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FLUCTUATIONS OF THE FRONT IN A ONE-DIMENSIONAL MODEL FOR THE SPREAD OF AN INFECTION[1]

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We study the following microscopic model of infection or epidemic reaction: red and blue particles perform independent nearest-neighbor continuous-time symmetric random walks on the integer lattice $\mathbb{Z}$ with jump rates D_R for red particles and D_B for blue particles, the interaction rule being that blue particles turn red upon contact with a red particle. The initial condition consists of i.i.d. Poisson particle numbers at each site, with particles at the left of the origin being red, while particles at the right of the origin are blue. We are interested in the dynamics of the front, defined as the rightmost position of a red particle. For the case $D_R = D_B$, Kesten and Sidoravicius established that the front moves ballistically, and more precisely that it satisfies a law of large numbers. Their proof is based on a multi-scale renormalization technique, combined with approximate sub-additivity arguments. In this paper, we build a renewal structure for the front propagation process, and as a corollary we obtain a central limit theorem for the front when $D_R = D_B$. Moreover, this result can be extended to the case where $D_R > D_B$, up to modifying the dynamics so that blue particles turn red upon contact with a site that has previously been occupied by a red particle. Our approach extends the renewal structure approach developed by Comets, Quastel and Ramírez for the so-called frog model, which corresponds to the $D_B = 0$ case.

1. Introduction. Consider the following microscopic model of infection or epidemic reaction on the integer lattice $\mathbb{Z}$. There are two types of particles: red and blue, both moving as independent, continuous-time, symmetric, nearest-neighbor random walks, with total jump rate D_R for red particles and D_B for blue particles. The interaction rule between particles is the following: when a red particle jumps to a site where there are blue particles, all of them immediately become red particles; when a blue particle jumps to a site where there are red particles, it immediately becomes a red particle. The initial condition is the following: at time zero, each site in $\mathbb{Z}$ bears a random number of particles whose distribution is Poisson with parameter $\rho > 0$, the numbers of particles at distinct sites being independent. Moreover, particles at the left of the origin (including the origin) are red,

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while particles at the right of the origin are blue. We are interested in the asymptotic behavior of the rightmost site r_t occupied by a red particle at time t, which we call the *front*.

Such particle systems have received attention in the physical literature, as microscopic stochastic models which, in the limit of a large average number of particles per lattice site, yield reaction–diffusion equations describing the propagation of a front, the prototypical example being the Fisher–Kolmogorov–Petrovsky–Piscounov equation; see, for example, [13–16]. We refer to [18] for an extensive review of the subject from a theoretical physics perspective.

On the other hand, according to [10], this model was suggested within the mathematics community by Frank Spitzer around 1980, but rigorous mathematical results describing the behavior of the front have been difficult to obtain.

Indeed, the only two special cases for which ballisticity of the front and a law of large numbers have been mathematically established are the following:

- $D_R > D_B = 0$; this is the so-called *frog model* [1, 19]. Beyond ballisticity and the law of large numbers, a central limit theorem and a large deviations principle have also been obtained [3, 5].
- $D_R = D_B > 0$; this model will be referred to as the *single-rate KS infection model*, after Kesten and Sidoravicius [10, 12], where "single rate" emphasizes the fact that red and blue particles share the same jump rate.

Specifically, in [10], it is shown that the front moves ballistically, in the sense that there exist two constants C_1, C_2 such that a.s.

$$0 < C_2 \le \liminf_{t\to+\infty} t^{-1} r_t \le \limsup_{t\to+\infty} t^{-1} r_t \le C_1 < +\infty. \tag{1}$$

This result is strengthened in [12] where it is shown that there exists $0 < v_* < +\infty$ such that a.s.,

$$\lim_{t\to+\infty} t^{-1} r_t = v_*. \tag{2}$$

Analogous results hold on $\mathbb{Z}^d$ for arbitrary $d \ge 1$, with (2) being the one-dimensional version of a general shape theorem proved in [12]. Here, we are interested in the fluctuations of r_t, and the first main result of the present paper is the following.

THEOREM 1. *For the single-rate KS infection model, there exists a (nonrandom) number $0 < \sigma_*^2 < +\infty$ such that, as ε goes to zero,*

$$B_t^\varepsilon := \varepsilon^{1/2}\big(r_{\varepsilon^{-1}t} - \varepsilon^{-1} v_* t\big), \qquad t \ge 0,$$

converges in law on the Skorohod space to a Brownian motion with variance σ_*^2.

Note that the method used to derive the above results also yields the convergence to an invariant distribution of the environment of particles as seen from the front.

For the general case in which $D_R \neq D_B$, an upper bound on the speed similar to the one in (1) is proved in [10], but no corresponding lower bound is available, so that even ballisticity as described in (1) is an open question. We now introduce a slight variation upon this model for which, when $D_R > D_B > 0$, it is indeed possible to derive results similar to those that hold for the single-rate model. This variation consists in making the infectious power of red particles *remanent*, in the sense that a blue particle turns red not only when it is in contact with a red particle, but as soon as it is located at a site that has previously been occupied by a red particle. We call this model the *remanent KS infection model*. In this context, it is natural to define the position of the front at time t as the rightmost position ever occupied by a red particle up to time t. We can then prove the two following results.

THEOREM 2. *For the remanent KS infection model with* $0 < D_B \leq D_R$, *there exists* $0 < v_\star < +\infty$ *such that a.s.*,

$$\lim_{t \to +\infty} t^{-1} r_t = v_\star.$$

THEOREM 3. *For the remanent KS infection model with* $0 < D_B \leq D_R$, *there exists a* (*nonrandom*) *number* $0 < \sigma_\star^2 < +\infty$ *such that*, *as* ε *goes to zero*,

$$B_t^\varepsilon := \varepsilon^{1/2}\big(r_{\varepsilon^{-1}t} - \varepsilon^{-1} v_\star t\big), \qquad t \geq 0,$$

converges in law on the Skorohod space to a Brownian motion with variance $\sigma_\star^2$.

Our strategy to prove Theorems 1, 2, 3 is based on the definition of a renewal structure, extending the approach developed by Comets, Quastel and Ramírez in [5] to study the frog model ($D_R > D_B = 0$).

Here, a renewal structure is a sequence of a.s. finite random times $0 =: \kappa_0 < \kappa_1 < \kappa_2 < \cdots$ such that:

- the r.v.s $(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})_{n \geq 0}$ are independent,
- the r.v.s $(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})_{n \geq 1}$ are identically distributed,
- $\mathbb{E}(\kappa_2 - \kappa_1)^2 < +\infty$ and $\mathbb{E}(r_{\kappa_2} - r_{\kappa_1})^2 < +\infty$.

Given such a renewal structure, the law of large numbers and the central limit theorem for r_t can be derived in a standard way, applying to r_{κ_n} the corresponding results for sums of i.i.d. square-integrable random variables, then approximating r_t by $r_{\kappa_{n_t}}$, where $n_t := \sup\{n \geq 1; \kappa_n \leq t\}$.

The core of the work lies in finding an appropriate definition for the renewal structure, and then proving the required tail-estimates. In the present context, the idea is to find random times κ_n that satisfy the following two conditions: (i) the history of the front after time κ_n does not depend (up to translation) on the future trajectories of particles located below r_{κ_n} at time κ_n, and (ii) the distribution of particles located above r_{κ_n} at time κ_n is fixed (up to translation).

In [5], condition (i) is achieved by considering times after which the front remains forever above a (space–time) straight line, while particles lying below the front at these times remain forever below the straight-line. For the frog model, condition (ii) is then automatically satisfied, since the distribution of blue particles above the front[4] is fixed, due to the fact that blue particles do not move. This is no longer true in the more complex case when both red and blue particles move, since the distribution of particles located above r_t at a time where the front jumps then depends upon the whole past of the process. As a consequence, new ideas are required to define a proper renewal structure in this context. We achieve (ii) by extending the trajectories of our random walks infinitely far into the past, looking at times before which the front always lies below a straight line, while particles lying above the front at these times have remained above the straight line for their whole past history. A key role in the corresponding argument is played by the invariance properties of the Poisson distribution of particles, which allows the construction of the time-reversal of the random walk trajectories and the analysis of the distribution of the blue particles in terms of this time-reversal.

Once the renewal structure is defined, it is necessary to obtain tail estimates for the random variables κ_1, r_{κ_1}, and $\kappa_{n+1} - \kappa_n$ and $r_{\kappa_{n+1}} - r_{\kappa_n}$ for $n \geq 1$. To this end, we adapt some of the techniques used in [5], especially the use of martingale methods to control the behavior of systems of independent random walks. It turns out that some of the more involved steps in the proof given in [5], that were needed to control the accumulation of particles below the front, are replaced in the present paper by a softer and (hopefully) more transparent argument.

Let us point out one important technical difference between the frog model and the infection models considered here: ballistic lower bounds for the position of the front are easy to obtain in the case of the frog model, while they seem to be very difficult[5] for infection models where both red and blue particles move. In fact, the lower bound part[6] of (1) is the main result of [10], and is obtained through a quite demanding multi-scale renormalization argument. We do not provide an independent proof of ballisticity here, and instead have to rely on the estimate proved in [10]. Still, at least in the one-dimensional case, our renewal structure approach provides an alternative way of deriving the law of large numbers (2) (already proved in [12]) from the coarser ballisticity estimate obtained in [10]. The original proof in [12] is based on an approximate sub-additivity argument, and relies too on the ballisticity estimates proved in [10]. Note also that the only

[4]Strictly speaking, this is true only when the front hits a position above its past record.
[5]By contrast, ballistic upper bounds are relatively easy to obtain.
[6]More precisely, a quantitative version of it.

missing ingredient to make our proofs of Theorems 2 and 3 work when $D_R > D_B > 0$ in the nonremanent case, is a lower bound on the speed comparable to the one established in [10] for the single-rate model.[7]

A natural question concerns our specific choice for the Poisson initial distribution of particles. One can take advantage of the fact that the random variables $(\kappa_{i+1} - \kappa_i, r_{\kappa_{i+1}} - r_{\kappa_i})_{i\geq 1}$ are independent from the initial configuration of particles at the left of the origin to show that our results are still valid if one starts with a Poisson distribution of particles conditioned upon a nonzero probability event concerning only the initial configuration of particles at the left of the origin. For instance, we can prescribe the initial numbers of particles below zero at any given finite number of sites. Still, it seems necessary to use the Poisson distribution of particles as a reference probability measure, so it is unclear how we could extend our results to, say, an arbitrary initial condition with suitable decay of the number of particles at infinity.

One should note that, strictly speaking, the initial distribution of particles we have described is not exactly the same as the one considered by Kesten and Sidoravicius. Indeed, in [10, 12], the initial condition is obtained by adding a deterministic finite and nonzero number of red particles placed arbitrarily, to a configuration formed by an i.i.d. Poisson number of particles at each site of $\mathbb{Z}$. For the single-rate KS model on $\mathbb{Z}$, it is irrelevant for the value of r_t whether particles initially at the left of r_0 are red or blue, so the only difference lies in the added red particles. Using the previous remark on the possibility to condition the initial configuration by the numbers of particles at a finite set of sites, we see that our results in fact include the kind of initial configurations considered in [10, 12].

One should also note that the results of [10, 12] are stated in terms of $\sup_{s\in[0,t]} r_s$ rather than r_t (when specialized to the one-dimensional case). It clearly makes no difference for results on the scale of the law of large numbers, since particles move sub-ballistically. Although such an argument cannot be used for the central limit theorem, it turns out that, with our definition of the renewal structure, $r_{\kappa_n} = \sup_{s\in[0,\kappa_n]} r_s$, so that the CLT holds for either r_t or $\sup_{s\in[0,t]} r_s$.

Finally, note that our results do not say anything on the case $D_R < D_B$. The only available results for a model of this kind are those of [11], where a version of the infection model with $0 = D_R < D_B$ is considered, and it is shown that, for sufficiently small ρ, the asymptotic velocity of the front is zero, while it is conjectured that a positive asymptotic velocity is obtained for sufficiently large ρ.

Let us also mention that other regeneration approaches have been considered within the context of random walks in dynamic random environment (see, e.g., [2, 7, 9]).

The rest of the paper is organized as follows. In Section 2, we give a formal construction of the random process associated with the single-rate KS infection

[7] Specifically, we would need a result analogous to Proposition 9 in Section 4.

model, together with statements of its main structural properties. Section 3 provides the definition of the renewal structure, and its key structural properties are stated and proved there, save for the estimates on the tail, which form the content of Section 4. Finally, Section 5 briefly explains how to extend the previous results to the case of the remanent KS infection model with $D_R > D_B$. For the sake of readability, some technical points are not discussed in detail, and we refer to the arXiv version [4] of the present work for a more thorough treatment of these points.

2. Formal construction of the single-rate process. In this section, we describe the construction of the single-rate process, in two steps. First, we construct, on appropriate spaces, the dynamics of systems of independent random walks, without any reference to a possible interaction between them. We then state important structural properties of the dynamics, such as the strong Markov property, or the invariance with respect to space–time shifts of the Poisson distribution on the space of trajectories. Finally, we define the infection process as a function of these random walks, together with the corresponding notion of red and blue particles.

2.1. *Reference spaces.* It is convenient to assign a label to each particle in the system, so that a particle can be uniquely identified by its label. More precisely, we assume that each particle is labelled by an element of the interval [0, 1], in such a way that no two particles share the same label. As a consequence, a configuration of particles at a given time can be represented by a family

$$w = \big(w(x), x \in \mathbb{Z}\big),$$

where, for all x, $w(x)$ is a (possibly empty) subset of [0, 1], representing the labels of the particles located at site x.

Given $\theta > 0$, introduce the space $\mathbb{S}_\theta$ of all configurations of labelled particles $w = (w(x), x \in \mathbb{Z})$ satisfying $w(x) \cap w(y) = \varnothing$ whenever $x \neq y$, and $\sum_{x\in\mathbb{Z}} |w(x)| e^{-\theta|x|} < +\infty$. Throughout this paper, $\mathbb{S}_\theta$ is our reference space for particle configurations, where θ is assumed to be a given positive real number. The specific value of θ used in the proofs is made precise later [see (15)], and the construction we now develop is valid for any $\theta > 0$.

To define a distance on $\mathbb{S}_\theta$, we first define a distance on the set of all finite subsets of elements of [0, 1]. Consider two such subsets $a = \{a_1 > \cdots > a_p\}$, and $b = \{b_1 > \cdots > b_q\}$. If $p < q$, define $a_i := 0$ for $p + 1 \le i \le q$; if $p > q$, define $b_i := 0$ for $q + 1 \le i \le p$. Then define the distance between a and b by

$$d(a, b) := |q - p| + \sum_{i=1}^{\max(p,q)} |b_i - a_i|.$$

We now define a distance d_θ on $\mathbb{S}_\theta$ by

$$d_\theta(w_1, w_2) := \sum_{x\in\mathbb{Z}} d\big(w_1(x), w_2(x)\big) e^{-\theta|x|}.$$

Let us turn to the description of particle trajectories. A priori, the model consists only of particles moving after time zero. However, the definition of the regeneration structure involves the extension of their trajectories to negative time indices, so we start from the beginning with a space allowing the description of trajectories with a time-index in $\mathbb{R}$. A pair (W, u), where $W = (W_t)_{t\in\mathbb{R}}$ is a càdlàg function from $\mathbb{R}$ to $\mathbb{Z}$ with nearest-neighbor jumps (i.e., ± 1), and $u \in [0, 1]$, is called a (labelled) particle path, with u being the label of the particle whose path is described by W. In the sequel, we often call such a pair (W, u) a particle, instead of a particle path.

Given a finite or countable set ψ of particle paths with pairwise distinct labels, and a time coordinate $t \in \mathbb{R}$, we define the configuration of labelled particles $X_t(\psi) = (X_t(\psi)(x))_{x\in\mathbb{Z}}$ by

$$X_t(\psi)(x) := \{u; W_t = x, (W, u) \in \psi\}.$$

In words, $X_t(\psi)(x)$ is the set of labels of particle paths that are located at x at time t. Our reference space for the trajectories of the particles in the system is the set Ω formed by all the sets ψ of particle trajectories such that $t \mapsto X_t(\psi)$ is a càdlàg function from $\mathbb{R}$ to $(\mathbb{S}_\theta, d_\theta)$, and such that no two particle paths jump at the same time. We endow Ω with the cylindrical σ-algebra $\mathcal{F}$ generated by all the maps $\psi \mapsto X_t(\psi)$ from Ω to $\mathbb{S}_\theta$ equipped with the Borel sets associated with the metric d_θ. For all $t \in \mathbb{R}$, we define $\mathcal{F}_t := \sigma(X_s, s \in]-\infty, t])$. For all $x \in \mathbb{Z}$ and $t \in \mathbb{R}$, the space–time shift $\pi_{x,t}$ on Ω is defined by the fact that $\pi_{x,t}(\psi)$ is the set of particle paths obtained from ψ by replacing each path $((W_s)_{s\in\mathbb{R}}, u)$ by $((W_{s-t} - x)_{s\in\mathbb{R}}, u)$. We also consider the space $\mathcal{D}_+$ as the space of càdlàg maps from $[0, +\infty[$ to $\mathbb{S}_\theta$. Both spaces are equipped with their respective cylindrical σ-algebras. Finally, we denote by Ψ the canonical map on Ω, that is, $\Psi(\psi) := \psi$, so that, whenever we consider a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, the notation Ψ stands for the random set of particle trajectories in the system.

2.2. *Reference probability* $\mathbb{P}_w$. To each $w \in \mathbb{S}_\theta$, we associate a probability measure $\mathbb{P}_w$ on $(\Omega, \mathcal{F})$ describing the evolution of a system of independent particles starting in configuration w at time 0.

Fix $w \in \mathbb{S}_\theta$, and, for all x, write $w(x)$ as an ordered tuple

$$w(x) = \{u(x, 1) > \cdots > u(x, |w(x)|)\},$$

and define

$$A := \{(x, i); x \in \mathbb{Z}, 1 \le i \le |w(x)|\}.$$

Consider an i.i.d. family of random walks $Z = (Z(x, i), (x, i) \in A)$ where, for every $(x, i) \in A$, $Z(x, i) = (Z_t(x, i))_{t\in\mathbb{R}}$ is a two-sided continuous-time random walk on $\mathbb{Z}$, starting at x at time zero, and evolving in both positive and negative time directions, with symmetric nearest-neighbor steps, and constant jump rate equal to 2. We view $Z(x,i)$ as a random variable taking values in the space of càdlàg paths from $\mathbb{R}$ to $\mathbb{Z}$ equipped with the cylindrical σ-algebra. It can be checked that, up to a modification on a set of probability zero, the family of random paths $\{(Z(x,i), u(x,i)); (x,i) \in A\}$ is a random variable taking values in $(\Omega, \mathcal{F})$, so that we can define

$$\mathbb{P}_w := \text{distribution of } \{(Z(x,i), u(x,i)); (x,i) \in A\} \qquad \text{on } (\Omega, \mathcal{F}).$$

The expectation with respect to $\mathbb{P}_w$ is denoted by $\mathbb{E}_w$.

