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RELATIVE COMPLEXITY OF RANDOM WALKS IN RANDOM SCENERY IN THE ABSENCE OF A WEAK INVARIANCE PRINCIPLE FOR THE LOCAL TIMES

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We answer a question of Aaronson about the relative complexity of Random Walks in Random Sceneries driven by either aperiodic two-dimensional random walks, two-dimensional Simple Random walk, or by aperiodic random walks in the domain of attraction of the Cauchy distribution. A key step is proving that the range of the random walk satisfies the Fölner property almost surely.

1. Introduction. The notion of entropy was introduced to ergodic theory by Kolmogorov as an isomorphism invariant. That is, if two measure preserving systems are measure-theoretically isomorphic then their entropy is the same. It was later shown in the seminal paper of Ornstein [26] that two Bernoulli automorphisms, transformations isomorphic to a shift of an i.i.d. sequence, are isomorphic if and only if their entropies coincide. Therefore, entropy is a complete invariant for Bernoulli automorphisms. In an attempt to understand whether two zero entropy systems are isomorphic, Ferenczi [16], Katok and Thouvenot [18] and others introduced notions of measure theoretic complexity, which roughly measure the rate of growth of information.

Aaronson [1] recently introduced relative complexity which is a relativised notion of complexity. He calculated the relative complexity of Random Walk in Random Scenery (RWRS), where the jump random variable is integer-valued, centred, aperiodic and in the domain of attraction of an α-stable distribution with $1 < \alpha \leq 2$. The main tool used there is Borodin's weak invariance principle for the local times [6, 7]. Random walks in random scenery are examples of non-Bernoulli K-automorphisms [17], and the relative complexity was conjectured by Thouvenot to be an isomorphism invariant for them. Indeed Austin [2] recently introduced the *bi-covering number* as an isomorphism invariant, and used the weak convergence of local times to show that for the class of random walks in random scenery with jump distribution of finite variance, the bi-covering number grows at the same rate as Aaronson's relative complexity.

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For the purpose of this Introduction, we now describe the classical random walk in random scenery. Let $X_1, X_2, \dots$ be i.i.d. $\mathbb{Z}^d$-valued random variables, the *jump process*, and $S_n := \sum_{k=1}^n X_k$ the corresponding random walk. The scenery is a field of i.i.d. random variables $\{Z_j\}_{j\in\mathbb{Z}^d}$, independent of $\{X_i\}$. The joint process (X_n, Z_{S_n}) is then known as a random walk in random scenery. The relative complexity of Aaronson in that case is heuristically the rate of growth of the information arising from the scenery for most realizations of $X_1, X_2, \dots$.

In the case where S_n is the simple random walk on the integers, it follows from the local central limit theorem that the range of the random walk at time n, $R(n) := \{S_j : 1 \le j \le n\}$, is almost surely of order constant times $\sqrt{n}$. The range of the random walk is related to this problem since $\{Z_{S_j} : j \in [1, n]\} = \{Z_k : k \in R(n)\}$. It then appears that the rate of growth of information arising from the scenery should be of the order of $\exp(H(Z_0)\cdot \#R(n))$, where $H(Z_0)$ is the Shannon entropy of Z_0. Thus, for this example one would expect that for most w, the relative complexity is of the order of $\exp(c_w\sqrt{n}H(Z_0))$. A precise formulation of this statement, in terms of nontrivial distributional limits, was verified in Aaronson [1].

In this paper, we treat random walks in random sceneries driven by the simple random walk in $\mathbb{Z}^2$, by aperiodic, recurrent, $\mathbb{Z}^2$-valued random walks with finite variance, or by an aperiodic, recurrent, integer-valued random walk in the domain of attraction of the Cauchy distribution. Since the limiting distributions do not have local times, Aaronson's and Austin's methods do not apply. For these types of RWRS, Kesten and Spitzer [20] conjectured that when $\mathrm{var}(Z_0) < \infty$, there exists a sequence $a_n \to \infty$ such that $\frac{1}{a_n}\sum_{k=1}^{nt} Z_{S_n}$ converges weakly to a Brownian motion. This was shown to be true by Bolthausen [5] when S_n is the planar simple random walk and by the first author and Utev [12] for the case of the Cauchy distribution. Bolthausen's argument was generalised by Černy [9] and the ideas there were a major inspiration for us. Since there is no weak invariance principle for the local time, Černy's argument relies on the asymptotic behaviour of the self-intersection local times (see Section 4) in order to prove that "for most of the points in $R(n)$, the local time up to time n, is greater than a constant times $\log(n)$"; see Theorem 4.1 for a precise formulation. We refine this method to prove a result of independent interest, namely that the range of the random walk is almost surely a Fölner sequence (Theorem 4.2). With these two theorems at hand, we can proceed by a simplified argument to deduce the main result, Theorem 3.1, which answers Aaronson's question about the relative complexity of this type of RWRS. We think that this simpler and softer method can be used to calculate the relative complexity of other RWRS's such as those in [3, 27].

The paper is organised as follows. In Section 2, we provide the relevant definitions and results from ergodic theory. Section 3 contains the precise formulation of RWRS and the main results. In Section 4, we state and prove the results we need for the random walk and its range. Section 5 contains the proof of the main theorem. For the sake of completeness, we include an Appendix with proofs of some standard facts about the random walks we consider, and a formulation of Karamata's Tauberian theorem.

2. Preliminaries.

2.1. *Relative complexity over a factor.* Let $(X, \mathcal{B}, m)$ be a standard probability space and $T : X \to X$ a m-preserving transformation. Denote by $\mathfrak{B}(X)$ the collection of all measurable countable partitions of X. In order to avoid confusion with notions from probability, we will denote the partitions by Greek letters $\beta \in \mathfrak{B}(X)$ and the atoms of β by $\beta^1, \beta^2, \dots, \beta^{\#\beta}, \#\beta \in \mathbb{N} \cup \{\infty\}$. A partition β is a *generating partition* if the smallest σ-algebra containing $\{T^{-n}\beta : n \in \mathbb{Z}\}$ is $\mathcal{B}$. For $\beta \in \mathfrak{B}(X)$ and $n \in \mathbb{N}$, let

$$\beta_0^n := \bigvee_{j=0}^{n} T^{-j}\beta = \left\{\bigcap_{j=0}^{n} T^{-j}b_j : b_1, \dots, b_n \in \beta\right\}.$$

The *β-name* of a point $x \in X$ is the sequence $\beta(x) \in (\#\beta)^{\mathbb{Z}}$ defined by

$$\beta_n(x) = i \quad \text{if and only if} \quad T^n x \in \beta^i.$$

The *(T, β, n)-Hamming pseudo-metric* on X is defined by

$$\bar{d}_n^{(\beta)}(x, y) = \frac{1}{n}\#\bigl\{k \in \{0, \dots, n-1\} : \beta_k(x) \neq \beta_k(y)\bigr\}.$$

That is two points $x, y \in X$ are $\bar{d}_n^{(\beta)}$-close if for most of the $k's$ in $\{0, \dots, n-1\}$, $T^k x$ and $T^k y$ lie in the same partition element of β. An ϵ-ball in the Hamming pseudo-metric will be denoted by

$$B(n, \beta, x, \epsilon) := \bigl\{y \in X : \bar{d}_n^{(\beta)}(x, y) < \epsilon\bigr\}.$$

This pseudo-metric was used in [16, 18] to define complexity sequences and slow-entropy-type invariants. It was shown, for example, by Katok and Thouvenot [18] that if the growth rate of the complexity sequence is of order e^{hn} with $h > 0$, then h equals the entropy of X and by Ferenczi [16] that T is isomorphic to a translation of a compact group if and only if the complexity is of lesser order than any sequence which grows to infinity. In this paper, we will be interested with the relativised versions of these invariants which were introduced in Aaronson [1].

A T-invariant sub-σ-algebra $\mathcal{C} \subset \mathcal{B}$ is called a *factor*. An equivalent definition in ergodic theory is a probability preserving transformation $(Y, \tilde{\mathcal{C}}, \nu, S)$ with a (measurable) factor map $\pi : X \to Y$ such that $\pi T = S\pi$ and $\nu = m \circ \pi^{-1}$, in this case $\mathcal{C} = \pi^{-1}\tilde{\mathcal{C}}$.

Given a factor $\mathcal{C} \subset \mathcal{B}$, $n \in \mathbb{N}$, $\beta \in \mathfrak{B}(X)$ and $\epsilon > 0$, we define a $\mathcal{C}$-measurable random variable $\mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon) : X \to \mathbb{N}$ by

$$\mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon)(x) := \min\left\{\#F : F \subset X, m\left(\bigcup_{z \in F} B(n, \beta, z, \epsilon)\middle|\mathcal{C}\right)(x) > 1 - \epsilon\right\},$$

where $m(\cdot|\mathcal{C})$ denotes the conditional measure of m with respect to $\mathcal{C}$. The sequence of random variables $\{\mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon)\}_{n=1}^{\infty}$ is called the *relative complexity* of (T, β) with respect to β given $\mathcal{C}$.

Given a sequence of random variables $Y_n, n \in \mathbb{N}$ taking values in $[0, \infty]$ we write $Y_n \xrightarrow[n\to\infty]{\mathfrak{D}} Y$ to denote "Y_n converges to Y in distribution" and $Y_n \xrightarrow[n\to\infty]{m} Y$ to denote convergence in probability.

The *upper entropy dimension of* T *given* $\mathcal{C}$ is defined by

$$\overline{\mathrm{EDim}}(T, \mathcal{C}) := \inf\left\{t > 0 : \frac{\log \mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon)}{n^t} \xrightarrow[n\to\infty, \epsilon\to 0]{m} 0, \forall \beta \in \mathfrak{B}(X)\right\}$$

and the *lower entropy dimension of* T *given* $\mathcal{C}$ is

$$\underline{\mathrm{Edim}}(T, \mathcal{C}) = \sup\left\{t > 0 : \exists \beta \in \mathfrak{B}(X), \frac{\log \mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon)}{n^t} \xrightarrow[n\to\infty, \epsilon\to 0]{m} \infty\right\}.$$

In case $\underline{\mathrm{Edim}}(T, \mathcal{C}) = \overline{\mathrm{Edim}}(T, \mathcal{C}) = a$, we write $\mathrm{Edim}(T, \mathcal{C}) = a$ and call this quantity the *entropy dimension of* T *given* $\mathcal{C}$.

The next theorem is a special case of [1], Theorem 2, when $\{n_k\}_{k=1}^{\infty} = \mathbb{N}$.

