_n)$ of A. Indeed, it is true for $n \le 3$, as in this case the cycle must be semi-elementary. For $n > 3$, we make an induction proof. Assume that for any $n' \in [\![n-1]\!]$ and any cycle $(j_1, \dots, j_{n'})$ of A: $\prod_{k=1}^{n'} a_{j_k j_{k+1}} > 0$. If $(i_1, \dots, i_n)$ is a semi-elementary cycle, $\prod_{k=1}^{n} a_{i_k i_{k+1}} > 0$ by (3.1). If not, there exist distinct p and q, not neighbours in the cycle, $1 \le p+1 < q \le n$, and such that $a_{i_p i_q} \neq 0$ and $a_{i_q i_p} \neq 0$. Hence, we have

$$\begin{aligned}\prod_{k=1}^{n} a_{i_k i_{k+1}} = {} & \frac{1}{a_{i_p i_q} a_{i_q i_p}} \biggl(\biggl(\prod_{k \in [\![p,q-1]\!]} a_{i_k i_{k+1}} \biggr) a_{i_q i_p} \biggr) \\ & \times \biggl(\biggl(\prod_{k \in [\![1,n]\!] \setminus [\![p,q-1]\!]} a_{i_k i_{k+1}} \biggr) a_{i_p i_q} \biggr).\end{aligned}$$

Note that (i_p, i_q) is an elementary cycle, hence $a_{i_p i_q} a_{i_q i_p} > 0$.

$(\prod_{k \in [\![p,q-1]\!]} a_{i_k i_{k+1}}) a_{i_q i_p}$ and $(\prod_{k \in [\![1,n]\!] \setminus [\![p,q-1]\!]} a_{i_k i_{k+1}}) a_{i_p i_q}$ are positive by induction hypothesis. Consequently, $(i_1, i_2, \dots, i_n)$ is also positive.

Since any cycle is the finite union of simple cycles, the above argument works for any cycle. Hence, any cycle (simple or not) is positive.

For $i_1, \dots, i_n$ in $[\![d]\!]$, either $(i_1, \dots, i_n)$ is a cycle or there exists $k \in [\![n]\!]$ such that $a_{i_k i_{k+1}} = 0$ (if $k = n$, i_{n+1} still denotes i_1). Hence, in both cases, we have $\prod_{k=1}^{n} a_{i_k i_{k+1}} \ge 0$.

For any $\sigma \in \Sigma_n$, $\prod_{k=1}^{n} a_{i_k i_{\sigma(k)}}$ is the product of $\nu(\sigma)$ terms, each term corresponding to a cycle of σ (which is not necessarily a cycle of A). Thanks to the above, one obtains: $\prod_{k=1}^{n} a_{i_k i_{\sigma(k)}} \ge 0$. Consequently, for any $n_1, \dots, n_d \in \mathbb{N}$ and $\beta > 0$, $\mathrm{per}_\beta\, A[n_1, \dots, n_d]$ is a sum of nonnegative terms and, therefore, is nonnegative.

This implies, thanks to (2.3), that A is β-positive for all $\beta > 0$.

Necessity. Assume that A is β-positive for all $\beta > 0$. Then, for any n in $\mathbb{N}^*$, $i_1, \dots, i_n$ in $[\![d]\!]$, we have $\sum_{\sigma \in \Sigma_n} \beta^{\nu(\sigma)} \prod_{k=1}^{n} a_{i_k i_{\sigma(k)}} \ge 0$. Dividing by β, and letting β tend to 0, one obtains

$$\sum_{\sigma \in \Sigma_n : \nu(\sigma)=1} \prod_{k=1}^{n} a_{i_k i_{\sigma(k)}} \ge 0. \tag{3.2}$$

We show now by induction the following property for every $n > 0$:

$P(n)$: For any simple cycle $(i_1, \dots, i_n)$ of A, $\mathrm{sgn}(\prod_{k=1}^{n} a_{i_k i_{k+1}}) = 1$.

$P(1)$ is true. Fix $n > 1$ and assume $P(p)$ is true for all $p \in [\![n-1]\!]$.

Let $(i_1, \dots, i_n)$ be a simple cycle. If there is no other simple cycle whose set of vertices is $\{i_1, \dots, i_n\}$, then we get directly (3.1) from (3.2).

Otherwise, there is another simple cycle $(j_1, \dots, j_n)$ having $\{i_1, \dots, i_n\}$ for set of vertices ($\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\}$).

Suppose that $(\prod_{k=1}^n a_{i_k i_{k+1}})$ and $(\prod_{k=1}^n a_{j_k j_{k+1}})$ have opposite signs. Without loss of generality, suppose that

$$\operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_k i_{k+1}}\right) = 1 = -\operatorname{sgn}\left(\prod_{k=1}^{n} a_{j_k j_{k+1}}\right). \tag{3.3}$$

As $\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\}$, there exist σ in Σ_n such that $(j_1, \dots, j_n) = (i_{\sigma(1)}, \dots, i_{\sigma(n)})$.

For r, s in $[\![n]\!]$ such that $r > s$, and $(u_q)_{1 \le q \le n}$ sequence of real numbers, we use the following convention:

$$\prod_{q=r}^{s} u_q = \prod_{q=r}^{n} u_q \times \prod_{q=1}^{s} u_q. \tag{3.4}$$

Fix k in $[\![n]\!]$,

- either $\sigma(k+1) = \sigma(k) + 1$, and since $\operatorname{sgn}(\prod_{k=1}^n a_{i_k i_{k+1}}) = 1$, one obtains

$$\operatorname{sgn}(a_{i_{\sigma(k)} i_{\sigma(k+1)}}) = \prod_{q=\sigma(k+1)}^{\sigma(k)-1} \operatorname{sgn}(a_{i_q i_{q+1}}); \tag{3.5}$$

- either $\sigma(k+1) \neq \sigma(k) + 1$, and in this case one obtains (3.5) by induction hypothesis.

Consequently, we have

$$\begin{aligned}
\operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_{\sigma(k)} i_{\sigma(k+1)}}\right) &= \operatorname{sgn}\left(\prod_{k=1}^{n} \prod_{q=\sigma(k+1)}^{\sigma(k)-1} a_{i_q i_{q+1}}\right) \\
&= \operatorname{sgn}\left(\prod_{k=0}^{n-1} \prod_{q=\sigma(n-k+1)}^{\sigma(n-k)-1} a_{i_q i_{q+1}}\right).
\end{aligned}$$

Using (3.4), for each k, $(i_q, \sigma(n-k+1) \le q \le \sigma(n-k))$ is made of one piece of the cycle $(i_q,\ 1 \le q \le n)$. Besides note that the first piece ($k = 0$) starts at the index $i_{\sigma(n-k+1)} = i_{\sigma(1)}$, and that the last piece ($k = n-1$) ends at the index $i_{\sigma(n-k)} = i_{\sigma(1)}$. Hence, there exists a positive integer r such that

$$\begin{aligned}
\operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_{\sigma(k)} i_{\sigma(k+1)}}\right) &= \operatorname{sgn}\left[\left(\prod_{q=\sigma(1)}^{\sigma(1)-1} \operatorname{sgn}(a_{i_q i_{q+1}})\right)^r\right] \\
&= \operatorname{sgn}\left[\prod_{q=1}^{n} \operatorname{sgn}(a_{i_q i_{q+1}})\right]^r = 1.
\end{aligned}$$

Consequently, we have

$$\operatorname{sgn}\left(\prod_{k=1}^{n} a_{j_k j_{k+1}}\right) = \operatorname{sgn}\left(\prod_{k=1}^{n} a_{i_{\sigma(k)} i_{\sigma(k+1)}}\right) = 1,$$

which contradicts (3.3). Hence, $(\prod_{k=1}^{n} a_{i_k i_{k+1}})$ and $(\prod_{k=1}^{n} a_{j_k j_{k+1}})$ have the same sign.

Using (3.2), we have a nonnegative sum of terms having the same sign. Therefore, each term of the sum is nonnegative.

Hence, $P(n)$ is true for all n, which establishes the necessity part. □

THEOREM 3.2. *An irreducible matrix A is β-positive for all $\beta > 0$ iff it is signature similar to a nonnegative matrix.*

PROOF. In the sufficiency part of the proof of Theorem 3.1, we have shown that all the semi-elementary cycles of A are positive iff all the cycles of A are positive. Theorem 3.2 is hence a consequence of Theorem 3.1 and Lemma 2.2. □

COROLLARY 3.3. *An irreducible matrix A is β-positive for all $\beta > 0$ iff the elementary cycles of $A + A^t$ are positive and for all $i, j \in [\![d]\!]$, $a_{ij}a_{ji} \geq 0$.*

PROOF. Assume that A is β-positive for all $\beta > 0$. Thanks to Theorem 3.2, $A + A^t$ is signature similar to a nonnegative matrix. Hence, all the elementary cycles of $A + A^t$ are positive. Besides thanks to Proposition 3.7(ii) in [26], we also have for all $i, j \in [\![d]\!]$, $a_{ij}a_{ji} \geq 0$.

Conversely, if all the elementary cycles of $A + A^t$ are positive, then so are the cycles of $A + A^t$ (as $A + A^t$ is symmetric). If additionally, for all i, j in $[\![d]\!]$, $a_{ij}a_{ji} \geq 0$, the sign of any semi-elementary cycle of A is the sign of the corresponding cycle in $A + A^t$ and, therefore, it is positive. Hence, by Theorem 3.1, A is β-positive. □

4. Links between β-permanentality and β-positivity. Remember that for a fixed $\beta > 0$, a $d \times d$-matrix A is β-permanental if $\det(I + ZA)^{-\beta}$ is the Laplace transform of a nonnegative random vector. Vere-Jones has obtained the following NSC for the realization of β-permanentality (Proposition 4.5 in [26]):

Fix $\beta > 0$. A matrix A is β-permanental iff for every $\alpha \geq 0$, $\det(I + \alpha A) > 0$ *and $A(I + \alpha A)^{-1}$ is β-positive.*

By continuity, the proposition below reformulates this NSC.

PROPOSITION 4.1. *Fix $\beta > 0$. A matrix A is β-permanental iff for every $\alpha \geq$* 0, *$(I + \alpha A)$ is nonsingular and $A(I + \alpha A)^{-1}$ is β-positive.*

To establish his NSC, Vere-Jones notes that

$$\det\bigl(I-(Z-\alpha I)A\bigr)^{-\beta}=\det(I+\alpha A)^{-\beta}\det\bigl(I-ZA(I+\alpha A)^{-1}\bigr)^{-\beta}, \tag{4.1}$$

and actually bases his proof on the following result that we will use several times.

PROPOSITION 4.2. *Fix $\beta>0$. A matrix A is β-permanental iff for every $\alpha\geq 0$, the multivariate power series* $\det(I-(Z-\alpha I)A)^{-\beta}$ *in $z_1,\dots,z_d$ contains only nonnegative coefficients.*

To justify the result presented in Proposition 4.2, Vere-Jones refers to a "multivariate analogue of Feller's complete monotonicity property for Laplace transform." But this result can also be seen as a consequence of the Bernstein–Haussdorf–Widder–Choquet theorem (presented as Theorem 2.2 in [22]) and first proved by Choquet ([5], Théorème 10).

