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JUMP-TYPE HUNT PROCESSES GENERATED BY LOWER BOUNDED SEMI-DIRICHLET FORMS

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Let E be a locally compact separable metric space and m be a positive Radon measure on it. Given a nonnegative function k defined on $E \times E$ off the diagonal whose anti-symmetric part is assumed to be less singular than the symmetric part, we construct an associated regular lower bounded semi-Dirichlet form η on $L^2(E; m)$ producing a Hunt process X^0 on E whose jump behaviours are governed by k. For an arbitrary open subset $D \subset E$, we also construct a Hunt process $X^{D,0}$ on D in an analogous manner. When D is relatively compact, we show that $X^{D,0}$ is censored in the sense that it admits no killing inside D and killed only when the path approaches to the boundary. When E is a d-dimensional Euclidean space and m is the Lebesgue measure, a typical example of X^0 is the stable-like process that will be also identified with the solution of a martingale problem up to an η-polar set of starting points. Approachability to the boundary ∂D in finite time of its censored process $X^{D,0}$ on a bounded open subset D will be examined in terms of the polarity of ∂D for the symmetric stable processes with indices that bound the variable exponent $\alpha(x)$.

1. Introduction. Let E be a locally compact separable metric space equipped with a metric d, m be a positive Radon measure with full topological support and $k(x, y)$ be a nonnegative Borel measurable function on the space $E \times E \setminus$ diag, where diag denotes the diagonal set $\{(x, x) : x \in E\}$. A purpose of the present paper is to construct Hunt processes on E and on its subsets with jump behaviors being governed by the kernel k by using general results on a lower bounded semi-Dirichlet form on $L^2(E; m)$.

The inner product and the norm in $L^2(E; m)$ are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Let $\mathcal{F}$ be a dense linear subspace of $L^2(E; m)$ such that $u \wedge 1 \in \mathcal{F}$ whenever $u \in \mathcal{F}$. A (not necessarily symmetric) bilinear form η on $\mathcal{F}$ is called a *lower bounded closed form* if the following three conditions are satisfied: we set $\eta_\beta(u, v) = \eta(u, v) + \beta(u, v)$, $u, v \in \mathcal{F}$. There exists a $\beta_0 \geq 0$ such that:

(B.1) (lower boundedness); for any $u \in \mathcal{F}$, $\eta_{\beta_0}(u, u) \geq 0$.
(B.2) (sector condition); for any $u, v \in \mathcal{F}$,

$$|\eta(u, v)| \leq K\sqrt{\eta_{\beta_0}(u, u)} \cdot \sqrt{\eta_{\beta_0}(v, v)}$$

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for some constant $K \geq 1$.

(B.3) (completeness); the space $\mathcal{F}$ is complete with respect to the norm $\eta_\alpha^{1/2}(\cdot,\cdot)$ for some, or equivalently, for all $\alpha > \beta_0$.

For a lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E;m)$, there exist unique semigroups $\{T_t; t > 0\}$, $\{\widehat{T}_t; t > 0\}$ of linear operators on $L^2(E;m)$ satisfying

$$(T_t f, g) = (f, \widehat{T}_t g), \qquad f, g \in L^2(E;m), \|T_t\| \leq e^{\beta_0 t}, \|\widehat{T}_t\| \leq e^{\beta_0 t}, t > 0, \tag{1.1}$$

such that their Laplace transforms G_α and $\widehat{G}_\alpha$ are determined for $\alpha > \beta_0$ by

$$G_\alpha f, \widehat{G}_\alpha f \in \mathcal{F}, \qquad \eta_\alpha(G_\alpha f, u) = \eta_\alpha(u, \widehat{G}_\alpha f) = (f, u),$$
$$f \in L^2(E;m), u \in \mathcal{F}.$$

See the first part of Section 3 for more details. $\{T_t; t > 0\}$ is said to be *Markovian* if $0 \leq T_t f \leq 1, t > 0$, whenever $f \in L^2(E;m), 0 \leq f \leq 1$. It was shown by Kunita [15] that the semigroup $\{T_t; t > 0\}$ is Markovian if and only if

$$Uu \in \mathcal{F} \quad \text{and} \quad \eta(Uu, u - Uu) \geq 0 \qquad \text{for any } u \in \mathcal{F}, \tag{1.2}$$

where Uu denotes the unit contraction of u: $Uu = (0 \vee u) \wedge 1$. A lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E;m)$ satisfying (1.2) will be called a *lower bounded semi-Dirichlet form* on $L^2(E;m)$. The term "semi" is added to indicate that the dual semigroup $\{\widehat{T}_t; t > 0\}$ may not be Markovian although it is positivity preserving. As we shall see in Section 3 for a lower bounded semi-Dirichlet form η which is regular in the sense stated below, if the associated dual semigroup $\{\widehat{T}_t; t > 0\}$ were Markovian, or equivalently, if m were excessive, then η is necessarily a nonnegative definite closed form, namely, β_0 in conditions (B.1), (B.3) [resp., (B.2)] can be retaken to be 0 (resp., 1).

A lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$ is said to be *regular* if $\mathcal{F} \cap C_0(E)$ is uniformly dense in $C_0(E)$ and η_α-dense in $\mathcal{F}$ for $\alpha > \beta_0$, where $C_0(E)$ denotes the space of continuous functions on E with compact support. Carrillo-Menendez [8] constructed a Hunt process properly associated with any regular lower bounded semi-Dirichlet form on $L^2(E;m)$ by reducing the situation to the case where η is nonnegative definite. We shall show in Section 4 that a direct construction is possible without such a reduction.