We now quote two key properties of the family $(\mathbb{P}_w, w \in \mathbb{S}_\theta)$, namely the strong Markov property and the invariance of the Poisson initial distribution $\mathbb{P}_\nu$ with respect to space–time shifts.

PROPOSITION 1. *The strong Markov property holds for our process*: *for every $w \in \mathbb{S}_\theta$, every nonnegative $(\mathcal{F}_t)_{t\geq 0}$-stopping time T, and bounded measurable function F on $\mathcal{D}_+$, one has that, on $\{T < +\infty\}$,*

$$\mathbb{E}_w\big(F\big((X_{T+t})_{t\geq 0}\big)|\mathcal{F}_T\big) = \mathbb{E}_{X_T}\big(F\big((X_t)_{t\geq 0}\big)\big) \qquad \mathbb{P}_w\text{-a.s.} \tag{3}$$

We now turn to the invariance properties of the Poisson distribution of particles with respect to the dynamics. Consider an i.i.d. family $N = (N_x)_{x\in\mathbb{Z}}$ of Poisson processes on [0, 1], with intensity ρ. With probability one, $(N_x)_{x\in\mathbb{Z}} \in \mathbb{S}_\theta$, and we call ν the probability distribution on $\mathbb{S}_\theta$ induced by N. The probability measure $\mathbb{P}_\nu$ defined by $\mathbb{P}_\nu(\cdot) := \int_{\mathbb{S}_\theta} \mathbb{P}_w(\cdot)\, d\nu(w)$ is the reference measure we use to describe the dynamics starting from a Poisson initial distribution of particles.

PROPOSITION 2. *The probability distribution $\mathbb{P}_\nu$ on Ω is invariant with respect to the space–time shifts $\pi_{x,t}$.*

2.3. *Infection dynamics.* We now formally define the infection dynamics, through random variables defined on $(\Omega, \mathcal{F})$. Given a system of independent random walks specified by an element of Ω, we define the corresponding front dynamics, using the fact that particle initially at the left of the origin are red, while particles initially at the right of the origin are blue.

We start by defining the sequence $(T_k)_{k\geq 0}$, which characterizes the sequence of times at which the front moves (upward or downward). First, let $T_0 := 0$, $\mathfrak{r}_0 := \sup\{x \leq 0; \exists (W,u) \in \Psi, W_0 = x\}$ (with the convention $\inf \varnothing = -\infty$) and define inductively the families of random variables $(T_\ell)_{\ell\geq 0}$ and $(\mathfrak{r}_\ell)_{\ell\geq 0}$ as follows. Consider $t > T_\ell$. We say that t is upward if there exists $(W,u) \in \Psi$ such that $W_{t-} = \mathfrak{r}_\ell$ and $W_t = \mathfrak{r}_\ell + 1$. We say that t is downward if there exists $(W,u) \in \Psi$ such that $W_{t-} = \mathfrak{r}_\ell$, $W_t = \mathfrak{r}_\ell - 1$, and $X_{t-}(\mathfrak{r}_\ell) = \{u\}$. Then let

$$T_{\ell+1} := \inf\{t > T_\ell;\ t \text{ is upward or downward}\},$$

with the convention that $\inf \varnothing = +\infty$. By the fact that paths are càdlàg in $\mathbb{S}_\theta$, one must have that $T_{\ell+1} > T_\ell$ when $T_\ell < +\infty$. Provided that $T_{\ell+1} < +\infty$, one must also have that $T_{\ell+1}$ is indeed a upward or downward time. In the upward case, we let $\mathfrak{r}_{\ell+1} := \mathfrak{r}_\ell + 1$. In the downward case, we let $\mathfrak{r}_{\ell+1} := \mathfrak{r}_\ell - 1$. Now r_t is defined on each interval $[T_\ell, T_{\ell+1}[$ by $r_t := \mathfrak{r}_\ell$. From the results in [10], one has that, for all $k \geq 1$, $T_k < +\infty$, and $\sup_\ell T_\ell = +\infty$, almost surely with respect to $\mathbb{P}_\nu$. For the sake of definiteness, we set $\mathfrak{r}_\ell := +\infty$ if $T_\ell = +\infty$, and $r_t := +\infty$ for $t \geq \sup_\ell T_\ell$.

In the sequel, we say that a time $t > 0$ is a jump time for the front if it is one of the times $T_1, T_2, \dots$ at which the position of the front either increases or decreases by one unit.

For all $0 < t < +\infty$, we denote by B_t the subfamily of particle paths corresponding to particles that are blue at time t, that is,

$$B_t := \{(W, u) \in \Psi; \forall s \in [0, t[, W_s > r_s\}.$$

Similarly, the subfamily of paths associated with particles that are red at time t is

$$R_t := \{(W, u) \in \Psi; \exists s \in [0, t[, W_s \leq r_s\}.$$

We extend the definition by setting $B_0 := \{(W, u) \in \Psi; W_0 \geq 0\}$ and $R_0 := \{(W, u) \in \Psi; W_0 < 0\}$. One checks that, with these definitions, for all $0 < t < +\infty$, r_t corresponds $\mathbb{P}_\nu$-a.s. to the position of the rightmost red particle at time t.

In the sequel, we shall use the following σ-algebras. First, given $t \geq 0$, $\mathcal{F}_t^R$ is defined by[8]

$$\mathcal{F}_t^R := \sigma\big((W_s, u); s \leq t, (W, u) \in R_t\big).$$

Informally, $\mathcal{F}_t^R$ contains the information relative to the trajectories of particles that are red at time t, up to time t. If T is a nonnegative random variable on $(\Omega, \mathcal{F})$, we also define[9]

$$\mathcal{F}_T^R := \sigma(T, r_T) \vee \sigma\big((W_s, u); s \leq T, (W, u) \in R_T\big).$$

Similarly, we let

$$\mathcal{G}_t^R := \sigma\big((W_s, u); s \in \mathbb{R}, (W, u) \in R_t\big).$$

Informally, $\mathcal{G}_t^R$ contains the information relative to the full trajectories of the particles that are red at time t. When T is a nonnegative random variable, we also define

$$\mathcal{G}_T^R := \sigma(T, r_T) \vee \sigma\big((W_s, u); s \in \mathbb{R}, (W, u) \in R_T\big).$$

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[8]Formally, $\mathcal{F}_t^R$ is generated by all the random variables of the form

$$\#\big(R_t \cap \{(W, u); W_s = k, u \in [a, b]\}\big),$$

where $k \in \mathbb{Z}$, $0 \leq a < b \leq 1$, and $s \leq t$.

[9]Formally, $\mathcal{F}_T^R$ is generated by all the random variables of the form

$$\mathbf{1}(s \leq T) \times \#\big(R_T \cap \{(W, u); W_s = k, u \in [a, b]\}\big),$$

where $k \in \mathbb{Z}$, $0 \leq a < b \leq 1$, and $s \in \mathbb{R}$.

3. Regeneration structure.

3.1. *Definition of* $(\kappa_n)_{n\geq 0}$. We now define the regeneration structure that is used to prove the central limit theorem. Remember that it is based on straight lines drawn on the space–time plane. In the sequel, α is a strictly positive real number corresponding to the slope of these straight lines.

Consider an upward jump time $t > 0$. We say that t is a *backward sub-α time* if $r_t > \alpha t$ and if, for all $0 \leq s < t$, one has $r_s < r_t - \alpha(t-s)$. We say that t is a *backward super-α time* if, for any (W, u) in B_t, and for all $s < t$, one has $W_s \geq r_t - \alpha(t-s)$. If t is both a backward sub-α and super-α time, we say that t is a *backward α time*. We say that t is a *forward sub-α time* if, for all $(W, u) \in R_t$ such that $W_t \leq r_t - 1$, one has that $W_s \leq r_t - 1 + \alpha(s-t)$ for all $s > t$, and if the particle (W, u) making the front jump at time t remains at r_t during the time-interval $[t, t+\alpha^{-1}]$, and then satisfies the inequality $W_s \leq r_t - 1 + \alpha(s-t)$ for all $s \geq t + \alpha^{-1}$. We say that t is a *forward super-α time* if, for all $s > t$, one has $r_s \geq r_t + \lfloor \alpha(s-t) \rfloor$, and if, moreover, there exists $(W, u) \in B_t$ such that $W_s = r_t$ for all $s \in [t, t+\alpha^{-1}]$. If t is both a forward sub-α and super-α time, we say that t is a *forward α time*. Finally, if t is both a forward and backward α time, we say that t is an *α-separation time*. We extend the definition of a backward super-α time and of a forward super-α time by allowing $t = 0$ in the above definitions. These definitions are illustrated in Figure 1.

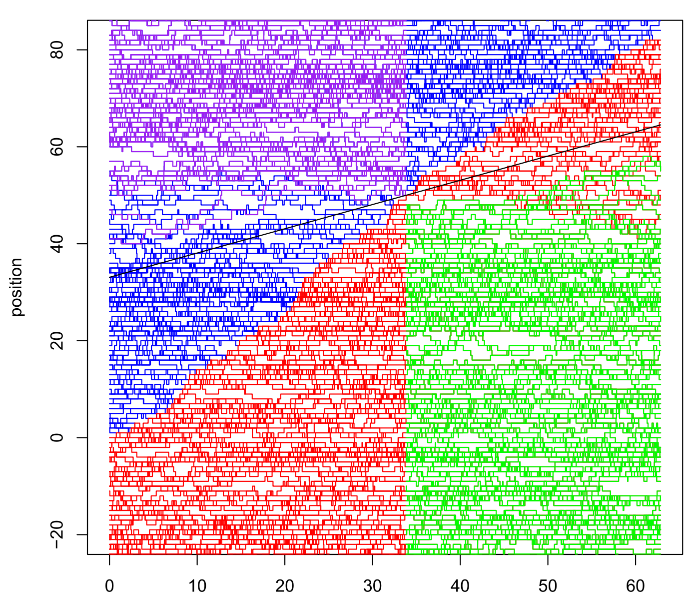

FIG. 1. *A realization of the KS infection model with an α separation time t. Posterior (resp., prior) to the forward α time t, green (resp., purple) is used instead of red (resp., blue) to draw the trajectories of particles that lie below (resp., above) r_t at time t.*

One then defines the renewal structure by letting $\kappa_0 := 0$, and inductively:

$$\kappa_{i+1} := \inf\{T_j > \kappa_i; T_j \text{ is an } \alpha\text{-separation time}\}.$$

Before discussing why the above definition indeed yields a renewal structure for the model, let us briefly explain why it is at least conceivable that such a sequence of α-separation times exists. First, note that the existence of backward sub-α times is a direct consequence of the front moving ballistically, provided that α is chosen in such a way that $\alpha < \liminf t^{-1} r_t$. Similarly, ballisticity of the front with speed strictly greater than α also yields the existence of forward super-α times. On the other hand, for a system of independent random walks whose distribution at time 0 is characterized by i.i.d. Poisson numbers of particles at every site $x \leq 0$, the maximum position occupied at time $t \geq 0$ by a random walk in the system, grows only sub-linearly as a function of t. This provides at least a heuristic justification of why forward sub-α times exist, and a symmetric argument can be made for backward super-α times, by invoking time-reversal and the reversibility of the Poisson distribution of particles with respect to the dynamics of independent random walks. With a mild dose of faith, the simultaneous occurrence of these four properties at a single time t should look plausible. Mathematical arguments giving rigorous content to this heuristic line of reasoning are found in Section 4.

3.2. *Key properties of* $(\kappa_n)_{n\geq 1}$. The key properties of the sequence $(\kappa_n)_{n\geq 1}$ are stated in the following theorem.

THEOREM 4. *With respect to* $\mathbb{P}_\nu$, *the r.v.s* $(\kappa_n)_{n\geq 0}$ *are a.s. finite and*:

- *the r.v.s* $(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})_{n\geq 0}$ *are independent*,
- *the r.v.s* $(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})_{n\geq 1}$ *are identically distributed*,
- $\mathbb{E}(\kappa_2 - \kappa_1)^2 < +\infty$ *and* $\mathbb{E}(r_{\kappa_2} - r_{\kappa_1})^2 < +\infty$.

Given Theorem 4, it is more or less standard to derive Theorem 1, approximating r_t by $r_{\kappa_{n_t}}$, where $n_t := \sup\{n \geq 0; \kappa_n \leq t\}$. Note that, due to the definition of κ, one has $r_{\kappa_{n_t}} \leq r_t \leq r_{\kappa_{n_t+1}}$, which eases the corresponding approximation argument. We do not give the details here (see, e.g., [5]).

The proof of Theorem 4 relies on two distinct results, stated below as Propositions 3 and 4. Proposition 3 deals with structural properties of $(\kappa_n)_{n\geq 0}$, while Proposition 4 provides tail estimates.

PROPOSITION 3. *For all* $n \geq 1$, *one has the following properties*:

(a) *the r.v.s* $\kappa_1, \dots, \kappa_n$ *and* $r_{\kappa_1}, \dots, r_{\kappa_n}$ *are measurable with respect to* $\mathcal{G}^R_{\kappa_n}$.

(b) *on* $\{\kappa_n < +\infty\}$, *the conditional distribution of* $(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})$ *with respect to* $\mathcal{G}^R_{\kappa_n}$ *is the distribution*[10] *of* (κ_1, r_{κ_1}) (B_0) *with respect to* $\mathbb{P}_\nu$, *conditioned on* $t = 0$ *being a backward and forward super-*α *time for* B_0.

[10]This is a slight abuse of terminology, since, strictly speaking, B_0 is only $\mathbb{P}_\nu$-a.s. equal to a random variable from $(\Omega, \mathcal{F})$ to itself.

Note that, in the formulation of Proposition 3 above, the fact that $t = 0$ is a backward and forward super-α time for B_0 means that the assumptions characterizing a backward and forward super-α time are satisfied when the set of particle trajectories taken into account is restricted to B_0, that is, with Ψ replaced by B_0. In particular, the relevant front dynamics here is $(r_t(B_0))_{t\geq 0}$, not $(r_t)_{t\geq 0}$. Also, implicit in this formulation is the fact that the conditioning event has a nonzero probability, which is proved in Corollary 1.

PROPOSITION 4. *For small enough α (depending on ρ), there exists $\theta > 0$ such that $\mathbb{E}_\nu(\kappa_1^2) < +\infty$ and $\mathbb{E}_\nu(r_{\kappa_1}^2) < +\infty$.*

Deducing Theorem 4 from Propositions 3 and 4 is straightforward. The rest of this section is devoted to the proof of Proposition 3, while Proposition 4, is proved in Section 4.

3.3. *Structural properties*: *Proof of Proposition* 3. For the sake of readability, the proof of Proposition 3 is divided into a sequence of four steps.

Step 1 establishes the measurability condition (a) in Proposition 3. This is a classical step when dealing with renewal structures, although a nontrivial one since the κ_ns look infinitely far into the future of the trajectories. It merely reflects the consistency across the κ_ns of the various comparison conditions involving parallel space–time lines of slope α. Step 2 is similar, establishing that, broadly speaking, going from κ_n to κ_{n+1} is equivalent to going from 0 to κ_1, keeping only the trajectories of particles that are blue at time κ_n and applying a space–time translation sending (r_{κ_n}, κ_n) to $(0, 0)$. Step 3 explicitly characterizes the distribution of particle trajectories that are blue at time T_k, conditional on the trajectories of particles that are red at time T_k. This is a key result, relying on the invariance properties of the distribution $\mathbb{P}_\nu$. Step 4 builds on this result to characterize the distribution of particle trajectories that are blue at time κ_n, conditional on the trajectories of particles that are red at time κ_n.

3.3.1. *Step* 1: *Measurability with respect to $\mathcal{G}^R_{\kappa_n}$.* We now prove statement (a) in Proposition 3, that is, the fact that, for all $n \geq 1$, the r.v.s $\kappa_1, \dots, \kappa_n$ and $r_{\kappa_1}, \dots, r_{\kappa_n}$ are measurable with respect to $\mathcal{G}^R_{\kappa_n}$.

First, note that the measurability of κ_n and r_{κ_n} with respect to $\mathcal{G}^R_{\kappa_n}$ is a direct consequence of the definition of $\mathcal{G}^R_{\kappa_n}$. Also, with our conventions, the result is obvious on $\{\kappa_n = +\infty\}$, so we may assume that $\{\kappa_n < +\infty\}$.

From the definition of the infection dynamics, particle paths (W, u) outside R_{κ_n} have no influence on the front jumps between time 0 and κ_n, so that the history of the front up to time κ_n is exactly the same as the one that would be obtained if there were no other particle paths in the system besides those in R_{κ_n}. As a consequence, the jump times $T_1 < \cdots < T_\ell = \kappa_n$ that lie between time 0 and κ_n, are measurable with respect to $\mathcal{G}^R_{\kappa_n}$. What remains to be proved is that, for every jump time T_i such that $1 \le i \le \ell - 1$, it is possible to tell whether T_i is a backward/forward sub/super-α time, using only the information contained in $\mathcal{G}^R_{\kappa_n}$, which is not a priori obvious since the definition of each $\kappa_1, \dots, \kappa_{n-1}$, imposes some conditions on every particle trajectory in the system, so we have to check that, as far as particles in B_{κ_n} are concerned, these conditions are subsumed by those already imposed by the definition of κ_n:

- Whether T_i is a backward sub-α time involves only the history of the front up to time κ_n, so this condition is $\mathcal{G}^R_{\kappa_n}$-measurable.
- Whether T_i is a backward super-α time involves conditions on trajectories in $B_{T_i} \cap R_{\kappa_n}$, which are $\mathcal{G}^R_{\kappa_n}$-measurable, and conditions on trajectories in $B_{T_i} \cap B_{\kappa_n}$ which are automatically satisfied thanks to the fact that κ_n itself is a backward super-α time and the fact that $r_{T_i} \le r_{\kappa_n} - \alpha(\kappa_n - T_i)$ since κ_n is also a backward sub-α time.
- Whether T_i is a forward sub-α time involves the trajectories of paths in $R_{T_i} \subset R_{\kappa_n}$ only, so this condition is $\mathcal{G}^R_{\kappa_n}$-measurable.
- Whether T_i is a forward super-α time involves a condition of the front up to time κ_n, so this condition is $\mathcal{G}^R_{\kappa_n}$-measurable, which is $\mathcal{G}^R_{\kappa_n}$-measurable, plus a condition on the front posterior to κ_n which is automatically satisfied thanks to the fact that κ_n itself is a forward super-α time.