THEOREM 2.1 (Aaronson's generator theorem). *Let* $(X, \mathcal{B}, m, T)$ *be a measure preserving transformation and* $d_n > 0$ *a sequence.*

(a) *If there is a countable* T*-generator* $\beta \in \mathfrak{B}(X)$ *and a random variable* Y *on* $[0, \infty]$ *satisfying*

$$\frac{\log \mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon)}{d_n} \xrightarrow[n\to\infty, \epsilon\to 0]{\mathfrak{D}} Y.$$

Then for all T*-generating partitions* $\alpha \in \mathfrak{B}(X)$,

$$\frac{\log \mathcal{K}_{\mathcal{C}}(\alpha, n, \epsilon)}{d_n} \xrightarrow[n\to\infty, \epsilon\to 0]{\mathfrak{D}} Y.$$

(b) *If for some* $\beta \in \mathfrak{B}(X)$, *a generating partition for* T,

$$\frac{\log \mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon)}{n^t} \xrightarrow[n\to\infty, \epsilon\to 0]{m} 0,$$

then $\overline{\mathrm{EDim}}(T, \mathcal{C}) \le t$.

(c) *If for some partition* $\beta \in \mathfrak{B}(X)$,

$$\frac{\log \mathcal{K}_{\mathcal{C}}(\beta, n, \epsilon)}{n^t} \xrightarrow[n\to\infty, \epsilon\to 0]{m} \infty$$

then $\underline{\mathrm{Edim}}(T, \mathcal{C}) \ge t$.

2.2. *Basic ergodic theory for* $\mathbb{Z}^d$ *actions.* Let $(X, \mathcal{B}, m)$ be a standard probability space and G be a countable Abelian group. A measure preserving action of G on $(X, \mathcal{B}, m)$ is a map $\mathsf{S} : G \to \mathrm{Aut}(X, \mathcal{B}, m)$ such that for every $g_1, g_2 \in G$, $\mathsf{S}_{g_1 g_2} = \mathsf{S}_{g_1}\mathsf{S}_{g_2}$ and for all $g \in G$, $(\mathsf{S}_g)_* m = m$. The action is *ergodic* if there are no nontrivial S-invariant sets.

Given an ergodic G action on $(X, \mathcal{B}, m, \mathsf{S})$ and an increasing sequence F_n of subsets of G, one can define a sequence of averaging operators $A_n : L_2(X, \mathcal{B}, m) \circlearrowleft$ by

$$A_n(f) := \frac{1}{\# F_n} \sum_{g \in F_n} f \circ \mathsf{S}_g$$

and ask whether for all $f \in L_2(X, \mathcal{B}, m)$ one has $A_n(f) \to \int_X f \, \mathrm{d}m$ in L_2. The sequences of sets $\{F_n\}_{n=1}^{\infty}$ for which this is necessarily true are called *Fölner sequences* and they are characterised by the property that for every $g \in G$,

$$\frac{\#[F_n \triangle \{F_n + g\}]}{\# F_n} \xrightarrow[n \to \infty]{} 0.$$

In this work, we will be concerned with either actions of $G = \mathbb{Z}$ which is generated by one measure preserving transformation or $G = \mathbb{Z}^2$ which corresponds to two commuting measure preserving transformations. For a finite partition β of X, one defines the *entropy* of S with respect to β by

$$h(\mathsf{S}, \beta) := \lim_{n \to \infty} \frac{1}{n^d} H\Biggl(\bigvee_{j \in [0,n]^d \cap \mathbb{Z}^d} \mathsf{S}_j^{-1} \beta \Biggr),$$

where $H(\beta) = \sum_{j=1}^{\#\beta} m(\beta^i) \log m(\beta^i)$ is the *Shannon entropy* of the partition. The entropy of S is then defined by

$$h(\mathsf{S}) = \sup_{\beta \in \mathfrak{B}(X) : \beta \text{finite}} h(\mathsf{S}, \beta).$$

As in the case of a $\mathbb{Z}$-action, one says that β is a generating partition if the smallest sigma algebra containing $\bigvee_{j \in \mathbb{Z}^d} \mathsf{S}_j^{-1} \beta$ is $\mathcal{B}$. In an analogous way to the case of $\mathbb{Z}$-actions, it follows that if β is a generating partition for S then $h(\mathsf{S}) = h(\mathsf{S}, \beta)$ and if $h(\mathsf{S}) < \infty$ then there exist finite generating partitions [10, 19, 22].

3. Random walks in random sceneries and statement of main theorem. In what follows, we will be interested in a random walk in random scenery where the jump random variable $\xi \in \mathbb{Z}^2$ is in the domain of attraction of 2-dimensional Brownian Motion, or $\xi \in \mathbb{Z}$ and is in the domain of attraction of the Cauchy law. The reason that these two models are of most interest to us is that the limiting distribution does not have a local time process.

To be more precise, let $\xi, \xi_1, \xi_2, \dots$ be i.i.d. $\mathbb{Z}^d$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with characteristic function $\phi_\xi(t) := \mathbb{E}(e^{it \cdot \xi})$ for $t \in [-\pi, \pi]^d$, and that either:

A1 (1-*stable*) $\xi \in \mathbb{Z}$ and $\phi_\xi(t) = 1 - \gamma|t| + o(|t|)$ for $t \in [-\pi, \pi]$, for some $\gamma > 0$; or

A2 ξ is in $\mathbb{Z}^2$ and $\mathbb{E}|\xi|^2 < \infty$ with nonsingular covariance matrix Σ; equivalently $\phi_\xi(t) = 1 - \langle t, \Sigma t \rangle + o(|t|^2)$ for $t \in [-\pi, \pi]^2$.

In the above cases, the random walk is given by $S_n(\xi) := \xi_1 + \xi_2 + \cdots + \xi_n$. We will also assume that the random walk is *strongly aperiodic* in the sense that there is no proper subgroup L of $\mathbb{Z}^d$ such that $\mathbb{P}(\xi - x \in L) = 1$ for some $x \in \mathbb{Z}^d$.

We are also interested in the two-dimensional *simple random walk*, which has period 2 and is thus not covered by **A2** above.

A2$'$ $\xi \in \mathbb{Z}^2$ and $\mathbb{P}[\xi = e] = 1/4$ for $|e| = 1$. Then $\sqrt{\det(\Sigma)} = 1/2$.

Denote by μ_ξ the distribution of ξ. The base of the RWRS is then defined as $\Omega = (\mathbb{Z}^d)^{\mathbb{N}}$ the space of all $\mathbb{Z}^d$-valued sequences, $\mathbb{P} = \prod_{k=1}^{\infty} \mu_\xi$, the product measure, and $\sigma : \Omega \to \Omega$ the left shift on Ω defined by

$$(\sigma w)_n = w_{n+1}.$$

When $d = 2$, the random scenery is an ergodic probability preserving $\mathbb{Z}^2$- action $(Y, \mathcal{C}, \nu, \mathsf{S})$ and when $d = 1$ it is just an ergodic probability preserving transformation $\mathsf{S} : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$.

The skew product transformation on $Z = \Omega \times Y$, $\mathcal{B}_Z = \mathcal{B}_\Omega \otimes \mathcal{B}_Y$, $m = \mathbb{P} \times \nu$, defined by

$$T(w, y) = \big(\sigma w, \mathsf{S}_{w_1}(y)\big),$$

is the *random walk in random scenery* with scenery $(Y, \mathcal{C}, \nu, \mathsf{S})$ and *jump random variable* ξ.

REMARK 3.1. The results of Kalikow [17] were extended to more general RWRSs, including the planar case, by den Hollander and Steif [13]. In particular, they show that these RWRSs are not Bernoulli.

THEOREM 3.1. *Let* $(Z, \mathcal{B}_Z, m, T)$ *be* RWRS *with random scenery* $(Y, \mathcal{C}, \nu, \mathsf{S})$ *and jump random variable* ξ.

(a) *If* $d = 1$ *and* ξ *satisfies* **A1** *then for any generating partition* $\boldsymbol{\beta}$ *for* T,

$$\frac{\log(n)}{\pi \gamma n} \log \mathcal{K}_{\mathcal{B}_\Omega}(\beta, n, \epsilon) \xrightarrow{m} h(\mathsf{S}).$$

(b) *If* $d = 2$ *and* ξ *satisfies* **A2** *or* **A2**$'$ *then for any generating partition* $\boldsymbol{\beta}$ *for* T,

$$\frac{\log n}{2\pi \sqrt{\det(\Sigma)} n} \log \mathcal{K}_{\mathcal{B}_\Omega}(\beta, n, \epsilon) \xrightarrow{m} h(\mathsf{S}).$$

In particular, in both cases

$$\operatorname{Edim}(T, \mathcal{B}_\Omega) = 1.$$

REMARK 3.1. This theorem states that the rate of growth of the complexity is of order $\#R(n)$, where $R(n)$ is the range of the random walk up to time n. This conclusion is similar to the conclusion of Aaronson for the case where the random walk is in the domain of attraction of an α-stable random variable with $1 < \alpha \leq 2$. Our method of proof can apply to these cases as well. In addition, since we are not using the full theory of weak convergence of local times, one can hope that this method will apply also to a wider class of dependent jump distributions.

Two probability preserving transformations $(X_i, \mathcal{B}_i, m_i, T_i)$, $1 = 1, 2$, are relatively isomorphic over the factors $\mathcal{C}_i \subset \mathcal{B}_i$ if there exists a measurable isomorphism $\pi : (X_1, \mathcal{B}_1, m_1, T_1) \to (X_2, \mathcal{B}_2, m_2, T_2)$ such that $\pi^{-1}\mathcal{C}_2 = \mathcal{C}_1$. The following corollary follows from Theorem 3.1 together with [1], Corollary 4.

COROLLARY 3.1. *Suppose that* $(Z_i, \mathcal{B}_{Z_i}, m_i, T_i)$, $i = 1, 2$, *are two Random walks in random sceneries with strongly aperiodic* $\mathbb{Z}^2$*-valued jump random variable* ξ *which satisfy* **A2** *or* **A2**$'$ *and their sceneries* $\mathsf{S}^{(i)}$ *have finite entropies.*

If these two systems are isomorphic over their bases $\mathcal{B}_{\Omega_i}$, *then*

$$\sqrt{\det(\Sigma_1)}\,h\big(\mathsf{S}^{(1)}\big) = \sqrt{\det(\Sigma_2)}\,h\big(\mathsf{S}^{(2)}\big).$$

4. The range of the random walk. Let $R(n) = \{S(1), \dots, S(n)\}$, be the range of the random walk and for $x \in \mathbb{Z}^d$ define the local time,

$$l(n, x) = \sum_{j=1}^{n} \mathbf{1}\{S(j) = x\}.$$

Denote by $\mathcal{F}$ the σ-algebra generated by $\{\xi_n\}_{n=1}^{\infty}$.