The two next lemmas present stability properties for β-permanental matrices and β-positive matrices.

THEOREM 4.3. *Fix $\beta>0$:*

(i) *If A is β-positive matrix, then for any $\gamma\geq 0$, $A+\gamma I$ is also β-positive.*
(ii) *If A is β-permanental matrix, then for any $\gamma\geq 0$, $A+\gamma I$ is also β-permanental.*

PROOF. (i) Let A be a β-positive matrix and γ a nonnegative real number. We have

$$\det\bigl(I-Z(A+\gamma I)\bigr)^{-\beta}=\det(I-\gamma Z)^{-\beta}\det\bigl(I-Z(I-\gamma Z)^{-1}A\bigr)^{-\beta}.$$

This power series is both product and composition of power series with nonnegative coefficients. Therefore, it has only nonnegative coefficients and we can conclude that $A+\gamma I$ is β-positive.

(ii) The proof is similar to the previous one. For A β-permanental matrix and $\gamma>0$, we have, for any $\alpha\geq 0$,

$$\begin{aligned}
&\det\bigl(I-(Z-\alpha)(A+\gamma I)\bigr)^{-\beta}\\
&\quad=\det\bigl((1+\gamma\alpha)I-\gamma Z\bigr)^{-\beta}\\
&\qquad\times\det\bigl(I-(Z-\alpha)\bigl((1+\alpha\gamma)I-\gamma Z\bigr)^{-1}A\bigr)^{-\beta}\\
&\quad=\det\bigl((1+\gamma\alpha)I-\gamma Z\bigr)^{-\beta}\\
&\qquad\times\det\left(I-\left(\sum_{k=1}^{\infty}\frac{\gamma^{k-1}}{(1+\alpha\gamma)^{k+1}}Z^k-\frac{\alpha}{1+\alpha\gamma}\right)A\right)^{-\beta}.
\end{aligned} \tag{4.2}$$

Since A is β-permanental, this power series is both product and composition of power series with nonnegative coefficients (Proposition 4.2). Hence, it has only nonnegative coefficients and one concludes that $A+\gamma I$ is β-permanental. $\square$

The following lemma is a generalization of Lemma 2.5 in [18], which corresponds to (ii) with $\sigma \in]-\frac{1}{a_{kk}}, 0]$.

LEMMA 4.4. *Fix $\beta > 0$:*

(i) *If a matrix A is β-positive, the matrix $A^{(k)} + \sigma \bar{A}^{(k)}$ is also β-positive, for any $\sigma \geq 0$.*

(ii) *If a matrix A is β-permanental, the matrix $A^{(k)} + \sigma \bar{A}^{(k)}$ is also β-permanental, for any $\sigma \geq -\frac{1}{a_{kk}}$ if $a_{kk} \neq 0$ and for any real σ if $a_{kk} = 0$.*

PROOF. Without loss of generality, we assume: $k = d$.

If $\bar{A}^{(d)} = 0$, then (i) and (ii) are obviously satisfied. We hence suppose that $\bar{A}^{(d)} \neq 0$:

(i) Consider the matrix $I - ZA$. For each i in $[\![d-1]\!]$, we add to the ith row, $\frac{z_i a_{id}}{1 - z_d a_{dd}}$ times the last row. The determinant of the obtained matrix is unchanged and the $d-1$ first entries of the last columns of this matrix are 0. Therefore, we have

$$\begin{aligned}
\det(I - ZA)^{-\beta} &= \det\big((\delta_{ij} - z_i a_{ij})_{1\leq i,j\leq d}\big)^{-\beta} \\
&= (1 - z_d a_{dd})^{-\beta} \\
&\quad \times \det\left(\left(\delta_{ij} - z_i a_{ij} + \frac{z_i a_{id}}{1 - z_d a_{dd}}(-z_d a_{dj})\right)_{1\leq i,j\leq d-1}\right)^{-\beta} \\
&= (1 - z_d a_{dd})^{-\beta} \\
&\quad \times \det\left(\left(\delta_{ij} - z_i\left(a_{ij} + \frac{z_d}{1 - z_d a_{dd}} a_{id} a_{dj}\right)\right)_{1\leq i,j\leq d-1}\right)^{-\beta}.
\end{aligned}$$

Denote by $Z^{(d)}$ the matrix $\operatorname{diag}(z_1, \dots, z_{d-1})$ to obtain

$$\begin{aligned}
\det(I - ZA)^{-\beta} &= (1 - z_d a_{dd})^{-\beta} \\
&\quad \times \det\left(I - Z^{(d)}\left(A^{(d)} + \frac{z_d}{1 - z_d a_{dd}} \bar{A}^{(d)}\right)\right)^{-\beta}.
\end{aligned} \tag{4.3}$$

For a $d \times d$ matrix M, define: $\|M\| = \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{\|Mx\|}{\|x\|}$, where for any y in $\mathbb{R}^d$, $\|y\|$ denotes its Euclidian norm.

Fix $\sigma > 0$. Set $R_d = \frac{\sigma}{1 + \sigma a_{dd}}$ and for i in $[\![d-1]\!]$, $R_i = \frac{1}{\|A^{(d)}\| + 2\|\sigma \bar{A}^{(d)}\|}$.

Then for $z_1, \dots, z_d \in \mathbb{C}^d$ such that $|z_i| \leq R_i$, $1 \leq i \leq d$, we have

$$1 - z_d a_{dd} \neq 0$$

and

$$\det\left(I - Z^{(d)}\left(A^{(d)} + \frac{z_d}{1 - z_d a_{dd}} \bar{A}^{(d)}\right)\right) \neq 0$$

[indeed $\|Z^{(d)}(A^{(d)} + \frac{z_d}{1-z_d a_{dd}} \bar{A}^{(d)})\| < 1$].

This implies that for Z such that $|z_i| \leq R_i$, $1 \leq i \leq d$, the power series expansion of $\det(I - ZA)^{-\beta}$ converges. As A is β-positive, all the coefficients of this power series are nonnegative.

If we choose $z_d = R_d$, we get that all the coefficients of the power series expansion of $\det(I - Z^{(d)}(A^{(d)} + \sigma \bar{A}^{(d)}))$ are nonnegative.

Consequently, $A^{(d)} + \sigma \bar{A}^{(d)}$ is β-positive.

(ii) The identity (4.3) is still available. Since A is β-permanental, the function: $(z_1, z_2, \dots, z_d) \mapsto \det(I - ZA)^{-\beta}$, is *absolutely monotone* on the half space $\{z_1, \dots, z_d \in \mathbb{C} : \mathrm{Re}(z_1) < 0, \dots, \mathrm{Re}(z_d) < 0\}$. Equivalently, the function $(z_1, z_2, \dots, z_d) \mapsto \det(I + ZA)^{-\beta}$ is completely monotone on $\{z_1, \dots, z_d \in \mathbb{C} : \mathrm{Re}(z_1) > 0, \dots, \mathrm{Re}(z_d) > 0\}$.

For $\sigma > -\frac{1}{a_{dd}}$, set $z_d = \frac{\sigma}{1+\sigma a_{dd}}$ [we adopt the convention: $-\frac{1}{a_{dd}} = -\infty$ when $a_{dd} = 0$].

Hence, thanks to (4.3), the function $(z_1, \dots, z_{d-1}) \mapsto \det(I - Z^{(d)}(A^{(d)} + \sigma \bar{A}^{(d)}))^{-\beta}$ is absolutely monotone. Consequently, $A^{(d)} + \sigma \bar{A}^{(d)}$ is β-permanental.

If $a_{dd} \neq 0$, this result can be extended to the case $\sigma = -\frac{1}{a_{dd}}$ by continuity. □

THEOREM 4.5. *For a fixed $\beta > 0$, let A be a β-positive matrix with spectral radius ρ. Then for every $r > \rho$, the matrix $(rI - A)^{-1}$ is β-permanental.*

PROOF. For any $\alpha \geq 0$,

$$\begin{aligned}
&\det\bigl(I - (Z - \alpha)(rI - A)^{-1}\bigr)^{-\beta} \\
&\quad = \det(rI - A)^{\beta} \det\bigl((\alpha + r)I - Z - A\bigr)^{-\beta} \\
&\quad = \det(rI - A)^{\beta} \det\bigl((\alpha + r)I - Z\bigr)^{-\beta} \det\bigl(I - \bigl((\alpha + r)I - Z\bigr)^{-1} A\bigr)^{-\beta},
\end{aligned}$$

which is both product and composition of power series with nonnegative coefficients. Hence, it is a power series with nonnegative coefficients.

This is true for any $\alpha \geq 0$. Consequently, thanks to Proposition 4.2, the matrix $(rI - A)^{-1}$ is β-permanental. □

The proposition below shows that if a matrix A satisfies some stronger than β-positivity property, then for a big enough positive ρ, $A + \rho I$ is β-permanental.

PROPOSITION 4.6. *For $\beta, \gamma > 0$, let A be a* $\mathrm{d} \times d$*-matrix such that the multivariable Taylor series expansion in $z_1^{n_1}, \dots, z_d^{n_d}$ of* $\det(I - (Z - \gamma I)A)^{-\beta}$ *contains only nonnegative coefficients and is defined for all $|z_1|, \dots, |z_d| \leq \gamma$. Then there exists $\rho > 0$ such that the matrix $\rho I + A$ is β-permanental.*

PROOF. For $0 \leq \gamma' \leq \gamma$, the power series $\det(I - (Z - \gamma' I)A)^{-\beta}$ contains only nonnegative coefficients, also. One sets $\rho = \gamma^{-1}$ and easily checks, using an argument similar to (4.2) that $\det(I - (Z - \alpha)(\rho I + A))^{-\beta}$ is the product of two power series with only nonnegative coefficients.

Thanks to Proposition 4.2, the matrix $\rho I + A$ is hence β-permanental. $\square$

5. NSC to be β-permanental for all $\beta > 0$. A NSC for a covariance matrix to be β-permanental for all $\beta > 0$, has been established by Griffiths in [13]. Bapat [1] has then shown that for nonsingular matrices, this NSC characterizes symmetric inverse M-matrices. Eisenbaum and Kaspi [9] have then extended Bapat's result to the nonsymmetric case, but they use Griffiths and Milne's criterion and neglect the occurrence of zero entries. Vere-Jones (Proposition 4.7 in [26]) has extended Griffiths' NSC [13] to the nonsymmetric case and uses also Griffiths and Milne's criterion. However, Proposition 4.7 in [26] makes sense only under the additional assumption that the matrix $\mathrm{adj}(A)$ has no zero entry. Besides this assumption implies that: $\mathrm{rank}(A) \geq \dim(A) - 1$.

Under the assumption of irreducibility, Theorem 5.1 below contains the result of [9] and extends it to singular matrices.