Later on, the nonnegative definite semi-Dirichlet form was investigated by Ma, Oberbeck and Röckner [16] and Fitzsimmons [10] specifically in a general context of the quasi-regular Dirichlet form and the special standard process. However, in producing the forms η from nonsymmetric kernels k corresponding to a considerably wide class of jump type Hunt processes in finite dimensions whose dual semigroups need not be Markovian, we will be forced to allow positive β_0.

To be more precise, we set for $x, y \in E, x \neq y$,

$$k_s(x,y) := \tfrac{1}{2}\{k(x,y) + k(y,x)\} \quad \text{and} \quad k_a(x,y) := \tfrac{1}{2}\{k(x,y) - k(y,x)\}, \tag{1.3}$$

that is, the kernel $k_s(x,y)$ denotes the symmetrized one of k, while $k_a(x,y)$ represents the anti-symmetric part of k. We impose four conditions (2.1)–(2.4) on k_s and k_a stated below. Condition (2.1) on k_s is nearly optimal for us to work with the symmetric Dirichlet form (1.4) defined below, while conditions (2.2)–(2.4) require k_a to be less singular than k_s.

Let conditions (2.1)–(2.4) be in force on k. Denote by $C_0^{\mathrm{lip}}(E)$ the space of uniformly Lipschitz continuous functions on E with compact support. We also let

$$(1.4)\qquad \begin{cases} \mathcal{E}(u,v) := \displaystyle\iint_{E\times E\setminus \mathrm{diag}} \big(u(y)-u(x)\big)\big(v(y)-v(x)\big) \\ \qquad\qquad\qquad\qquad \times\, k_s(x,y)m(dx)m(dy), \\ \mathcal{F}^r = \{u \in L^2(E;m) : u \text{ is Borel measurable and } \mathcal{E}(u,u)<\infty\}. \end{cases}$$

$(\mathcal{E},\mathcal{F}^r)$ is a symmetric Dirichlet form on $L^2(E;m)$ and $\mathcal{F}^r$ contains the space $C_0^{\mathrm{lip}}(E)$. We denote by $\mathcal{F}^0$ the $\mathcal{E}_1$-closure of $C_0^{\mathrm{lip}}(E)$ in $\mathcal{F}^r$. $(\mathcal{E},\mathcal{F}^0)$ is then a regular Dirichlet form on $L^2(E;m)$ (cf. [13], Example 1.2.4, Theorem 3.1.1 and see also [23] and [24]).

For $u \in C_0^{\mathrm{lip}}(E)$ and $n \in \mathbb{N}$, the integral

$$(1.5)\qquad \mathcal{L}^n u(x) := \int_{\{y\in E : d(x,y)>1/n\}} \big(u(y)-u(x)\big)k(x,y)m(dy), \qquad x \in E,$$

makes sense. We prove in Proposition 2.1 and Theorem 2.1 in Section 2 that the finite limit

$$(1.6)\qquad \eta(u,v) = -\lim_{n\to\infty}\int_E \mathcal{L}^n u(x)v(x)m(dx) \qquad \text{for } u,v \in C_0^{\mathrm{lip}}(E),$$

exists, η extends to $\mathcal{F}^0\times\mathcal{F}^0$ and $(\eta,\mathcal{F}^0)$ is a lower bounded semi-Dirichlet form on $L^2(E;m)$ with parameter $\beta_0 = 8(C_1 \vee C_2C_3)(\geq 0)$ where C_1–C_3 are constants appearing in conditions (2.2)–(2.4). Furthermore, the form $\mathcal{E}$ is shown to be a *reference* (*symmetric Dirichlet*) *form* of η in the sense that, for each fixed $\alpha > \beta_0$,

$$(1.7)\qquad c_1\mathcal{E}_1(u,u) \leq \eta_\alpha(u,u) \leq c_2\mathcal{E}_1(u,u), \qquad u \in \mathcal{F}^0,$$

for some positive constants c_1, c_2 independent of $u \in \mathcal{F}^0$. Therefore, $(\eta,\mathcal{F}^0)$ becomes a regular lower bounded semi-Dirichlet form on $L^2(E;m)$ and gives rise to an associated Hunt process $X^0 = (X_t^0, P_x^0)$ on E. We call X^0 the *minimal Hunt process* associated with the form η. Equation (1.6) indicates that the limit of $\mathcal{L}^n$ in n plays a role of a pre-generator of X^0 informally.

If we define the kernel k^* by

$$(1.8)\qquad k^*(x,y) := k(y,x), \qquad x,y \in E, x \neq y,$$

and the form η^* by (1.5) and (1.6) with k^* in place of k, we have the same conclusions as above for η^* (Corollary 2.1 of Section 2). In particular, there exists a minimal Hunt process X^{0*} associated with the form η^*.

In the second half of Section 3, we are concerned with a killed dual semigroup $\{e^{-\beta t}\widehat{T}_t; t > 0\}$, which can be verified to be Markovian for a large $\beta > 0$ but only for a restricted subfamily of the forms η considered in Section 2 (lower order cases). For a higher order η, the killed dual semigroup may not be Markovian no matter how big β is. We shall also exhibit an example of a one-dimensional probability kernel k $[\int_{\mathbb{R}^1} k(x, y)\,dy = 1]$ with m being the Lebesgue measure, for which the associated semi-Dirichlet form η is not nonnegative definite and accordingly the associated dual semigroup itself is non-Markovian.