The following theorem extends [9], Theorem 2, to the case **A1**.

THEOREM 4.1. *Let* Y_n *be a point chosen uniformly at random from* $R(n)$, *that is*,

$$\mathbb{P}[Y_n = x|\mathcal{F}] = \frac{\mathbb{1}\{x \in R(n)\}}{\#R(n)}. \tag{4.1}$$

(i) *If* **A1** *holds*, *then*

$$\mathbb{P}\left[\pi\gamma \frac{l(n, Y_n)}{\log n} \geq u \middle| \mathcal{F}\right] \to \mathrm{e}^{-u} \qquad \text{a.s. as } n \to \infty. \tag{4.2}$$

(ii) *If* **A2** ([9], *Theorem* 2) *or* **A2**$'$ *holds*, *then*

$$\mathbb{P}\left[2\pi\sqrt{\det(\Sigma)} \frac{l(n, Y_n)}{\log n} \geq u \middle| \mathcal{F}\right] \to \mathrm{e}^{-u} \qquad \text{a.s. as } n \to \infty. \tag{4.3}$$

The following is the main result of this section.

THEOREM 4.2. *Suppose that* **A1**, **A2** *or* **A2**$'$ *holds*, *then* $R(n)$ *is almost surely a Fölner sequence*, *that is*, *almost surely for all* $j \in \mathbb{Z}^d$

$$\lim_{n\to\infty} \frac{\#[R(n)\triangle(R(n) + j)]}{\#R(n)} = 0. \tag{4.4}$$

4.1. *Auxiliary results.* Before we embark on the proofs of Theorems 4.1 and 4.2, we require several standard results. The next result is a direct consequence of strong aperiodicity and Assumptions **A1** and **A2**. Its proof is a standard application of Fourier inversion, and is included in the Appendix for the sake of completeness.

LEMMA 4.1. *Suppose that* **A1** *or* **A2** *holds. Then with* $\gamma_1 := \pi\gamma$ *and* $\gamma_2 := 2\pi\sqrt{|\Sigma|}$

$$\sup_w \mathbb{P}\big[S(m) = w\big] = O\left(\frac{1}{m}\right), \tag{4.5}$$

$$\mathbb{P}\big[S(m) = w\big] - \mathbb{P}\big[S(m) = 0\big] = O\left(\frac{|w|}{m^2}\right), \tag{4.6}$$

$$\mathbb{P}\big[S(m) = w\big] \sim \frac{1}{\gamma_d m}. \tag{4.7}$$

LEMMA 4.2. *Suppose that* **A1** *holds. Then as* $\lambda \uparrow 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\lambda\phi(t)\,\mathrm{d}t}{1 - \lambda\phi(t)} \sim \frac{1}{\pi\gamma}\log\left(\frac{1}{1-\lambda}\right). \tag{4.8}$$

Since simple random walk is not aperiodic, to prove Theorem 4.1 for the case **A2**$'$ we recall the following (see [23], Theorem 1.2.1).

LEMMA 4.3. *Under* **A2**$'$

$$\sup_x \mathbb{P}\big[S(m) = x\big] = O\left(\frac{1}{m}\right), \tag{4.9}$$

$$\sum_{k=0}^{n} \mathbb{P}[S_m = 0] \sim \frac{1}{\pi}\log n. \tag{4.10}$$

4.2. *Proof of Theorem* 4.1. The result under **A2** has been proven in [9]. We will therefore focus on the remaining cases, explaining how to adapt the arguments in [9]. We write C for a generic positive constant.

For $\alpha \geq 0$, define the *α-fold self-intersection local time* as follows:

$$L_n(\alpha) := \sum_{x\in\mathbb{Z}^d} l(n,x)^\alpha, \qquad \alpha > 0$$

$$L_n(0) := \lim_{\alpha\downarrow 0} L_n(\alpha) = \sum_{x\in\mathbb{Z}^d} \mathbf{I}\big\{l(n,x) > 0\big\} = \#R(n).$$

The first step of the method in [9] is to calculate the asymptotic behaviour of $L_n(\alpha)$ for $\alpha \in \mathbb{N}$. The next step is to use the aforementioned asymptotics to show that all integer moments of $2\pi\sqrt{|\Sigma|}l(n, Y_n)/\log(n)$, where Y_n is uniformly distributed in $R(n)$, converge to those of the exponential distribution with unit mean for almost every random walk path. Since the exponential distribution is uniquely determined by its integer moments the result follows.

The following Proposition 4.1 extends the estimates of [9] to the cases **A1** and **A2**$'$. See also the appendix [8] for the estimates in the case of assumption **A2** under the aperiodicity condition.

PROPOSITION 4.1. *For $d = 1, 2$, and any integer $k \geq 1$ if* **A1** *or* **A2**$'$ *holds then as $n \to \infty$*

$$\mathbb{E}L_n(k) \sim \frac{\Gamma(k+1)}{(\pi\gamma_d)^{k-1}} n(\log n)^{k-1}, \tag{4.11}$$

$$\operatorname{var}(L_n(k)) = O(n^2(\log n)^{2k-4}), \tag{4.12}$$

$$\lim_{n\to\infty} \frac{n(\log n)^{k-1}}{(\pi\gamma_d)^{k-1}} L_n(k) = \Gamma(k+1) \qquad \textit{almost surely}. \tag{4.13}$$

PROOF. Once (4.11) and (4.12) have been established, (4.13) follows for geometric subsequences by Chebyshev's inequality and the complete result by the same argument as in Černy [9], which uses in addition the monotonicity of $L_n(\alpha)$ with respect to n in order to interpolate.

Case **A1**: The estimate (4.12) is contained in Theorem 3 of Deligiannidis and Utev [11]. It remains to prove (4.11).

Similar to [9], we write

$$\begin{aligned} \mathbb{E}L_n(k) &= \sum_{j_1,\dots,j_k=0}^{n} \mathbb{P}[S_{k_1} = \cdots = S_{j_k}] \\ &= \sum_{b=1}^{k} \rho(b,k) \sum_{0\leq j_1<\cdots<j_b\leq n} \mathbb{P}[S_{j_1} = \cdots = S_{j_b}], \end{aligned}$$

where $\rho(k,k) = k!$, while the remaining factors will not be important.

Letting

$$M_n(b) := \Big\{(m_0,\dots,m_b) \in \mathbb{N}^{b+1} : m_1,\dots,m_{b-1} \geq 1, \sum m_i = n\Big\}, \tag{4.14}$$

we have by the Markov property

$$\begin{aligned} a_b(n) &:= \sum_{0\leq j_1<\cdots<j_b\leq n} \mathbb{P}[S_{j_1} = \cdots = S_{j_b}] \\ &= \sum_{m\in M_n(b)} \prod_{i=1}^{b-1} \mathbb{P}[S_{m_i} = 0]. \end{aligned} \tag{4.15}$$

Then for $\lambda \in [0, 1)$, by standard Fourier inversion

$$\begin{aligned}
\sum_{n=0}^{\infty} a_b(n)\lambda^n &= \sum_{n=0} \lambda^n \sum_{m\in M_n(b)} \prod_{i=1}^{b-1} \mathbb{P}[S_{m_i} = 0] \\
&= \sum_{m_0\geq 0} \sum_{m_1,\dots,m_{b-1}\geq 1}^{\infty} \sum_{n=0}^{\infty} \lambda^{m_0+\cdots+m_{b-1}+n} \prod_{i=1}^{b-1} \mathbb{P}[S_{m_i} = 0] \\
&= \sum_{m_0=0}^{\infty} \lambda^{m_0} \sum_{n=0}^{\infty} \lambda^n \prod_{i=1}^{b-1} \sum_{m_i=1}^{\infty} \lambda^{m_i} \mathbb{P}[S_{m_i} = 0] \\
&= \frac{1}{(1-\lambda)^2} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda\phi(t)}\right]^{b-1} \sim \frac{(\pi\gamma)^{1-b}}{(1-\lambda)^2} \log\left(\frac{1}{1-\lambda}\right)^{b-1},
\end{aligned}$$

as $\lambda \uparrow 1$, by Lemma 4.2. Then under **A1** (4.11) follows by Karamata's Tauberian theorem, given in Appendix B, since the sequence $a_b(n)$ is monotone increasing.

Case **A2′**: The estimate (4.11) follows from (4.15) and (4.10).

The proof of (4.12) can be adapted from [11]. The variance is given by

$$\begin{aligned}
\operatorname{var}(L_n(k)) = C(k) \sum_{i_1\leq\cdots\leq i_k} \sum_{l_1\leq\cdots\leq l_k} &\{\mathbb{P}[S(i_1) = \cdots = S(i_k); S(l_1) = \cdots = S(l_k)] \\
&- \mathbb{P}[S(i_1) = \cdots = S(i_k)]\mathbb{P}[S(l_1) = \cdots = S(l_k)]\}.
\end{aligned}$$

The terms where $l_1, \dots, l_k$ are not completely contained in any of the intervals $[i_j, i_{j+1}]$ can be bounded above by the positive term in the sum using (4.9). A similar, albeit more involved, calculation is performed in the proof of Proposition 4.2.