THEOREM 5.1. *An irreducible matrix A is β-permanental for all $\beta > 0$ iff A is signature similar to an element in the closure of the inverse M-matrices.*

PROOF. *Step* 1: Assume that A is nonsingular. Thanks to Vere-Jones characterization, if A is β-permanental for every β, then for every $\alpha \geq 0$, $I + \alpha A$ is invertible and $A(I + \alpha A)^{-1}$ is β-positive, for all $\beta > 0$.

Since A is irreducible and invertible, then A^{-1} is irreducible and $A^{-1} + \alpha I$ is also irreducible for any $\alpha \geq 0$. As $I + \alpha A$ is invertible, $(A^{-1} + \alpha I)$ is also invertible and we have $(A^{-1} + \alpha I)^{-1} = A(I + \alpha A)^{-1}$ is irreducible for any $\alpha \geq 0$. Using Theorem 3.2 for the matrix $A(I + \alpha A)^{-1}$, one obtains that for every $\alpha \geq 0$, that $I + \alpha A$ is invertible and $A(I + \alpha A)^{-1}$ is signature similar to a nonnegative matrix.

Set $B = A^{-1}$. There exists an irreducible matrix P with positive diagonal entries and $c > 0$ such that $B = cI - P$. One has $I - \frac{P}{c+\alpha} = (c + \alpha)^{-1} A^{-1}(I + \alpha A)$. Hence, for any $\alpha \geq 0$, $I - \frac{P}{c+\alpha}$ is invertible and $(I - \frac{P}{c+\alpha})^{-1}$ is signature similar to a matrix with nonnegative entries.

For α big enough, $(I - \frac{P}{c+\alpha})^{-1} = I + \frac{P}{c+\alpha} + \frac{F(\alpha)}{(c+\alpha)^2}$, where F is a bounded function in the vicinity of $+\infty$.

Choose now α_0 big enough such that

$$\min_{p_{ij} \neq 0} |p_{ij}| > \max_{i,j} \frac{|F_{ij}(\alpha_0)|}{c + \alpha_0}. \tag{5.1}$$

There exists a signature matrix S_{α_0} such that $S_{\alpha_0}(I - \frac{P}{c+\alpha_0})^{-1} S_{\alpha_0}$ has nonnegative entries. Hence, $S_{\alpha_0} P S_{\alpha_0} + \frac{S_{\alpha_0} F(\alpha_0) S_{\alpha_0}}{c+\alpha_0}$ has nonnegative off-diagonal entries. From

(5.1), we know that $\min_{(S_{\alpha_0}PS_{\alpha_0})_{ij}\neq 0}|(S_{\alpha_0}PS_{\alpha_0})_{ij}| > \max_{i,j}\frac{|(S_{\alpha_0}F(\alpha_0)S_{\alpha_0})_{ij}|}{c+\alpha_0}$. This implies that all the entries of $S_{\alpha_0}PS_{\alpha_0}$ are nonnegative.

Let λ_0 be the Perron–Frobenius eigenvalue of $S_{\alpha_0}PS_{\alpha_0}$. It is also an eigenvalue of P. If $\lambda_0 \geq c$, then for $\alpha = \lambda_0 - c$, one obtains that $A^{-1}(I+\alpha A) = B + \alpha I = \lambda_0 I - P$ is not invertible. This contradicts our assumption (i). Therefore, one must have $\lambda_0 < c$.

This implies that $S_{\alpha_0}BS_{\alpha_0} = cI - S_{\alpha_0}PS_{\alpha_0}$ is a nonsingular M-matrix. Consequently, A is signature similar to an inverse M-matrix.

The converse is a consequence of Theorem 4.5. Hence, Theorem 5.1 is established for nonsingular matrices.

Step 2: Assume that A is singular. There exists γ_0 such that for all $\gamma \in]0,\gamma_0[$, $A+\gamma I$ is invertible. Assume now that A is β-permanental for all β, then thanks to Theorem 4.3(ii), for all $\gamma > 0$, $A+\gamma I$ is β-permanental for all $\beta > 0$. Hence, for every $\gamma \in]0,\gamma_0[$, $A+\gamma I$ is signature similar to an inverse M-matrice.

We want to prove that there exists a signature matrix S such that for every $\gamma \in]0,\gamma_0[$, $S(A+\gamma I)S$ is an inverse M-matrix. For any $\gamma \in]0,\gamma_0[$, we denote by S_γ the signature matrix such that $S_\gamma(A+\gamma I)S_\gamma = S_\gamma AS_\gamma + \gamma I$ is an inverse M-matrix. Set: $\gamma_n = \gamma_0/n$. The sequence (S_{γ_n}) is a sequence of signature matrices. As the set of signature matrices with fixed size d is finite, there exists a signature matrix S such that $\{k \in \mathbb{N}^* : S_{\gamma_k} = S\}$ is infinite. Call this set J. The sequence $(SAS + \gamma_k I)_{k\in J}$ is a sequence of inverse M-matrices and converges to SAS.

Conversely, assume that there exists a signature matrix S and a sequence $(A_n)_{n\in\mathbb{N}}$ of inverse M-matrices such that SA_nS converges to A. If the simple limit of a sequence of Laplace transforms is continuous, then it is itself a Laplace transform. Consequently, A is also β-permanental for all $\beta > 0$. $\square$

REMARK 5.2. It follows from Theorem 1 in [10], and from the fact that a principal submatrix of an inverse M-matrix is still an inverse M-matrix, that if an irreducible matrix A belongs to the closure of inverse M-matrices, it can be written as follows:

$$A = D_1 PB[n_1,\dots,n_d]P^t D_2,$$

where D_1, D_2 are diagonal matrices with positive diagonal entries, P is a permutation matrix, B is a $d\times d$ inverse M-matrix and $n_1,\dots,n_d$ are positive integers.

Therefore, a matrix A is β-permanental for all $\beta > 0$, iff A has the above form.

Thanks to Theorem 5.1, we can establish the following theorem which represents a constraining necessary condition for an irreducible matrix to be β-permanental.

THEOREM 5.3. (i) *Let A be an irreducible matrix. If A is β-permanental for all $\beta > 0$, then A has no zero entry.*

(ii) *For a fixed* $\beta > 0$, *let* A *be an irreducible* β-*permanental* $d \times d$ *matrix. Then the zero entries of* A *are symmetric*, *that is*, *for any* i, j *in* $[\![d]\!]$, $a_{ij} = 0 \iff a_{ji} = 0$.

PROOF. (i) Denote by d the dimension of A. Thanks to Theorem 5.1, A is signature equivalent to $B = (b_{ij})_{1 \le i, j \le d}$ element of the closure of inverse M-matrices. The matrix B is also irreducible. For any given $i, j \in [\![d]\!]$, we show now that $b_{ij} > 0$.

An inverse M-matrix has the path product property (see [17] and [16]), that is, if B is an inverse M-matrix, we have for any integer $n \ge 3$ and $i_1, \dots, i_n \in [\![d]\!]$

$$\frac{\prod_{k=1}^{n-1} b_{i_k i_{k+1}}}{\prod_{k=2}^{n-1} b_{i_k i_k}} \le b_{i_1 i_n}.$$

By continuity, for any matrix B in the closure of inverse M-matrices, one has

$$\prod_{k=1}^{n-1} b_{i_k i_{k+1}} \le b_{i_1 i_n} \prod_{k=2}^{n-1} b_{i_k i_k}. \tag{5.2}$$

As B is irreducible, we chose $n, i_1, \dots, i_n$ such that $i_1, \dots, i_n$ is a path in $G(B)$ from i to j. Hence, we have: $\prod_{k=1}^{n-1} b_{i_k i_{k+1}} > 0$. Using (5.2), we obtain: $b_{ij} = b_{i_1 i_n} > 0$. We have proven that any irreducible matrix belonging to the closure of inverse M-matrices is entrywise positive. Consequently, the matrix B has no zero entry, which implies that A also has no zero entry.

(ii) We prove our claim by induction on d. For $d = 1, 2$, it is obviously true. For $d = 3$, by Corollary 6.3, A is either diagonally similar to a symmetric positive semidefinite matrix, or to an element of the closure of inverse M-matrices (we mention that the proof of Corollary 6.3 does not make use of Theorem 5.3). In the first case, our claim is clearly true. In the second case, according to part (i) of the theorem, A has no zero entry.

Now, we consider an arbitrary integer $d \ge 4$ and we assume that the claim of the theorem is valid for any $p \times p$ matrix, with $p \in [\![d-1]\!]$. Let A be an irreducible β-permanental $d \times d$ matrix and suppose that there exists i, j in $[\![d]\!]$ such that $a_{ij} = 0$ and $a_{ji} \ne 0$. We want to find a contradiction.

Choose k in $[\![d]\!]$ such that $k \ne i$ and $k \ne j$. By Lemma 4.4, for any $x > 0$, $A^{(k)} + x\bar{A}^{(k)}$ is a β-permanental $(d-1) \times (d-1)$ matrix. For $x > 0$ small enough, $A^{(k)} + x\bar{A}^{(k)}$ is also irreducible. Using the induction hypothesis, the zero entries of $A^{(k)} + x\bar{A}^{(k)}$ are symmetric, for $x > 0$ small enough. The (i, j)-entry of this matrix is: $a_{ij} + x a_{ik} a_{kj} = x a_{ik} a_{kj}$. Its (j, i)-entry is $a_{ji} + x a_{jk} a_{ki}$. For $x > 0$ small enough, this last entry is nonzero. Hence, by symmetry: $x a_{ik} a_{kj} \ne 0$. Therefore, we have: $a_{ji} a_{ik} a_{kj} \ne 0$, which implies that the principal 3×3 submatrix $A[\{i, j, k\} \times \{i, j, k\}]$ of A is irreducible. As A is β-permanental, $A[\{i, j, k\} \times \{i, j, k\}]$ is also β-permanental. By the induction hypothesis for $p = 3$, the zero entries of $A[\{i, j, k\} \times \{i, j, k\}]$ must be symmetric, which is a contradiction with the hypothesis $a_{ij} = 0$ and $a_{ji} \neq 0$. Therefore, $a_{ij} = 0$ implies $a_{ji} = 0$. Hence, our claim is established for every d. □

THEOREM 5.4. *Fix $\beta_0 > 0$. Let A be an irreducible β_0-permanental matrix. The matrix A is β-permanental for all $\beta > 0$, iff for any $\sigma \geq 0$, every 3×3 principal submatrix of $A(I + \sigma A)^{-1}$ is β-permanental for all $\beta > 0$.*

The above result is a direct consequence of the following proposition.

PROPOSITION 5.5. *Fix $\beta_0 > 0$. Let A be an irreducible β_0-permanental matrix. If any 3×3 principal submatrix of A is β-permanental for any $\beta > 0$, then A is β-positive for any $\beta > 0$.*

PROOF OF PROPOSITION 5.5. Using Theorem 3.2 and Lemma 2.2, we have to show that any cycle $(i_1, \dots, i_n)$ of A is positive, that is,

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{n-1} i_n} a_{i_n i_1} > 0. \tag{5.3}$$

Since A is β_0-permanental, we have (see Proposition 3.7 in [26]): $a_{ii} \geq 0$ for any i, and $a_{ij} a_{ji} \geq 0$ for $i \neq j$. Hence, (5.3) is satisfied for $n = 1$ and $n = 2$.