When $E = \mathbb{R}^d$ the d-dimensional Euclidean space and $m(dx) = dx$ the Lebesgue measure on it, we shall verify in Section 5 that our requirements (2.1)–(2.4) on the kernel $k(x, y)$ are fulfilled by

$$\begin{aligned} k^{(1)}(x, y) &= w(x)|x - y|^{-d-\alpha(x)}, \\ k^{(1)*}(x, y) &= w(y)|x - y|^{-d-\alpha(y)}, \qquad x, y \in \mathbb{R}^d, x \neq y, \end{aligned} \tag{1.9}$$

for $w(x)$ given by (5.1) and $\alpha(x)$ satisfying the bounds (5.2). A Markov process corresponding to $k^{(1)}$ is called a *stable-like process* and has been constructed by Bass [4] as a unique solution to a martingale problem. In this case, we shall prove that the minimal Hunt process associated with the corresponding form η is conservative and actually a solution to the same martingale problem, identifying it with the one constructed in [4] up to an η-polar set of starting points.

In Section 6, we consider an arbitrary open subset D of E. Define m_D by $m_D(B) = m(B \cap D)$ for any Borel set $B \subset E$. By replacing E and m with D and m_D, respectively, in (1.4), we obtain a symmetric Dirichlet form $(\mathcal{E}_D, \mathcal{F}_D^r)$ on $L^2(D; m_D)$. Denote by $\overline{D}$ the closure of D and by $C_0^{\mathrm{lip}}(\overline{D})$ the restriction to $\overline{D}$ of the space $C_0^{\mathrm{lip}}(E)$. We also denote by $C_0^{\mathrm{lip}}(D)$ the space of uniformly Lipschitz continuous functions on D with compact support in D. Let $\mathcal{F}_{\bar{D}}$ and $\mathcal{F}_D^0$ be the $\mathcal{E}_{D,1}$-closures of $C_0^{\mathrm{lip}}(\overline{D})$ and $C_0^{\mathrm{lip}}(D)$, respectively, in $\mathcal{F}_D^r$. Then $(\mathcal{E}_D, \mathcal{F}_{\bar{D}})$ is a regular symmetric Dirichlet form on $L^2(\overline{D}; m_D)$, while $(\mathcal{E}_D^0, \mathcal{F}_D^0)$ is a regular symmetric Dirichlet form on $L^2(D; m_D)$ where $\mathcal{E}_D^0$ is the restriction of $\mathcal{E}_D$ to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$.

By making the same replacement in (1.5) and (1.6), we get a form η_D on $C_0^{\mathrm{lip}}(\overline{D}) \times C_0^{\mathrm{lip}}(\overline{D})$, which extends to $\mathcal{F}_{\bar{D}} \times \mathcal{F}_{\bar{D}}$ to be a regular lower bounded semi-Dirichlet form on $L^2(\overline{D}; m_D)$ possessing $\mathcal{E}_D$ as its reference form, yielding an associated Hunt process $X^{\bar{D}}$ on $\overline{D}$. We also consider the restriction η_D^0 of η_D to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$ so that $(\eta_D^0, \mathcal{F}_D^0)$ is a regular lower bounded semi-Dirichlet form on $L^2(D; m_D)$ possessing $\mathcal{E}_D^0$ as its reference form. We shall show in Section 6 that the part process $X^{D,0}$ of $X^{\bar{D}}$ on D, namely, the Hunt process obtained from $X^{\bar{D}}$ by killing upon hitting the boundary ∂D, is properly associated with $(\eta_D^0, \mathcal{F}_D^0)$.

We shall also prove in Section 6 that $X^{\bar{D}}$ admits no jump from D to ∂D, and furthermore when D is relatively compact, $X^{\bar{D}}$ is conservative so that $X^{D,0}$ admits no killing inside D and its sample path is killed only when it approaches to the boundary ∂D. $X^{D,0}$ is accordingly different from the part process of X^0 on the set D in general because the sample path of X^0 may jump from D to $E \setminus D$ resulting in a killing inside D of its part process. By adopting k^* instead of k, we get in an analogous manner Hunt processes $X^{\bar{D}*}$ on $\overline{D}$ and $X^{D,0*}$ on D satisfying the same properties as above.

When $(\mathcal{E}, \mathcal{F}^r)$ is the Dirichlet form on $L^2(\mathbb{R}^d)$ of a symmetric stable process on $\mathbb{R}^d$, the space $\mathcal{F}^0$ is identical with $\mathcal{F}^r$. In this case, for an arbitrary open set $D \subset \mathbb{R}^d$, the symmetric Hunt process on D associated with $(\mathcal{E}_D^0, \mathcal{F}_D^0)$ is a *censored stable process* on D in the sense of Bogdan, Burdzy and Chen [7]. It was further shown in [7] that, if D is a d-set, then the space $\mathcal{F}_{\bar{D}}$ coincides with $\mathcal{F}_D^r$ so that the symmetric Hunt process on $\overline{D}$ associated with $(\mathcal{E}_D, \mathcal{F}_D^r)$ was called a *reflecting stable process* over $\overline{D}$.