Suppose then that $l_1, \dots, l_k \in [i_j, i_{j+1}]$ for some j, and by symmetry we can take $j = 1$. Define $M_n(2k)$ as in (4.14) and change variables to

$$\begin{aligned}
&i_1 = m_0, \qquad l_1 = m_0 + m_1, \qquad l_2 = m_0 + m_1 + m_2, \dots, l_k = m_0 + \cdots + m_k, \\
&i_2 = l_k + m_{k+1}, \dots, i_k = l_k + m_{k+1} + \cdots + m_{2k-1}.
\end{aligned}$$

Write $p(m) = \mathbb{P}[S(m) = 0]$ and $\bar{p}(m) = 1/(\pi m)$. The contribution of these terms is then

$$J_n(k) = C(k) \sum_{M_n(2k)} \prod_{\substack{1\leq j\leq 2k-1 \\ j\neq 1,k+1}} p(m_j) \times \{p(m_1 + m_{k+1}) - p(m_1 + \cdots + m_{k+1})\}.$$

By [24], Theorem 2.1.3, we have that

$$|p(m) + p(m+1) - 2\bar{p}(m)| \leq \frac{C}{m^2}.$$

Let $q := m_2 + \cdots + m_k$ and

$$M := n - \sum_{\substack{0 \le j \le 2k-1 \\ j \ne 1, k+1}} m_j.$$

Then

$$\begin{aligned}
&\left| \sum_{m_1+m_{k+1}=0}^{M} p(m_1+m_{k+1}) - p(m_1+m_{k+1}+q) \right| \\
&\quad \le \sum_{m_1=0}^{M} \sum_{m_{k+1}=0}^{[(M-m_1)/2]} \big| p(m_1+2m_{k+1}) + p(m_1+2m_{k+1}+1) \\
&\qquad - p(m_1+2m_{k+1}+q) - p(m_1+2m_{k+1}+1+q) \big| \\
&\quad \le \sum_{m_1=0}^{M} \sum_{m_{k+1}=0}^{[(M-m_1)/2]} \Big\{ \big| \bar{p}(m_1+2m_{k+1}) - \bar{p}(m_1+2m_{k+1}+q) \big| \\
&\qquad + \frac{C}{(m_1+2m_{k+1})^2} \Big\} \\
&\quad \le \sum_{m_1=0}^{M} \sum_{m_{k+1}=0}^{[(M-m_1)/2]} \left\{ \frac{q}{(m_1+2m_{k+1})(m_1+2m_{k+1}+q)} + \frac{C}{(m_1+2m_{k+1})^2} \right\} \\
&\quad \le \sum_{m_1+m_{k+1}=0}^{n} \left\{ \frac{q}{(m_1+m_{k+1})(m_1+m_{k+1}+q)} + \frac{C}{(m_1+m_{k+1})^2} \right\}.
\end{aligned}$$

In order to ease the notation, here and elsewhere in the paper we allow fractions of the form $1/m$ where m may be zero. In these cases, we treat the fraction as 1.

Going back to $J_n(k)$, we have

$$\begin{aligned}
J_n(k) &\le \sum_{M_n(2k)} \prod_{\substack{1 \le j \le 2k-1 \\ k \ne 1, k+1}} p(m_k) \times \left\{ \frac{m_2 + \cdots + m_k}{(m_1 + \cdots + m_{k+1})(m_1 + m_{k+1})} \right. \\
&\qquad \left. + \frac{C}{(m_1+m_{k+1})^2} \right\} \\
&= \sum_{M_n(2k)} \prod_{\substack{1 \le k \le 2k-1 \\ k \ne 1, k+1}} p(m_k) \frac{m_2 + \cdots + m_k}{(m_1 + \cdots + m_{k+1})(m_1 + m_{k+1})} \\
&\qquad + O\big(n(\log n)^{2k-2}\big) \\
&=: J_n'(k) + O\big(n(\log n)^{2k-2}\big).
\end{aligned}$$

By symmetry after we split the sum in the numerator and we combine $m = m_1 + m_{k+1}$,

$$J_n'(k) \le Ckn \sum_{m_1,\dots,m_{2k-1}=0}^{n} \frac{1}{m_3 \cdots m_{2k-1}(m_1 + m_{k+1})(m_1 + \cdots + m_{k+1})}$$
$$\le Cn(\log n)^{2k-4} \sum_{m,m_2=0}^{n} \frac{1}{m + m_2} \le Cn^2(\log n)^{2k-4}. \qquad \square$$

REMARK 4.1. A similar proof can be performed for any periodic random walk, by summing over the period.

Given Proposition 4.1, the proof of Theorem 4.1 is very similar to [9] and is thus omitted. We just point out that under **A1**

$$\frac{\log(n)}{\pi\gamma n} \#R(n) \to 1 \qquad \text{a.s. as } n \to \infty, \tag{4.16}$$

by a simple application of Result 2 in Le Gall and Rosen [25] with $\beta = d = 1$ and $s(n) \equiv 1$, after one notices that in our case the truncated *Green's function* satisfies

$$h(n) := \sum_{k=0}^{n} \mathbb{P}(S_k = 0) \sim \frac{\log(n)}{\pi\gamma}, \tag{4.17}$$

by Lemma 4.8 and Karamata's Tauberian theorem B.1.

For $\mathbf{A2}'$ note that [15], Theorem 4, states that almost surely

$$\frac{\log n}{\pi n} \#R(n) \to 1.$$

4.3. *Proof of the Fölner property of the range* (*Theorem* 4.2). Let $\alpha > 0$ and define

$$L_{n,w}(\alpha) := \sum_{x \in \mathbb{Z}^d} l(n,x)^\alpha l(n,x+w)^\alpha.$$

These quantities are of interest since

$$L_{n,w}(0) := \lim_{\alpha \downarrow 0} L_{n,w}(\alpha) = \sum_{x \in \mathbb{Z}^d} \mathbf{I}\big(l(n,x) > 0\big)\mathbf{I}\big(l(n,x+w) > 0\big)$$
$$= \#\big(R(n) \cap R(n) + w\big).$$

Using the above notation, the Fölner property (4.4) can be written as

$$\lim_{n\to\infty} \frac{L_{n,w}(0)}{L_{n,0}(0)} = 1, \qquad \text{a.s.}$$

We will use the following result.

PROPOSITION 4.2. *Assume* **A1** *or* **A2** *holds. For all $w \in \mathbb{Z}^d$ and $\alpha \in \mathbb{Z}$, $\alpha \geq 1$*

$$\frac{L_{n,w}(\alpha)}{n(\log n)^{2\alpha-1}} \to \begin{cases} \dfrac{\Gamma(2\alpha+1)}{(\pi\gamma)^{2\alpha-1}}, & \text{for } d=1, \\ \dfrac{\Gamma(2\alpha+1)}{(2\pi\sqrt{|\Sigma|})^{2\alpha-1}}, & \text{for } d=2, \end{cases}$$

almost surely as $n \to \infty$.

We first complete the proof of Theorem 4.2 and then we prove the above proposition.

PROOF OF THEOREM 4.2. We first treat the cases **A1** and **A2**.

Let Y_n be defined as in (4.1). Setting $\gamma_d = 2\pi\sqrt{|\Sigma|}$ for $d=2$ and $\gamma_d = \pi\gamma$ for $d=1$, define

$$W_n := \gamma_d^2 \frac{l(n, Y_n) l(n, Y_n + w)}{\log(n)^2}.$$

For integer α, by Proposition 4.2

$$\begin{aligned}\mathbb{E}[W_n^\alpha | \mathcal{F}] &= \frac{\gamma_d^{2\alpha}}{\#R(n)} \sum_x \frac{l(n,x)^\alpha l(n,x+w)^\alpha}{\log(n)^{2\alpha}} \\ &= \frac{\gamma_d^{2\alpha-1} L_{n,w}(\alpha)}{n \log(n)^{2\alpha-1}} \frac{\gamma_d n / \log(n)}{R(n)} \to \Gamma(2\alpha+1),\end{aligned}$$

almost surely. These are the moments of Y^2, where $Y \sim \mathrm{Exp}(1)$. Since

$$\limsup_{k\to\infty} \frac{\Gamma(1+2k)^{1/2k}}{2k} = \lim \frac{\Gamma(1+2k)^{1/2k}}{2k} = \mathrm{e}^{-1} < \infty,$$

these moments define a unique distribution on the positive real line (see [14]) and, therefore, $\mathbb{P}$-almost surely, we have that conditionally on $\mathcal{F}$, $W_n \to Y^2$ in distribution. Then

$$\begin{aligned}&\frac{\sum_x \mathbf{I}(l(n,x) > 0, l(n,x+w) > 0)}{\#R(n)} \\ &\quad = \lim_{\alpha\downarrow 0} \frac{\gamma_d^{2\alpha}}{R(n)} \sum_x \frac{l(n,x)^\alpha l(n,x+w)^\alpha}{\log(n)^{2\alpha}} \\ &\quad = \lim_{\alpha\downarrow 0} \mathbb{E}[W_n^\alpha | \mathcal{F}] \qquad \text{and by monotone convergence} \\ &\quad = \mathbb{E}\Big[\lim_{\alpha\downarrow 0} W_n^\alpha | \mathcal{F}\Big] = \mathbb{P}(W_n > 0 | \mathcal{F}),\end{aligned}$$

almost surely. This shows that

$$\frac{\sum_x \mathbf{I}(l(n,x) > 0, l(n,x+w) > 0)}{\#R(n)} = \mathbb{P}(W_n > 0 | \mathcal{F}) \xrightarrow[n\to\infty]{} \mathbb{P}(Y^2 > 0) = 1.$$

Simple Random Walk. For the simple random walk in $\mathbb{Z}^2$, notice that one can consider the *lazy* version of the random walk, where $\mathbb{P}[\xi' = 0] = 1/2$ while for $e \in \mathbb{Z}^2$, with $|e| = 1$ we have $\mathbb{P}[\xi' = e] = 1/4d$. Then the lazy simple random walk $S'_n := \sum_{i=1}^n \xi'$, is strongly aperiodic and satisfies **A2**$'$ and, therefore, letting $R'(n) := \{S'(0), \dots, S'(n)\}$ be the range of $\{S'(n)\}_n$ we have for all $w \in \mathbb{Z}^2$

$$\frac{\#(R'(n) \cap R'(n) + w)}{\#R'(n)} \to 1,$$

almost surely. Define recursively the successive jump times

$$T_0 := \min\{j \geq 1 : S'_j \neq S'_{j-1}\}, \qquad T_k := \min\{j > T_{k-1} : S'_j \neq S'_{j-1}\}.$$

Notice that the range of the simple random walk $R(n)$ is equal to the range of the lazy walk at the time of the nth jump, $R'(T_n)$. Therefore,

$$\frac{\#(R(n) \cap R(n) + w)}{\#R(n)} = \frac{\#(R'(T_n) \cap R'(T_n) + w)}{\#R'(T_n)} \to 1,$$

since $T_n \to \infty$ almost surely. $\square$

REMARK 4.2. Note that it is also possible to prove Theorem 4.2 under **A2**$'$ directly, by proving the corresponding version of Proposition 4.2 and then following the same argument as for **A2**. To adapt the variance calculation in Proposition 4.2 to the simple random walk, one has to sum first over the period similarly to the proof of Proposition 4.1.