3.3.2. *Step* 2: *From* κ_n *to* κ_{n+1}. We now state the following property: for all $n \ge 1$ $\{\kappa_n < +\infty\}$, the following identity holds:

$$(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n}) = (\kappa_1, r_{\kappa_1})\big(\pi_{r_{\kappa_n}, \kappa_n}(B_{\kappa_n})\big). \tag{4}$$

The meaning of the above identity is that $(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n})$ is identical to (κ_1, r_{κ_1}) applied to $\pi_{r_{\kappa_n}, \kappa_n}(B_{\kappa_n})$, that is, the system consisting only of trajectories of particles that are blue at time κ_n, to which a space–time translation has been applied so that (r_{κ_n}, κ_n) is sent to $(0, 0)$. The fact that the only trajectories playing a role are those in B_{κ_n} is a consequence of the forward α time property of κ_n: particles that are red at time κ_n do not have any influence on the evolution of the front after time κ_n, since their trajectories are confined below a space–time line of slope α, while the front is constrained to lie above this same line. Checking the backward/forward sub/super-α time conditions in a way similar to the proof of Step 1 above, precisely leads to identity (4) (the details can be found in [4]).

3.3.3. *Step* 3: *Distribution of* $\pi_{r_{T_k}, T_k}(B_{T_k})$ *given* $\mathcal{G}^R_{T_k}$ *under* $\mathbb{P}_\nu$. We now establish the fact that, under $\mathbb{P}_\nu$, the conditional distribution of $\pi_{r_{T_k}, T_k}(B_{T_k})$ with respect to $\mathcal{G}^R_{T_k}$, is the same as that of B_0, conditioned by the fact that every trajectory in B_0 avoids the space–time translated trajectory of the front, that is, $(r_{T_k + t} - r_{T_k})_{-T_k \le t < 0}$. More formally, we have the following.

PROPOSITION 5. *Let $F : \Omega \to \mathbb{R}$ denote a bounded measurable map. Then, for all $k \geq 1$, on the event that T_k is upward,*

$$\mathbb{E}_\nu\big(F\big(\pi_{r_{T_k},T_k}(B_{T_k})\big)|\mathcal{G}^R_{T_k}\big) = \xi\big(F, (r_{s+T_k} - r_{T_k})_{-T_k \leq s \leq 0}, -r_{T_k}\big) \qquad a.s.,$$

where,[11] *given a path $q = (q_s)_{t \leq s \leq 0}$ with values in $\mathbb{Z}$,*

$$\xi(F, q, x) := \mathbb{E}_\nu\big(F(B_0)|G(q, x)\big),$$

and

$$G(q, x) := \{\forall (W, u) \in B_0, W_t > x, \forall t \leq s < 0, W_s > q_s\}. \tag{5}$$

Figure 2 illustrates this conditioning. Note that the avoidance condition obviously has to be satisfied. Indeed, trajectories of particles that are blue at time T_k must have avoided the front during the time-interval $[0, T_k[$, for a particle in contact with the front at a time $t < T_k$ will necessary be red at time T_k. What is not

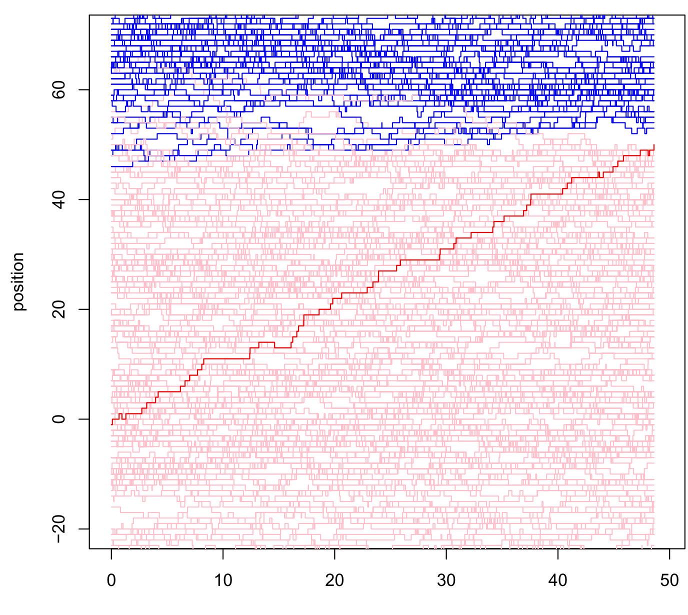

FIG. 2. *Realization of the KS infection model. Pink trajectories correspond to particles that are red at time t, that is, R_t, while blue trajectories correspond to particles that are blue at time t, that is, B_t. The trajectory of the front is drawn in red.*

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[11]Note that the definition makes sense since, as is easily checked, $\mathbb{P}_\nu(G(q, x)) > 0$ for all càdlàg path $q = (q_s)_{t \leq s \leq 0}$ with values in $\mathbb{Z}$, taking nearest-neighbor steps, such that $q_0 = 0$, $q_{0-} = -1$, and containing a finite number of jumps.

obvious is that the influence upon B_{T_k} of the history of the whole process up to time T_k, admits such a simple description.

The core idea underlying the proof of Proposition 5 is time-reversal. Instead of starting from the description of the initial configuration at time zero, and trying to understand how it evolves up to time t, we start from the configuration at time t, and express the relevant properties in terms of the time-reversed trajectories. This is where the reversibility of the Poisson distribution of particles with respect to systems of independent random walks plays a key role. Broadly speaking, one starts from a configuration at time t with i.i.d. Poisson numbers of particles at each site, and a prescribed value, say x, for r_t. One can then express the whole history between time 0 and time t of particles that are red at time t—including the trajectory of the front from time 0 to time t—in terms of random walk trajectories going backward from those sites that are $< x$. On the other hand, the history of particles that are blue at time t is described by an *independent* set of random walk trajectories going backward from sites $\geq x$, with the constraint that they must avoid the front between time 0 and time t. This informal description is made precise in the proof of Proposition 5 given below.

PROOF OF PROPOSITION 5. To avoid the main idea being obscured by technicalities, we start with a simplified argument, which is not completely valid since it involves conditioning upon the exact values taken by continuous random variables. If our model were a discrete-time one, this argument would immediately translate into a full proof. In our continuous-time framework, additional approximation arguments are needed, of which we give a sketch, referring to [4] for a full account of the technical details.

Consider $y \in \mathbb{Z}$, $t > 0$, $v \in [0, 1]$, and a càdlàg path $g = (g_s)_{0 \leq s \leq t}$ with values in $\mathbb{Z}$, taking nearest-neighbor steps, such that $g_t = y$, $g_{t-} = y - 1$. Now define the event $J = J(t, y, v, g)$ by

$$J = \{T_k = t, r_{T_k} = y, U_k = v, (r_s)_{0 \leq s \leq t} = (g_s)_{0 \leq s \leq t}\},$$

where U_k denotes the (random) label of the particle making the front move at time T_k. Then introduce a partition of the set of particle paths defined by

$$\Delta_+ = \{(W, u) \in \Psi; W_t > y \text{ or } (W_t = y \text{ and } u \neq v)\}$$

and $\Delta_- = \Psi \setminus \Delta_+$. Finally, let $(r'_s)_{0 \leq s \leq t}$ denote the position of the front generated by the particles in Δ_- up to time t, T'_k the kth time at which this front moves, and U'_k the label of the particle path making this front move at time T'_k. Introduce the events J' and A defined by

$$J' = \{T'_k = t, r'_{T'_k} = y, U'_k = v, (r'_s)_{0 \leq s \leq t} = (g_s)_{0 \leq s \leq t}\},$$

$$A = \{\forall (W, u) \in \Delta_+, W_0 > 0 \text{ and } \forall 0 \leq s < t, W_s > g_s\}.$$

The key identity to prove Proposition 5 is the following:

$$J = J' \cap A. \tag{6}$$

We first check the inclusion $J \subset J' \cap A$. Note that, on J, by definition Δ_+ coincides with B_t. As a consequence, the two avoidance conditions in A have to be satisfied since particles in Δ_+ have to be blue at time t. On the other hand, on J, we also have that Δ_- coincides with R_t. Since the history of the front up to time t is entirely prescribed by the dynamics of particles that are red at time t, the quantities r'_t, T'_k, U'_k must then coincide with r_t, T_k, U_k.

We now check the reverse inclusion $J' \cap A \subset J$. Let us prove that, on J, one has $r'_s = r_s$ for all $0 \le s \le t$. By contradiction, assume that there exists a time $s \le t$ such that $r'_s \ne r_s$, and let s_0 be the first such time. Due to the condition $W_0 > 0$ for particles in Δ_+, we must have $r'_0 = r_0$, so that $s_0 > 0$. Then the only possibility for r'_{s_0} not to be equal to r_{s_0} is that some particle in Δ_+ either makes r_s (and not r'_s) jump upward at time s_0, or prevents r_s (and not r'_s) to jump downward at time s_0. In turn, this implies that such a particle hits the position r'_s at some time $s \le s_0$, which is ruled out by the definition of A. Knowing that $r'_s = r_s$ for all $0 \le s \le t$, the other conditions in J are automatically satisfied.

Now, on $J = J' \cap A$, we have seen that $B_t = \Delta_+$ and $R_t = \Delta_-$. On the other hand, in view of the definition of Δ_+ and Δ_-, the invariance of the probability $\mathbb{P}_\nu$ with respect to space–time shifts (Proposition 2) entails that

$$\big(\pi_{y,t}(\Delta_+), \pi_{y,t}(\Delta_-)\big) \overset{d}{=} (B_0, R_0), \tag{7}$$

and we note that B_0 and R_0 are independent due to the Poisson structure of $\mathbb{P}_\nu$. Finally, we note that, from their definitions, J' is measurable w.r.t. Δ_- while A is measurable w.r.t. Δ_+, with the explicit representation in terms of $\pi_{y,t}(\Delta_+)$:

$$A = \big\{\forall (W,u) \in \pi_{y,t}(\Delta_+), W_{-t} > -y, \forall -t \le s' < 0, W_{s'} > g_{-t-s'} - y\big\}.$$

If J were an event with nonzero probability, we would readily deduce from the above results that, conditional upon $J = J' \cap A$, the random variable $\pi_{r_{T_k}, T_k}(B_{T_k}) = \pi_{y,t}(\Delta_+)$ is independent from $R_{T_k} = \Delta_-$, and follows the distribution of B_0 conditioned upon the event

$$\big\{\forall (W,u) \in B_0, W_{-t} > -y, \forall -t \le s' < 0, W_{s'} > g_{-t-s'} - y\big\}.$$

Moreover, from a countable decomposition into pairwise disjoint events of the form

$$\{T_k \text{ is upward}\} = \bigsqcup_{t,y,v,g} J(t,y,v,g),$$

we would then deduce the conclusion of the proposition. However, in our continuous-time framework, each event of the form $J(t,y,v,g)$ has zero probability, and a countable decomposition as above does not exist. To tackle this problem, we rely on discrete approximations, which allow us to recover the conclusion in the limit. More precisely, letting $T_k^{(\ell)} := 2^{-\ell}(\lceil 2^\ell T_k \rceil)$ and $U_k^{(m)} := 2^{-m}(\lceil 2^m U_k \rceil)$, we can perform a countable decomposition based on the values of $T_k^{(\ell)}$ and $U_k^{(m)}$, but it then becomes necessary to show that the contributions of various undesirable events (e.g., between time T_k and $T_k^{(\ell)}$) lead to a vanishing contribution when we take the limits $\ell, m \to +\infty$. Moreover, to get decent convergence properties, we have to use regularity properties of the Markov semigroup of $(X_t)_t$, and characterize the conditional distribution of $\pi_{r_{T_k}, T_k}(B_{T_k})$ through the conditional expectation of sufficiently nice functionals F, the appropriate choice being $F = f_1(X_{t_1}) \times \cdots \times f_p(X_{t_p})$, where $t_1 < \cdots < t_p$, and $f_1, \dots, f_p$ are bounded and uniformly continuous on $(\mathbb{S}_\theta, d_\theta)$. $\square$

3.3.4. *Step* 4: *Distribution of* $\pi_{r_{\kappa_n}, \kappa_n}(B_{\kappa_n})$ *given* $\mathcal{G}_{\kappa_n}^R$ *under* $\mathbb{P}_\nu$. We now establish the fact that, under $\mathbb{P}_\nu$, on the event $\{\kappa_n < +\infty\}$, the conditional distribution of $\pi_{r_{\kappa_n}, \kappa_n}(B_{\kappa_n})$ with respect to $\mathcal{G}_{\kappa_n}^R$, is the same as that of B_0, conditioned by the event H defined as[12]

$$H := \{t = 0 \text{ is a backward and forward super-}\alpha \text{ time for } B_0\}. \tag{8}$$

More formally, we prove that, for all bounded measurable map $F : \Omega \to \mathbb{R}$, one has that, on $\{\kappa_n < +\infty\}$,

$$\mathbb{E}_\nu\big(F\big(\pi_{r_{\kappa_n}, \kappa_n}(B_{\kappa_n})\big) | \mathcal{G}_{\kappa_n}^R\big) = \mathbb{E}_\nu\big(F(B_0)\mathbf{1}_H\big) \qquad \text{a.s.} \tag{9}$$

The key idea to go from Step 3 to the present result consists in showing that, when $T_k = \kappa_n$, the avoidance condition that results from conditioning B_{T_k} by R_{T_k}, is subsumed by the condition that κ_n is both a backward super- and sub-α time.

Consider $C \in \mathcal{G}_{\kappa_n}^R$ such that $C \subset \{\kappa_n < +\infty\}$, and write the decomposition

$$\mathbb{E}_\nu\big(F\big(\pi_{r_{\kappa_n}, \kappa_n}(B_{\kappa_n})\big)\mathbf{1}_C\big) = \sum_{k \geq 1} \mathbb{E}_\nu\big(F\big(\pi_{r_{T_k}, T_k}(B_{T_k})\big)\mathbf{1}_C \mathbf{1}(\kappa_n = T_k)\big). \tag{10}$$

We first note that, for each $k \geq 1$, the event $\{\kappa_n = T_k\}$ corresponds to $t = 0$ being a backward and forward super-α time for $\pi_{r_{T_k}, T_k}(B_{T_k})$, plus a set of conditions bearing only on $R(T_k)$, that is, the trajectories of particles that are red at time T_k, and implying among other things that T_k is a backward sub-α time. As a consequence, one can write $\{\kappa_n = T_k\} = H_k \cap J_k$, where $J_k \in \mathcal{G}_{T_k}^R$ and $H_k := \{t = 0 \text{ is a backward and forward super-}\alpha \text{ time for } \pi_{r_{T_k}, T_k}(B_{T_k})\}$, and with $J_k \subset \{T_k \text{ is a backward sub-}\alpha \text{ time}\}$. Moreover, since $C \in \mathcal{G}_{\kappa_n}^R$, one can write $C \cap \{\kappa_n = T_k\} = D_k \cap \{\kappa_n = T_k\}$, where $D_k \in \mathcal{G}_{T_k}^R$. Putting things together, we obtain the identity

$$\mathbb{E}_\nu\big(F\big(\pi_{r_{T_k}, T_k}(B_{T_k})\big)\mathbf{1}_C \mathbf{1}(\kappa_n = T_k)\big) = \mathbb{E}_\nu\big(F\mathbf{1}_H\big(\pi_{r_{T_k}, T_k}(B_{T_k})\big)\mathbf{1}_{D_k \cap J_k}\big). \tag{11}$$

[12] Corollary 1 proves that $\mathbb{P}_\nu(H) > 0$.

We can now invoke Proposition 5, using the fact that $D_k, J_k \in \mathcal{G}^R_{T_k}$, leading to the identity

$$\mathbb{E}_\nu\big(F\mathbf{1}_H\big(\pi_{r_{T_k},T_k}(B_{T_k})\big)\mathbf{1}_{D_k\cap J_k}\big) = \mathbb{E}_\nu(K_k\mathbf{1}_{D_k\cap J_k}), \tag{12}$$

where we have set $K_k := \xi(F\mathbf{1}_H, (r_{s+T_k} - r_{T_k})_{-T_k\le s\le 0}, -r_{T_k})$.

We now make the key observation that, for a path $q = (q_s)_{t\le s\le 0}$ such that $q_0 = 0$ and $q_s < \alpha s$ for all $t \le s < 0$, and x such that $x < \alpha t$, the event that $t = 0$ is a backward super-α time implies the event $G(q,x)$, so that in particular $H \subset G(q,x)$. As a consequence, we can write

$$\xi(F\mathbf{1}_H, q, x) = \mathbb{E}_\nu\big(F(B_0)\mathbf{1}_H(B_0)|G(q,x)\big) = \frac{\mathbb{E}_\nu(F(B_0)\mathbf{1}_H)}{\mathbb{P}_\nu(G(q,x))}. \tag{13}$$

By the fact that, on J_k, T_k is a backward sub-α time, we can precisely apply the above observation to the path $q = (r_{s+T_k} - r_{T_k})_{-T_k\le s\le 0}$ and the position $x = -r_{T_k}$. Putting together (10), (11), (12), (13), we obtain that

$$\mathbb{E}_\nu\big(F\big(\pi_{r_{\kappa_n},\kappa_n}(B_{\kappa_n})\big)\mathbf{1}_C\big) = \mathbb{E}_\nu\big(F(B_0)\mathbf{1}_H\big) \times h(C), \tag{14}$$

where $h(C)$ is an expression depending on the event C but not on F. Using (14) with the choice $F \equiv 1$ shows that $h(C) = \mathbb{P}_\nu(C)$, so that (14) indeed proves identity (9).

3.3.5. *Conclusion.* Part (a) of Proposition 3 has been proved in Step 1. As for part (b), we know from Step 2 that, on the event $\{\kappa_n < +\infty\}$, we have

$$(\kappa_{n+1} - \kappa_n, r_{\kappa_{n+1}} - r_{\kappa_n}) = (\kappa_1, r_{\kappa_1})\big(\pi_{r_{\kappa_n},\kappa_n}(B_{\kappa_n})\big).$$

From Step 4, the conditional distribution of $\pi_{r_{\kappa_n},\kappa_n}(B_{\kappa_n})$ with respect to $\mathcal{G}^R_{\kappa_n}$ is that of B_0 conditioned by the event that $t = 0$ is a backward and forward super-α time for B_0, which yields the desired result.

4. Estimates on the renewal structure.

4.1. *Overview.* This section is devoted to the proof of Proposition 4. To control the tail of the random variables κ_1 and r_{κ_1}, we rely on a sequence of stopping times $S_1 \le D_1 \le D_2 \le S_2 \le \cdots$, where S_n is the nth attempt at obtaining an α-separation time, and, in case this attempt fails, D_n is the time at which the failure is detected, while $D_n = +\infty$ if the attempt is successful. As a consequence, one has that $\kappa_1 \le S_{\mathfrak{K}}$, where $\mathfrak{K} := \inf\{n \ge 1; D_n = +\infty\}$, and our approach consists in bounding the tail of the number $\mathfrak{K}$, and the tail of the increments $S_{n+1} - S_n$ and $r_{S_{n+1}} - r_{S_n}$ on the event that $D_n < +\infty$.

The organization of this section is the following. In Section 4.2, we give the precise definition of the random variables S_n and D_n, while Section 4.3 lists the various parameters, assumptions and conventions, used in subsequent estimates.