For the nonsymmetric kernel $k^{(1)}$ on $\mathbb{R}^d$ as (1.9), associated Hunt processes $X^{D,0}$, $X^{D,0*}$ on an arbitrary open set $D \subset \mathbb{R}^d$ may well be called *censored stable-like processes* in view of the stated properties of them. However, it is harder in this case to identify the space $\mathcal{F}_{\bar{D}}$ with $\mathcal{F}_D^r$, and accordingly we call the associated Hunt processes $X^{\bar{D}}$, $X^{\bar{D}*}$ over $\overline{D}$ *modified reflecting stable-like processes* analogously to the Brownian motion case (cf. [11]). At the end of Section 6, we give sufficient conditions in terms of the upper and lower bounds of the variable exponent $\alpha(x)$ for the approachability in finite time of the censored stable-like processes to the boundary.

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2. Construction of a lower bounded semi-Dirichlet form from k. Throughout this section, we make the following assumptions on a nonnegative Borel measurable function $k(x, y)$ on $E \times E \setminus \mathrm{diag}$:

$$M_s \in L^2_{\mathrm{loc}}(E; m) \qquad \text{for } M_s(x) = \int_{y \neq x} \bigl(1 \wedge d(x, y)^2\bigr) k_s(x, y) m(dy), \quad x \in E, \tag{2.1}$$

$$C_1 := \sup_{x \in E} \int_{d(x,y) \geq 1} |k_a(x, y)| m(dy) < \infty, \tag{2.2}$$

and there exists a constant $\gamma \in (0, 1]$ such that

$$C_2 := \sup_{x \in E} \int_{d(x,y) < 1} |k_a(x, y)|^{\gamma} m(dy) < \infty, \tag{2.3}$$

and furthermore, for some constant $C_3 \geq 0$,

$$|k_a(x, y)|^{2-\gamma} \leq C_3 k_s(x, y) \qquad \text{for any } x, y \in E \text{ with } 0 < d(x, y) \leq 1. \tag{2.4}$$

For each $n \in \mathbb{N}$, define $\mathcal{L}^n u$ for $u \in C_0^{\mathrm{lip}}(E)$ by (1.5) and $\eta^n(u,v)$ for $u, v \in C_0^{\mathrm{lip}}(E)$ by

$$\eta^n(u,v) := -\int_E \mathcal{L}^n u(x) v(x) m(dx), \tag{2.5}$$

the integral on the right-hand side being absolutely convergent by (2.1). We note that any $u \in C_0^{\mathrm{lip}}(E)$ belongs to the domain $\mathcal{F}^r$ of the form $\mathcal{E}$ defined by (1.4). In fact, if we denote by K the support of u, then $\mathcal{E}(u,u)$ is dominated by twice the integral of $(u(x)-u(y))^2 k_s(x,y)m(dx)m(dy)$ on $K \times E$, which is finite by (2.1).

$\mathcal{E}(u,v)$ admits also an alternative expression for $u, v \in C_0^{\mathrm{lip}}(E)$,

$$\mathcal{E}(u,v) := \iint_{E\times E\setminus \mathrm{diag}} \big(u(y)-u(x)\big)\big(v(y)-v(x)\big)k(x,y)m(dx)m(dy),$$

because the right-hand side of the above can be seen to be equal to the same integral with $k(y,x)$ in place of $k(x,y)$ by interchanging the variables x, y, and we arrive at the expression in (1.4) by averaging. In particular, $\mathcal{E}(u,v) = \lim_{n\to\infty} \mathcal{E}^n(u,v)$ for $u, v \in C_0^{\mathrm{lip}}(E)$ where

$$\mathcal{E}^n(u,v) := \iint_{d(x,y)>1/n} \big(u(y)-u(x)\big)\big(v(y)-v(x)\big)k(x,y)m(dx)m(dy). \tag{2.6}$$

PROPOSITION 2.1. *Assume* (2.1)–(2.4). *Then for all $u, v \in C_0^{\mathrm{lip}}(E)$, the limit*

$$\eta(u,v) = \lim_{n\to\infty} \eta^n(u,v)$$

exists. Moreover, the limit has the following expression:

$$\eta(u,v) = \frac{1}{2}\mathcal{E}(u,v) + \iint_{y\neq x} \big(u(x)-u(y)\big)v(y)k_a(x,y)m(dx)m(dy), \tag{2.7}$$

where $\mathcal{E}$ is defined by (1.4) *and the integral on the right-hand side is absolutely convergent.*

PROOF. For $u, v \in C_0^{\mathrm{lip}}(E)$, we have

$$\begin{aligned}
\eta^n(u,v) - \eta^n(v,u) &= -\iint_{d(x,y)>1/n} \big(u(y)-u(x)\big)v(x)k(x,y)m(dx)m(dy) \\
&\quad + \iint_{d(x,y)>1/n} \big(v(y)-v(x)\big)u(x)k(x,y)m(dx)m(dy) \\
&= -\iint_{d(x,y)>1/n} u(y)v(x)k(x,y)m(dx)m(dy) \\
&\quad + \iint_{d(x,y)>1/n} v(y)u(x)k(x,y)m(dx)m(dy) \\
&= 2\iint_{d(x,y)>1/n} u(x)v(y)k_a(x,y)m(dx)m(dy),
\end{aligned}$$