PROOF OF PROPOSITION 4.2. First, we prove the result for $\alpha \in \mathbb{N}$ and then we extend it to the general case $\alpha \geq 0$. For $\alpha \in \mathbb{N}$, we have

$$\begin{aligned} L_{n,w}(\alpha) &= \sum_{x \in \mathbb{Z}^2} \left(\sum_{i=0}^n \mathbf{I}(S_i = x) \right)^\alpha \left(\sum_{i=0}^n \mathbf{I}(S_i = x + w) \right)^\alpha \\ &= \sum_{x \in \mathbb{Z}^2} \sum_{i_1, \dots, i_\alpha = 0}^n \mathbf{I}\big[S(i_1) = \dots = S(i_\alpha) = x\big] \\ &\quad \times \sum_{k_1, \dots, k_\alpha = 0}^n \mathbf{I}\big[S(k_1) = \dots = S(k_\alpha) = x + w\big] \\ &= \sum_{i_1, \dots, i_{2\alpha} = 0}^n \mathbf{I}\big\{S(i_1) = \dots = S(i_\alpha) = S(i_{\alpha+1}) - w = \dots = S(i_{2\alpha}) - w\big\}, \end{aligned}$$

which for $w = 0$ corresponds to the term $L_n(2\alpha)$. Then we can rewrite $L_{n,w}(\alpha)$ as

$$\begin{aligned} L_{n,w}(\alpha) &= \sum_{\beta=1}^{2\alpha} \sum_{j=(\beta-\alpha)\vee 0}^{\alpha \wedge \beta} \sum_{\boldsymbol{\epsilon} \in E(\beta, j)} j!(\beta - j)! \\ &\quad \times \sum_{0 \leq i_1 < \dots < i_\beta \leq n} \mathbf{I}\big\{S(i_1) + \epsilon_1 w = \dots = \dots = S(i_\beta) + \epsilon_\beta w\big\}, \end{aligned} \tag{4.18}$$

where the third sum is over the set

$$E(\beta, j) := \Big\{ \boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_\beta) \in \{-1, 0\}^\beta : \sum |\epsilon_i| = j \Big\}.$$

Expectation of $L_{n,w}(\alpha)$. For given β, n and $\boldsymbol{\epsilon} \in E(\beta, j)$, we have using the Markov property

$$\begin{aligned} \alpha(\boldsymbol{\epsilon}, \beta, n) &:= \mathbb{E} \sum_{0 \le i_1 < \cdots < i_\beta \le n} \mathbf{I}\{S(i_1) + \epsilon_1 w = \cdots = \cdots = S(i_\beta) + \epsilon_\beta w\} \\ &= \sum_{m \in M_n(\beta)} \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = (\epsilon_i - \epsilon_{i+1}) w]. \end{aligned}$$

Next, we show that the asymptotic behaviour does not actually depend on w or $\boldsymbol{\epsilon}$. In this direction, we rewrite

$$\begin{aligned} \alpha(\boldsymbol{\epsilon}, \beta, n) = {} & \sum_{m \in M_n} \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = 0] \\ & + \sum_{m \in M_n} \left\{ \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = (\epsilon_i - \epsilon_{i+1}) w] - \prod_{i=1}^{\beta-1} \mathbb{P}[S(m_i) = 0] \right\} \\ =:{} & \alpha(\mathbf{0}, \beta, n) + \mathcal{E}(\boldsymbol{\epsilon}, \beta, n, w), \end{aligned}$$

and we claim that $\mathcal{E}(\beta, n, w) = o(\alpha(\mathbf{0}, \beta, n))$ as $n \to \infty$.

Letting $\delta_i = \epsilon_i - \epsilon_{i+1}$ we telescope the product to get

$$\begin{aligned} &\mathcal{E}(\boldsymbol{\epsilon}, \beta, n, w) \\ &\quad = \sum_{m \in M_n} \left\{ \prod_{i=1}^{\beta-1} \mathbb{P}[S_{m_i} = \delta_i w] - \prod_{i=1}^{\beta-1} \mathbb{P}[S_{m_i} = 0] \right\} \\ &\quad = \sum_{j=0}^{\beta-1} \sum_{m \in M_n} \prod_{i=1}^{\beta-1-j} \mathbb{P}[S(m_i) = \delta_i w] \times [\mathbb{P}[S(m_{\beta-1-j+1}) = \delta_{\beta-1-j+1} w] \\ &\qquad - \mathbb{P}[S(m_{\beta-1-j+1}) = 0]] \prod_{l=\beta-1-j+2}^{\beta-1} \mathbb{P}[S(m_l) = 0], \end{aligned} \tag{4.19}$$

where implicitly the indices are not allowed to exceed their corresponding ranges.

We analyse the first term in detail to obtain

$$\begin{aligned} &\left| \sum_{m \in M_n} \prod_{i=1}^{\beta-2} \mathbb{P}[S(m_i) = \delta_i w] \times [\mathbb{P}[S(m_{\beta-1}) = \delta_{\beta-1} w] - \mathbb{P}[S(m_{\beta-1}) = 0]] \right| \\ &\qquad \le \sum_{m_0, \dots, m_{\beta-2}=0}^{n} \prod_{i=1}^{\beta-2} \mathbb{P}[S(m_i) = \delta_i w] \end{aligned}$$

$$\times \sum_{m_{\beta-1}=0}^{n} \left|\mathbb{P}\left[S(m_{\beta-1}) = \delta_{\beta-1} w\right] - \mathbb{P}\left[S(m_{\beta-1}) = 0\right]\right|$$

$$\leq \sum_{m_0,\ldots,m_{\beta-2}0}^{n} \prod_{i=1}^{\beta-2} \mathbb{P}\left[S(m_i) = \delta_i w\right]\left[1 + \sum_{m_{\beta-1}=1}^{\infty} \frac{C}{m^2}\right] = O\left(n \log(n)^{\beta-2}\right),$$

by Lemma 4.1. The remaining errors are very similar.

The asymptotic behaviour of $\alpha(\mathbf{0}, \beta, n)$ follows from [9] for $d = 2$ and Lemma 4.2 for $d = 1$ and is given by

$$\alpha(\mathbf{0}, \beta, n) \sim n\left(\frac{\log n}{\gamma_d}\right)^{\beta-1},$$

where $\gamma_1 := \pi\gamma$ and $\gamma_2 := 2\pi\sqrt{\det(\Sigma)}$. Going back to (4.18), we see that the leading term corresponds to $\beta = 2\alpha$, while from the above discussion we can replace the terms $\alpha(\boldsymbol{\epsilon}, 2\alpha, n)$ by $\alpha(\mathbf{0}, 2\alpha, n)$. Since $\#E(2\alpha, \alpha)(\alpha!)^2 = \Gamma(2\alpha + 1)$, we conclude that

$$\mathbb{E}L_{n,w}(\alpha) = \mathbb{E} \sum_{x \in \mathbb{Z}^d} l(n, x)^{\alpha} l(n, x + w)^{\alpha} \sim \Gamma(2\alpha + 1) n \left(\frac{\log n}{\gamma_d}\right)^{2\alpha-1}.$$

Variance of $L_{n,w}(\alpha)$. To compute the variance, we will follow the approach developed in [11]. First, notice that

$$\begin{aligned}
&\mathbb{E}L_{n,w}(\alpha)^2 \\
&\quad = \mathbb{E} \sum_{i_1,\ldots,i_{2\alpha}=0}^{n} \mathbf{I}\left(S(i_1) = \cdots = S(i_\alpha) = S(i_{\alpha+1}) - w = \cdots = S(i_{2\alpha}) - w\right) \\
&\qquad \times \sum_{j_1,\ldots,j_{2\alpha}=0}^{n} \mathbf{I}\left(S(j_1) = \cdots = S(j_\alpha) = S(j_{\alpha+1}) - w = \cdots = S(j_{2\alpha}) - w\right).
\end{aligned}$$

Let A_m, A'_m be 0 or 1 according to whether there is a w or not in the mth increment. Then

$$\begin{aligned}
\operatorname{var}\left(L_{n,w}(\alpha)\right) = \sum_{k_1,\ldots,k_{2\alpha}} \sum_{l_1,\ldots,l_{2\alpha}} \Big\{&\mathbb{P}\left[S(k_1) = S(k_2) + A_2 w = \cdots = S(k_{2\alpha}) + A_{2\alpha} w;\right. \\
&\left. S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w\right] \\
&- \mathbb{P}\left[S(k_1) = S(k_2) + A_2 w = \cdots = S(k_{2\alpha}) + A_{2\alpha} w\right] \\
&\times \mathbb{P}\left[S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w\right]\Big\}.
\end{aligned}$$

As we shall see, the presence of w does not affect the asymptotic behaviour. The main role is played by the interlacement of the sequences $\mathbf{k} = (k_1, \ldots, k_{2\alpha})$ and $\mathbf{l} = (l_1, \ldots, l_{2\alpha})$. In order to define the interlacement index $v(\mathbf{k}, \mathbf{l})$, of two sequences $\mathbf{k} = (k_1, \dots, k_r)$ and $\mathbf{l} = (l_1, \dots, l_s)$, let $\mathbf{j}$ be the combined sequence of length $r + s$, where ties between elements of $\mathbf{k}$ and $\mathbf{l}$ are counted twice. We also define $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_{r+s})$, where $\epsilon_i = 1$ if the ith element of the combined sequence is from $\mathbf{k}$ and 0 if it is from $\mathbf{l}$; that is, $\epsilon_1 = 1$ if $j_i \in \mathbf{k}$ and 0 otherwise. Then we define the *interlacement index*,

$$v(\mathbf{k}, \mathbf{l}) = v(k_1, \dots, k_r; l_1, \dots, l_s) := \sum_{i=1}^{r+s-1} |\epsilon_{i+1} - \epsilon_i|, \tag{4.20}$$

which counts the number of times $\mathbf{k}$ and $\mathbf{l}$ cross over.

When $v = 1$, then the contribution is zero by the Markov property. The main contribution will be from $v = 2$. Similar to [11], the contributions of terms with $v \geq 3$ can be bounded above by just considering the positive part, $\mathbb{E}L_{n,w}(\alpha)^2$. Let us first treat this case leaving $v = 2$ for later.