For $n = 3$, note that since (i_1, i_2, i_3) is a cycle, $A[\{i_1, i_2, i_3\}]$ is irreducible; besides $A[\{i_1, i_2, i_3\}]$ is β-permanental for all $\beta > 0$. Thanks to Theorem 5.1(ii), $A[\{i_1, i_2, i_3\}]$ is hence diagonally similar to an element of the closure of inverse M-matrices. Consequently, $A[\{i_1, i_2, i_3\}]$ is diagonally similar to a nonnegative matrix. This implies that: $a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_1} \geq 0$, and (5.3) is satisfied for $n = 3$.

For $n > 3$, we make an induction proof. Assume that (5.3) is satisfied for n ($n \geq 3$), we show that (5.3) is satisfied for $n + 1$.

Let $(i_1, i_2, \dots, i_{n+1})$ be a cycle of A. We have $a_{i_1 i_2} \cdots a_{i_n i_{n+1}} a_{i_{n+1} i_1} \neq 0$. Thanks to Theorem 5.3, we know that $a_{i_1 i_2}$, $a_{i_2 i_1}$, $a_{i_2 i_3}$ and $a_{i_3 i_2}$ are not equal to 0. Consequently, the matrix $A[\{i_1, i_2, i_3\}]$ is irreducible. But $A[\{i_1, i_2, i_3\}]$ is also β-permanental for all $\beta > 0$. Using Theorem 5.3(i), we know that $A[\{i_1, i_2, i_3\}]$ has no zero entry. This implies that $a_{i_1 i_3} a_{i_3 i_1} \neq 0$, and we can write

$$a_{i_1 i_2} \cdots a_{i_n i_{n+1}} a_{i_{n+1} i_1} = \frac{1}{a_{i_1 i_3} a_{i_3 i_1}} (a_{i_1 i_2} a_{i_2 i_3} a_{i_3 i_1})(a_{i_1 i_3} a_{i_3 i_4} \cdots a_{i_n i_{n+1}} a_{i_{n+1} i_1})$$

which is a product of three positive terms, by the induction hypothesis. □

6. Dimension 3. In [18], Kogan and Marcus establish a NSC for a nonsingular 3×3-matrix to be β-permanental for a fixed $\beta > 0$. Here, we establish a NSC for a 3×3-matrix to be β-positive for a fixed β. In the corollary below, we also extend their result to singular matrices.

THEOREM 6.1. *Fix $\beta > 0$. Let A be an irreducible 3×3-matrix which is not diagonally similar to a symmetric matrix. Then A is β-positive if and only if it is signature similar to a nonnegative matrix.*

Theorem 6.1 does not consider irreducible 3×3-matrices which are diagonally similar to a symmetric matrix. In Section 7.2, we give some conditions for such matrices to be 1-positive. These matrices are not necessarily positive semidefinite.

PROOF OF THEOREM 6.1. Suppose that A is β-positive and is not signature similar to a nonnegative matrix. We know by assumption that A is neither diagonally similar to a symmetric matrix. By Lemma 2.4, we have

$$a_{12}a_{23}a_{31} \neq a_{13}a_{32}a_{21} \tag{6.1}$$

and by Lemma 2.2, we have $a_{12}a_{23}a_{31} < 0$ or $a_{13}a_{32}a_{21} < 0$. We may assume without loss of generality that

$$a_{12}a_{23}a_{31} < 0. \tag{6.2}$$

Besides by Lemma 4.4, $A^{(3)} + \sigma \bar{A}^{(3)}$ must be β-positive, for all $\sigma \geq 0$, which implies by Proposition 3.7 in [26] that

$$(a_{12} + \sigma a_{13}a_{32})(a_{21} + \sigma a_{23}a_{31}) \geq 0 \qquad \forall \sigma \geq 0. \tag{6.3}$$

Because of (6.1), one cannot have $a_{12} = a_{13}a_{32} = 0$, or $a_{21} = a_{23}a_{31} = 0$.

Set $\sigma_0 = -\frac{a_{21}}{a_{23}a_{31}}$. Note that $\sigma_0 \geq 0$. If $\sigma_0 > 0$, then because of (6.3), one has for every $\varepsilon > 0$: $a_{23}a_{31}(a_{12} + (\sigma_0 + \varepsilon)a_{13}a_{32}) \geq 0$ and $a_{23}a_{31}(a_{12} + (\sigma_0 - \varepsilon) \times a_{13}a_{32}) \leq 0$. Therefore, we must have: $a_{12} + \sigma_0 a_{13}a_{32} = 0$. This condition is equivalent to: $a_{12}a_{23}a_{31} = a_{13}a_{32}a_{21}$, which contradicts the assumption of Theorem 6.1. Hence, $\sigma_0 = 0$, which means that $a_{21} = 0$. But thanks to (6.3) this implies that, for every $\sigma > 0$,

$$a_{12}a_{23}a_{31} + \sigma a_{13}a_{32}a_{23}a_{31} \geq 0.$$

Letting σ tend to 0, one obtains $a_{12}a_{23}a_{31} \geq 0$, which contradicts (6.2).

Consequently, A must be signature similar to a nonnegative matrix.

Conversely, if A is signature similar to a nonnegative matrix, then clearly thanks to (2.2), A is β-positive for any given $\beta > 0$. $\square$

REMARK 6.2. Thanks to Proposition 4.1, Kogan and Marcus NSC can be easily deduced from Theorem 6.1. Conversely, thanks to Theorem 4.5, one can also deduce Theorem 6.1 from Kogan and Marcus NSC.

COROLLARY 6.3. *Fix $\beta > 0$. Let A be an irreducible 3×3 matrix. If A is β-permanental, then*:

– *either A is diagonally similar to a positive semidefinite symmetric matrix,*
– *or A is signature similar to an element of the closure of inverse M-matrices.*

PROOF. Assume that the matrix A is β-permanental. The eigenvalues of A have nonnegative real part because $z \mapsto \det(I - zA)^{-\beta}$ must be analytic for $\mathrm{Re}(z) < 0$ (see Vere-Jones in [26], Proposition 4.6). If A is diagonally similar to a symmetric matrix, then A must have real nonnegative eigenvalues only. Hence, A is diagonally similar to a positive semidefinite symmetric matrix.

Thanks to Vere-Jones characterization (Proposition 4.1), for all $\alpha \geq 0$, $I + \alpha A$ is invertible and $A(I + \alpha A)^{-1}$ is β-positive. If A is not diagonally similar to a symmetric matrix, then $A(I + \alpha A)^{-1}$ is neither diagonally similar to a symmetric matrix. Indeed, we have $A(I + \alpha A)^{-1} = \alpha^{-1} I - \alpha^{-1}(I + \alpha A)^{-1}$. Then using Theorem 6.1 for every $\alpha \geq 0$, $A(I + \alpha A)^{-1}$ is signature similar to a matrix with nonnegative entries. Consequently, for every $\alpha \geq 0$, $A(I + \alpha A)^{-1}$ is β'-positive for any $\beta' > 0$. Hence, thanks to Vere-Jones characterization, A is β'-permanental for every $\beta' > 0$. Therefore, by Theorem 5.1(ii) A is signature similar to an element of the closure of inverse M-matrices. $\square$

REMARK 6.4. The permanent of an M-matrix is always nonnegative (see, e.g., [4, 11] or [12]), and consequently it is so for any principal submatrix of an M-matrix. It is hence natural to ask whether M-matrices are 1-positive.

Fix $d \geq 3$, and consider a $d \times d$ M-matrix, $A = (a_{ij})_{1 \leq i,j \leq d}$, with no zero off-diagonal entry and not diagonally similar to a symmetric matrix. Then thanks to Lemma 2.4, A has a nonsymmetric cycle. The length of a nonsymmetric cycle is always strictly greater than 2. Since the off-diagonal entries of A are nonzero, one can always choose three vertices i, j, k in this cycle such that they form a nonsymmetric cycle: $a_{ij}a_{jk}a_{ki} \neq a_{ji}a_{kj}a_{ik}$.

If for some $\beta > 0$, A was β-positive, then its 3×3-principal submatrix corresponding to the vertices i, j and k, would be β-positive, also. Since this principal submatrix is not diagonally similar to a symmetric matrix and has no zero entry, it should be, according Theorem 6.1, signature similar to a nonnegative matrix. This last claim can not be true. We conclude that such an M-matrix A cannot be β-positive, whatever the value of $\beta > 0$.

7. Beyond dimension 3. Fix $\beta > 0$. In view of the results of the previous section, it is natural to ask whether in dimension $d > 3$, there exists an irreducible β-positive $d \times d$-matrix which is not diagonally similar to a symmetric matrix nor signature similar to a nonnegative matrix.

This question is the analogue for β-positive matrices of the question generated by Kogan and Marcus work [18] on β-permanental 3×3-matrices. Namely, does there exist an irreducible β-permanental $d \times d$-matrix which is not diagonally similar to a symmetric positive semidefinite matrix nor signature similar to an element of the closure of inverse M-matrices?

We consider matrices that can be written as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{7.1}$$

where:

- A_{11} is a $p \times p$ $(1 \le p \le d-1)$ symmetric positive semidefinite square matrix,
- A_{12} and A_{21} are (not necessarily square) rank 1 matrices such that the block matrix $(A_{12}, (A_{21})^t)$ can be written $(\gamma_j C_i)_{\substack{1\le i\le p\\ 1\le j\le 2d-2p}}$, where $\gamma_1, \dots, \gamma_{2d-2p}$ are nonnegative real numbers,
- A_{22} is a square nonnegative matrix.

We make use of these matrices to answer positively to the two questions (Section 7.1) but also to find some necessary conditions for a given matrix to be 1-positive (Section 7.2).

7.1. *Positive answer to the two questions.*

THEOREM 7.1. *The matrices that satisfy* (7.1), *are* 1*-positive.*

PROOF. We prove Theorem 7.1 in two steps.

Step 1. We show that if for any A_{11} symmetric positive semidefinite matrix, any rank 1 matrices A_{12} and A_{21} such that $A_{12} = (A_{21})^t$:

$$\operatorname{per}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \ge 0, \tag{7.2}$$

then Theorem 7.1 is proved.

Indeed, first note that if A has the form (7.1), then for any $n_1, \dots, n_d \in \mathbb{N}$ such that $n_1 + \cdots + n_d \ge 1$, $A[n_1, \dots, n_d]$ has also the form (7.1). Hence, thanks to (2.2), to show that any matrix A satisfying (7.1) is 1-positive, it is sufficient to show that for any matrix A satisfying (7.1) $\operatorname{per} A \ge 0$.