and further

$$
\begin{aligned}
\eta^n(u,v)&+\eta^n(v,u)\\
&=-\iint_{d(x,y)\ge 1/n}\big(u(y)-u(x)\big)v(x)k(x,y)m(dx)m(dy)\\
&\quad-\iint_{d(x,y)\ge 1/n}\big(v(y)-v(x)\big)u(x)k(x,y)m(dx)m(dy)\\
&=\iint_{d(x,y)\ge 1/n}\big(u(y)-u(x)\big)\big(v(y)-v(x)\big)k(x,y)m(dx)m(dy)\\
&\quad-\iint_{d(x,y)\ge 1/n}\big(u(y)-u(x)\big)v(y)k(x,y)m(dx)m(dy)\\
&\quad-\iint_{d(x,y)\ge 1/n}\big(v(y)-v(x)\big)u(x)k(x,y)m(dx)m(dy)\\
&=\mathcal{E}^n(u,v)-2\iint_{d(x,y)\ge 1/n}u(y)v(y)k_a(x,y)m(dx)m(dy).
\end{aligned}
$$

By adding up the obtained identities, we get for $u,v\in C_0^{\mathrm{lip}}(E)$,

$$
\begin{aligned}
2\eta^n(u,v)=\mathcal{E}^n(u,v)+2\iint_{d(x,y)>1/n}&\big(u(x)-u(y)\big)v(y)\\
&\times k_a(x,y)m(dx)m(dy).
\end{aligned}
\tag{2.8}
$$

Since $\mathcal{E}^n(u,v)$ converges to $\mathcal{E}(u,v)$ as $n\to\infty$, it remains to see that the second term of the right-hand side also converges absolutely as $n\to\infty$ for each $u,v\in C_0^{\mathrm{lip}}(E)$.

From the Schwarz inequality and (2.2), we see that

$$
\begin{aligned}
&\iint_{d(x,y)>1/n}\big|\big(u(x)-u(y)\big)v(y)k_a(x,y)\big|m(dx)m(dy)\\
&\quad\le\iint_{1/n<d(x,y)<1}|u(x)-u(y)|\cdot|v(y)||k_a(x,y)|^{\gamma/2}\\
&\qquad\qquad\times|k_a(x,y)|^{1-\gamma/2}m(dx)m(dy)\\
&\qquad+\iint_{d(x,y)\ge 1}|u(x)-u(y)|\cdot|v(y)|k_s(x,y)^{1/2}|k_a(x,y)|^{1/2}m(dx)m(dy)\\
&\quad\le\sqrt{\iint_{1/n<d(x,y)<1}\big(u(x)-u(y)\big)^2|k_a(x,y)|^{2-\gamma}m(dx)m(dy)}\\
&\qquad\times\sqrt{\iint_{1/n<d(x,y)<1}v(y)^2|k_a(x,y)|^{\gamma}m(dx)m(dy)}\\
&\qquad+\sqrt{C_1}\|v\|\sqrt{\iint_{d(x,y)\ge 1}\big(u(x)-u(y)\big)^2k_s(x,y)m(dx)m(dy)}.
\end{aligned}
$$

So, by making use of assumptions (2.3) and (2.4) and an elementary inequality $\sqrt{A}+\sqrt{B}\le\sqrt{2}\sqrt{A+B}$ holding for $A\ge 0$ and $B\ge 0$, we have

$$\iint_{d(x,y)>1/n}\big|\big(u(x)-u(y)\big)v(y)k_a(x,y)\big|m(dx)m(dy)$$
$$\le\sqrt{2}\sqrt{C_1\vee C_2C_3}\|v\|\cdot\sqrt{\mathcal{E}^n(u,u)}.$$

Then taking $n\to\infty$,

$$\iint_{y\neq x}\big|\big(u(x)-u(y)\big)v(y)k_a(x,y)\big|m(dx)m(dy)$$
$$\le\sqrt{2}\sqrt{C_1\vee C_2C_3}\|v\|\cdot\sqrt{\mathcal{E}(u,u)}<\infty$$

as was to be proved. $\square$

For $u,v\in C_0^{\mathrm{lip}}(E)$, set

$$\eta_\beta(u,v)=\eta(u,v)+\beta(u,v),\qquad\beta>0,$$

and

$$B(u,v):=\iint_{x\neq y}\big(u(x)-u(y)\big)v(y)k_a(x,y)m(dx)m(dy). \tag{2.9}$$

Then equation (2.7) reads

$$\eta(u,v)=\tfrac{1}{2}\mathcal{E}(u,v)+B(u,v),\qquad u,v\in C_0^{\mathrm{lip}}(E), \tag{2.10}$$

while we get from the proof of the preceding proposition

$$|B(u,v)|\le C_4\|v\|\sqrt{\mathcal{E}(u,u)}, \tag{2.11}$$

where $C_4=\sqrt{2}\cdot\sqrt{C_1\vee C_2C_3}$. Now we put $\beta_0:=4(C_4)^2=8(C_1\vee C_2C_3)$.