In Section 4.4, we prove elementary results on the hitting times and hitting probabilities of a straight line by a system of independent random walks. Section 4.5 is devoted to an extension of the quantitative ballisticity estimates obtained in [10] to the case where the initial distribution of particles is restricted to locations above $x = 0$. Section 4.6 contains an analogue of Proposition 5 suited to the definition of the stopping time S_n. Finally, Section 4.7 combines these ingredients to prove the tail estimates on the renewal structure, that is, Proposition 4.

4.2. *Definition of S_n and D_n.* We define by induction the sequence of stopping times on which our estimates on the renewal structure are based. Besides α, the definition involves two integer parameters $\mathscr{C} \geq 1$ and $L \geq 1$, and the following notion: given $0 \leq s < t$, we say that t is an (s, α)-*crossing time* if there exists $k \in \{1, 2, \ldots\}$ such that $r_v < r_s + k + \alpha(v - s)$ for all $v \in [s, t[$ and $r_t \geq r_s + k + \alpha(t - s)$.

To initialize the induction, let $D_0 := 0$ and $\Upsilon_0 := \varnothing$. Now, for $n \geq 1$, assume that the random variables D_{n-1}, Υ_{n-1} have already been defined, and let S'_n be the infimum of the $t > D_{n-1}$ such that:

- t is a backward sub-α time;
- $\Upsilon_{n-1} \subset R_t$;
- B_t contains at least $\mathscr{C}$ particles (W, u) such that $W_t = r_t$.

Then define S_n as the infimum of the $t > S'_n$ such that:

- t is a backward sub-α time;
- $]S'_n, t[$ contains a number of (S'_n, α)-crossing times at least equal to L;
- B_t contains at least $\mathscr{C}$ particles (W, u) such that $W_t = r_t$.

We use the notation (W^{*n}, u^{*n}) for the particle that makes the front jump at time S_n, and define the subset $R^*_{S_n} := R_{S_n} \setminus \{(W^{*n}, u^{*n})\}$. If S_n is a backward super-α time, then $\Upsilon_n := \varnothing$ and D_n is defined as the infimum of the $t > S_n$ such that *a least one* of the following five conditions holds:

(1) $r_t < r_{S_n} + \lfloor \alpha(t - S_n) \rfloor$,
(2) $t \leq S_n + \alpha^{-1}$ and there is no $(W, u) \in B_{S_n}$ such that $W_{S_n} = r_{S_n}$ and W remains at r_{S_n} during $[S_n, S_n + t]$,
(3) $W_t > r_{S_n} - 1 + \alpha(t - S_n)$ for some $(W, u) \in R^*_{S_n}$,
(4) $t \leq S_n + \alpha^{-1}$ and $W^{*n}_t \neq r_{S_n}$,
(5) $t > S_n + \alpha^{-1}$ and $W^{*n}_t > r_{S_n} - 1 + \alpha(t - S_n)$.

Note that (1) and (2) detect the potential failure of S_n to be a forward super-α time, while (3)–(4)–(5) detect the potential failure of S_n to be a forward sub-α time.

On the other hand, if S_n is not a backward super-α time, consider the set of particle paths $(W, u) \in B_{S_n}$ such that there exists $t < S_n$ for which $W_t < r_{S_n} - \alpha(S_n - t)$. Among this set, consider the pair $(W^{(n)}, u^{(n)})$ such that (W_{S_n}, u) is the smallest

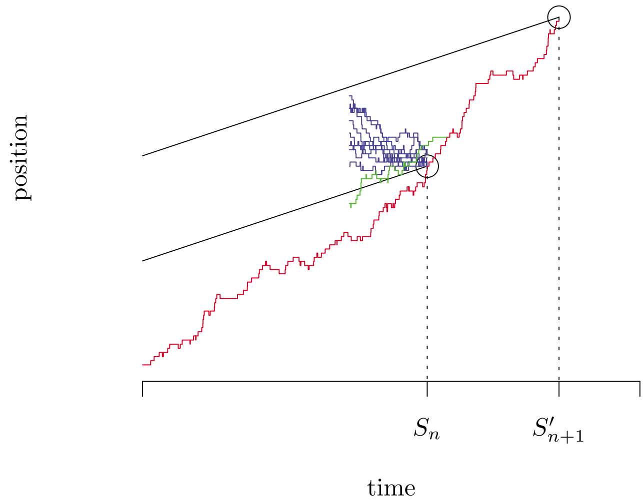

FIG. 3. *From S_n to S'_{n+1} when S_n fails to be a backward super-α time (in this case $D_n = S_n$). Only the most relevant portions of trajectories are shown. The trajectory of the front r_t is depicted in red, while blue is used for the trajectories of blue particles, except for the witness trajectory $W^{(n)}$, which is drawn in green. Circles are used at locations where the number of particles is assumed to be $\geq \mathscr{C}$.*

with respect to the lexicographical order,[13] and define $\Upsilon_n := \{(W^{(n)}, u^{(n)})\}$ and $D_n := S_n$, so that, when nonempty, the set Υ_n can be though of as containing a witness trajectory for the fact that S_n failed to be a backward super-α time.

The reason why S_n is not taken equal to S'_n is a technical one, and comes from the fact that, when proving tail estimates, one needs to have some "room" between S'_n and S_n so that, at time S_n, the configuration of particles below the front has "smoothed out" the irregularities that may be present at time S'_n.

Some of the above definitions are illustrated in Figures 3, 4 and 5.

4.3. *List of parameters, assumptions and conventions*. Let us recapitulate the list of parameters encountered so far: $D_R = D_B > 0$ is the common jump rate of red and blue particles, $\rho > 0$ is the average number of particles per site in the initial Poisson distribution of particles, $\theta > 0$ is a parameter characterizing the space $\mathbb{S}_\theta$ of particle configurations we work with, $\alpha > 0$ is the slope of space–time lines involved in the definition of the renewal structure, $\mathscr{C}$ and L are two additional integer parameters involved in the definition of S_n and D_n given above. While $D_R = D_B$ and ρ are fixed parameters of the model, $\theta, \alpha, \mathscr{C}, L$ can be chosen at our convenience. One more parameter $\beta > 0$ will play a role in the following proofs.

[13]Remember that (x_1, u_1) is smaller than (x_2, u_2) with respect to the lexicographical order if $x_1 < x_2$, or $x_1 = x_2$ and $u_1 < u_2$.

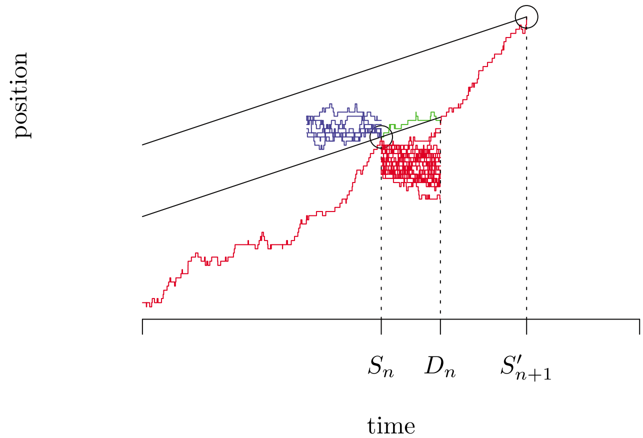

FIG. 4. *From S_n to S'_{n+1} when S_n is a backward super-α time but condition* (1) *is realized first. Only the most relevant portions of trajectories are shown. The trajectory of the front r_t is depicted in red, except for the part causing condition* (1), *which is drawn in green. Circles are used at locations where the number of particles is assumed to be $\geq \mathscr{C}$.*

Here are the assumptions on the various parameters that we assume to hold throughout the sequel:

$$\begin{cases} 0 < \alpha < \beta < (1/3)C_2(\rho/4), \\ \alpha\theta - 2(\cosh\theta - 1) > 0, \end{cases} \tag{15}$$

where $C_2 > 0$ is defined in Proposition 7, which is adapted from [10]. Such a choice of parameters is always possible by choosing first α and β, then θ close

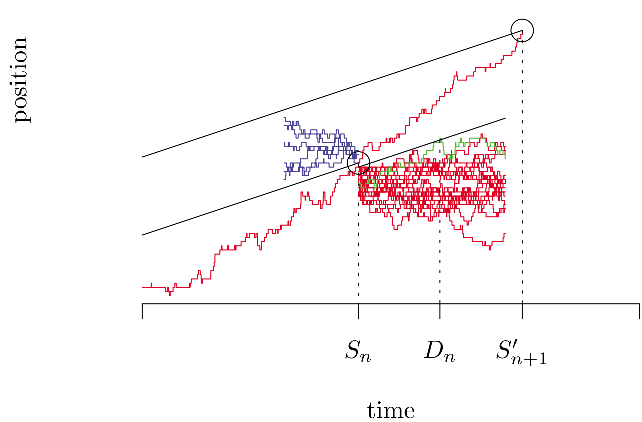

FIG. 5. *From S_n to S'_{n+1} when S_n is a backward super-α time but condition* (3) *is realized first. The trajectory of the front r_t is depicted in red, which is also used for red particles, except the trajectory causing condition* (3), *which is drawn in green. Circles are used at locations where the number of particles is assumed to be $\geq \mathscr{C}$.*

enough to zero, using the fact that $\cosh(\theta) = 1 + o(\theta)$ when θ goes to zero. In addition to (15), we shall have to assume that $\mathscr{C}$ is large enough, and also that L is large enough (depending on $\mathscr{C}$). These assumptions on $\mathscr{C}$ and/or L will always be made explicit in the sequel.

We now explain our convention for constants: what we call constants in the rest of this section may depend on $\rho, \alpha, \beta, \theta$, but unless otherwise mentioned, not on $\mathscr{C}$ or L. As a rule, we use $c_1, c_2, \ldots$ to denote constants whose range of validity extends throughout the section, and are used in the statement of propositions or lemmas. On the other hand, we use $d_1, d_2, \ldots$ to denote constants that are purely local to proofs.

4.4. *Hitting of a straight line by random walks.* Starting with an initial configuration of particles $w \in \mathbb{S}_\theta$, we establish two bounds on the hitting time and probability of a straight line of slope α by one of the random walks whose initial position is ≤ 0. In both cases, the key quantity is the "exponential norm" ϕ_θ defined by

$$\phi_\theta(w) := \sum_{x \leq 0} \sum_{u \in w(x)} e^{\theta x}.$$

Lemma 1 below gives (a) an upper bound on the probability that the hitting time is finite and $\geq t$, and shows (b) that an upper bound on the value of $\phi_\theta(w)$ translates into a lower bound on the probability that none of the random walks ever hits the straight line.

LEMMA 1. *The following bounds hold*:

(a) *For all $w \in \mathbb{S}_\theta$, and all $t \geq 0$,*

$$\mathbb{P}_w\big(\exists (W, u)\ \exists s \geq t,\ W_0 \leq 0,\ W_s \geq \alpha s\big) \leq \phi_\theta(w) e^{-\mu t},$$

where $\mu := \alpha\theta - 2(\cosh\theta - 1) > 0$ [see (15)].

(b) *For all $K > 0$, there exists $g(K) > 0$ such that, for all $w \in \mathbb{S}_\theta$ such that $\phi_\theta(w) \leq K$, one has*

$$\mathbb{P}_w\big(\forall (W, u) \text{ such that } W_0 \leq 0, \forall t > 0, \text{ one has } W_t < \alpha t\big) \geq g(K).$$

The proof of the lemma is based on the following elementary result for a single random walk, which we state and prove first.

LEMMA 2. *Let $(\zeta_s)_{s \geq 0}$ be a continuous-time simple symmetric random walk on $\mathbb{Z}$ with total jump rate 2 starting at $x \leq 0$, with respect to a probability measure P_x. Then for all $t \geq 0$*

$$P_x(\exists s \geq t; \zeta_s \geq \alpha s) \leq e^{\theta x} e^{-\mu t}.$$

PROOF. For all $s \geq 0$, let $M_s := e^{\theta\zeta_s - 2(\cosh(\theta)-1)s}$, and $T := \inf\{s \geq t; \zeta_s \geq \alpha s\}$. Then $(M_s)_{s\geq 0}$ is a càdlàg martingale, and T is a stopping time, so that, for all finite $K > 0$, one has

$$E_x(M_{T\wedge K}) = E_x(M_0) = e^{\theta x}. \tag{16}$$

Now we have that $\liminf_{K\to+\infty} M_{T\wedge K} \geq M_T \mathbf{1}(T < +\infty)$, so that, by Fatou's lemma and (16),

$$E_x\big(M_T \mathbf{1}(T < +\infty)\big) \leq e^{\theta x}. \tag{17}$$

Now, by definition of T, one has that, on $\{T < +\infty\}$,

$$M_T \geq e^{\theta\alpha T - 2(\cosh(\theta)-1)T} = e^{\mu T} \geq e^{\mu t}, \tag{18}$$

where the last inequality comes from the fact that $\mu > 0$ and $T \geq t$. The result now follows from combining (17) and (18). $\square$

PROOF OF LEMMA 1. First note that part (a) of the lemma is a direct consequence of Lemma 2 and of the union bound over each particle. We now prove part (b), using the notation already appearing in the statement of Lemma 2.

Let us choose $\theta' > \theta$ such that $\mu' := \alpha\theta' - 2(\cosh(\theta') - 1) > 0$ (this is possible since $\mu > 0$), and observe that Lemma 2 holds with θ', μ' instead of θ, μ. We deduce that, for all $x < 0$, we have

$$\mathfrak{p}(x) \leq e^{\theta' x}, \tag{19}$$

where

$$\mathfrak{p}(x) := P_x(\exists s > 0; \zeta_s \geq \alpha s).$$

Moreover, we must have $\mathfrak{p}(0) < 1$, for otherwise we could prove that

$$P_0\Big(\limsup_{t\to+\infty} \zeta_t/t \geq \alpha\Big) = 1,$$

which would contradict the law of large numbers. Since all the random walks in our model evolve independently, we can rewrite the probability we want to bound from below as

$$\prod_{x\leq 0} \big(1 - \mathfrak{p}(x)\big)^{|w(x)|}.$$

Now the inequality $\phi_\theta(w) \leq K$ implies that, for all $x \leq 0$, one has that

$$\big|w(x)\big| \leq e^{-\theta x} K. \tag{20}$$

As a consequence, we have the bound

$$\prod_{x\leq 0} \big(1 - \mathfrak{p}(x)\big)^{|w(x)|} \geq \bigg(\prod_{x\leq 0} \big(1 - \mathfrak{p}(x)\big)^{e^{-\theta x} K}\bigg).$$

In view of (19) and of the fact that $\theta' > \theta$, we have that $\sum_{x\leq 0} e^{-\theta x} e^{\theta' x} < +\infty$, so the right-hand side of the above inequality is > 0, and depends only on K. $\square$

4.5. *Ballisticity estimates.* We start by recalling the quantitative ballisticity estimates derived in [10].

PROPOSITION 6. *There exist a constant $C_1(\rho) > 0$ and a constant c_1, depending on ρ and $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_\nu\big(r_t \geq C_1(\rho)t\big) \leq c_1 \exp(-t).$$

PROPOSITION 7. *There exists a constant $C_2(\rho) > 0$ such that, for all $K > 0$, there exists a constant c_2, depending on ρ and K, such that, for every $t > 0$,*

$$\mathbb{P}_\nu\big(r_t \leq C_2(\rho)t\big) \leq c_2 t^{-K}.$$

Note that, strictly speaking, Propositions 6 and 7 do not appear in [10], which uses slightly different definitions for the front and the initial condition. However, they are rather easily derived from Theorems 1 and 2 in [10].

When trying to control the tail of κ_1, we encounter situations where ballisticity estimates similar to those appearing in Propositions 6 and 7 are needed, but where the initial condition consists only of particles located at the right of the origin. To be specific, we define $\nu_{\mathscr{C},+}$ to be the probability distribution on $\mathbb{S}_\theta$ obtained by starting from the Poisson distribution ν, removing every particle whose location is < 0, and conditioning the number of particles located at $x = 0$ to be $\geq \mathscr{C}$. The corresponding distribution on the space of trajectories is denoted by $\mathbb{P}_{\nu_{\mathscr{C},+}}$.

Adapting the upper bound (Proposition 6) with $\nu_{\mathscr{C},+}$ instead of ν turns out to be rather straightforward, since removing red particles from the initial condition cannot increase the position of the front [see equation (21)]. The precise result we need in the sequel is an easy corollary from this adaptation, and we quote it without proof.

PROPOSITION 8. *Let $C_1'(\rho) := C_1(\rho) + 1$. There exist strictly positive constants c_3, c_4, with c_3 depending on $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C},+}}\big(\exists s \geq 0; r_s \geq C_1'(\rho)\max(s,t)\big) \leq c_3 \exp(-c_4 t).$$

On the other hand, adapting the lower bound (Proposition 7) requires more work. The precise result we have is the following, with $\beta > 0$ being defined in (15).

PROPOSITION 9. *There exist constants $c_5, c_6 > 0$, with c_5 depending on $\mathscr{C}$, such that, for every $t > 0$,*

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(\exists s \geq t; r_s \leq \alpha s) \leq c_5 t^{-c_6 \cdot \mathscr{C}}.$$

The following corollary to Proposition 9, whose proof is given in the last part of the present subsection, shows that the super-α time condition for B_0, which constitutes "half" of the conditions involved in the definition of the renewal structure, has indeed a positive probability with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$.

COROLLARY 1. *For all large enough $\mathscr{C}$,*

$$\mathbb{P}_{\nu_{\mathscr{C},+}}\big(\{t=0 \text{ is a backward and forward super-}\alpha \text{ time for } B_0\}\big) > 0.$$

The rest of this subsection is mainly devoted to the proof of Proposition 9, which is based on coupling the evolution of the front with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$ with a modified version of the dynamics, then using a symmetrization trick to enable a comparison with an initial configuration consisting of i.i.d. Poisson numbers of particles on the whole of $\mathbb{Z}$.

4.5.1. *Step* 1: *Monotone couplings*. We start by defining a modified version of the infection dynamics. In the modified version, the front is at zero at time zero and, after time zero, the dynamics is defined as the original one, with the difference that the front is never allowed to go below level zero (i.e., a jump that would make the front go below zero for the original dynamics has no effect on the front in the modified dynamics). We call $(\hat{r}_s)_{s\geq 0}$ the trajectory of the corresponding front.

Our first statement is that both the original front r_t and the modified front $\hat{r}_t$ are nondecreasing with respect to the addition of particles in the system. Indeed, we claim that, for all $\psi_1, \psi_2 \in \Omega$ such that $\psi_1 \subset \psi_2$, one has, for all $t \geq 0$,

$$r_t(\psi_1) \leq r_t(\psi_2) \tag{21}$$

and

$$\hat{r}_t(\psi_1) \leq \hat{r}_t(\psi_2). \tag{22}$$

The proof of (21) consists in observing that, by definition, $r_0(\psi_1) \leq r_0(\psi_2)$, and that, since only nearest-neighbor steps can be performed, the trajectories of both fronts must meet before crossing each other. Assuming that at a time s one has $r_s(\psi_1) = r_s(\psi_2)$, and calling t the next time at which any of the fronts jumps, the assumption that $\psi_1 \subset \psi_2$ entails that, if t is upward for ψ_1, it is also upward for ψ_2, while, if t is downward for ψ_2, it is also downward for ψ_1, so that $r_t(\psi_1) \leq r_t(\psi_2)$ in any case. We argue similarly to prove (22).