Case $v \geq 3$. Letting $\rho(\alpha)$ denote combinatorial factors, the contribution to $\mathbb{E}L_{n,w}(\alpha)^2$ from the terms with interlacement $v \geq 3$ is trivially bounded above by

$$\begin{aligned} I_n(w, \alpha) &:= \rho(\alpha) \sum_{k_1,\dots,k_{2\alpha}} \sum_{l_1,\dots,l_{2\alpha}} \mathbb{P}\big[S(k_1) = S(k_2) + A_2 w = \cdots = S(k_{2\alpha}) + A_{2\alpha} w; \\ &\qquad S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w\big] \\ &= \rho(\alpha) \sum_{k_1,\dots,k_{2\alpha}} \sum_{l_1,\dots,l_{2\alpha}} \sum_x \mathbb{P}\big[S(k_1) = \cdots = S(k_{2\alpha}) + A_{2\alpha} w; \\ &\qquad S(l_1) = S(k_1) - x, S(l_1) = S(l_2) + A'_2 w = \cdots = S(l_{2\alpha}) + A'_{2\alpha} w\big], \end{aligned}$$

where $A_i, A'_i \in \mathbb{Z}$ and may vary from line to line. Let $(j_1, \dots, j_{4\alpha})$ denote the combined sequence, allowing for matches. Changing variables

$$\begin{aligned} &j_1 = m_0, \qquad j_2 = m_0 + m_1, \dots, j_{4\alpha} = m_0 + \cdots + m_{4\alpha-1}, \\ &n = m_0 + \cdots + m_{4\alpha}, \end{aligned}$$

with $m_0, \dots, m_{4\alpha} \geq 0$, we get

$$\begin{aligned} I_n(w, \alpha) \leq \rho(\alpha) \sum_{m_0,\dots,m_{4\alpha-1} \geq 0} \sum_x \mathbb{P}\big[&S(m_1) = S(m_1 + m_2) + A_2 w + \delta_2 x \\ &= \cdots = S(m_1 + \cdots + m_{4\alpha}) + A_{4\alpha} w + \delta_{4\alpha} x\big], \end{aligned}$$

where $\delta_i := \epsilon_i - \epsilon_{i+1} \in \{-1, 0, +1\}$, and $\boldsymbol{\epsilon}$ is defined as earlier. A simple application of the Markov property results in

$$I_n(w, \alpha) \leq \rho(\alpha) n \sum_{m_1,\dots,m_{4\alpha-1} \geq 0} \sum_x \prod_{k=1}^{4\alpha-1} \mathbb{P}\big[S(m_k) = (\delta_{k-1} - \delta_k) x + A_k w\big],$$

where the factor n resulted from the free index m_0. Notice that since v is the number of interlacements, exactly $u := 4\alpha - 1 - v$ of the δ's are 0, and thus by (4.17)

$$I_n(w,\alpha) \le Cn \log(n)^{4\alpha-1-v} \sum_{j_1,\dots,j_v} \sum_x \prod_{t=1}^{v} \mathbb{P}[S(j_t) = \delta_t' x + A_t w], \tag{4.21}$$

where $\delta_t' \in \{-1,+1\}$. Letting

$$D_{n,v} := \sum_{j_1,\dots,j_v=0}^{n} \sum_x \prod_{k=1}^{v} \mathbb{P}[S(j_k) = \delta_k' x + A_k w],$$

notice that

$$D_{n,v} \le D_{n,v-1} \sum_{j_v} \sup_y \mathbb{P}[S(j_v) = y] \le C D_{n,v-1} \sum_{j_v=1}^{n} \frac{1}{j_v} \le C \log(n) D_{n,v-1}.$$

Repeating we arrive at $D_{n,v} \le C \log(n)^{v-3} D_{n,3}$ and, therefore,

$$I_n(w,\alpha) \le Cn \log(n)^{4\alpha-4} D_{n,3}.$$

To complete our study of the $v \ge 3$ case, we now treat the term $D_{n,3}$,

$$\begin{aligned} D_{n,3} &\le C \sum_{i \le j \le k} \sum_x \mathbb{P}[S(i) = \delta_i' x + A_i w] \times \mathbb{P}[S(j) = \delta_j' x + A_j w] \\ &\quad \times \mathbb{P}[S(k) = \delta_k' x + A_k w] \\ &\le C \sum_{i \le j \le k} \Bigl(\sup_y \mathbb{P}[S(j) = y]\Bigr) \sup_y \mathbb{P}[S(i+k) = y]. \end{aligned}$$

By symmetry and Lemma 4.1,

$$\begin{aligned} D_{n,3} &\le C \sum_{1 \le i \le j \le k \le n} \frac{1}{j} \frac{1}{i+k} \le C \sum_{m_1=1}^{n} \sum_{m_2,m_3=0}^{n} \frac{1}{m_1+m_2} \frac{1}{2m_1+m_2+m_3} \\ &\le C \sum_{m_1=1}^{n} \sum_{m_2=0}^{n} \frac{1}{m_1+m_2} \log\Bigl(1 + \frac{n}{m_1+m_2}\Bigr) \le C \sum_{j=1}^{2n} \log\Bigl(1 + \frac{n}{j}\Bigr) \\ &\le C \int_{x=1}^{2n} \log\Bigl(1 + \frac{n}{x}\Bigr) \mathrm{d}x \le n \int_{1/2}^{n} \log(1+y) \frac{\mathrm{d}y}{y^2} \le Cn. \end{aligned}$$

Therefore, $D_{n,3} = O(n)$, and thus the total contribution of the terms with $v \ge 3$ is $O(n^2 \log(n)^{4\alpha-4})$.

Case $v = 2$. Letting $M_n(4\alpha)$ be defined as usual, we have for some q that $l_1, \dots, l_{2\alpha} \in [k_q, k_{q+1}]$. Denoting by $J_n(w,\alpha)$ the contribution of a single term with $v = 2$

$$
\begin{aligned}
J_n(w,\alpha) = \sum_{M_n(4\alpha)} \prod_{\substack{1\le k\le 4\alpha-1\\ k\ne q,q+2\alpha}} &\mathbb{P}[S(m_k) = A_k w] \\
\times &\big[\mathbb{P}\big(S(m_q) + S(m_{q+2\alpha}) = K_1 w\big) \\
- &\mathbb{P}\big(S(m_q) + \cdots + S(m_{q+2\alpha}) = K_2 w\big)\big],
\end{aligned}
\tag{4.22}
$$

where K_1, K_2 are integers determined by $\mathbf{k}, \mathbf{l}$ and their interlacement. By (4.7), it follows that

$$
\begin{aligned}
&J_n(w,\alpha) \\
&\quad\le Cn\log(n)^{2\alpha-2} \sum_{p_0,\dots,p_{2\alpha}} \frac{1}{p_2\cdots p_{2\alpha}} \left[\frac{1}{p_0+p_1} - \frac{1}{p_0+p_1+\cdots+p_{2\alpha}}\right] \\
&\quad= Cn\log(n)^{2\alpha-2} \sum_{p_0,\dots,p_{2\alpha}} \frac{p_2+\cdots+p_{2\alpha}}{p_2\cdots p_{2\alpha}(p_0+p_1)(p_0+p_1+\cdots+p_{2\alpha})} \\
&\quad\le C\alpha n\log(n)^{2\alpha-2} \sum_{p_0,\dots,p_{2\alpha}} \frac{1}{p_3\cdots p_{2\alpha}(p_0+p_1)(p_0+p_1+p_2+\cdots+p_{2\alpha})} \\
&\quad\le C\alpha n\log(n)^{2\alpha-2} \sum_{p_2,\dots,p_{2\alpha}} \sum_{j=0}^{2n} \frac{1}{p_3\cdots p_{2\alpha}(j+p_2+\cdots+p_{2\alpha})} \\
&\quad\le C\alpha n\log(n)^{2\alpha-2}\log(n)^{2\alpha-3+1} \sum_{p_1,p_2=0}^{n} \frac{1}{p_1+p_2} \le C\alpha n^2\log(n)^{4\alpha-4}.
\end{aligned}
$$

Thus, the total contribution of terms with interlacement index $v = 2$ is $O(n^2\log(n)^{4\alpha-4})$.

To complete the proof of Proposition 4.2, we first use Chebyshev's inequality to prove convergence along subsequences $n = \lfloor \rho^k \rfloor$, for $0 < \rho < 1$. We can fill in the gaps following the standard trick, as in [9]. $\square$

5. Proof of Theorem 3.1. Our proof follows closely the outline of the proof of [1] and [18]. The main difference in our approach is that we are using the almost sure Fölner property of the range and that we substitute the role of the local times with Theorem 4.1. In the following, we assume that the entropy of S is finite. The case of infinite entropy can be easily derived by the same method.

Fix a finite generator β for S, the existence of which is a consequence of Krieger's Finite Generator Theorem [22] for $d = 1$ and [10, 19] for $d = 2$. Let $\alpha = \{[x_1] : x \in \Omega\}$ be the partition of Ω according to the first coordinate. The partition $\Upsilon := \alpha \times \beta$ is a countable generating partition of $\Omega \times Y$ for T. Thus, by

Aaronson's Generator Theorem (Theorem 2.1), we need to show that

$$\frac{\log n}{n} \log \mathcal{K}_{\mathcal{B}_\Omega \times Y}(\Upsilon, n, \epsilon) \xrightarrow{m} \pi h(\mathsf{S}) \cdot \begin{cases} \gamma, & d = 1, \mathbf{A1}, \\ 2\sqrt{\det \Sigma}, & d = 2, \mathbf{A2}, \mathbf{A2}'. \end{cases}$$

For $a_0, a_1, \ldots, a_n \in \Upsilon$, we write

$$[a_0, a_1, \ldots, a_n] := \bigcap_{j=0}^{n} T^{-j} a_j$$

and the $\bar{d}_n$ metric on $\bigvee_{j=0}^{n-1} T^{-j}\Upsilon$,

$$\bar{d}_n\big([a_0, a_1, \ldots, a_{n-1}], [a_0', a_1', \ldots, a_{n-1}]\big) := \frac{\#\{0 \le j \le n-1 : a_j \ne a_j'\}}{n}.$$

Since $T^n(w, y) = (\sigma^n w, \mathsf{S}_{\sum_{j=1}^n w_j}(y))$, it is straightforward to check that for all $n \in \mathbb{N}$ and $(w, y) \in \Omega \times Y$,

$$\left(\bigvee_{j=0}^{n-1} T^{-j}\Upsilon\right)(w, y) = \big[w_0^{n-1}\big] \times \beta_{R_n(w)}(y),$$

where $\beta_{R_n(w)}(y) := (\bigvee_{l \in R_n(w)} \mathsf{S}_l^{-1}\beta)(y)$ and $R_n(w) := \{\sum_{j=1}^{l} w_j : 1 \le l \le n\}$ is the range of the random walk up to time n. For $n \in \mathbb{N}$, define $\Pi_n : \Omega \to 2^{\bigvee_{j=0}^{n-1} T^{-j}\Upsilon}$ by

$$\Pi_n(w) := \left\{a \in \left(\bigvee_{j=0}^{n-1} T^{-j}\Upsilon\right) : m\big(a|\mathcal{B}_\Omega \times Y\big)(w) > 0\right\}.$$

These are the partition elements seen by w. The function

$$\Phi_{n,\epsilon}(x) := \min\left\{\#F : F \subset \Pi_n(x), m\left(\bigcup_{a \in F} a \middle| \mathcal{B}_\Omega \times Y\right) > 1 - \epsilon\right\},$$

is an upper bound for $\mathcal{K}_{\mathcal{B}_\Omega \times \mathcal{Y}}(\Upsilon, n, \epsilon)(x)$ since in the definition of $\Phi_{n,\epsilon}$ we are using all sequences in $\Pi_n(x)$ on their own and not grouping them into balls.