From equation (2.10) and the bound (2.11), we have for $u\in C_0^{\mathrm{lip}}(E)$,

$$\eta_{\beta_0}(u,u)=\tfrac{1}{4}\mathcal{E}_{\beta_0}(u,u)+\tfrac{1}{4}\mathcal{E}(u,u)+\tfrac{3}{4}\beta_0\|u\|^2+B(u,u)$$
$$\ge\tfrac{1}{4}\mathcal{E}_{\beta_0}(u,u)+\sqrt{3}C_4\sqrt{\mathcal{E}(u,u)}\|u\|+B(u,u)\ge\tfrac{1}{4}\mathcal{E}_{\beta_0}(u,u).$$

Further, for $u,v\in C_0^{\mathrm{lip}}(E)$,

$$|\eta(u,v)|\le\tfrac{1}{2}|\mathcal{E}(u,v)|+|B(u,v)|$$
$$\le\tfrac{1}{2}\sqrt{\mathcal{E}(u,u)}\sqrt{\mathcal{E}(v,v)}+C_4\|v\|\sqrt{\mathcal{E}(u,u)}$$
$$\le\tfrac{1}{2}\big(\sqrt{\mathcal{E}(v,v)}+2C_4\|v\|\big)\sqrt{\mathcal{E}(u,u)}$$
$$\le\tfrac{\sqrt{2}}{2}\sqrt{\mathcal{E}_{\beta_0}(v,v)}\sqrt{\mathcal{E}_{\beta_0}(u,u)}.$$

So it also follows that

$$|\eta(u,v)| \le 2\sqrt{2}\sqrt{\eta_{\beta_0}(u,u)}\sqrt{\eta_{\beta_0}(v,v)} \tag{2.12}$$

and

$$\tfrac{1}{4}\mathcal{E}_{\beta_0}(u,u) \le \eta_{\beta_0}(u,u) \le \tfrac{2+\sqrt{2}}{2}\mathcal{E}_{\beta_0}(u,u), \qquad u,v \in C_0^{\mathrm{lip}}(E). \tag{2.13}$$

Let $\mathcal{F}^0$ be the $\mathcal{E}_1$-closure of $C_0^{\mathrm{lip}}(E)$ in $\mathcal{F}^r$. Since $\mathcal{F}^0$ is complete with respect to $\mathcal{E}_\alpha$ for any $\alpha > 0$, the estimates obtained above readily lead us to the first conclusion of the following theorem.

THEOREM 2.1. *Assume* (2.1)–(2.4). *Then the form η defined by Proposition* 2.1 *extends from $C_0^{\mathrm{lip}}(E) \times C_0^{\mathrm{lip}}(E)$ to $\mathcal{F}^0 \times \mathcal{F}^0$ to be a lower bounded closed form on $L^2(E;m)$ satisfying* (B.1)–(B.3) *with $\beta_0 = 8(C_1 \vee C_2C_3)$, $K = 2\sqrt{2}$ and possessing $(\mathcal{E},\mathcal{F}^0)$ as a reference form in the sense of* (1.7).

Furthermore, the pair $(\eta,\mathcal{F}^0)$ is a regular lower bounded semi-Dirichlet form on $L^2(E;m)$.

We note that the above constant β_0 is equal to 0 if k is symmetric: $k(x,y) = k(y,x)$, $(x,y) \in E \times E \setminus \mathrm{diag}$.

PROOF OF THEOREM 2.1. It suffices to prove the contraction property (1.2) for the present pair $(\eta,\mathcal{F}^0)$. We first show this for $u \in C_0^{\mathrm{lip}}(E)$. Note that $Uu \in C_0^{\mathrm{lip}}(E)$ and, for $n \in \mathbb{N}$,

$$\begin{aligned}
&\eta^n(Uu, u - Uu)\\
&\quad = -\iint_{d(x,y)>1/n} \big(Uu(y) - Uu(x)\big)\big(u(x) - Uu(x)\big)k(x,y)m(dx)m(dy)\\
&\quad = \iint_{\{d(x,y)>1/n\}\cap\{x\,:\,u(x)\ge 1\}} \big(1 - Uu(y)\big)\big(u(x) - 1\big)k(x,y)m(dx)m(dy)\\
&\qquad - \iint_{\{d(x,y)>1/n\}\cap\{x\,:\,u(x)\le 0\}} Uu(y)u(x)k(x,y)m(dx)m(dy)\\
&\quad \ge 0.
\end{aligned}$$

Then, we have by Proposition 2.1

$$\eta(Uu, u - Uu) = \lim_{n\to\infty} \eta^n(Uu, u - Uu) \ge 0.$$

Following a method in [17], Lemma 4.9, we next prove (1.2) for any $u \in \mathcal{F}^0$. Choose a sequence $\{u_\ell\} \subset C_0^{\mathrm{lip}}(E)$ which is $\mathcal{E}_1$-convergent to u. Then

$$\|Uu_\ell - Uu\| \to 0, \qquad \ell \to \infty, \tag{2.14}$$

because U is easily seen to be a continuous operator from $L^2(E;m)$ to $L^2(E;m)$. Fix $\alpha > \beta_0$. We then get from (1.7) the boundedness

$$\sup_\ell \eta_\alpha(Uu_\ell, Uu_\ell) \le C_2 \sup_\ell \mathcal{E}_1(u_\ell, u_\ell) < \infty.$$

On the other hand, using the dual resolvent $\widehat{G}_\alpha$ associated with the lower bounded closed form $(\eta, \mathcal{F}^0)$, we see from equation (3.1) below that, for any $g \in L^2(E;m)$,

$$\eta_\alpha(Uu_\ell, \widehat{G}_\alpha g) = (Uu_\ell, g) \to (Uu, g) = \eta_\alpha(Uu, \widehat{G}_\alpha g), \qquad \ell \to \infty.$$