Next, we claim that the modified front always dominates the original front, in the sense that, for all $\psi \in \Omega$, one has, for all $t \geq 0$,

$$r_t(\psi) \leq \hat{r}_t(\psi), \tag{23}$$

which can be proved with an argument quite similar to that used for (21).

Finally, we prove the monotonicity of the modified front with respect to a symmetrization of trajectories that we implement through a map $\mathscr{T}:\Omega \to \Omega$. Consider a pair (W,u). If $W_0 \geq 0$, then let $W^\times := W$. On the other hand, if $W_0 < 0$, consider $\tau := \inf\{s > 0; W_s = 0\}$, and let $W_s^\times := -W_s$ on $]-\infty, \tau[$ and $W_s^\times := W_s$ on $[\tau, +\infty[$. Now let

$$\mathscr{T}(\psi) := \big\{(W^\times, u); (W,u) \in \psi\big\}.$$

We can now state the monotonicity property: for all $\psi \in \Omega$, and for all $t \geq 0$,

$$\hat{r}_t(\psi) \leq \hat{r}_t \circ \mathscr{T}(\psi). \tag{24}$$

To prove (24), we argue as in the proof of (21), so that it is enough to prove that, if $\hat{r}_s = \hat{r}_s \circ \mathscr{T}$, and if t denotes the first time after s at which any of the fronts jumps, one has $\hat{r}_t \leq \hat{r}_t \circ \mathscr{T}$. Assume that t is upward for $\hat{r}$. Then by definition the corresponding random walk W is such that $W_s \geq 0$, so that $W^{\times}$ coincides with W on $[s, +\infty[$, and so t is also upward for $\hat{r} \circ \mathscr{T}$. On the other hand, if t is downward for $\hat{r} \circ \mathscr{T}$, then the common location of the fronts has to be ≥ 1, and there must be at least one $(W, u) \in \Psi$ such that $W_s = W_s^{\times} = \hat{r}_s$. In fact, there cannot be more than one such (W, u), since otherwise t could not be downward for $\hat{r} \circ \mathscr{T}$. As a consequence, there is only one such (W, u), and t must also be downward for $\hat{r}$.

4.5.2. *Step* 2: *Comparison between distributions*. Given $t \geq 0$, let $t_0 := t/3$, $t_1 := (2/3)t$, and, for $s \in [t_1, t]$, define $r_s^{(1)} := \hat{r}_{s-t_1} \circ \pi_{0,t_1}$. Then define four distributions $\nu_+, \nu_1, \nu_2, \nu_3$ on $\mathbb{S}_\theta$, in the following way. First, let ν_+ be the probability distribution on $\mathbb{S}_\theta$ obtained by starting from the Poisson distribution ν, then removing every particle whose location is < 0. Then let $(N_x^{(1)})_{x \in \mathbb{Z}}$ denote an independent family of Poisson processes on [0, 1], where, for all $x \in \mathbb{Z}$, the rate of $N_x^{(1)}$ is equal to $\rho p_{t_1}(x, \mathbb{N})$, with $p_{t_1}(x, \mathbb{N}) := \sum_{y \in \mathbb{N}} p_{t_1}(x, y)$. Define ν_1 as the distribution induced by $(N_x^{(1)})_{x \in \mathbb{Z}}$ on $\mathbb{S}_\theta$. Define also $(N_x^{(2)})_{x \in \mathbb{Z}}$ to be an independent family of Poisson processes on [0, 1], where the rate of $N_x^{(2)}$ is $\rho/2$ for $x \geq 1$, $\rho/4$ for $x = 0$, and 0 for $x < 0$, and define ν_2 as the distribution induced by $(N_x^{(2)})_{x \in \mathbb{Z}}$. Finally, define ν_3 exactly as ν, with the difference that the constant value of the rate is equal to $\rho/4$ instead of ρ.

We now claim that

$$r_{t_0}(\mathbb{P}_{\nu_3}) \prec r_t^{(1)}(\mathbb{P}_{\nu_{\mathscr{C},+}}), \tag{25}$$

where $\prec$ denotes stochastic domination between probability measures on $\mathbb{R}$. We also use stochastic domination on $\mathbb{S}_\theta$ equipped with the order relation induced by inclusion between sets, that is, $w_1 \leq w_2$ when $w_1(x) \subset w_2(x)$ for all $x \in \mathbb{Z}$.

To begin with, one checks that ν_+ is stochastically dominated by $\nu_{\mathscr{C},+}$. As a consequence, the distribution of X_{t_1} with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$ stochastically dominates ν_1. Using (22), we deduce that

$$r_t^{(1)}(\mathbb{P}_{\nu_+}) \prec r_t^{(1)}(\mathbb{P}_{\nu_{\mathscr{C},+}}). \tag{26}$$

Then observe that ν_1 is the distribution of X_{t_1} with respect to $\mathbb{P}_{\nu_+}$, so that

$$\hat{r}_{t_0}(\mathbb{P}_{\nu_1}) = r_t^{(1)}(\mathbb{P}_{\nu_+}). \tag{27}$$

Then ν_2 is stochastically dominated by ν_1, since, for all $x \geq 0$, we have $p_{t_1}(x, \mathbb{N}) \geq 1/2$. By (22), we deduce that

$$\hat{r}_{t_0}(\mathbb{P}_{\nu_2}) \prec \hat{r}_{t_0}(\mathbb{P}_{\nu_1}). \tag{28}$$

We also have that the image of the probability measure $\mathbb{P}_{\nu_3}$ by the map $\mathscr{T}$ is $\mathbb{P}_{\nu_2}$, so that, by (24),

$$\hat{r}_{t_0}(\mathbb{P}_{\nu_3}) \prec \hat{r}_{t_0}(\mathbb{P}_{\nu_2}). \tag{29}$$

Using (23), we finally deduce that

$$r_{t_0}(\mathbb{P}_{\nu_3}) \prec \hat{r}_{t_0}(\mathbb{P}_{\nu_3}). \tag{30}$$

Putting together (26), (27), (28), (29), (30), we see that (25) is proved.

4.5.3. *Step* 3: *Sojourn above zero*. To successfully exploit the comparisons established in Step 2, we need to control the probability, with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$, that the front remains for a substantial amount of time above the origin. Our claim is that there exist constants $c_7, c_8 > 0$, with c_7 depending on $\mathscr{C}$, such that, for every $t > 0$,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}\Bigl(\inf_{s\in[(2/3)t,t]} r_s \le 0\Bigr) \le c_7 t^{-c_8\cdot\mathscr{C}}. \tag{31}$$

We now proceed to the (essentially self-contained and elementary) proof of (31). Let $t_0 := t/3$. Then fix a real number $0 < v < \sqrt{2/3}$, and define $y_{t_0} := \lfloor v(t_0 \log t_0)^{1/2}\rfloor$ and $\varepsilon(t_0) := \frac{t_0^{-v^2/4}}{v(\log t_0)^{1/2}}$. Let $(\zeta_s)_{s\ge 0}$ denote a continuous-time simple symmetric random walk with total jump rate 2 starting at site x, with respect to a probability measure P_x. By a standard local limit theorem,[14] we have that, as t goes to infinity,

$$P_0(\zeta_{t_0} \le -y_{t_0}) \sim d_1 \varepsilon(t_0), \tag{32}$$

where d_1 is a positive constant. Using the reflection principle, we deduce that there exists a strictly positive constant d_2 such that, for large t,

$$P_0\Bigl(\inf_{s\in[0,t_0]} \zeta_s \le -y_{t_0}\Bigr) \le d_2 \varepsilon(t_0).$$

Now let $\mathfrak{Z}_s$ denote the supremum of the positions at time s of the particle paths that are located at the origin at time zero, and let C_1 denote the event that $\mathfrak{Z}_s > -y_{t_0}$ for all $s \in [0, t_0]$. Since the number of these particle paths is at least $\mathscr{C}$, we deduce that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_1^c) \le d_2^{\mathscr{C}} \varepsilon(t_0)^{\mathscr{C}}. \tag{33}$$

Now let $z_{t_0} := \lfloor \varepsilon(t_0)^{-3}\rfloor$, and consider the number $\mathfrak{N}$ of particle paths whose location at time zero lies in the interval $[0, z_{t_0}]$. Let C_2 denote the event that $\mathfrak{N}$ is

[14] See, for example, [8], Chapter XVI, Section 6 on Large Deviations for the case of a discrete-time random walk. The continuous-time follows easily, by controlling the fluctuations of the number of steps performed by the walk.

at least equal to $\rho z_{t_0}/2$. By standard large deviations bounds for Poisson random variables (see, e.g., [6]), we have that, for all large t,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_2^c) \le \exp(-d_3 z_{t_0}), \tag{34}$$

for some strictly positive constant d_3. Now define $\mathfrak{N}'$ to be the number of particle paths that:

(a) start at an initial position in $[0, z_{t_0}]$;
(b) hit $-y_{t_0}$ during the time-interval $[0, t_0]$;
(c) hit 0 after having hit $-y_{t_0}$ and before time $2t_0$.

For a particle starting in $[0, z_{t_0}]$, the probability to hit $-y_{t_0}$ during $[0, t_0]$ is larger than or equal to $q_{t_0} := P_{z_{t_0}}(\inf_{s\in[0,t_0]} \zeta_s \le -y_{t_0})$. Moreover, using the symmetry of the walk, we see that, starting from $-y_{t_0}$, the probability for the walk to hit 0 before time t_0 is larger than or equal to q_{t_0}. As a consequence, given $\mathfrak{N}$, the distribution of $\mathfrak{N}'$ stochastically dominates a binomial distribution with parameters $\mathfrak{N}$ and $q_{t_0}^2$. Moreover, as t goes to infinity, $z_{t_0} = o(t_0^{1/2})$ and $y_{t_0} z_{t_0} = o(t_0)$ due to the fact that $v^2 < 2/3$, so that (32) is also valid for $P_{z_{t_0}}(\zeta_{t_0} \le -y_{t_0})$, from which we deduce that, for large t,

$$q_{t_0} \ge d_4 \varepsilon(t_0),$$

where d_4 is a strictly positive constant. Define C_3 to be the event that $\mathfrak{N}' \ge \mathfrak{N} q_{t_0}^2/2$. Using standard (see, e.g., [17]) large deviations bounds for binomial random variables, we deduce from the preceding discussion that for all large enough t,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_2 \cap C_3^c) \le \exp(-d_5 \varepsilon(t_0)^{-1}), \tag{35}$$

for some strictly positive constant d_5. Now consider the intervals of the form $[2t_0 + k, 2t_0 + k + 1]$, for $0 \le k \le \lfloor t_0 \rfloor$. Then consider a random walk satisfying conditions (a) to (c) above, stopped at the first time it hits the origin after having hitted $-y_{t_0}$; by definition, this time is $\le 2t_0$. By symmetry, the probability that this walk is above 0 at time $2t_0 + k$ is $\ge 1/2$, and the probability that it then remains above 0 during the whole interval $[2t_0 + k, 2t_0 + k + 1]$ is larger than some strictly positive constant d_6. As a consequence, for each of the intervals we consider, the probability that none of the random walks that satisfy (a) to (c) lies above zero for the duration of the interval is conditional upon $\mathfrak{N}'$, bounded above by $(1 - d_6)^{\mathfrak{N}'}$. Now define C_4 as the event that, for every $s \in [2t, 3t]$, there exists at least one random walk satisfying (a) to (c) whose position at time s is ≥ 0. Using the union bound over all the intervals, whose total number is $\le t_0 + 1$, we obtain that for all large enough t,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(C_2 \cap C_3 \cap C_4^c) \le (t_0 + 1)\exp(-d_7 \varepsilon(t_0)^{-1}). \tag{36}$$

We now observe that, on $C_1 \cap C_2 \cap C_3 \cap C_4$, one must have $r_s \ge 0$ for all $s \in [2t_0, 3t_0] = [(2/3)t, t]$. Indeed, we know that the front always lies above the maximum position of the particles initially at zero. By C_1, the front lies above $-y_{t_0}$ during the interval $[0, t_0]$. As a consequence, any particle path satisfying (a) and (b) must hit the front before time t_0. For that reason, on C_4, the front lies above 0 during the interval $[2t_0, 3t_0]$. Now using (33), (34), (35), (36), we have that, for large enough t, the probability of the complement of $C_1 \cap C_2 \cap C_3 \cap C_4$ is bounded above by

$$d_2^{\mathscr{C}} \varepsilon(t_0)^{\mathscr{C}} + \exp(-d_3 z_{t_0}) + \exp(-d_5 \varepsilon(t_0)^{-1}) + (t_0 + 1) \exp(-d_7 \varepsilon(t_0)^{-1}),$$

and the first term dominates the others when t_0 is large.

4.5.4. *Conclusion.* We now put together the different pieces leading to the proof of Proposition 9. Remember from Step 2 the notation $t_1 = (2/3)t$ and $r_s^{(1)} = \hat{r}_{s-t_1} \circ \pi_{0,t_1}$, and let $C := \{r_s \geq 0 \text{ for all } s \in [t_1, t]\}$. Our first claim is that

(37) $$\text{on the event } C, \text{ one has that } r_s^{(1)} \leq r_s \qquad \text{for all } s \in [t_1, t].$$

Indeed, on C, one has that $r_{t_1}^{(1)} \leq r_{t_1}$ since by definition $r_{t_1}^{(1)} = 0$. We argue as in the proof of (21), and assume that $s_0 \in [t_1, t]$ is such that $r_{s_0}^{(1)} = r_{s_0}$. Since, on C, the jumps that affect both fronts between time s_0 and time t are exactly the same, one must have that $r_s^{(1)} = r_s$ for all $s \in [s_0, t]$. This proves the claim.

Now by (37), we have that, on C, $r_t^{(1)} \leq r_t$, so that $\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t \leq \beta t)$ is bounded above by $\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t^{(1)} \leq \beta t) + \mathbb{P}_{\nu_{\mathscr{C},+}}(C^c)$. Thanks to (31), we have that $\mathbb{P}_{\nu_{\mathscr{C},+}}(C^c) \leq a_1 t^{-a_2 \cdot \mathscr{C}}$. On the other hand, by (25), the distribution of $r_t^{(1)}$ with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$ stochastically dominates that of r_{t_0} with respect to $\mathbb{P}_{\nu_3}$. Using Proposition 7 with $K := \mathscr{C}$, and the fact that β is chosen such that $\beta < (1/3)C_2(\rho/4)$, we have that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t^{(1)} \leq \beta t) \leq \mathbb{P}_{\nu_3}(r_{t_0} \leq \beta t) \leq \mathbb{P}_{\nu_3}(r_{t_0} \leq C_2(\rho/4)t_0) \leq c_2 t_0^{-\mathscr{C}}.$$

We have thus proved a bound of the desired form for $\mathbb{P}_{\nu_{\mathscr{C},+}}(r_t \leq \beta t)$. Going from such a bound to a similar one for $\mathbb{P}_{\nu_{\mathscr{C},+}}(\exists s \geq t; r_s \leq \alpha s)$ is easy, and we omit the details (see, e.g., the proof of Corollary 1 for related ideas).

4.5.5. *Proof of Corollary* 1. We now prove Corollary 1. Let G denote the event that $t = 0$ is a backward super-α time, and observe that $\mathbb{P}_{\nu_{\mathscr{C},+}}(G) > 0$, using Lemma 1 and the symmetry of the distribution of our random walks. For $n \geq 1$, define $A_{n,1} := \{r_n \geq \beta n\}$ and let $A_{n,2}$ denote the event that the particle at the front at time n with the smallest label remains above level $\alpha(n+1)$ during the time-interval $[n, n+1]$. For $k \geq 1$, introduce the event $A^{(k)} := \{r_t \geq \alpha t \text{ for all } t \geq k\}$, and note that $\bigcap_{n \geq k}(A_{n,1} \cap A_{n,2}) \subset A^{(k)}$. By Proposition 9, one has that $\mathbb{P}_{\nu_{\mathscr{C},+}}(A_{n,1}^c) \leq c_5 n^{-c_6 \cdot \mathscr{C}}$. Then, using, for example, a variance bound for the random walk, one has that, for large enough n, $\mathbb{P}_{\nu_{\mathscr{C},+}}(A_{n,1} \cap A_{n,2}^c) \leq d_1 n^{-2}$, for some constant $d_1 > 0$. As a consequence, we have that, for all large enough $\mathscr{C}$, there exists $k \geq 1$ such that $\sum_{n \geq k} \mathbb{P}_{\nu_{\mathscr{C},+}}(A_{n,1}^c \cup A_{n,2}^c) < \mathbb{P}_{\nu_{\mathscr{C},+}}(G)$. We thus have that

$\mathbb{P}_{\nu_{\mathscr{C},+}}(G \cap \bigcap_{n \ge k}(A_{n,1} \cap A_{n,2})) > 0$, whence the fact that $\mathbb{P}_{\nu_{\mathscr{C},+}}(G \cap A^{(k)}) > 0$. Let U_0 denote the largest label of a particle path (W, u) such that $W_0 = 0$ (if there is no such particle path, we set $U_0 := 0$). We deduce from the fact that $\mathbb{P}_{\nu_{\mathscr{C},+}}(G \cap A^{(k)}) > 0$ the existence of a $u_0 < 1$ such that

$$\mathbb{P}_{\nu_{\mathscr{C},+}}\big(G \cap A^{(k)} \cap \{U_0 \le u_0\}\big) > 0. \tag{38}$$

Now let Ψ_0 denote the subset of Ψ obtained by removing all particle paths (W, u) such that $W_0 = 0$ and $u > u_0$. We deduce from (38) that

$$\mathbb{P}_{\nu_+}\big(G(\Psi_0) \cap A^{(k)}(\Psi_0) \cap \{|X_0(\Psi_0)| \ge \mathscr{C}\}\big) > 0,$$

with the convention that, for $D \in \mathcal{F}$, $D(\Psi_0)$ denotes the event that $\mathbf{1}_D(\Psi_0) = 1$. Now introduce the event A' that:

- there exists a particle path (W, u) such that $u > u_0$ and $W_s = 0$ for $s \in [0, \alpha^{-1}]$, and another particle path (W, u) such that $u > u_0$, $W_s \ge \lfloor \alpha s \rfloor$ for all $s \in [0, k]$;
- every particle path (W, u) such that $W_0 = 0$ and $u > u_0$ satisfies $W_s > \alpha s$ for $s < 0$.

One clearly has that $\mathbb{P}_{\nu_+}(A') > 0$, and that the two events A' and $G(\Psi_0) \cap A^{(k)}(\Psi_0) \cap \{|X_0(\Psi_0)| \ge \mathscr{C}\}$ are independent with respect to $\mathbb{P}_{\nu_+}$, and [using (21)], that $A^{(k)}(\Psi_0) \cap A'$ implies that 0 is a forward super-αtime for B_0.