Assume (7.2). Let A be a matrix having the form (7.1). As we can exchange simultaneously rows and columns without changing the value of the permanent, we have

$$\operatorname{per} A = \operatorname{per}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \operatorname{per}\begin{pmatrix} A_{11} & \alpha_1 C & \cdots & \alpha_q C & 0 \\ \beta_1 C^t & & & & \\ \vdots & & \mathbf{A}_{22} & & \\ \beta_{q'} C^t & & & & \\ 0 & & & & \end{pmatrix},$$

where C is a nonzero column vector, q, q' are positive integers, $\alpha_1, \dots, \alpha_q$, $\beta_1, \dots, \beta_{q'}$ are positive real numbers and 0 are zero matrices with the appropriate dimension (having no column—when A_{12} has no zero column—and/or no row—when A_{21} has no zero row).

One obtains: $\operatorname{per} A = (\prod_{i=1}^{q} \alpha_i)(\prod_{i=1}^{q'} \beta_i) \operatorname{per} B$, with

$$B = (b_{ij})_{1\le i,j\le d} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & C & \cdots & C & 0 \\ C^t & & & & \\ \vdots & & \mathrm{A}'_{22} & & \\ C^t & & & & \\ 0 & & & & \end{pmatrix},$$

where $B_{11} = A_{11}$, A'_{22} is a block square matrix obtained from A_{22} by first dividing its ith row by β_i for $1 \le i \le q'$, then dividing its jth column by α_j for $1 \le j \le q$.

We show now that $\operatorname{per} B \ge 0$.

For $I \subset [\![d]\!]$, I^c denotes $[\![d]\!] \setminus I$.

We have

$$\begin{aligned} \operatorname{per} B &= \sum_{\sigma\in\Sigma_d} \prod_{i=1}^{d} b_{i\sigma(i)} = \sum_{\substack{I,J\subset[\![p+1,d]\!]:\\ |I|=|J|}} \sum_{\substack{\sigma\in\Sigma_d:\\ \{i>p:\sigma(i)>p\}=I=\sigma^{-1}(J)}} \prod_{i\in I} b_{i\sigma(i)} \prod_{i\in I^c} b_{i\sigma(i)} \\ &= \sum_{\substack{I,J\subset[\![p+1,d]\!]:\\ |I|=|J|}} \Biggl(\sum_{\sigma\in\Sigma(I,J)} \prod_{i\in I} b_{i\sigma(i)} \Biggr) \Biggl(\sum_{\substack{\sigma\in\Sigma(I^c,J^c):\\ \sigma(I^c\cap[\![p+1,d]\!])\subset[\![p]\!]}} \prod_{i\in I^c} b_{i\sigma(i)} \Biggr) \\ &= \sum_{\substack{I,J\subset[\![p+1,d]\!]:\\ |I|=|J|}} \operatorname{per} B[I\times J] \operatorname{per} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix} [I^c \times J^c]. \end{aligned}$$

Remark that A'_{22} is a nonnegative matrix. Hence, for all $I, J \subset [\![p+1,d]\!]$: $\operatorname{per} B[I\times J] \ge 0$.

If $I, J \subset [\![p+1,d]\!]$, then $[\![p]\!] \subset I^c$ and $[\![p]\!] \subset J^c$. Set

$$K = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix} [I^c \times J^c].$$

In case K satisfies the assumption of (7.2) (e.g., when $q = q'$ and $I = J$), then $\operatorname{per} K \ge 0$. But it might happen that for some choices of (I, J), K does not satisfy the assumption of (7.2). When it is so, either K contains a zero row or a zero column, and hence, $\operatorname{per} K = 0$.

We finally obtain $\operatorname{per} B \ge 0$, which completes Step 1.

Step 2. We show now that (7.2) is true.

Let A be a square matrix satisfying

$$A = (a_{ij})_{1\le i,j\le d} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix}$$

with A_{11} symmetric positive semidefinite $p \times p$-matrix, A_{12}, A_{21} such that $A_{21} = (A_{12})^t$ and 0 zero square matrix.

We have to show that $\operatorname{per} A \geq 0$

$$\operatorname{per} A = \sum_{\sigma \in \Sigma_d} \prod_{i=1}^{d} a_{i\sigma(i)} = \sum_{\substack{I,J \subset [\![p]\!]: \\ |I|=|J|}} \sum_{\substack{\sigma \in \Sigma_d: \\ \{i \leq p: \sigma(i) \leq p\} = I = \sigma^{-1}(J)}} \prod_{i \in I} a_{i\sigma(i)} \prod_{i \in I^c} a_{i\sigma(i)}$$

$$= \sum_{\substack{I,J \subset [\![p]\!]: \\ |I|=|J|}} \Bigl(\sum_{\sigma \in \Sigma(I,J)} \prod_{i \in I} a_{i\sigma(i)} \Bigr) \Bigl(\sum_{\substack{\sigma \in \Sigma(I^c, J^c): \\ \sigma(I^c \cap [\![p]\!]) \subset [\![p+1, d]\!]}} \prod_{i \in I^c} a_{i\sigma(i)} \Bigr)$$

$$= \sum_{\substack{I,J \subset [\![p]\!]: \\ |I|=|J|}} \operatorname{per} A[I \times J] \operatorname{per} \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} [I^c \times J^c].$$

For $I, J \subset [\![p]\!]$, if $|[\![p]\!] \setminus I| (= |[\![p]\!] \setminus J|) \neq d - p$,

$$\operatorname{per} \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} [I^c \times J^c] = 0.$$

Hence, we have that if $2p < d$, $\operatorname{per} A = 0$ and if $2p \geq d$,

$$\operatorname{per} A = \sum_{\substack{I,J \subset [\![p]\!]: \\ |I|=|J|=2p-d}} \operatorname{per} A_{11}[I \times J] \operatorname{per} A_{12}\bigl[([\![p]\!] \setminus I) \times [\![d-p]\!]\bigr]$$
$$\times \operatorname{per} A_{21}\bigl[[\![d-p]\!] \times ([\![p]\!] \setminus J)\bigr]$$

which leads to

$$\operatorname{per} A = \sum_{\substack{I,J \subset [\![p]\!]: \\ |I|=|J|=2p-d}} \operatorname{per} A_{11}[I \times J] \operatorname{per} A_{12}\bigl[([\![p]\!] \setminus I) \times [\![d-p]\!]\bigr] \times \operatorname{per} A_{12}\bigl[([\![p]\!] \setminus J) \times [\![p]\!]\bigr]. \tag{7.3}$$

The case $I = J = \varnothing$ being trivial, assume that: $2p > d$, and set: $k = 2p - d$. In view of (7.3), to show that $\operatorname{per} A \geq 0$, it is sufficient to prove that for any positive semidefinite $p \times p$-matrix B, the matrix $(\operatorname{per} B[I \times J])_{I,J \subset [\![p]\!]: |I|=|J|=k}$ is positive semidefinite.

To establish the latest, we recall some fundamental results of linear algebra (see, e.g., Bathia's book [2], pages 12–19).

The k-fold tensor product space of $\mathbb{R}^d$, denoted $\bigotimes^k(\mathbb{R}^p)$, is endowed with the inner product

$$\langle x_1 \otimes \cdots \otimes x_k | y_1 \otimes \cdots \otimes y_k \rangle = \langle x_1, y_1 \rangle \cdots \langle x_k, y_k \rangle$$

for $x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{R}^p$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^p$.

For any positive semidefinite real $p \times p$ matrix B, consider $\bigotimes^k B$, the k-fold tensor product of B on $\bigotimes^k(\mathbb{R}^p)$. It is a linear operator on $\bigotimes^k(\mathbb{R}^p)$ defined as follows:

$$\bigotimes^k B(x_1 \otimes \cdots \otimes x_k) = Bx_1 \otimes \cdots \otimes Bx_k$$

for $x_1, \ldots, x_k \in \mathbb{R}^p$.

If $\epsilon_1, \dots, \epsilon_p$ is an orthonormal eigenvector base of B in $\mathbb{R}^p$ and if $\lambda_1, \dots, \lambda_p$ are its corresponding eigenvalues, $(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_k})_{1 \le i_1, \dots, i_k \le p}$ is an orthonormal eigenvector base of $\bigotimes^k B$ in $\bigotimes^k (\mathbb{R}^p)$ and $(\lambda_{i_1} \cdots \lambda_{i_k})_{1 \le i_1, \dots, i_k \le p}$ are its corresponding eigenvalues. Consequently, since B is positive semidefinite, $\bigotimes^k B$ is also positive semidefinite.

For $x_1, \dots, x_k \in \mathbb{R}^p$, their symmetric tensor product $x_1 \vee \cdots \vee x_k$ is defined by

$$x_1 \vee \cdots \vee x_k = \frac{1}{\sqrt{k!}} \sum_{\sigma \in \Sigma_k} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k)}.$$

Denote by $(e_1, \dots, e_p)$ the canonical base of $\mathbb{R}^p$ and for $I = \{i_1, \dots, i_k\} \subset [\![p]\!]$, set: $e_I = e_{i_1} \vee \cdots \vee e_{i_k}$. In $\bigotimes^k (\mathbb{R}^p)$, we consider the subspace F spanned by the orthonormal family $(e_I)_{I \subset [\![p]\!]: |I| = k}$. This family is actually an orthonormal base of F. Let p_F denote the orthogonal projection from $\bigotimes^k (\mathbb{R}^p)$ onto F, then $p_F \circ (\bigotimes^k B)$ is represented by a positive semidefinite matrix in this base.

For $I, J \subset [\![p]\!]$ such that $|I| = |J| = k$, the (I, J) entry of this matrix is equal to $\langle e_I, (\bigotimes^k B) e_J \rangle$. We have

$$\begin{aligned}
\left\langle e_I, \left(\bigotimes^k B\right) e_J \right\rangle &= \langle e_{i_1} \vee \cdots \vee e_{i_k} | B e_{j_1} \vee \cdots \vee B e_{j_k} \rangle \\
&= \frac{1}{k!} \sum_{\sigma, \sigma' \in \Sigma_p} \langle e_{i_{\sigma(1)}}, B e_{j_{\sigma'(1)}} \rangle \cdots \langle e_{i_{\sigma(k)}}, B e_{j_{\sigma'(k)}} \rangle \\
&= \sum_{\sigma \in \Sigma_p} \langle e_{i_1}, B e_{j_{\sigma(1)}} \rangle \cdots \langle e_{i_k}, B e_{j_{\sigma(k)}} \rangle \\
&= \operatorname{per} B[I \times J].
\end{aligned}$$

This proves that $(\operatorname{per} B[I \times J])_{I, J \subset [\![p]\!]: |I| = |J| = k}$ is positive semidefinite. Therefore, $\operatorname{per} A \ge 0$ and Theorem 7.1 is proved. $\square$

The following proposition provides a positive answer to the first question.