To get a lower bound, introduce

$$\mathcal{Q}_{n,\epsilon}(x) := \max\big\{\#\big\{z \in \Pi_n(x) : \bar{d}_n(a, z) \le \epsilon\big\} : a \in \Pi_n(x)\big\}$$

to be the maximal cardinality of elements of $\Pi_n(x)$ at a $\bar{d}_n$ ball centred at some $a \in \Pi_n(x)$. It then follows that

$$\mathcal{K}_{\mathcal{B}_\Omega \times Y}(P, n, \epsilon)(x) \ge \frac{\Phi_{n,\epsilon}(x)}{\mathcal{Q}_{n,\epsilon}(x)}.$$

Therefore, the proof is separated into two parts. First, we prove that

$$\frac{\log n}{n} \log \Phi_{n,\epsilon} \xrightarrow{m} \pi h(\mathsf{S}) \cdot \begin{cases} \gamma, & d = 1, \mathbf{A1}, \\ 2\sqrt{\det \Sigma}, & d = 2, \mathbf{A2}, \mathbf{A2}', \end{cases} \tag{5.1}$$

and the second part consists of showing that

$$\frac{\log n}{n} \log \mathcal{Q}_{n,\epsilon}(x) \xrightarrow{m} 0. \tag{5.2}$$

We will deduce (5.1) from the following Shannon–Mcmillan–Breiman theorem.

LEMMA 5.1. *For $\mathbb{P}$-almost every $w \in \Omega$,*

$$-\frac{\log n}{n} \log \nu\bigl(\beta_{R_n(w)}(y)\bigr) \xrightarrow{m} \pi h(\mathsf{S}) \cdot \begin{cases} \gamma, & d = 1, \mathbf{A1}, \\ 2\sqrt{\det \Sigma}, & d = 2, \mathbf{A2}, \mathbf{A2}', \text{ as } n \to \infty. \end{cases}$$

PROOF. Let $d \in \{1, 2\}$. By Theorem 4.2, for $\mathbb{P}$-almost every w, the range $\{R_n(w)\}$ is a Fölner sequence for $\mathbb{Z}^d$. Whence by Kieffer's Shannon–McMillan–Breiman theorem [21], for $\mathbb{P}$-a.e. w,

$$-\frac{1}{\#R_n(w)} \log \nu\bigl(\beta_{R_n(w)}(y)\bigr) \xrightarrow[n\to\infty]{\nu} h(\mathsf{S})$$

and thus by Fubini,

$$-\frac{1}{\#R_n(w)} \log \nu\bigl(\beta_{R_n(w)}(y)\bigr) \xrightarrow[n\to\infty]{m} h(\mathsf{S}).$$

Notice that $h(\mathsf{S}, \beta) = h(\mathsf{S})$ since β is a generating partition. Since by [15] and (4.16),

$$\frac{\log n}{n} \#R_n(w) \xrightarrow[n\to\infty]{\text{a.s.}} \pi \begin{cases} \gamma, & d = 1, \mathbf{A1}, \\ 2\sqrt{\det \Sigma}, & d = 2, \mathbf{A2}, \mathbf{A2}', \end{cases}$$

the conclusion of the lemma follows. □

To keep the notation short, write

$$\mathrm{b}_d(n) := \frac{\pi n}{\log(n)} \begin{cases} \gamma, & d = 1, \mathbf{A1}, \\ 2\sqrt{\det \Sigma}, & d = 2, \mathbf{A2}, \mathbf{A2}'. \end{cases}$$

PROOF OF (5.1). Let $\epsilon > 0$ and for $n \in \mathbb{N}$, $x \in \Omega$ let

$$H_{n,x,\epsilon} := \bigl\{y \in Y : \nu(\beta_{R_n(x)})(y) = e^{-\mathrm{b}_d(n) h(\mathsf{S})(1\pm\epsilon)}\bigr\}.$$

By Lemma 5.1, there exists N_ϵ such that for all $n > N_\epsilon$, $\exists G_{n,\epsilon} \in \mathcal{B}_\Omega$ so that $\mathbb{P}(G_{n,\epsilon}) > 1 - \epsilon$ and for all $x \in G_{n,\epsilon}$,

$$\nu(H_{n,x,\epsilon}) > 1 - \frac{\epsilon}{2}. \tag{5.3}$$

For $x \in G_{n,\epsilon}$, set $F_{n,x,\epsilon} := \{\beta_{R_n(x)}(y) : y \in H_{n,x,\epsilon}\}$. Since

$$\min\bigl\{\log \nu(a) : a \in F_{n,x,\epsilon}\bigr\} > -\mathrm{b}_d(n) h(\mathsf{S})(1 + \epsilon),$$

one has by a standard counting argument that for $x \in G_{n,\epsilon}$

$$\log \Phi_{n,\epsilon}(x) \le \log \# F_{n,x,\epsilon} \le \mathrm{b}_d(n) h(\mathsf{S})(1+\epsilon).$$

On the other hand, it follows from (5.3) that for small ϵ and $x \in G_{n,\epsilon}$, if $F \subset \Pi_n(x)$ with $m(\bigcup_{a\in F} a|\mathcal{B}_\Omega \times Y)(x) > 1-\epsilon$, then for large n

$$\# F \ge \frac{1-3\epsilon/2}{\max\{\log \nu(a) : a \in F_{n,x,\epsilon}\}} \ge \frac{e^{\mathrm{b}_d(n)h(\mathsf{S})(1-\epsilon)}}{2}.$$

Thus, for every $x \in G_{n,\epsilon}$ with n large,

$$\log \Phi_{n,\epsilon}(x) \ge \mathrm{b}_d(n)h(\mathsf{S})(1-\epsilon) + \log(1/2) \ge \mathrm{b}_d(n)h(\mathsf{S})(1-2\epsilon).$$

The conclusion follows since

$$m\big(\big[\log \Phi_{n,\epsilon}(x) = \mathrm{b}_d(n)h(\mathsf{S})(1\pm 2\epsilon)\big]\big) \ge \mathbb{P}(G_{n,\epsilon}) \xrightarrow[n\to\infty,\epsilon\to 0]{} 1. \qquad \square$$

5.1. *Proof of Equation* (5.2). Let $\varepsilon > 0$ and choose $\delta > 0$ such that

$$2H(3\delta/2) + 3\delta \log(\#\beta) < \varepsilon,$$

where for $0 < p < 1$,

$$H(p) = -p\log_2(p) - (1-p)\log_2(1-p),$$

is the entropy appearing in the Stirling approximation for the binomial coefficients. It follows from Theorem 4.1 that there exists $\mathsf{c} > 0$ such that for all large n, the sets

$$A_{\delta,n} = \left\{ w \in \Omega : \frac{\#\{x \in R_n(w) : l(n,x)(w) > \mathsf{c}\log(n)\}}{\# R_n(w)} > 1-\delta \right\},$$

satisfy $\mathbb{P}(A_{\delta,n}) > 1-\delta$. Since $\# R_n(w) \sim \mathrm{b}_d(n)$ almost surely we can assume further that for all $w \in A_{\delta,n}$, $\# R_n(w) \lesssim 2\mathrm{b}_d(n)$.

Since $\Pi_n(w) \subset [w_0^{n-1}] \times \beta_{R_n(w)}$, we can define a map $\mathbf{z} : \Pi_n(w) \to \beta_{R_n(w)}$ by

$$a =: \big[w_0^{n-1}\big] \times \mathbf{z}(a).$$

For $z \in \beta_{R_n(w)}$ and $j \in R_n(w)$, denote by z_j the element of β such that $z \subset \mathsf{S}_j^{-1}\beta$.

LEMMA 5.2. *For large $n \in \mathbb{N}$ and $w \in A_{\delta,n}$, if $a, a' \in \Pi_n(w)$ then*

$$\#\{j \in R_n(w) : \mathbf{z}(a)_j \ne \mathbf{z}(a')_j\} \le \mathrm{b}_d(n)\left(\frac{\bar{d}_n(a,a')}{\hat{\mathsf{c}}} + 2\delta\right),$$

where $\hat{\mathsf{c}} := \mathsf{c}\cdot \mathrm{b}_d(n)\log(n)/n$.

PROOF. Define

$$K_n(w) := \{j \in R_n(w) : \mathbf{z}(a)_j \neq \mathbf{z}(a')_j\}$$

and

$$F_n(w) := \{j \in R_n(w) : l(n,x)(w) \geq \mathtt{c}\log(n)\}.$$

Then $K_n \subset (K_n \cap F_n) \cup F_n^c$ and, therefore, since $w \in A_{\delta,n}$,

$$\begin{aligned}\#K_n(w) &\leq \#(K_n \cap F_n)(w) + \#F_n^c(w)\\ &\leq \#(K_n \cap F_n)(w) + \delta \#R_n(w)\\ &\lesssim \#(K_n \cap F_n)(w) + 2\delta \mathrm{b}_d(n).\end{aligned}$$

Finally,

$$\begin{aligned}\#(K_n \cap F_n) &\leq \frac{1}{\mathtt{c}\log(n)} \sum_{j \in F_n(w)} l(n,j)\mathbf{1}_{K_n(w)}\\ &\leq \frac{1}{\mathtt{c}\log(n)} \#\{0 \leq i \leq n-1 : \mathbf{z}(a)_{s_i(w)} \neq \mathbf{z}(a')_{s_i(w)}\}\\ &= \frac{n}{\mathtt{c}\log(n)} \bar{d}_n(a,a').\end{aligned}$$

The conclusion follows. $\square$

PROOF OF (5.2). First, we show that for n large enough so that $A_{\delta,n}$ is defined,

$$\max_{w \in A_{\delta,n}} \log \mathcal{Q}_{n,\hat{\mathtt{c}}\delta}(w) \leq \varepsilon b_d(n).$$

To see this, first notice that by Lemma 5.2 for every $a \in \Pi_n(w)$,

$$\begin{aligned}&\{a' \in \Pi_n(w) : \bar{d}_n(a,a') \leq \hat{\mathtt{c}}\delta\}\\ &\qquad \subset \{\mathbf{z} \in \beta_{R_n(w)} : \#\{j \in R_n(w) : \mathbf{z}(a)_j \neq \mathbf{z}_j\} \leq 3\delta \mathrm{b}_d(n)\}.\end{aligned}$$

Thus, for $w \in A_{\delta,n}$, using the Stirling approximation for the Binomial and $\#R_n(w) \lesssim 2\mathrm{b}_d(n)$,

$$\begin{aligned}\log \mathcal{Q}_{n,\hat{\mathtt{c}}\delta}(w) &\leq \log\left[\binom{\#R_n(w)}{3\delta \mathrm{b}_d(n)} (\#\beta)^{3\delta \mathrm{b}_d(n)}\right]\\ &\lesssim 3\mathrm{b}_d(n)\delta \log(\#\beta) + \log\binom{2\mathrm{b}_d(n)}{3\delta \mathrm{b}_d(n)}\\ &\sim \mathrm{b}_d(n)[3\delta \log(\#\beta) + 2H(3\delta/2)]\\ &\leq \varepsilon \mathrm{b}_d(n).\end{aligned}$$

This shows that for large n,

$$\mathbb{P}\big(\log \mathcal{Q}_{n,\hat{c}\delta} > 2\varepsilon \mathrm{b}_d(n)\big) \le \mathbb{P}\big(A^c_{\delta,n}\big) \le \delta,$$

and thus we have completed the proof of (5.2). □

As was mentioned before, Theorem 3.1 follows from (5.1) and (5.2).