4.6. *Conditional distribution of* $\pi_{r_{S_n}, S_n}(B_{S_n})$. We now give an analogue of Proposition 5 describing the conditional distribution of $\pi_{r_{S_n}, S_n}(B_{S_n})$ given $\mathcal{G}^R_{S_n}$. The reason why the result is not exactly the same as in Proposition 5 is that the definition of S_n involves additional conditions on trajectories in B_{S_n}, beyond those saying that the trajectory of a particle that is blue at time S_n has to avoid the front between time 0 and S_n.

Indeed, one first (mild) condition is that the number of particles located at site r_{S_n} at time S_n has to be $\ge \mathscr{C}$. More importantly, for each $1 \le k < n$, we record whether S_k is or is not a backward super-α time, and this leads to the following two types of conditions on B_{S_n}:

(a) when S_k is a backward super-α time, all the trajectories in B_{S_n} have to lie above a space–time half-line of slope α extending from (r_{S_k}, S_k) in the past direction;

(b) when S_k is not a backward super-α time, all the trajectories (W, u) in B_{S_n} such that (W_{S_k}, u) is less than $(W^{(k)}_{S_k}, u^{(k)})$ with respect to the lexicographical order, have to lie above a space–time half-line of slope α extending from (r_{S_k}, S_k) in the past direction, where $(W^{(k)}, u^{(k)})$ denotes the witness trajectory contained in Υ_k.

Note that we have defined S_n in such a way that the witness trajectories contained in those Υ_k for which $1 \le k \le n-1$ and S_k is not a backward super-α time, are included in R_{S_n}, so that the above conditions can indeed be expressed using only the information available in $\mathcal{G}^R_{S_n}$. Moreover, one checks that, provided that S_n is a backward super- and sub-α time, the additional conditions (a) and (b) are automatically satisfied. Since S_n is by definition always a backward sub-α time, conditions (a) and (b) will in fact be satisfied as soon as S_n is a backward super-α time.

Adapting the arguments leading to Proposition 5, we thus obtain the following result.

PROPOSITION 10. *For any bounded measurable map $F : \Omega \to \mathbb{R}$, and all $n \geq 1$, one has that, on $\{S_n < +\infty\}$,*

$$\mathbb{E}_\nu\big(F\big(\pi_{r_{S_n},S_n}(B_{S_n})\big)|\mathcal{G}^R_{S_n}\big) = \eta\big(F, Q^{(n)}\big) \qquad a.s.,$$

where $Q^{(n)}$ is a $\mathcal{G}^R_{S_n}$-measurable random variable, and where

$$\eta(F, \mathfrak{q}) := \mathbb{E}_\nu\big(F(B_0)|\mathfrak{G}(\mathfrak{q}) \cap \{\Xi_0 = 1\}\big), \tag{39}$$

where $\mathfrak{G}(\mathfrak{q})$ is an event that serves to encode conditions (a) *and* (b) *discussed above on $\pi_{r_{S_n},S_n}(B_{S_n})$, and has the property that*

$$G \subset \mathfrak{G}(\mathfrak{q}), \tag{40}$$

where $G := \{t = 0$ is a backward super-α time$\}$.

Although exact, the description of the distribution of $\pi_{r_{S_n},S_n}(B_{S_n})$ given by Proposition 10 may not be very handy to prove explicit bounds. Fortunately, we have observed that the complicated explicit conditions on B_{S_n} are automatically satisfied if one assumes that S_n is also a backward super-α time. As a result, the following comparison between the (conditional) distribution of $\pi_{r_{S_n},S_n}(B_{S_n})$ and the much simpler distribution $\mathbb{P}_{\nu_{\mathscr{C},+}}$, can be derived from Proposition 10.

COROLLARY 2. *For any nonnegative bounded measurable map $F : \Omega \to \mathbb{R}$, the following bound holds for all $n \geq 1$, on $\{S_n < +\infty\}$:*

$$\mathbb{E}_\nu\big(F\big(\pi_{r_{S_n},S_n}(B_{S_n})\big)|\mathcal{G}^R_{S_n}\big) \leq c_9 \mathbb{E}_{\nu_{\mathscr{C},+}}(F) \qquad a.s.,$$

where c_9 is a positive constant depending on $\mathscr{C}$.

PROOF. From (39) and (40), we have that, for nonnegative F,

$$\eta(F, \mathfrak{q}) \leq \frac{\mathbb{E}_\nu(F(B_0)\mathbf{1}_{\Xi_0=1})}{\mathbb{P}_\nu(G \cap \{\Xi_0 = 1\})} \leq c_9 \mathbb{E}_{\nu_{\mathscr{C},+}}(F),$$

with

$$c_9 := \frac{\mathbb{P}_\nu(\Xi_0 = 1)}{\mathbb{P}_\nu(G \cap \{\Xi_0 = 1\})},$$

using Lemma 1 to establish that $\mathbb{P}_\nu(G \cap \{\Xi_0 = 1\}) > 0$. The conclusion now follows from Proposition 10. $\square$

4.7. *Tail bounds*. We are now ready to prove the tail bounds that are necessary to control the regeneration times. Let us describe how the proof is organized. On the whole, the tools at our disposal are the following:

- comparison of the distribution of $\pi_{r_{S_n},S_n}(B_{S_n})$ with $\nu_{\mathscr{C},+}$,
- bounds on the ballistic behavior of the front with respect to $\mathbb{P}_{\nu_{\mathscr{C},+}}$,
- bounds on the hitting time/probability of a straight line of slope α by a system of random walks, controlled by ϕ_θ.

A first step, using these tools, consists in proving tail bounds for the random variables $S_{n+1} - S_n$ and $r_{S_{n+1}} - r_{S_n}$ on the event that $D_n < +\infty$, conditional upon $\mathcal{F}^R_{S_n}$. Basically, the idea is that, due to the ballisticity of the front, the failure of S_n to be an α-separation time can be detected by looking at trajectories "around" (S_n, r_{S_n}), while, for the same reason, the post-D_n conditions that characterize S'_{n+1} and S_{n+1} have to be satisfied within a "short" interval of time. The bounds are stated in Proposition 11, where the time intervals $[S_n, S'_n]$ and $[S'_n, S_{n+1}]$ are dealt with separately.

Apart from well-controlled deterministic quantities, these bounds involve the random quantity $\mathcal{M}_n$, defined for all $n \geq 1$, on the event $\{S_n < +\infty\}$, by

$$\mathcal{M}_n := \sum_{(W,u)\in R^*_{S_n}} e^{-\theta(r_{S_n} - W_{S_n})}, \tag{41}$$

where $R^*_{S_n}$ is defined in Section 4.2. Broadly speaking, $\mathcal{M}_n$ measures the accumulation of particles below the front position r_{S_n}, and the appearance of such a quantity in the bounds comes from the necessity to control the forward sub-α time property at time S_n, which is done with the help of the function ϕ_θ applied to the particles whose positions are $\leq r_{S_n}$ at time S_n.

To get rid of this random term and obtain deterministic tail bounds, we need to control the evolution of $\mathcal{M}_n$. Thus, the second step in the proof consists in establishing an affine induction inequality (Proposition 12) for the conditional expectation of $\mathcal{M}_n$, whose coefficient can be made < 1 for a suitable choice of the parameter L. The third, easier step, consists in proving tail bounds in the case $n = 1$ (i.e., for S_1, $\mathcal{M}_1$, etc.), stated in Proposition 13. With this step completed, the previous results can be combined to prove a uniform bound (Proposition 14) on the conditional expectation of $\mathcal{M}_n$ given that $D_{n-1} < +\infty$, which is the missing ingredient needed to obtain the suitable deterministic tail bounds on the renewal structure that imply finiteness of the second moments.

Before we enter the various steps of the proof, let us first note that, thanks to Proposition 9 and to the fact that $\alpha < \beta$, we know that, for $n \geq 0$, when $D_n < +\infty$, one almost surely has that $S_{n+1} < +\infty$.

4.7.1. *Step* 1: *Tail bounds conditional on* $\mathcal{F}^R_{S_n}$. The following proposition lists the various bounds we have.

PROPOSITION 11. *For all $n \geq 1$, for all $t > 0$ and $K > 0$, following bounds hold on $\{S_n < +\infty\}$*:

$$\begin{aligned} &\mathbb{P}_\nu(S'_{n+1} - S_n \geq t, D_n < +\infty | \mathcal{F}^R_{S_n}) \\ &\quad \leq e^\theta \mathcal{M}_n e^{-c_{10} t} + c_{11} t^{-c_{12}\mathscr{C}} \qquad \text{a.s.}, \end{aligned} \tag{42}$$

$$\begin{aligned} &\mathbb{P}_\nu(r_{S'_{n+1}} - r_{S_n} \geq K, D_n < +\infty | \mathcal{F}^R_{S_n}) \\ &\quad \leq \mathcal{M}_n e^{-c_{13} K} + c_{14} K^{-c_{15}\mathscr{C}} \qquad \text{a.s.}, \end{aligned} \tag{43}$$

$$\mathbb{P}_\nu(S_{n+1} - S'_{n+1} \geq t, D_n < +\infty | \mathcal{F}^R_{S'_{n+1}}) \leq c_{16} t^{-c_{17}\mathscr{C}} \qquad \text{a.s.}, \tag{44}$$

$$\mathbb{P}_\nu(r_{S_{n+1}} - r_{S'_{n+1}} \geq K, D_n < +\infty | \mathcal{F}^R_{S'_{n+1}}) \leq c_{18} K^{-c_{19}\mathscr{C}} \qquad \text{a.s.}, \tag{45}$$

where $c_{10}, \ldots, c_{19}$ are strictly positive constants, with c_{11}, c_{14} depending on $\mathscr{C}$, and c_{16}, c_{18} depending on $\mathscr{C}$ and L.

The proof of Proposition 11 relies on controlling the numbers of α-crossings in the relevant time intervals, so for $n \geq 1$, let $\mathcal{N}_n$ and $\mathcal{N}'_n$, respectively, denote the number of (S_n, α)-crossings contained in the time-interval $[S_n, S'_{n+1}]$, and the number of (S'_{n+1}, α)-crossing times contained in the time-interval $[S'_{n+1}, S_{n+1}]$. The key estimates on these variables are given in the following lemma.

LEMMA 3. *One has the following bounds*: *for all $n \geq 1$, for all $K > 0$, on $\{S_n < +\infty\}$*,

$$\mathbb{P}_\nu(\mathcal{N}_n \geq K, D_n < +\infty | \mathcal{F}^R_{S_n}) \leq e^\theta \mathcal{M}_n e^{-c_{20} K} + c_{21} K^{-c_{22}\mathscr{C}} \qquad \text{a.s.}, \tag{46}$$

$$\mathbb{P}_\nu(\mathcal{N}'_n \geq K, D_n < +\infty | \mathcal{F}^R_{S'_n}) \leq c_{23} K^{-c_{24}\mathscr{C}} \qquad \text{a.s.}, \tag{47}$$

where $c_{20}, \ldots, c_{24}$ are strictly positive constants, with c_{21} depending on $\mathscr{C}$ and c_{21} depending on $\mathscr{C}$ and L.

PROOF OF LEMMA 3. Let S''_{n+1} denote the infimum of the $t > D_n$ such that:

- t is a backward sub-α time;
- $\Upsilon_n \subset R_t$.

Let $\mathcal{N}^{(1)}_n$ and $\mathcal{N}^{(2)}_n$ denote, respectively, the numbers of (S_n, α)-crossings contained in the time-interval $[S_n, S''_{n+1}[$ and in the time-interval $[S''_{n+1}, S'_{n+1}]$, so that

$$\mathcal{N}_n = \mathcal{N}^{(1)}_n + \mathcal{N}^{(2)}_n. \tag{48}$$

Our first claim is that there exists a constant $d_1 < 1$, depending on $\mathscr{C}$, such that, for all $\ell \geq 1$,

$$\mathbb{P}_\nu(\mathcal{N}^{(2)}_n \geq \ell, D_n < +\infty | \mathcal{G}^R_{S_n}) \leq d_1^\ell \qquad \text{a.s.} \tag{49}$$

Assume that $D_n < +\infty$, and denote by $\tau_1, \tau_2, \dots$ the successive backward sub-α times posterior to S''_{n+1} (with $\tau_1 := S''_{n+1}$), and let $J := \inf\{j \geq 1; \Xi_{\tau_j} = 1\}$ (remember that $\Xi_t = 1$ means that there are at least $\mathscr{C}$ particles located at site r_t in B_t). By definition, we have $S'_{n+1} = \tau_J$. Since S_n is a backward sub-α time, any (S_n, α)-crossing in $[S''_{n+1}, S'_{n+1}]$ is a backward sub-α time, so we have

$$\mathcal{N}_n^{(2)} \leq J. \tag{50}$$

Now using an argument similar to the one leading to Proposition 10, we have that, for all $i \geq 1$, on $\{D_n < +\infty\}$, the distribution of $\pi_{r_{\tau_i}, \tau_i}(B_{\tau_i})$ conditional upon $\mathcal{G}^R_{\tau_i}$ is that of B_0 conditioned upon an event containing G, so that, on $\{D_n < +\infty\}$, one has the bound

$$\mathbb{P}_\nu\big(\Xi_{\tau_i} = 1 | \mathcal{G}^R_{\tau_i}\big) \geq \mathbb{P}_\nu\big(\{\Xi_0 = 1\} \cap G\big) \qquad \text{a.s.}$$

Since for all $i \geq 2$, the random variables $\Xi_{\tau_1}, \dots, \Xi_{\tau_{i-1}}$ are measurable with respect to $\mathcal{G}^R_{\tau_i}$, we deduce that, on $\{D_n < +\infty\}$,

$$\mathbb{P}_\nu\big(J \geq \ell | \mathcal{G}^R_{S''_{n+1}}\big) \leq \big(1 - \mathbb{P}_\nu\big(\{\Xi_0 = 1\} \cap G\big)\big)^\ell \qquad \text{a.s.} \tag{51}$$

Combining (50) and (51), we deduce (49), using also the fact that[15] $\mathcal{G}^R_{S_n} \subset \mathcal{G}^R_{S''_{n+1}}$ since $S_n \leq S''_{n+1}$ and S_n is $\mathcal{G}^R_{S''_{n+1}}$-measurable.

Now consider the event $\mathcal{N}_n^{(1)} > \ell$. Start with the case where S_n is not a backward super-α time, and call H_n the corresponding event. We first bound the probability that $W^{(n)}_{S_n} > r_{S_n} + \ell/2$. From Lemma 2, a random walk starting at $x \geq 0$ at time zero has a probability bounded above by $e^{-\theta x}$ to ever cross at a negative time the half-line of slope α starting at $(0, 0)$. Using Corollary 2 and the union bound over all the particle paths in B_{S_n}, we deduce that

$$\mathbb{P}_\nu\big(H_n, W^{(n)}_{S_n} > r_{S_n} + \ell/2 | \mathcal{G}^R_{S_n}\big) \leq c_9 \sum_{x > \ell/2} e^{-\theta x} \rho.$$

We deduce that

$$\mathbb{P}_\nu\big(H_n, W^{(n)}_{S_n} > r_{S_n} + \ell/2 | \mathcal{G}^R_{S_n}\big) \leq d_2 e^{-d_3 \ell}, \tag{52}$$

where d_2 and d_3 are strictly positive constants, with d_2, d_3 depending on $\mathscr{C}$. On the other hand, assume that $W^{(n)}_{S_n} \leq r_{S_n} + \ell/2$. Assume also that $\mathcal{N}_n^{(1)} > \ell$, and let t denote the time of the ℓth (S_n, α)-crossing posterior to S_n. By definition of $\mathcal{N}_n^{(1)}$, we must have that $(W^{(n)}, u^{(n)}) \notin R_t$, whence $W^{(n)}_t \geq r_t \geq r_{S_n} + \ell + \alpha(t - S_n)$.

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[15] Note that this property is not obvious. It is a consequence that it is enough to look at trajectories in R_{S_n} to check the backward super-α time property for S_j where $j \leq n - 1$. See the discussion before Proposition 10.

Since $W^{(n)}_{S_n} \le r_{S_n} + \ell/2$, this implies that $W^{(n)}_t \ge W^{(n)}_{S_n} + \ell/2 + \alpha(t - S_n)$. Using again Corollary 2, Lemma 2 and the union bound, we deduce that

$$\mathbb{P}_\nu\big(\mathcal{N}_n^{(1)} > \ell, H_n, W^{(n)}_{S_n} \le r_{S_n} + \ell/2 | \mathcal{G}^R_{S_n}\big) \le c_9\bigg(e^{-\theta\ell/2}\rho_{\mathscr{C}} + \sum_{1\le x\le \ell/2} e^{-\theta\ell/2}\rho\bigg),$$

where $\rho_{\mathscr{C}}$ denotes the expected value of a Poisson random variable of parameter ρ conditioned upon being $\ge \mathscr{C}$ (we have to use $\rho_{\mathscr{C}}$ since, under $\nu_{\mathscr{C},+}$, the number of particles at the origin has a Poisson distribution conditioned by taking a value $\ge \mathscr{C}$). We deduce that there exists a strictly positive constant d_4 depending on $\mathscr{C}$ and a strictly positive constant d_5 such that

$$\mathbb{P}_\nu\big(\mathcal{N}_n^{(1)} > \ell, H_n, W^{(n)}_{S_n} \le r_{S_n} + \ell/2 | \mathcal{G}^R_{S_n}\big) \le d_4 e^{-d_5\ell}. \tag{53}$$

Now consider the case where S_n is a backward super-α time. In this case, $\Upsilon = \varnothing$ and, by definition of S''_{n+1}, $\mathcal{N}_n^{(1)}$ is also the number of (S_n, α)-crossings contained in the time-interval $[S_n, D_n]$. Introduce $t_0 := \ell/C'_1(\rho)$ (remember that $C'_1(\rho)$ is defined in Proposition 16), assuming that ℓ is large enough so that $t_0 > \alpha^{-1}$, and consider the cases $D_n - S_n > t_0$ and $D_n - S_n \le t_0$ separately. Assume first that $D_n - S_n \le t_0$, and let t denote the time of the ℓth (S_n, α)-crossing posterior to S_n. The fact that $\mathcal{N}_n^{(1)} > \ell$ implies that $t < D_n$, while $r_t \ge r_{S_n} + \ell$. Moreover, since $t < D_n$, r_t is in fact equal to $r_{S_n} + r_{t-S_n}(\pi_{r_{S_n}, S_n}(B_{S_n}))$ since, by definition, particles in R_{S_n} cannot influence the front between time S_n and time D_n. As a consequence, $r_{t-S_n}(\pi_{r_{S_n}, S_n}(B_{S_n})) \ge \ell$, while $t - S_n \le t_0$. Using Corollary 2 and Proposition 8, we deduce that

$$\mathbb{P}_\nu\big(\mathcal{N}_n^{(1)} \ge \ell, H_n^c, D_n - S_n \le t_0 | \mathcal{G}^R_{S_n}\big) \le c_3 e^{-c_4 t_0}. \tag{54}$$

On the other hand, using again the fact that particles in R_{S_n} cannot influence the front prior between time S_n and time D_n, we see that, if $D_n - S_n > t_0$, at least one of the three following events must occur, according to which of the five conditions defining D_n corresponds to the smallest time (note that our assumption that $t_0 > \alpha^{-1}$ rules out (2) and (4)): for some $t \ge S_n + t_0$, $r_t(\pi_{r_{S_n}, S_n}(B_{S_n})) < \lfloor \alpha(t - S_n)\rfloor$, or there exists a particle path $(W, u) \in R_{S_n}$ such that $W_{S_n} \le r_{S_n} - 1$ and a $t \ge t_0$ such that $W_{S_n + t} \ge r_{S_n} - 1 + \alpha(t - S_n)$, or there exists a $t \ge t_0$ such that $W^{*n}_t > r_{S_n} - 1 + \alpha t$, while $W^{*n}_{S_n + \alpha^{-1}} = r_{S_n}$. Using Corollary 2, Proposition 9, Lemma 1 and Lemma 2, and the strong Markov property[16] at time S_n and $S_n + \alpha^{-1}$, we deduce by the union bound that

$$\begin{aligned}
&\mathbb{P}_\nu\big(\mathcal{N}_n^{(1)} \ge \ell, H_n^c, t_0 < D_n - S_n < +\infty | \mathcal{F}^R_{S_n}\big) \\
&\quad \le c_9 c_5 t^{-c_6 \cdot \mathscr{C}} + e^\theta \mathcal{M}_n e^{-\mu t_0} + e^{-\mu(t_0 - \alpha^{-1})}.
\end{aligned} \tag{55}$$

[16]The Markov property of the dynamics holds with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$, not $(\mathcal{F}^R_t)_{t\ge 0}$. Here, we use the fact that $\mathcal{F}^R_{S_n} \subset \mathcal{F}_{S_n}$, and also that $\mathcal{F}^R_{S_n} \subset \mathcal{G}^R_{S_n}$.