PROPOSITION 7.2. *Let A be a square matrix satisfying the following condition*:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{7.4}$$

where A_{11} is a symmetric positive semidefinite matrix with at least one off-diagonal negative entry, A_{12} and A_{21} are entrywise positive matrices such that $A_{12} = (A_{21})^t$ has rank 1, *and A_{22} is a nonsymmetric square nonnegative matrix.*

Then A is 1*-positive and is not diagonally similar to a symmetric matrix nor to a nonnegative matrix.*

PROOF. Since A satisfies (7.4), A satisfies also (7.1). Hence, thanks to Theorem 7.1, we know that A is 1-positive.

Denote by d the dimension of A and by p the dimension of A_{11}.

As A_{11} has at least an off-diagonal negative entry, there exist i, j in $[\![p]\!]$, with $i \neq j$, such that $a_{ij} = a_{ji} < 0$. As A_{22} is not symmetric, there exist k, l in $[\![p+1, d]\!]$, with $k \neq l$, such that $a_{kl} \neq a_{lk}$. Note that (i, j, k, l) is a nonsymmetric cycle of A. Consequently, A is not diagonally similar to a symmetric matrix.

A_{22} is a nonnegative matrix and $a_{kl} \neq a_{lk}$, thus we have either $a_{kl} > 0$ or $a_{lk} > 0$. Hence, either (i, j, k, l) or (i, j, l, k) is a negative cycle. It follows that the matrix A is not diagonally similar to a nonnegative matrix. $\square$

The following theorem answers positively to the second question.

THEOREM 7.3. *For every $d \geq 4$, there exists nonsingular 1-permanental $d \times d$ matrices that are not diagonally similar to a symmetric matrix, nor diagonally similar to an inverse M-matrix.*

PROOF. Let A be a matrix satisfying (7.4). Let r be a positive real number greater than the spectral radius of A. From Theorem 4.5, $(rI - A)^{-1}$ is 1-permanental.

By Proposition 7.2, A is not diagonally similar to a symmetric matrix, hence neither is $(rI - A)^{-1}$.

Assume that $(rI - A)^{-1}$ is diagonally similar to an inverse M-matrix. Then there exists a nonsingular diagonal matrix D, a positive real number c and a nonnegative matrix Q such that $(rI - A)^{-1} = D^{-1}(cI - Q)^{-1}D$. One obtains $DAD^{-1} = (r - c)I + Q$.

This implies that all the off-diagonal entries of DAD^{-1} are nonnegative. Besides, the diagonal entries of DAD^{-1} have the same sign as those of A, and thus they are nonnegative. Consequently, all the entries of DAD^{-1} are nonnegative.

This leads to a contradiction, as, by Proposition 7.2, A is not diagonally similar to a matrix with nonnegative entries.

Therefore, $(rI - A)^{-1}$ is not diagonally similar to an inverse M-matrix. $\square$

COROLLARY 7.4. *Let A be a square matrix satisfying the following condition*:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{7.5}$$

where the matrix A_{11} is symmetric, the matrices A_{12} and A_{21} have positive entries only and such that the block matrix $(A_{12}, (A_{21})^t)$ has rank 1, and A_{22} is a square matrix with positive entries only.

Then there exists $\gamma > 0$ such that $A + \gamma I$ is 1-permanental.

The above corollary of Theorem 7.1 gives another way to answer positively to the second question. Indeed, choose A satisfying both (7.5) and (7.4), and choose $\gamma > 0$ such that $A + \gamma I$ is 1-permanental. Then, thanks to Proposition 7.2, $A + \gamma I$ is not diagonally equivalent to a symmetric matrix nor to a nonnegative matrix. This last property implies that $A + \gamma I$ can not be diagonally equivalent to an inverse M-matrix.

PROOF OF COROLLARY 7.4. First note that for $\gamma > 0$ sufficiently big, $A_{11} + \gamma I$ is symmetric positive definite. Hence, we can assume in this proof that A_{11} is symmetric positive definite.

We just have to show that such a matrix A satisfies the assumption of Proposition 4.6. Thanks to (4.1), it is sufficient to check that for $\alpha > 0$ small enough, $A(\alpha) = A(I + \alpha A)^{-1}$ is 1-positive. Since matrices with the form (7.5) are 1-positive, it is hence sufficient to prove that $A(\alpha)$ has also the form (7.5) for $\alpha > 0$ small enough.

Assume that A is nonsingular. For $\alpha > 0$ small enough, $I + \alpha A$ is nonsingular and $A(I + \alpha A)^{-1} = (A^{-1} + \alpha I)^{-1}$.

For any real nonsingular matrix B such that

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

with B_{11} nonsingular symmetric matrix, B_{12} and B_{21} matrices such that the block matrix $(B_{12}, (B_{21})^t)$ has rank 1, we have

(∗) $\quad (B^{-1})_{11}$ *is a symmetric matrix, and* $((B^{-1})_{12}, ((B^{-1})_{21})^t)$ *has rank* 1.

Indeed, denote by $(B')^{-1}$ the Schur complement of B_{11}, that is, $(B')^{-1} = B_{22} - B_{21}(B_{11})^{-1}B_{12}$. Then it well known that

$$B^{-1} = \begin{pmatrix} (B^{-1})_{11} & (B^{-1})_{12} \\ (B^{-1})_{21} & (B^{-1})_{22} \end{pmatrix},$$

with

$$(B^{-1})_{11} = (B_{11})^{-1} + (B_{11})^{-1} B_{12} B' B_{21} (B_{11})^{-1},$$
$$(B^{-1})_{12} = -(B_{11})^{-1} B_{12} B',$$
$$(B^{-1})_{21} = -B' B_{21} (B_{11})^{-1}.$$

As $\operatorname{rank}(B_{12}, (B_{21})^t) = 1$, we can write: $(B_{12})_{ij} = \alpha_j C_i$ and $(B_{21})_{ij} = \beta_i C_j$. One obtains $B_{12} B' B_{21} = (C_i C_j \sum_{kl} \alpha_k \beta_l b'_{kl})_{i,j}$, which is symmetric. Therefore, $(B^{-1})_{11}$ is symmetric.

Besides, $\operatorname{rank}(B_{12}, (B_{21})^t) = 1$ and $((B^{-1})_{21})^t = -(B_{11})^{-1}(B_{21})^t(B')^t$.

Hence, $((B^{-1})_{12}, ((B^{-1})_{21})^t)$ has rank 1.

We can remove the assumption that B_{11} is invertible by continuity (consider $B+\epsilon I$ instead, and let ϵ tend to 0).

We make use of the fact that B satisfies ($*$) in two cases. First, the case $B=A$, then the case $B=A^{-1}+\alpha I$. This proves that $A(\alpha)=(A^{-1}+\alpha I)^{-1}$ has the form

$$\begin{pmatrix} A(\alpha)_{11} & A(\alpha)_{12} \\ A(\alpha)_{21} & A(\alpha)_{22} \end{pmatrix},$$

with $A(\alpha)_{11}$ symmetric invertible square matrix, $A(\alpha)_{12}$ and $A(\alpha)_{21}$ matrices such that the block matrix $(A(\alpha)_{12}, (A(\alpha)_{21})^t)$ has rank 1.

As $A(\alpha)$ tends to A when α tends to 0, for α small enough, $A(\alpha)_{12}$, $A(\alpha)_{12}$ and $A(\alpha)_{22}$ contain only positive entries and $A(\alpha)_{11}$ is symmetric positive definite. Hence, $A(\alpha)$ has the form (7.5).

In case A is singular, one can use the previous argument for $A+\varepsilon I$ (where $\varepsilon > 0$, small enough) instead of A and let then ε tend to 0. □

7.2. *Some conditions for* 1*-positivity.* The following proposition shows that Theorem 7.1 is no longer valid if one removes the assumption that A_{11} is positive semidefinite, even if one assumes instead that A_{11} is 1-positive.

PROPOSITION 7.5. *Let $A=(a_{ij})_{1\le i,j\le 3}$ be a symmetric 3×3-matrix such that A is not signature similar to a nonnegative matrix and $a_{33}=0$. Then A is* 1*-positive iff*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is positive semidefinite.

PROOF. Thanks to Lemma 2.3, since A is not signature similar to a nonnegative matrix, A has a negative cycle. As A is symmetric, one hence must have $a_{12}a_{23}a_{31} = a_{13}a_{32}a_{21} < 0$.

Assume that A is 1-positive. A is diagonally similar to

$$\begin{pmatrix} a_{11}/a_{13}^2 & a_{12}/(a_{13}a_{23}) & 1 \\ a_{12}/(a_{13}a_{23}) & a_{22}/a_{23}^2 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

To show that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

is positive semidefinite, is equivalent to show that

$$\begin{pmatrix} a_{11}/a_{13}^2 & a_{12}/(a_{13}a_{23}) \\ a_{12}/(a_{13}a_{23}) & a_{22}/a_{23}^2 \end{pmatrix}$$

is positive semidefinite.

Hence, we may assume without loss of generality that $a_{13} = a_{31} = a_{23} = a_{32} = 1$ and $a_{12} = a_{21} < 0$.

Thanks to (2.5), we have

$$\operatorname{per} A[n_1, n_2, n_3] = \sum_{\substack{\sum_i k_{ij} = n_j \\ \sum_j k_{ij} = n_i}} \left(\prod_{i,j=1}^{3} a_{ij}^{k_{ij}} \frac{\prod_{i=1}^{3} (n_i!)^2}{\prod_{i,j=1}^{3} k_{ij}!} \right)$$

for $(n_1, n_2, n_3) \in \mathbb{N}^3 \setminus \{(0, 0, 0)\}$.

As $a_{33} = 0$, only the terms of this sum with $k_{33} = 0$ are nonzero. We choose to take $n_3 = n_1 + n_2 - 1$. For this choice, the above sum contains only four terms, corresponding to:

- $k_{11} = 1, k_{12} = k_{21} = k_{22} = k_{33} = 0, k_{13} = k_{31} = n_1 - 1, k_{23} = k_{32} = n_2$,
- $k_{12} = 1, k_{11} = k_{21} = k_{22} = k_{33} = 0, k_{13} = n_1 - 1, k_{31} = n_1, k_{23} = n_2, k_{32} = n_2 - 1$,
- $k_{21} = 1, k_{11} = k_{12} = k_{22} = k_{33} = 0, k_{13} = n_1, k_{31} = n_1 - 1, k_{23} = n_2 - 1, k_{32} = n_2$,
- $k_{22} = 1, k_{11} = k_{12} = k_{21} = k_{33} = 0, k_{13} = k_{31} = n_1, k_{23} = k_{32} = n_2 - 1$.

Therefore, we have

$$\operatorname{per} A[n_1, n_2, n_1 + n_2 - 1] = \big((n_1 + n_2 - 1)!\big)^2 \big(n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22}\big)$$

for any $(n_1, n_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}$.

As A is 1-positive, we get that $n_1^2 a_{11} + 2n_1 n_2 a_{12} + n_2^2 a_{22} \geq 0$, for any $(n_1, n_2) \in \mathbb{N}^2 \setminus \{(0, 0)\}$. By dividing by any positive integer, we obtain that it is also true for any $(n_1, n_2) \in \mathbb{Q}_+^2 \setminus \{(0, 0)\}$ and by continuity it is also true for any $n_1, n_2 \in \mathbb{R}_+$. As $a_{12} < 0$, it is true for any $n_1, n_2 \in \mathbb{R}$.