Since $\{\widehat{G}_\alpha g : g \in L^2(E,m)\}$ is η_α-dense in $\mathcal{F}^0$, we can conclude by making use of the above η_α-bound of $\{Uu_\ell\}$ and the sector condition (B.2) that $\{Uu_\ell\}$ is η_α-weakly convergent to Uu as $\ell \to \infty$. In particular, by the above η_α-bound and (B.2) again, we have

$$\eta_\alpha(Uu_\ell, u_\ell) \to \eta_\alpha(Uu, u), \qquad \ell \to \infty. \tag{2.15}$$

We consider the dual form $\widehat{\eta}$ and the symmetrizing form $\tilde{\eta}$ of η defined by

$$\widehat{\eta}(u,v) = \eta(v,u), \qquad \tilde{\eta}(u,v) = \tfrac{1}{2}\big(\eta(u,v) + \eta(v,u)\big), \qquad u, v \in \mathcal{F}^0.$$

In the same way as above, we can see that $\{Uu_\ell\}$ converges as $\ell \to \infty$ to Uu $\widehat{\eta}_\alpha$-weakly and consequently $\tilde{\eta}_\alpha$-weakly. Since $(\tilde{\eta}_\alpha, \mathcal{F}^0)$ is a nonnegative definite symmetric bilinear form, it follows that

$$\begin{aligned} \eta_\alpha(Uu, Uu) = \tilde{\eta}_\alpha(Uu, Uu) &\le \liminf_{\ell\to\infty} \tilde{\eta}_\alpha(Uu_\ell, Uu_\ell) \\ &= \liminf_{\ell\to\infty} \eta_\alpha(Uu_\ell, Uu_\ell). \end{aligned} \tag{2.16}$$

We can then obtain (1.2) for $u \in \mathcal{F}^0$ from (2.14), (2.15) and (2.16) as

$$\begin{aligned} \eta(Uu, u - Uu) &\ge \lim_{\ell\to\infty} \eta(Uu_\ell, u_\ell) - \liminf_{\ell\to\infty} \eta(Uu_\ell, Uu_\ell) \\ &= \limsup_{\ell\to\infty} \eta(Uu_\ell, u_\ell - Uu_\ell) \ge 0. \end{aligned}$$

□

For the kernel k^* defined by (1.8), we have obviously

$$k_s^*(x,y) = k_s(x,y) \quad \text{and} \quad k_a^*(x,y) = -k_a(x,y), \qquad x, y \in E, x \ne y. \tag{2.17}$$

Hence, if the kernel $k(x,y)$ satisfies (2.1)–(2.4), so does the kernel $k^*(x,y)$. Define η^* as in Proposition 2.1 with $k^*(x,y)$ in place of $k(x,y)$. The same calculations made above for $k(x,y)$ remain valid for $k^*(x,y)$. Note also that the domain $\mathcal{F}^{0*}$ is the same as $\mathcal{F}^0$ since the symmetric form $\mathcal{E}^*$ defined by k^* is also the same as $\mathcal{E}$. Thus, we can have the following corollary.

COROLLARY 2.1. *Assume conditions (2.1)–(2.4) hold. Then the pair $(\eta^*, \mathcal{F}^0)$ is also a regular lower bounded semi-Dirichlet form on $L^2(E;m)$.*

3. Markov property of dual semigroups. First, we fix a general lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(E;m)$ satisfying (B.1)–(B.3) and make several remarks on it. The last condition (B.3) is equivalent to

(B.3)′ $(\tilde{\eta}_{\beta_0}, \mathcal{F})$ is a closed symmetric form on $L^2(E;m)$,

where $\tilde{\eta}$ denotes the symmetrization of the form $\eta : \tilde{\eta}(u,v) = \frac{1}{2}(\eta(u,v)+\eta(v,u))$. η_{β_0} is therefore a coercive closed form in the sense of [17], Definition 2.4, so that, by [17], Theorem 2.8, there exist uniquely two families of linear bounded operators $\{G_\alpha\}_{\alpha>\beta_0}, \{\widehat{G}_\alpha\}_{\alpha>\beta_0}$ on $L^2(E;m)$ such that, for $\alpha > \beta_0$, $G_\alpha(L^2(E;m))$, $\widehat{G}_\alpha(L^2(E;m)) \subset \mathcal{F}$ and

$$\eta_\alpha(G_\alpha f, u) = (f, u) = \eta_\alpha(u, \widehat{G}_\alpha f), \qquad f \in L^2(E;m), u \in \mathcal{F}. \tag{3.1}$$

In particlular, G_α and $\widehat{G}_\alpha$ are mutually adjoint:

$$(G_\alpha g, f) = (g, \widehat{G}_\alpha f), \qquad f, g \in L^2(E;m), \alpha > \beta_0. \tag{3.2}$$

We call $\{G_\alpha; \alpha > \beta_0\}$ (resp., $\{\widehat{G}_\alpha; \alpha > \beta_0\}$) the *resolvent* (resp., *dual resolvent*) associated with $(\eta, \mathcal{F})$.

Accordingly we see in exactly the same way as the proof of Theorem 2.8 of [17] that there exist strongly continuous contraction semigroups $\{S_t; t > 0\}, \{\widehat{S}_t; t > 0\}$ of linear operators on $L^2(E;m)$ such that, for $\alpha > 0$, $f \in L^2(E;m)$,

$$G_{\beta_0+\alpha} f = \int_0^\infty e^{-\alpha t} S_t f \, dt, \qquad \widehat{G}_{\beta_0+\alpha} f = \int_0^\infty e^{-\alpha t} \widehat{S}_t f \, dt.$$

We then set $T_t = e^{\beta_0 t} S_t$, $\widehat{T}_t = e^{\beta_0 t} \widehat{S}_t$ to get strongly continuous semigroups $\{T_t; t > 0\}, \{\widehat{T}_t; t > 0\}$ satisfying

$$G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f \, dt, \qquad \widehat{G}_\alpha f = \int_0^\infty e^{-\alpha t} \widehat{T}_t f \, dt, \qquad \alpha > \beta_0, \tag{3.3}$$

as well as (1.1).