Putting together (48), (49), (52), (53), (54) and (55), we deduce the first part of the lemma, that is, the bound (46).

To prove (47), we define S'''_{n+1} as the infimum of the $t > S'_{n+1}$ such that t is a backward sub-α time and $]S'_{n+1}, t[$ contains at least L (S'_{n+1}, α)-crossing times, and let $\tau'_1, \tau'_2, \dots$ denote the successive backward sub-α times posterior to S'''_{n+1} (with $\tau'_1 := S'''_{n+1}$), and let $I := \inf\{i \geq 1, \Xi_{\tau'_i} = 1\}$. We have by definition that $\mathcal{N}'_n \leq L + I$. Arguing exactly as in the proof of (46), we can prove a bound of the form

$$\mathbb{P}_\nu\big(I \geq \ell, D_n < +\infty | \mathcal{G}^R_{S'_{n+1}}\big) \leq d_6^\ell \qquad \text{a.s.,} \tag{56}$$

where $d_6 < 1$ is a constant depending on $\mathscr{C}$, which leads to the desired bound on the tail of $\mathcal{N}'_n$. □

We now give the proof of Proposition 11, which heavily relies on Lemma 3 we have just proved.

PROOF OF PROPOSITION 11. We start with the proof of (42). Assume that $S'_{n+1} - S_n \geq t$. If $r_{S_n+t} \geq r_{S_n} + \beta t$, we deduce that there exist at least $\lfloor \frac{(\beta-\alpha)t}{2} \rfloor$ distinct (S_n, α)-crossing times in $[S_n, S_n + t]$, whence the fact that $\mathcal{N}_n \geq \lfloor \frac{(\beta-\alpha)t}{2} \rfloor$. On the other hand, using (21), Proposition 9 and Corollary 2, we see that

$$\mathbb{P}_\nu\big(r_{S_n+t} - r_{S_n} \leq \beta t, D_n < +\infty | \mathcal{F}^R_{S_n}\big) \leq c_9 c_5 t^{-c_6 \cdot \mathscr{C}} \qquad \text{a.s.}$$

The conclusion now follows from (46). To prove (43), we note that, by definition of α-crossing times, we have

$$r_{S'_{n+1}} \leq r_{S_n} + \alpha\big(S'_{n+1} - S_n\big) + \mathcal{N}_n + 1.$$

The result then follows from combining (42) and (46). The proof of (44) is similar to that of (42), using (47) instead of (46), Proposition 9 and an analog of Corollary 2 for S'_{n+1}. The proof of (45) goes as the proof of (43), building on (44) and (47). □

4.7.2. *Step* 2: *Evolution of* $\mathcal{M}_n$. The main estimate we prove is the following affine induction inequality for $\mathcal{M}_n$.

PROPOSITION 12. *For all $n \geq 1$, and all large enough $\mathscr{C}$, one has the following bounds*:

$$\mathbb{E}_\nu\big(\mathcal{M}_{n+1}\mathbf{1}(D_n < +\infty) | \mathcal{F}^R_{S_n}\big) \leq c_{25} e^{-\theta L} \mathcal{M}_n + c_{26},$$

where c_{25} is a strictly positive constant depending on $\mathscr{C}$, and c_{26} is a strictly positive constant depending on $\mathscr{C}$ and L.

The proof of 12 involves a martingale inequality, proved in Lemma 4, and quantitative bounds on $\mathcal{L}_n^{(1)}$:= number of particle paths in $B_{S_n} \cap R_{S'_{n+1}}$ and $\mathcal{L}_n^{(2)}$:= number of particle paths in $B_{S'_{n+1}} \cap R_{S_{n+1}}$, derived in Lemma 5 with the help of the results proved in Step 1.

LEMMA 4. *Consider $w \in \mathbb{S}_\theta$ such that there is at least one particle at site* 0. *Let T be an $(\mathcal{F}_t)_{t\geq 0}$ stopping time such that T is a backward sub-α time and* $]0, T[$ *contains a number of* $(0,\alpha)$*-crossing times at least equal to* $m \geq 0$. *Then one has the following bound*:

$$\mathbb{E}_w\Biggl(\sum_{(W,u)\in R_{0+}} e^{-\theta(r_T - W_T)}\mathbf{1}(T < +\infty)\Biggr) \leq e^{-\theta m}\phi_\theta(w).$$

PROOF. Consider $(W,u) \in R_{0+}$, and, for all $s \geq 0$, let

$$M_s := e^{\theta W_s - 2(\cosh(\theta)-1)s}.$$

Then $(M_s)_{s\geq 0}$ is a càdlàg martingale. Since T is a stopping time, we have, for all finite $K > 0$, that

$$\mathbb{E}_w(M_{T\wedge K}) = \mathbb{E}_w(M_0) = e^{\theta W_0}. \tag{57}$$

Now we have that $\liminf_{K\to+\infty} M_{T\wedge K} \geq M_T \mathbf{1}(T < +\infty)$, so that, by Fatou's lemma and (57),

$$\mathbb{E}_w\bigl(M_T \mathbf{1}(T < +\infty)\bigr) \leq e^{\theta W_0}. \tag{58}$$

Now, from our assumptions on T and the fact that $r_0 = 0$, one has that, on $\{T < +\infty\}$, $r_T \geq \alpha T + m$. Using the fact that, by (15), $2(\cosh(\theta) - 1) \leq \alpha\theta$, we deduce that

$$2\bigl(\cosh(\theta) - 1\bigr)T \leq 2\bigl(\cosh(\theta) - 1\bigr)\left(\frac{r_T - m}{\alpha}\right) \leq \theta(r_T - m).$$

Writing

$$-\theta r_T + \theta W_T = -\theta r_T + 2\bigl(\cosh(\theta) - 1\bigr)T - 2\bigl(\cosh(\theta) - 1\bigr)T + \theta W_T,$$

we finally deduce that, on $\{T < +\infty\}$,

$$-\theta r_T + \theta W_T \leq -\theta m - 2\bigl(\cosh(\theta) - 1\bigr)T + \theta W_T.$$

In view of (58), we deduce that

$$\mathbb{E}_w\bigl(e^{-\theta(r_T - W_T)}\mathbf{1}(T < +\infty)\bigr) \leq e^{-\theta m}\mathbb{E}_w\bigl(M_T \mathbf{1}(T < +\infty)\bigr) \leq e^{-\theta m}e^{\theta W_0}.$$

The result now follows from summing the above inequality over all $(W,u) \in R_{0+}$. $\square$

LEMMA 5. *For all $n \geq 1$, and all large enough $\mathscr{C}$, one has the following bounds*:

$$\mathbb{E}_\nu\big(\mathcal{L}_n^{(1)}\mathbf{1}(D_n < +\infty)|\mathcal{F}_{S_n}^R\big) \leq c_{27} + c_{28}\mathcal{M}_n \qquad a.s., \tag{59}$$

$$\mathbb{E}_\nu\big(\mathcal{L}_n^{(2)}\mathbf{1}(D_n < +\infty)|\mathcal{F}_{S_n}^R\big) \leq c_{29} \qquad a.s., \tag{60}$$

where c_{27} and c_{28} are strictly positive constants depending on $\mathscr{C}$ and c_{29} is a strictly positive constant depending on $\mathscr{C}$ and L.

PROOF. We start with the proof of (59). Assume that $S'_{n+1} \leq S_n + t$ and that $r_{S'_{n+1}} \leq r_{S_n} + K$ for some $t, K > 0$. We can then bound $\mathcal{L}_n^{(1)}$ by counting the total number of particle paths (W, u) in B_{S_n} for which there exists $s \in [S_n, S_n + t]$ such that $W_s \in [r_{S_n}, r_{S_n} + K]$. This number includes all the particle paths (W, u) in B_{S_n} such that $W_{S_n} \in [r_{S_n}, r_{S_n} + K]$, plus the particle paths in B_{S_n} such that $W_{S_n} \geq r_{S_n} + K + 1$ that hit level $r_{S_n} + K$ during the time-interval $[S_n, S_n + t]$. Assume that we start with $\mathbb{P}_{\nu_{\mathscr{C},+}}$, and let $\mathcal{K}_1$ denote the number of particle paths (W, u) in B_0 such that $W_0 \in [0, K]$, while $\mathcal{K}_2$ denotes the number of particle paths in B_0 such that $W_0 \geq r_0 + K + 1$ that hit level K during the time-interval $[0, t]$. By standard properties of the Poisson distribution, we see that $\mathcal{K}_2$ is a Poisson random variable with distribution ρg, where

$$g := \sum_{x \geq K+1} P_x\Big(\inf_{s\in[0,t]} \zeta_s \leq K\Big).$$

Using the reflection principle, we see that $g \leq g'$, where

$$g' = 2 \sum_{x \geq K+1} P_x(\zeta_t \leq K).$$

Now using translation invariance, we can rewrite

$$g' = 2\sum_{x\geq 1} P_x(\zeta_t \leq 0) = 2\sum_{x\geq 1} P_0(x + \zeta_t \leq 0) = 2E_0\big(-\zeta_t\mathbf{1}(\zeta_t \leq -1)\big).$$

Using Schwarz's inequality, we deduce that $g' \leq 2\sqrt{2t}$. On the other hand, $\mathcal{K}_1$ is the sum of Ξ_0, whose distribution is that of a Poisson random variable of parameter ρ conditioned to be $\geq \mathscr{C}$, and of a Poisson random variable of parameter ρK, these two variables being independent, and independent from $\mathcal{K}_2$. Using Corollary 2, we deduce that, for some strictly positive constant d_1 depending on $\mathscr{C}$,

$$\mathbb{P}_\nu\big(\mathcal{L}_n^{(1)} \geq \ell, S'_{n+1} \leq S_n + t, r_{S'_{n+1}} \leq r_{S_n} + K|\mathcal{G}_{S_n}^R\big) \leq d_1 a_{t,K}(\ell), \tag{61}$$

where $a_{t,K}(\ell)$ denotes the probability for a Poisson random variable with parameter $\rho(K + 1 + 2\sqrt{2t})$ to be $\geq \ell$. Now consider two strictly positive constants b_1, b_2 with $\rho b_1 < 1$. Note that, for $K := b_1\ell$ and $t := b_2\ell$, one has, by standard large deviations bounds for Poisson random variables (see, e.g., [6]), that for all $\ell \geq 1$,

$$a_{t,K}(\ell) \leq d_2 e^{-d_3 \ell}, \tag{62}$$

where d_2, d_3 are strictly positive constants. Combining (42) and (43), we deduce that, on $\{S_n < +\infty\}$,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}\big(\mathcal{V}_n(\ell)^c, D_n < +\infty | \mathcal{F}^R_{S_n}\big) \leq e^{\theta} \mathcal{M}_n e^{-d_4 \ell} + d_5 \ell^{-d_6 \mathscr{C}} \qquad \text{a.s.}, \tag{63}$$

with

$$\mathcal{V}_n(\ell) := \{S'_{n+1} \leq S_n + b_2 \ell\} \cap \{r_{S'_{n+1}} \leq r_{S_n} + b_1 \ell\},$$

and where d_4, d_5, d_6 are strictly positive constants, d_5 depending on $\mathscr{C}$. Combining (61), (62) and (63), we deduce (59).

To prove (60), we use the same argument as in the proof of (59), with (44) and (45) replacing (42) and (43), respectively. $\square$

We are now ready to prove the affine induction inequality on $\mathcal{M}_n$.

PROOF OF PROPOSITION 12. Define

$$\mathcal{M}'_{n+1} := \sum_{(W,u) \in R^*_{S'_{n+1}}} \exp\big(-\theta (r_{S'_{n+1}} - W_{S'_{n+1}})\big),$$

where $R^*_{S'_{n+1}}$ is defined as the set $R_{S'_{n+1}}$ from which we remove the particle path that makes the front climb at time S'_{n+1}. By definition, we have that $\mathcal{M}'_{n+1} \leq \mathcal{A}^{(1)} + \mathcal{A}^{(2)}$, with

$$\mathcal{A}^{(1)} := \sum_{(W,u) \in R_{S_n}} \exp\big(-\theta (r_{S'_{n+1}} - W_{S'_{n+1}})\big)$$

and

$$\mathcal{A}^{(2)} := \sum_{(W,u) \in B_{S_n} \cap R_{S'_{n+1}}} \exp\big(-\theta (r_{S'_{n+1}} - W_{S'_{n+1}})\big).$$

First, using the fact that for each $(W,u) \in R_{S'_{n+1}}$, one has $W_{S'_{n+1}} \leq r_{S'_{n+1}}$, we have the bound

$$\mathcal{A}^{(2)} \leq \mathcal{L}^{(1)}_n. \tag{64}$$

Now using Lemma 4, we deduce that

$$E_\nu\big(\mathcal{A}^{(1)} \mathbf{1}(D_n < +\infty) | \mathcal{F}^R_{S_n}\big) \leq \mathcal{M}_n + 1, \tag{65}$$

where the $+1$ term comes from the fact that the definition of $\mathcal{M}_n$ involves the particles in $R^*_{S_n}$, not R_{S_n}, so we have to add the contribution to $\mathcal{A}^{(1)}$ of the particle path (W^{*n}, u^{*n}), which we bound by 1. Now we have that $\mathcal{M}_{n+1} \le \mathcal{B}^{(1)} + \mathcal{B}^{(2)}$, with

$$\mathcal{B}^{(1)} := \sum_{(W,u)\in R_{S'_{n+1}}} \exp\bigl(-\theta(r_{S_{n+1}} - W_{S_{n+1}})\bigr)$$

and

$$\mathcal{B}^{(2)} := \sum_{(W,u)\in B_{S'_{n+1}}\cap R_{S_{n+1}}} \exp\bigl(-\theta(r_{S_{n+1}} - W_{S_{n+1}})\bigr).$$

As in (64), we have the bound

$$\mathcal{B}^{(2)} \le \mathcal{L}_n^{(2)}. \tag{66}$$

On the other hand, using Lemma 4, we deduce that

$$E_\nu\bigl(\mathcal{B}^{(1)}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S'_{n+1}}\bigr) \le e^{-\theta L}\mathcal{M}'_{n+1} + 1. \tag{67}$$

Combining (64), (65), (66), (67) and using the fact that $\mathcal{F}^R_{S_n} \subset \mathcal{F}^R_{S'_{n+1}}$, we deduce that, on $\{S_n < +\infty\}$,

$$\begin{aligned}\mathbb{E}_\nu\bigl(\mathcal{M}_{n+1}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}\bigr) \le e^{-\theta L}\mathcal{M}_n + e^{-\theta L} + 1 \\ + e^{-\theta L}\mathbb{E}_\nu\bigl(\mathcal{L}_n^{(1)}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}\bigr) \\ + \mathbb{E}_\nu\bigl(\mathcal{L}_n^{(2)}\mathbf{1}(D_n < +\infty)|\mathcal{F}^R_{S_n}\bigr).\end{aligned}$$

The conclusion now follows from Lemma 5. □

4.7.3. *Step* 3: *Tail bounds for* $n = 1$. So far, we have proved results dealing with the behavior of the system during the time-interval $[S_n, S_{n+1}]$, for $n \ge 1$. The case of the interval $[0, S_1]$ is a little bit different since it starts at time $D_0 = 0$, where not all the properties of times $S_n, n \ge 1$ are met. However, the distribution of (R_0, B_0) is exactly known, and, in this case, Proposition 7 directly yields the estimates that we obtained by a combination of Proposition 9 and Corollary 2 in the case $[S_n, S_{n+1}]$. We merely state the relevant results, whose proofs are similar (and simpler) than those given in steps 1 and 2.

PROPOSITION 13. *One has the following bounds*:

$$\mathbb{P}_\nu\bigl(S'_1 \ge t\bigr) \le c_{30}t^{-c_{31}\mathscr{C}} \qquad a.s., \tag{68}$$

$$\mathbb{P}_\nu\bigl(S_1 - S'_1 \ge t\bigr) \le c_{32}t^{-c_{33}\mathscr{C}} \qquad a.s., \tag{69}$$

$$\mathbb{E}_\nu(\mathcal{M}_1) \le c_{34}, \tag{70}$$

where $c_{30}, \dots, c_{34}$ *are strictly positive constants*, *with* c_{30} *depending on* $\mathscr{C}$, *and* c_{32}, c_{34} *depending on* $\mathscr{C}$ *and* L.

4.7.4. *Conclusion.* We now put together the different pieces established in the previous steps. The first result is a uniform bound on the conditional expectation of $\mathcal{M}_n$ given that the first $n-1$ attempts at obtaining an α-separation time have failed.