This proves that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is positive semidefinite.

Conversely, if the matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is positive semidefinite, then by Theorem 7.1, A is 1-positive. $\square$

The following proposition shows that Theorem 7.1 is no longer valid if one removes the assumption that $\operatorname{rank}(A_{12}, (A_{21})^t) = 1$.

PROPOSITION 7.6. *Consider the following matrix A*:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{7.6}$$

where A_{11} is a symmetric $p \times p$-matrix with negative off-diagonal entries only ($p \geq 2$), A_{12} and A_{21} are matrices with positive entries only, such that $A_{12} = (A_{21})^t$, and A_{22} is a nonnegative $(d-p) \times (d-p)$-matrix such that none of its principal 2×2-submatrices is symmetric ($d - p \geq 2$).

Then, if A is 1-positive, it is necessary that A_{12} has rank 1.

PROOF. Assume that $A = (a_{ij})_{1 \leq i,j \leq d}$ is 1-positive and that A_{12} has not rank 1. Since $A_{12} \neq 0$, rank(A_{12}) must be strictly greater than greater than 1. Hence, there exist $i, j \in [\![p]\!]$, with $i \neq j$, and $k, l \in [\![p+1, d]\!]$, with $k \neq l$, such that

$$\operatorname{rank}\begin{pmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{pmatrix} > 1.$$

But $A[\{i, j, k, l\}]$ is still 1-positive. We are going to obtain a contradiction by showing that this fact implies that

$$\operatorname{rank}\begin{pmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{pmatrix} \leq 1.$$

Set:

$$B = A\big[\{i, j, k, l\}\big] = (b_{qr})_{1 \leq q,r \leq 4} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where B_{11}, B_{12}, B_{21} and B_{22} are 2×2 submatrices.

According to the assumptions, we have $b_{12} = b_{21} < 0$, and $b_{34} \neq b_{43}$. Without loss of generality, we may assume $b_{34} \neq 0$ (otherwise we exchange the indexes 3 and 4).

We also have that $B_{12} = (B_{21})^t$ is an entrywise positive matrice.

Besides $B^{(4)}$ is 1-positive and irreducible. Note that the cycle $(1, 2, 3)$ of $B^{(4)}$ is a negative cycle. Thanks to Theorem 6.1 and Lemma 2.2, one concludes that $B^{(4)}$ must be diagonally similar to a symmetric matrix. This implies $b_{23}b_{31} = b_{13}b_{32}$. Using the same argument for $B^{(3)}$, we obtain $b_{24}b_{41} = b_{14}b_{42}$.

From Lemma 4.4, $B^{(4)} + x\bar{B}^{(4)}$ is also 1-positive for any $x > 0$.

For x small enough, the cycle $(1, 2, 3)$ of $B^{(4)} + x\bar{B}^{(4)}$ is negative. Then, from Theorem 6.1 and Lemma 2.2, this cycle must be symmetric. This implies that for x small enough

$$\begin{aligned}(b_{12} + xb_{14}b_{42})&(b_{23} + xb_{24}b_{43})(b_{31} + xb_{34}b_{41}) \\ &= (b_{13} + xb_{14}b_{43})(b_{32} + xb_{34}b_{42})(b_{21} + xb_{24}b_{41})\end{aligned}$$

and hence for every real x.

We have two polynomials that must have the same roots with same multiplicity, which leads to

$$\left\{\frac{b_{14}b_{42}}{b_{12}}, \frac{b_{24}b_{43}}{b_{23}}, \frac{b_{34}b_{41}}{b_{31}}\right\} = \left\{\frac{b_{14}b_{43}}{b_{13}}, \frac{b_{34}b_{42}}{b_{32}}, \frac{b_{24}b_{41}}{b_{21}}\right\},$$

where the equality is between multisets (the multiplicity is taken into account).

As $\frac{b_{14}b_{42}}{b_{12}} = \frac{b_{24}b_{41}}{b_{21}}$ and $\frac{b_{24}b_{43}}{b_{23}} \neq \frac{b_{34}b_{42}}{b_{32}}$ [indeed $b_{43} \neq b_{34}$ and $(b_{24}, b_{23}) = (b_{42}, b_{32})$], one shows that

$$\frac{b_{24}b_{43}}{b_{23}} = \frac{b_{14}b_{43}}{b_{13}} \quad \text{and} \quad \frac{b_{34}b_{41}}{b_{31}} = \frac{b_{34}b_{42}}{b_{32}}.$$

Hence, one obtains $\frac{b_{23}}{b_{13}} = \frac{b_{24}}{b_{14}}$, which implies that

$$\operatorname{rank}\begin{pmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{pmatrix} = 1. \qquad \square$$

We have mentioned that Theorem 6.1 does not consider 3×3-matrices which are diagonally similar to a symmetric matrices. The proposition below shows that symmetric 1-positive matrices are not necessarily positive semidefinite.

PROPOSITION 7.7. *For $\alpha, \beta, \gamma > 0$, define the matrix*

$$A = \begin{pmatrix} 1 & -\alpha & \beta \\ -\alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}. \tag{7.7}$$

If $\alpha \leq 1$, or $\beta \leq 1$, or $\gamma \leq 1$, then A is 1-positive.

PROOF. If $\alpha \leq 1$, the proposition is a special case of Theorem 7.1 with $d = 3$ and $a_{12} = a_{21} = -\alpha$, $a_{13} = a_{31} = \beta$, $a_{23} = a_{32} = \gamma$ and $A_{22} = (1)$.

Besides, the roles of α, β and γ are symmetric. Indeed, a matrix in the form (7.7) is diagonally similar to

$$\begin{pmatrix} 1 & \alpha & -\beta \\ \alpha & 1 & \gamma \\ -\beta & \gamma & 1 \end{pmatrix}.$$

Then by exchanging together the first and the second rows and the first and the second column, we can see that the parts of α and β can be exchanged. In the same way, the parts of α and γ can be exchanged. The parts of β and γ can also be exchanged by simply intertwining together the second and third rows and the second and third columns.

Hence, if we have $\beta \leq 1$ or $\gamma \leq 1$, we obtain the same conclusion: A is 1-positive. $\square$

The following theorem is an extension of Theorem 7.1 which corresponds to the case $n_1 = \dim(A_{11})$ and $n_i = 1$, for every i in $\{2, 3, \ldots, d\}$.

THEOREM 7.8. *Let B be a $n \times n$ written as the following block matrix*:

$$B = (B_{ij})_{1 \leq i, j \leq d},$$

where for every i, j *in* $[\![d]\!]$, B_{ij} *is a* $n_i \times n_j$ *matrix* (*the nonnegative integers* $n_1, \dots, n_d$ *are such that* $n_1 + \cdots + n_d = n$).

If B *satisfies the three following conditions*:

(i) *for any* i *in* $[\![d]\!]$, B_{ii} *is positive semidefinite*;
(ii) *for any* i, j *in* $[\![d]\!]$, *if* $i \neq j$ *then* B_{ij} *is a nonnegative matrix*;
(iii) *for any* i *in* $[\![d]\!]$, *the* $n_i \times 2(n - n_i)$ *matrix written with* $2(d-1)$ *blocks of columns* $(B_{ij}, (B_{ji})^t)_{\substack{1 \le j \le d \\ j \neq i}}$ *has rank* 1;

then B *is* 1-*positive*.

PROOF. Denote by b_{ij} the (i, j)-entry of B.

Step 1: We first establish Theorem 7.8 under the assumption that none of the off-diagonal blocks of B has a zero entry (i.e., for every i, j, $i \neq j$, B_{ij} has no zero entry).

We prove that there exists a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ with positive diagonal entries, such that $B = DCD$, with $C = (c_{ij})_{1 \le i,j \le n} = (C_{ij})_{1 \le i,j \le d}$, where:

(1) For every i, j in $[\![d]\!]$, C_{ij} is a $n_i \times n_j$-matrix.
(2) For every i in $[\![d]\!]$, C_{ii} is positive semidefinite positive.
(3) For any i, j in $[\![d]\!]$, if $i \neq j$ then all the entries of C_{ij} are equal and positive.

For $1 \le l \le n_1$, set: $d_l = b_{n_1+1l}$. For $n_1 < l \le n$ set: $d_l = b_{1l}$. Define C by

$$C = D^{-1} B D^{-1}.$$

For $1 \le l \le n_1$, $c_{n_1+1,l} = \frac{1}{b_{1,n_1+1}}$, which is positive and does not depend on l.

For $n_1 < l \le n$, $c_{1l} = \frac{1}{b_{n_1+1,1}}$, which is positive and does not depend on l.

As for all $i \in [\![d]\!]$, the $n_i \times 2(n - n_i)$ matrix made of $2(d-1)$ blocks of columns $(B_{ij}, (B_{ji})^t)_{\substack{1 \le j \le d \\ j \neq i}}$ has rank 1, hence $(C_{ij}, (C_{ji})^t)_{\substack{1 \le j \le d \\ j \neq i}}$ has rank 1 as well.

Since $c_{n_1+1,i}$ $(1 \le i \le n_1)$ is positive and does not depend on i and $(C_{1j}, (C_{j1})^t)_{2 \le j \le d}$ has rank 1, then for any $j \in [\![2, n]\!]$, C_{1j} has identical rows with positive entries and C_{j1} has identical columns with positive entries.

As $c_{1l}, n_1 < l \le n$ is positive and does not depend on l and as, for any $i \in [\![2, d]\!]$, $(C_{ij}, (C_{ji})^t)_{\substack{1 \le j \le d \\ j \neq i}}$ has rank 1, and this implies that, for any $j \in [\![n]\!] \setminus \{i\}$, C_{ij} has identical rows with positive entries and C_{ji} has identical columns with positive entries.

Finally, for any i, j in $[\![d]\!]$ such that $i \neq j$, C_{ij} has identical rows and columns, hence all its entries are identical and positive. As B_{ii} (i in $[\![d]\!]$) is positive semidefinite, so is C_{ii}. Therefore, C satisfies (1), (2) and (3).

To prove that the matrix B is 1-positive, it is sufficient to prove that C is 1-positive. Since C satisfies (2) and (3), so does $C[n_1, \dots, n_n]$. Consequently, it is sufficient to prove that $\operatorname{per} C \ge 0$.

For any $i \neq j$ ($1 \leq i, j \leq d$), denote by γ_{ij} the constant value of the entries of the matrix C_{ij}.