APPENDIX A: APPENDIX: PROOFS OF AUXILIARY RESULTS

PROOF OF LEMMA 4.1. We only prove the second statement, the first being simpler. For $d = 1$ and any $\epsilon > 0$ by strong aperiodicity for $|t| > \epsilon$, it is true that $|\phi(t)| < C(\epsilon) < 0$. Therefore,

$$\begin{aligned}\big|\mathbb{P}\big[S(m) = 0\big] - \mathbb{P}[S_m = w]\big| &\le \int_{-\pi}^{\pi} |1 - \mathrm{e}^{\mathrm{i}tw}|\big|\phi(t)\big|^m \,\mathrm{d}t \\ &\le \int_{|t|<\epsilon} |1 - \mathrm{e}^{\mathrm{i}tw}|\big|\phi(t)\big|^m \,\mathrm{d}t + 4\pi C(\epsilon)^m,\end{aligned}$$

where the second term decays exponentially. For the first term, we have since $\phi(t) = 1 - \gamma|t| + o(|t|)$, for ϵ small enough and $|t| < \epsilon$,

$$\big|\phi(t)\big| \le \big|1 - \gamma|t|\big| + D(\epsilon)|t| \le 1 - \frac{\gamma}{2}|t|.$$

Therefore,

$$\begin{aligned}\int_{|t|<\epsilon} |1 - \mathrm{e}^{\mathrm{i}tw}|\big|\phi(t)\big|^m \,\mathrm{d}t &\le C \int_{|t|<\epsilon} |t||w|\left(1 - \frac{\gamma}{2}|t|\right)^m \mathrm{d}t \\ &= C|w| \int_{t=0}^{\epsilon} t\left(1 - \frac{\gamma t}{2}\right)^m \mathrm{d}t \\ &\le C|w| \int_{t=0}^{\epsilon} t \exp\left(-\frac{m\gamma t}{2}\right) \mathrm{d}t \le C\frac{|w|}{m^2}.\end{aligned}$$

We prove (4.7) for $d = 1$. By (4.6) it suffices to consider $w = 0$. For the moment fix a small $\epsilon > 0$. Then, by aperiodicity, for $|t| > \epsilon$, there exists $\rho(\epsilon) \in (0, 1)$, such that $|\phi(t)| < \rho(\epsilon)$. Thus,

$$\begin{aligned}\mathbb{P}\big[S(n) = 0\big] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(t)^n \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} \phi(t)^n \,\mathrm{d}t + O\big(\rho(\epsilon)^n\big) \\ &= \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon} \big[1 - \gamma|t| + R(t)\big]^n \,\mathrm{d}t + O\big(\rho(\epsilon)^n\big) \\ &=: I(n, \epsilon) + O\big(\rho(\epsilon)^n\big).\end{aligned}$$

Since $R(t) = o(t)$, for $|t| < \epsilon$, we can find $C(\epsilon)$ such that $|R(t)| \le C(\epsilon)|t|$ and such that $C(\epsilon) \to 0$ as $\epsilon \to 0$.

Therefore, letting $\gamma_1(\epsilon) := \gamma(1 + C(\epsilon))$

$$I(n,\epsilon) \geq \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\big[1-\gamma|t| - C(\epsilon)|t|\big]^n\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\epsilon}\big[1-\gamma_1(\epsilon)t\big]^n\,\mathrm{d}t$$

$$= \frac{1}{\pi\gamma_1(\epsilon)}\int_0^{\gamma_1(\epsilon)\epsilon}[1-t]^n\,\mathrm{d}t = \frac{1}{\pi\gamma_1(\epsilon)}\left\{\frac{1}{n+1} - \frac{[1-\gamma_1(\epsilon)\epsilon]^{n+1}}{n+1}\right\}.$$

Since for $\epsilon > 0$ small enough, we have $0 < 1 - \gamma_1(\epsilon)\epsilon < 1$ we compute

$$\liminf_{n\to\infty} n\mathbb{P}\big[S(n) = 0\big] \geq \frac{1}{\pi\gamma_1(\epsilon)}.$$

On the other hand, we also have

$$I(n,\epsilon) \leq \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\big[1-\gamma|t| + C(\epsilon)|t|\big]^n\,\mathrm{d}t = \frac{1}{\pi}\int_0^{\epsilon}\big[1-\gamma_2(\epsilon)t\big]^n\,\mathrm{d}t,$$

where $\gamma_2(\epsilon) = 1 - C(\epsilon)$. Thus,

$$I(n,\epsilon) \leq \frac{1}{\pi}\int_0^{\epsilon}\big[1-\gamma_2(\epsilon)t\big]^n\,\mathrm{d}t = \frac{1}{\pi\gamma_2(\epsilon)}\int_0^{\gamma_2(\epsilon)\epsilon}[1-t]^n\,\mathrm{d}t$$

$$= \frac{1}{\pi\gamma_2(\epsilon)}\left\{\frac{1}{n+1} - \frac{[1-\gamma_2(\epsilon)\epsilon]^{n+1}}{n+1}\right\}.$$

For $\epsilon > 0$ small enough, we have that $1 - \gamma_2(\epsilon)\epsilon \in (0, 1)$ and, therefore, we obtain that

$$\limsup_{n\to\infty} n\mathbb{P}\big[S(n) = 0\big] \leq \frac{1}{\pi\gamma_2(\epsilon)}.$$

Since $\epsilon > 0$ is can be arbitrarily small and $\gamma_1(\epsilon), \lim\gamma_2(\epsilon) \to \gamma$, (4.7) follows.

For $d = 2$, the proof is similar, using polar coordinates. □

PROOF OF LEMMA 4.2. Let $\delta > 0$ be arbitrary but small. Then

$$\frac{1}{2\pi}\int_{t=-\pi}^{\pi}\frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda\phi(t)} = \frac{1}{2\pi}\int_{|t|\leq\delta}\frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda\phi(t)} + \frac{1}{2\pi}\int_{\pi\geq|t|>\delta}\frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda\phi(t)}.$$

By strong aperiodicity for small enough $\delta > 0$, there exists a small positive constant $D(\delta)$ such that $|\phi(t)| < 1 - D(\delta)$ when $|t| > \delta$. Thus,

$$\left|\frac{1}{2\pi}\int_{\pi\geq|t|>\delta}\frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda\phi(t)}\right| \leq C D(\delta)^{-1},$$

for all $\lambda \leq 1$. Also

$$\frac{1}{2\pi}\int_{|t|\leq\delta}\frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda\phi(t)} = \frac{1}{2\pi}\int_{|t|\leq\delta}\frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda(1-\gamma|t|)} + I(\lambda,\delta),$$

where a standard argument using **A1** and the strong aperiodicity shows that there exists $r(\delta) = o_\delta(1)$, as $\delta \to 0$ such that

$$\bigl|I(\lambda, \delta)\bigr| \le r(\delta) \log\left(\frac{1}{1-\lambda}\right).$$

Finally, as $\lambda \uparrow 1$ it is easily seen that

$$\begin{aligned}\frac{1}{2\pi}\int_{|t|\le\delta} \frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda(1-\gamma|t|)} &\sim \frac{1}{\pi}\int_{t=0}^{\delta} \frac{\mathrm{d}t}{1-\lambda+\lambda\gamma t} \\ &= \frac{1}{\pi}\int_{t=0}^{\delta} \frac{\mathrm{d}t}{1-\lambda+\lambda\gamma t} \sim \frac{1}{\pi\gamma}\log\left(\frac{1}{1-\lambda}\right).\end{aligned}$$

Therefore, as $\lambda \uparrow 1$

$$\begin{aligned}\frac{1}{2\pi}\int_{t=-\pi}^{\pi} \frac{\lambda\phi(t)\,\mathrm{d}t}{1-\lambda\phi(t)} &= \frac{1}{\pi\gamma}\log\left(\frac{1}{1-\lambda}\right)\bigl(1+O\bigl(r(\delta)\bigr)\bigr)+O(1) \\ &\sim \frac{1}{\pi\gamma}\log\left(\frac{1}{1-\lambda}\right),\end{aligned}$$

since δ is arbitrarily small and $r(\delta) \to 0$ as $\delta \to 0$. $\square$

APPENDIX B: KARAMATA'S TAUBERIAN THEOREM FOR POWER SERIES

A function $h : \mathbb{R} \to \mathbb{R}$ is *slowly varying* at ∞ if for any $\lambda > 0$

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = 1.$$

The case we have in mind is $h(x) = \log x$.

For two functions (resp., sequences), write $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x) = 1$.

THEOREM B.1 (Corollary 1.7.3 in [4]). *If $a_n \ge 0$ and the power series $I(\lambda) = \sum_{n\ge 0} a_n\lambda^n$ converges for $\lambda \in [0, 1)$, then for $c, \rho \ge 0$ and a slowly varying function h,*

$$\sum_{k=0}^{n} a_k \sim cn^\rho h(n)/\Gamma(1+\rho)$$

as $n \to \infty$, if and only if

$$I(\lambda) \sim h\left(\frac{1}{1-\lambda}\right)\frac{c}{(1-\lambda)^\rho},$$

as $\lambda \to 1$. If in addition $c\rho > 0$ and a_n is eventually monotone, both are equivalent to

$$a_n \sim cn^{\rho-1}h(n)/\Gamma(\rho), \qquad \text{as } n \to \infty.$$

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