PROPOSITION 14. *For all large enough $\mathscr{C}$, and all large enough L (depending on $\mathscr{C}$), there exists $c_{35} < +\infty$, depending on $\mathscr{C}$ and L, such that, for all $n \geq 1$,*

$$\mathbb{E}_\nu(\mathcal{M}_n | D_{n-1} < +\infty) \leq c_{35}.$$

PROOF. For $n = 1$, the result is just (70). Consider $n \geq 1$, and write

$$\mathbb{E}_\nu(\mathcal{M}_{n+1} | D_n < +\infty) = \frac{\mathbb{E}_\nu(\mathcal{M}_{n+1}\mathbf{1}(D_n < +\infty) | D_{n-1} < +\infty)}{\mathbb{P}_\nu(D_n < +\infty | D_{n-1} < +\infty)}. \tag{71}$$

Using Proposition 12 and the fact that the event $D_{n-1} < +\infty$ is measurable with respect to $\mathcal{F}^R_{S_n}$, we deduce that

$$\mathbb{E}_\nu\big(\mathcal{M}_{n+1}\mathbf{1}(D_n < +\infty) | D_{n-1} < +\infty\big) \leq c_{25} e^{-\theta L} \mathbb{E}_\nu(\mathcal{M}_n | D_{n-1} < +\infty) + c_{26}.$$

On the other hand, observe that there exists a strictly positive constant d_1 such that

$$\mathbb{P}_\nu(D_n < +\infty | D_{n-1} < +\infty) \geq d_1, \tag{72}$$

considering, for example, the probability for the particle that makes the front climb at time S_n to cross at a time $> S_n$ the half-line of slope α starting at (S_n, r_{S_n}). Combining (71) and (72), we deduce that

$$\mathbb{E}_\nu(\mathcal{M}_{n+1} | D_n < +\infty) \leq d_1^{-1} c_{25} e^{-\theta L} \mathbb{E}_\nu(\mathcal{M}_n | D_{n-1} < +\infty) + d_1^{-1} c_{26}.$$

When L is large enough so that $d_1^{-1} c_{25} e^{-\theta L} < 1$, we deduce, using also (70), that the sequence $(\mathbb{E}_\nu(\mathcal{M}_n | D_{n-1} < +\infty))_{n \geq 1}$ is bounded. $\square$

We are now ready to prove our main estimate on the regeneration structure, namely, Proposition 4.

PROOF OF PROPOSITION 4. In this proof, we assume that $\mathscr{C}$ and L are large enough so that all the previous results hold. Remember the definition $\mathfrak{K} = \inf\{n \geq 1; D_n = +\infty\}$. Our first claim is that, for some strictly positive constant d_1 depending on $\mathscr{C}$ and L, for all $k \geq 1$,

$$\mathbb{P}_\nu(\mathfrak{K} \geq k) \leq d_1^k. \tag{73}$$

From Corollary 1, we have that

$$d_2 := \mathbb{P}_{\nu_{\mathscr{C},+}}\big(\{0 \text{ is a forward super } \alpha\text{-time for } B_0\} \cap G\big) > 0.$$

Consider $n \geq 1$. Using Proposition 10 and the fact that

$$G \subset G\big(Q^{(n)}, (r_{s+S_n} - r_{S_n})_{-S_n \leq s \leq 0}, -r_{S_n}\big),$$

we deduce that, on $\{D_{n-1} < +\infty\}$, one has that

$$\mathbb{P}_\nu\big(S_n \text{ is a forward and backward super } \alpha\text{-time for } B_{S_n} \,|\mathcal{G}^R_{S_n}\big) \geq d_2 \qquad \text{a.s.}$$

On the other hand, the event that S_n is a forward sub α-time is measurable with respect to $\mathcal{G}^R_{S_n}$. Call d_3 the probability for a random walk starting at zero to remain at zero during the time-interval $[0, \alpha^{-1}]$ and then to satisfy $W_s \leq \alpha s - 1$ for all $s \geq \alpha^{-1}$, which is > 0 by Lemma 2. Using Lemma 1, we deduce that, for arbitrary $K > 0$, on $\{D_{n-1} < +\infty\}$,

$$\mathbb{P}_\nu\big(S_n \text{ is a forward sub } \alpha\text{-time}|\mathcal{F}^R_{S_n}\big) \geq d_3 g(K)\mathbf{1}(\mathcal{M}_n \leq K) \qquad \text{a.s.}$$

We deduce that

$$\mathbb{P}_\nu(D_n < +\infty|D_{n-1} < +\infty) \geq d_2 d_3 g(K)\mathbb{P}_\nu(\mathcal{M}_n \leq K|D_{n-1} < +\infty).$$

By Proposition 14, we have that $\mathbb{E}_\nu(\mathcal{M}_n|D_{n-1} < +\infty) \leq c_{35}$, so that, by Markov's inequality, $\mathbb{P}_\nu(\mathcal{M}_n \leq 2c_{35}|D_{n-1} < +\infty) \geq 1/2$. Setting $K := 2c_{35}$ and $d_1 := 1/2(d_2 d_3 g(K))$, we see that (73) is proved.

Now observe that, by definition, $S_{\mathfrak{K}}$ is an α-separation time. As a consequence, we have that $\kappa_1 \leq S_{\mathfrak{K}}$. Writing $S_{\mathfrak{K}} := S_1 + \sum_{k=1}^{\mathfrak{K}-1}(S_{k+1} - S_k)$, we deduce that for all t and $n \geq 1$,

$$\begin{aligned}\mathbb{P}_\nu(\kappa_1 \geq t) \leq{}& \mathbb{P}_\nu(\mathfrak{K} > n) + \mathbb{P}_\nu(S_1 \geq t/n)\\ &+ \sum_{k=1}^{n-1}\mathbb{P}_\nu(S_{k+1} - S_k \geq t/n, D_k < +\infty).\end{aligned} \tag{74}$$

Let $t' := t/n$. Using (68) and (69), we deduce that

$$\mathbb{P}_\nu(S_1 \geq t/n) \leq c_{30}\big(t'/2\big)^{-c_{31}\mathscr{C}} + c_{32}\big(t'/2\big)^{-c_{33}\mathscr{C}}. \tag{75}$$

On the other hand, one has that

$$\begin{aligned}&\mathbb{P}_\nu(S_{k+1} - S_k \geq t/n, D_k < +\infty)\\ &\qquad \leq \mathbb{P}_\nu(S_{k+1} - S_k \geq t/n, D_k < +\infty|D_{k-1} < +\infty),\end{aligned}$$

and, using (42), (44) and Proposition 14, we deduce that

$$\begin{aligned}&\mathbb{P}_\nu(S_{k+1} - S_k \geq t/n, D_k < +\infty)\\ &\qquad \leq c_{35}e^{-c_{10}(t'/2)} + c_{11}\big(t'/2\big)^{-c_{12}\mathscr{C}} + c_{16}\big(t'/2\big)^{-c_{17}\mathscr{C}}.\end{aligned} \tag{76}$$

Choosing, for example, $n := \lceil t^{1/2}\rceil$, and using (73), (75) and (76) to bound the terms in (74), we deduce that

$$\mathbb{P}_\nu(\kappa_1 \geq t) \leq d_4 t^{-d_5\mathscr{C}+1/2}, \tag{77}$$

where d_4 and d_5 are strictly positive constants, with d_4 depending on $\mathscr{C}$ and L. Choosing $\mathscr{C}$ large enough, this proves the fact that κ_1 has a finite second moment. Now write

$$\mathbb{P}_\nu(r_{\kappa_1} \geq \ell) \leq \mathbb{P}_\nu(\kappa_1 > t) + \mathbb{P}_\nu\Big(\sup_{s\in[0,t]} r_s \geq \ell\Big).$$

Choosing $t := \ell/C_1'(\rho)$, and using Proposition 8 and (77), we deduce that

$$\mathbb{P}_\nu(r_{\kappa_1} \geq \ell) \leq d_6 \ell^{-d_5\mathscr{C}+1/2}, \tag{78}$$

where d_6 is a strictly positive constant depending on $\mathscr{C}$ and L. Choosing $\mathscr{C}$ large enough, this proves the fact that r_{κ_1} has a finite second moment. □

5. Extension to the case $D_R > D_B$. In this section, we briefly explain how to extend the approach leading to Theorem 1 for the single-rate KS model, to the remanent KS model with $D_R \geq D_B$, leading to Theorems 2 and 3. The basic idea is to express the random walk trajectories (W, u) actually followed by particles in the model where $D_R > D_B$, as time-changed trajectories, with respect to random walk trajectories $(\mathcal{W}, u)$ with constant jump rate. The key observation, stated as Lemma 6 below, is that, due to the fact that $D_R > D_B$, the set of particle trajectories (W, u) that are blue at an upward time for the front, coincides with the set of particle trajectories $(\mathcal{W}, u)$ whose position at time t is above the position of the front (except for the one that makes the front climb at time t). Finally, the remanent character of the infection makes it possible to deduce ballisticity estimates for the front by coupling with a single-rate model.

We now rigorously define the infection dynamics of the remanent infection model for $D_R > D_B$, assuming without loss of generality that $D_B = 2$. To emphasize the similarities, we use as much as possible the same notation that were already used for the single-rate KS infection model.

We use a construction of the dynamics with $D_R > D_B = 2$ that uses random walk trajectories $(\mathcal{W}, u)$ with constant jump rate 2, for which our reference probability space for paths $(\mathcal{W}, u)$ is $(\Omega, \mathcal{F}, \mathbb{P}_w)$. As long as a particle is blue, it follows the corresponding trajectory in the usual way, while as soon as it is turned into a red particle, it starts following the trajectory with a speed multiplied by a factor $D_R/2$. As a result, the actual path (W, u) followed by a particle is related to the path $(\mathcal{W}, u) \in \Omega$ by a time-change, which we describe below.

Let us first define the trajectory of the front. Since, by definition, the front can only perform upward jumps, it makes sense to start with $r_0 := 0$, which leads to the simplification that $r_{T_k} := k$ for all $k \geq 0$. We start with $T_0 := 0$, $r_0 := 0$, and define inductively the sequence $(T_k)_{k\geq 0}$ together with the value of $(r_t)_{t\in[0,T_k]}$. Consider $t > T_\ell$. We say that t is upward if there exists $(\mathcal{W}, u) \in \Psi$ such that $\mathcal{W}_s \leq r_s$ for some $s \in [0, t[$ and such that $\mathcal{W}_{v-} = \ell$ and $\mathcal{W}_v = \ell + 1$, where

$$v := \tau + \frac{D_R}{2}(t - \tau), \qquad \tau := \inf\{s \in [0, t[; \mathcal{W}_s \leq r_s\}. \tag{79}$$

Then let

$$T_{\ell+1} := \inf\{t > T_\ell;\ t \text{ is upward}\},$$

and

$$r_t := \ell \qquad \text{on } [T_\ell, T_{\ell+1}[.$$

The sets R_t and B_t of red and blue particles at time t are then defined exactly as in the single-rate KS infection model, namely

$$B_t := \{(\mathcal{W}, u) \in \Psi; \forall s \in [0, t[, \mathcal{W}_s > r_s\},$$
$$R_t := \{(\mathcal{W}, u) \in \Psi; \exists s \in [0, t[, \mathcal{W}_s \le r_s\}.$$

We now properly define (W, u) as a time-changed version of $(\mathcal{W}, u)$. Using the notation defined in (79), we let $W_t := \mathcal{W}_t$ for $t \in [0, \tau]$ and $W_t := \mathcal{W}_v$ for $t > \tau$. This construction is illustrated in Figure 6.

We now make the following key remark.

LEMMA 6. *For all $k \ge 1$, the set B_{T_k} coincides with the set of $(\mathcal{W}, u) \in \Psi$ such that $\mathcal{W}_{T_k} \ge k$, minus the particle that makes the front climb at time T_k.*

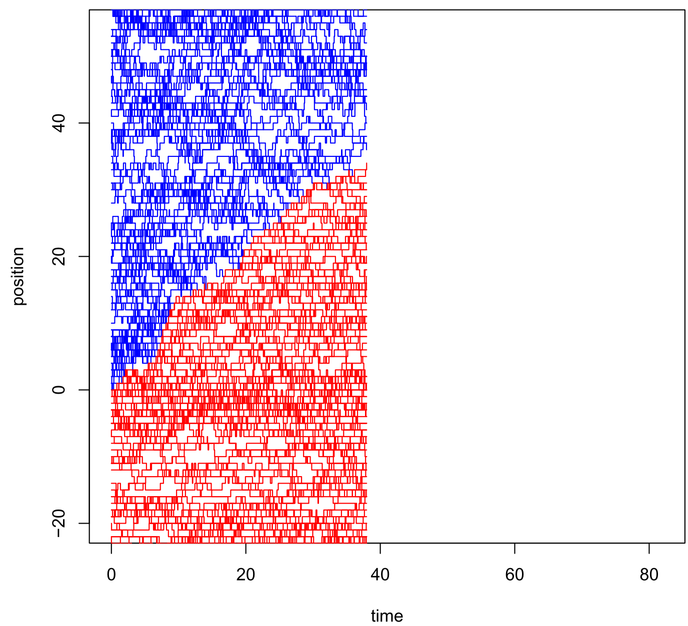
Note that the above result is an immediate consequence of the definition when $D_R = D_B$, but not in the present case, due to the time-change.

PROOF. One inclusion is immediate: a particle path $(\mathcal{W}, u)$ in B_{T_k} evolves using the jump rate $D_B = 2$ up to at least time T_k, so that $\mathcal{W}_{T_k}$ indeed corresponds to the position W_{T_k} of the corresponding particle at time T_k, and must by definition be $\ge k$. On the other hand, assume that a $(\mathcal{W}, u) \in R_{T_k}$ is such that $\mathcal{W}_{T_k} \ge k$, and hits (or lies below, to include particles in R_{0+}) the front for the first time at a time $\tau < T_k$. Introduce the time $t := \tau + (T_k - \tau)\frac{2}{D_R}$. Since $D_R > D_B = 2$, we have $t < T_k$, and by definition one has $W_t = \mathcal{W}_{T_k} \ge k$, whence the existence of a red particle above k at a time $< T_k$, which contradicts the definition of T_k. □

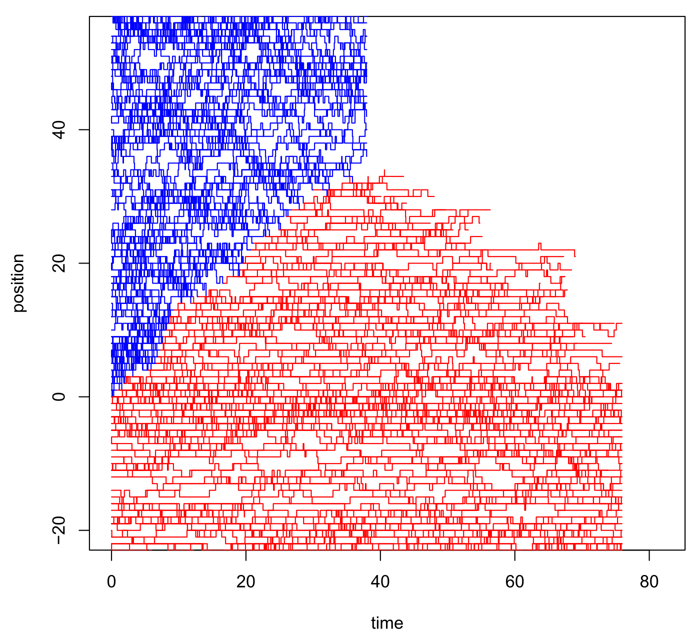
One now defines the renewal structure exactly as for the single-rate KS infection model, but with the time-changed trajectories W replacing the trajectories $\mathcal{W}$. Similarly, we can define

$$\mathcal{F}_t^R := \sigma((W_s, u); s \le t, (\mathcal{W}, u) \in R_t),$$
$$\mathcal{F}_T^R := \sigma(T, r_T) \vee \sigma((W_s, u); s \le t, (\mathcal{W}, u) \in R_T),$$
$$\mathcal{G}_t^R := \sigma((W_s, u); s \in \mathbb{R}, (\mathcal{W}, u) \in R_t),$$
$$\mathcal{G}_T^R := \sigma(T, r_T) \vee \sigma((W_s, u); s \in \mathbb{R}, (\mathcal{W}, u) \in R_T).$$

Note that it does not matter whether we define the σ-algebras $\mathcal{G}_t^R$ using the original or time-changed trajectories, since in both cases the history of the front up to time



(a) Actual trajectories



(b) Time-changed trajectories with constant jump rate $D_B = 1$

FIG. 6. *Realization of the KS infection model with* $D_R = 2$ *and* $D_B = 1$. *The actual evolution of the process is shown in* (a). *The evolution of the corresponding time-changed trajectories with a constant jump rate* $D_B = 1$ *is shown in* (b).

t is measurable, due to the fact that the σ-algebra includes the full trajectories (and not just the trajectories up to time t). The same remark is valid for $\mathcal{G}_T^R$, where T is a nonnegative random time. With the help of Lemma 6, and of the fact that, for any $(\mathcal{W}, u) \in B_{T_k}$, one has $W_s = \mathcal{W}_s$ for all $s \leq T_k$, it is then possible to re-prove Propositions 3 in exactly the same way as for the single-rate KS infection model.

The key advantage of introducing remanence in the model is that, when $D_R > D_B = 2$, a comparison holds with the single rate model with jump rate equal to 2.

LEMMA 7. *Let ${}^{(1)}r_t$ denote the front of the single-rate KS model with rate* 2, *and ${}^{(2)}r_t$ denote the front of the remanent KS model. If $D_R > D_B = 2$, one has that ${}^{(1)}r_t \leq {}^{(2)}r_t$ for any t.*

The above lemma, combined with Proposition 9, yields the key ballisticity estimate needed to reprove the estimates of Section 4 for the remanent KS infection model. The two additional results we need are the following: a version of the strong Markov property restricted to R_T, and an upper bound on the speed exactly similar to Proposition 6. Specifically, we have the following.

PROPOSITION 15. *The strong Markov property holds for our process*: *for all $w \in \mathbb{S}_\theta$, all nonnegative $(\mathcal{F}_t^R)_{t\geq 0}$-stopping time T, and bounded measurable function F on $\mathcal{D}_+$, one has that*, *on* $\{T < +\infty\}$,

$$\mathbb{E}_w\big(F\big(X(R_T)\big)|\mathcal{F}_T^R\big) = \mathbb{E}_{X_T(R_T)}\big(F(X)\big) \qquad \mathbb{P}_w\text{-a.s.}, \tag{80}$$

where we use the notation $X := (X_t)_{t\geq 0}$.

PROPOSITION 16. *For the remanent KS infection model*, *there exist a constant $C_1^\star(\rho) > 0$ and a constant c_{36}*, *depending on ρ and $\mathscr{C}$*, *such that*, *for every $t > 0$*,

$$\mathbb{P}_{\nu_{\mathscr{C},+}}\big(r_t \geq C_1^\star(\rho)t\big) \leq c_{36}\exp(-t).$$

It is then possible to reprove all the estimates of Section 4, the only difference being that, at some places, estimates for a random walk with constant jump rate 2 have to be replaced by estimates for a random walk whose jump rate may change from $D_B = 2$ to $D_R > 2$ at some time-point. These estimates are obtained by a simple comparison with a random walk with constant jump rate equal to D_R. One then obtains Proposition 4, leading to the proof of the law of large numbers (Theorem 2), and the central limit theorem (Theorem 3).

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