From (2.4), we have

$$\begin{aligned}
\operatorname{per} C &= \sum_{\substack{\sum_i k_{ij}=n_j \\ \sum_j k_{ij}=n_i}} \sum_{\substack{|I_{ij}|=|J_{ij}|=k_{ij} \\ \bigcup_i J_{ij}=[\![n_j]\!] \\ \bigcup_j I_{ij}=[\![n_i]\!]}} \left(\prod_{i,j=1}^{d} \operatorname{per} C_{ij}[I_{ij} \times J_{ij}] \right) \\
&= \sum_{\substack{\sum_i k_{ij}=n_j \\ \sum_j k_{ij}=n_i}} \left(\prod_{\substack{i,j=1 \\ i\neq j}}^{d} k_{ij}!(\gamma_{ij})^{k_{ij}} \right) \\
&\quad \times \sum_{\substack{I_{ii}, J_{ii} \subset [\![n_i]\!] \\ |I_{ii}|=|J_{ii}|=k_{ii} \\ 1\leq i\leq d}} \sum_{\substack{\forall i\neq j, |I_{ij}|=|J_{ij}|=k_{ij} \\ \bigcup_{i\neq j} J_{ij}=[\![n_j]\!]\setminus J_{jj} \\ \bigcup_{j\neq i} I_{ij}=[\![n_i]\!]\setminus I_{ii}}} \left(\prod_{i=1}^{d} \operatorname{per} C_{ii}[I_{ii} \times J_{ii}] \right).
\end{aligned}$$

For fixed k_{ij} ($1 \leq i, j \leq d$),

$$\begin{aligned}
\Delta = \#\Big\{ &(I_{ij}, J_{ij})_{\substack{1\leq i,j\leq d \\ i\neq j}} : \forall i \neq j, |I_{ij}| = |J_{ij}| = k_{ij}; \\
&\forall j, \bigcup_{i\neq j} J_{ij} = [\![n_j]\!] \setminus J_{jj}; \forall i, \bigcup_{j\neq i} I_{ij} = [\![n_i]\!] \setminus I_{ii} \Big\}
\end{aligned}$$

does not depend on the choice of I_{ii}, J_{ii} with the conditions $|I_{ii}| = |J_{ii}| = k_{ii}$ ($1 \leq i \leq d$). Hence, we have

$$\begin{aligned}
\operatorname{per} C &= \sum_{\substack{\sum_i k_{ij}=n_j \\ \sum_j k_{ij}=n_i}} \left(\Delta \prod_{\substack{i,j=1 \\ i\neq j}}^{d} k_{ij}!(\gamma_{ij})^{k_{ij}} \right) \sum_{\substack{I_{ii}, J_{ii} \subset [\![n_i]\!] \\ |I_{ii}|=|J_{ii}|=k_{ii} \\ 1\leq i\leq d}} \left(\prod_{i=1}^{d} \operatorname{per} C_{ii}[I_{ii} \times J_{ii}] \right) \\
&= \sum_{\substack{\sum_i k_{ij}=n_j \\ \sum_j k_{ij}=n_i}} \left(\Delta \prod_{\substack{i,j=1 \\ i\neq j}}^{d} k_{ij}!(\gamma_{ij})^{k_{ij}} \right) \left(\prod_{i=1}^{d} \left(\sum_{\substack{I,J \subset [\![n_i]\!] \\ |I|=|J|=k_{ii}}} \operatorname{per} C_{ii}[I \times J] \right) \right),
\end{aligned}$$

which is nonnegative, because for any i in $[\![d]\!]$ $(\operatorname{per} C_{ii}[I \times J])_{I,J\subset[\![n_i]\!]:|I|=|J|=k_{ii}}$ is positive semidefinite (see the argument developed in Step 2 of the proof of Theorem 7.1).

Step 2: To relax the assumption of no zero entry in the off-diagonal blocks, we show now that B is the limit as ϵ tends to 0 of matrices B_ϵ such that for every $\epsilon > 0$: $B_\epsilon = ((B_\epsilon)_{ij})_{1\leq i,j\leq d}$, where the matrix $(B_\epsilon)_{ij}$ has the same size as B_{ij} and:

- For any i in $[\![d]\!]$, $(B_\epsilon)_{ii}$ is positive semidefinite.
- For any i, j in $[\![d]\!]$, if $i \neq j$ then $(B_\epsilon)_{ij}$ has only positive entries.

- For any i in $[\![d]\!]$, the $n_i \times 2(n-n_i)$ matrix written with $2(d-1)$ blocks of columns $((B_\epsilon)_{ij}, ((B_\epsilon)_{ji})^t)_{\substack{1\le j\le d \\ j\neq i}}$ has rank 1.

Apart from being positive semidefinite, the matrices B_{ii}, $i \in [\![d]\!]$ have no part in the proof. Hence, without loss of generality, we may assume that for i in $[\![d]\!]$, $B_{ii} = 0$.

Assume that B has at least one zero entry in an off-diagonal block.

From B, we now build a matrix $B_\epsilon^{(1)}$ that has a number of zero entries strictly smaller than the number of zero entries of B, satisfies (i) and (ii) and such that $(B_\epsilon^{(1)})_{\epsilon>0}$ converges to B as ϵ tends to 0.

There exist $i_0, j_0 \in [\![d]\!]$ with $i_0 \neq j_0$ such that $B_{i_0j_0}$ has a zero entry, denote by (k_0, l_0) the indices in B of this zero entry: $b_{k_0l_0} = 0$.

We are always in one of the three following cases:

Case 1—$b_{k_0l} = b_{lk_0} = 0$, $\forall l \in [\![n]\!]$,
Case 2—$b_{kl_0} = b_{l_0k} = 0$, $\forall k \in [\![n]\!]$,
Case 3—$B_{i_0j_0} = 0$.

Indeed, suppose that we are not in Case 1 nor in Case 2, then as $\operatorname{rank}(B_{i_0j}, (B_{ji_0})^t)_{\substack{j\in[\![d]\!] \\ j\neq i_0}} = \operatorname{rank}(B_{j_0j}, (B_{jj_0})^t)_{\substack{j\in[\![d]\!] \\ j\neq j_0}} = 1$, we have $b_{kl_0} = 0$ for all $k \in [\![1 + \sum_{q=1}^{i_0-1} n_q, \sum_{q=1}^{i_0} n_q]\!]$ and $b_{k_0l} = 0$ for all $l \in [\![1 + \sum_{q=1}^{j_0-1} n_q, \sum_{q=1}^{j_0} n_q]\!]$. We also have that there exists k in $[\![d]\!]$ such that $b_{kl_0} \neq 0$ or $b_{l_0k} \neq 0$. As $\operatorname{rank}(B_{j_0j}, (B_{jj_0})^t)_j = 1$, this implies $B_{i_0j_0} = 0$, which is Case 3.

If we are in Case 1, there exists k in $[\![n_{i_0}]\!]$ such that the kth row of the matrix $(B_{i_0j}, (B_{ji_0})^t)_{1\le j\le d}$ is nonzero. $B_\epsilon^{(1)}$ is obtained from B by replacing its k_0th row by the $(\sum_{q=1}^{i_0-1} n_q + k)$th row of B multiplied by ϵ, and its k_0th column by the $(\sum_{q=1}^{i_0-1} n_q + k)$th column of B multiplied by ϵ. With this definition, it is easy to verify that $B_\epsilon^{(1)}$ has the properties (i) and (ii), and the number of its zero entries is strictly smaller than the number of zero entries of B.

If we are in Case 2, we do a similar construction, with l_0 instead of k_0.

If we are in Case 3, $B_\epsilon^{(1)}$ is obtained from B by replacing the submatrix $B_{i_0j_0}(=0)$ by $\epsilon K_{i_0} \times (K_{j_0})^t$, where K_{i_0} is a nonzero column of the matrix written with blocks of columns $(B_{i_0j}, (B_{ji_0})^t)_{1\le j\le d}$ and K_{j_0} is a nonzero column of the matrix written with blocks of columns $(B_{j_0j}, (B_{jj_0})^t)_{1\le j\le d}$. Note that $B_\epsilon^{(1)}$ satisfies Conditions (i) and (ii), and that the number of its zero entries is strictly smaller than the number of zero entries of B.

Moreover, in each Case, $B_\epsilon^{(1)}$ tends to B when ϵ tends to 0.

The submatrices of $B_\epsilon^{(1)}$, $(B_\epsilon^{(1)})_{ij}$, $1 \le i, j \le d$, are defined such that $B_\epsilon^{(1)} = ((B_\epsilon^{(1)})_{ij})_{1\le i,j\le d}$ and for every i, j, $(B_\epsilon^{(1)})_{ij}$ has the same size as the submatrix B_{ij}.

Define by induction $B_\epsilon^{(p)}$ from $B_\epsilon^{(p-1)}$, exactly as $B_\epsilon^{(1)}$ has been defined from B. This construction requires that $B_\epsilon^{(p-1)}$ has at least one zero entry in an off-diagonal block. We stop the construction at the first index $p_o \geq 1$ such that none of the off-diagonal block of $B_\epsilon^{(p_o)}$ has a zero entry. Set then $B_\epsilon = B_\epsilon^{(p_o)}$.

The matrix B_ϵ satisfies the three announced points, and as such is 1-positive thanks to Step 1. Since B_ϵ tends to B as ϵ tends to 0, B is 1-positive. □

COROLLARY 7.9. *Let B be a matrix satisfying all the assumptions of Theorem 7.8. Assume moreover that for any $i \neq j$, B_{ij} has no zero entry. Then there exists $\gamma > 0$ such that $B + \gamma I$ is 1-permanental.*

PROOF. To use Proposition 4.6, thanks to (4.1), it is sufficient to prove that for $\alpha > 0$ small enough, $B(\alpha) = B(I + \alpha B)^{-1}$ verifies also Conditions (i), (ii) and (iii) of Theorem 7.8.

As in the proof of Corollary 7.4, for $\alpha > 0$ small enough, $B(\alpha)_{11}$ is symmetric and the matrix written by block of columns $(B(\alpha)_{1j}, (B(\alpha)_{j1})^t)_{2\leq j\leq d}$ has rank 1. Similarly, for any i in $[\![d]\!]$ and for $\alpha > 0$ small enough, $B(\alpha)_{ii}$ is symmetric and the matrix written by block of columns $(B(\alpha)_{ij}, (B(\alpha)_{ij})^t)_{\substack{1\leq j\leq d\\ j\neq i}}$ has rank 1. Then, as $B(\alpha)$ tends to B when α tends to 0, $B(\alpha)$ fulfills Conditions (i), (ii) and (iii) for α small enough.

Corollary 7.9 is hence a consequence of Theorem 7.8 and Proposition 4.6. □

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UNIVERSITÉ PIERRE ET MARIE CURIE
LPMA – CASE 188
4, PLACE JUSSIEU
75252 PARIS CEDEX 05
FRANCE
E-MAIL: nathalie.eisenbaum@upmc.fr

UNIVERSITÉ PIERRE ET MARIE CURIE
LPMA – CASE 188
4, PLACE JUSSIEU
75252 PARIS CEDEX 05
FRANCE
AND
TELECOM PARISTECH
LTCI
46, RUE BARRAULT
75634 PARIS CEDEX 13
FRANCE
E-MAIL: franck.maunoury@upmc.fr