We call $\{T_t; t > 0\}$ (resp., $\{\widehat{T}_t; t > 0\}$) the *semigroup* (resp., dual semigroup) on $L^2(E;m)$ associated with the lower bounded closed form $(\eta, \mathcal{F})$. We introduce the *dual form* $\widehat{\eta}$ of η by

$$\widehat{\eta}(u,v) = \eta(v,u), \qquad u, v \in \mathcal{F}.$$

Then $(\widehat{\eta}, \mathcal{F})$ is a lower bounded closed form on $L^2(E;m)$ with which $\{\widehat{T}_t; t > 0\}$ and $\{\widehat{G}_\alpha; \alpha > \beta_0\}$ are the associated semigroup and resolvent, respectively.

Suppose $(\eta, \mathcal{F})$ is a lower bounded semi-Dirichlet form, namely, it satisfies the contraction property (1.2) additionally. As in the proof of the corollary to Theorem 4.1 of [15] or the proof of Theorem 4.4 of [17], we can then readily verify that the family $\{\alpha G_\alpha; \alpha > \beta_0\}$ is Markovian, which is in turn equivalent to the Markovian property of $\{T_t; t > 0\}$. Together with $\{T_t; t > 0\}$, its Laplace transform then determines a bounded linear operator G_α on $L^\infty(E;m)$ for every $\alpha > 0$

and $\{\alpha G_\alpha; \alpha > 0\}$ becomes Markovian. Further, $\{\widehat{T}_t; t > 0\}$ is positivity preserving in view of (1.1).

Suppose additionally that $(\eta, \mathcal{F})$ is regular. Then the associated Markovian semigroup and resolvent can be represented by the transition function $\{P_t; t > 0\}$ and the resolvent $\{R_\alpha; \alpha > 0\}$ of the associated Hunt process X specified in Theorem 2 of the next section: $P_t f = T_t f, t > 0$, and $R_\alpha f = G_\alpha f, \alpha > 0$, for any $f \in \mathcal{B}_b(E) \cap L^2(E; m)$. We call a σ-finite measure μ on E *excessive* relative to X if $\mu P_t \leq \mu$ for any $t > 0$. The next lemma was already observed in Silverstein [20].

LEMMA 3.1. *Let η be a regular lower bounded semi-Dirichlet form on $L^2(E; m)$.*

(i) *The following three conditions are mutually equivalent*:
   1. *m is excessive relative to X.*
   2. *The dual semigroup $\{\widehat{T}_t : t > 0\}$ is Markovian.*
   3. *$\eta(u - Uu, Uu) \geq 0$ for anyu $\in \mathcal{F}$.*

(ii) *If one of the three conditions in* (i) *is satisfied*, *then η is nonnegative definite and the constant β_0 in conditions* (B.1), (B.3) [*resp*., (B.2)] *can be retaken to be* 0 (*resp*., 1).

PROOF. (i) 3 is the Markovian criterion (1.2) for the dual semigroup. If 2 is satisfied, then for any $f \in L^2(E; m)$ with $0 \leq f \leq 1$, $0 \leq \widehat{T}_t f \leq 1$ so that $(f, P_t h) = (\widehat{T}_t f, h) \leq (1, h)$ for any $h \in \mathcal{B}_+ \cap L^2(E; m)$, from which 1 follows. The converse can be shown similarly.

(ii) By the Schwarz inequality,

$$(R_\alpha f(x))^2 \leq R_\alpha 1(x) R_\alpha f^2(x) \leq \frac{1}{\alpha} R_\alpha f^2(x), \qquad x \in E,\ f \in \mathcal{B}_b(E) \cap L^2(E; m).$$

Assuming 1 of (i), an integration with respect to m yields $\alpha^2 \|G_\alpha f\|^2 \leq \|f\|^2$, the L^2-contraction property of αG_α. In view of [17], Theorem 2.13, $\eta(u, u) = \lim_{\alpha \to \infty} \alpha(u - \alpha G_\alpha u, u) u \in \mathcal{F}$, which particularly implies that $\eta(u, u) \geq 0, u \in \mathcal{F}$, and $\{\eta_\alpha; \alpha > 0\}$ become equivalent on $\mathcal{F}$. $\square$

We now return to the setting of the preceding section that $(\eta, \mathcal{F}^0)$ is defined in terms of the kernel k satisfying conditions (2.1)–(2.4). By Proposition 2.1, $\widehat{\eta}(u, v) = \frac{1}{2}\mathcal{E}(v, u) + B(v, u)$ where B is defined by (2.9) on $\mathcal{F}^0 \times \mathcal{F}^0$. On the other hand, we have from (2.17) that $\eta^*(u, v) = \frac{1}{2}\mathcal{E}(u, v) - B(u, v)$ and consequently

$$\widehat{\eta}(u, v) = \eta^*(u, v) + \big(B(u, v) + B(v, u)\big), \qquad u, v \in \mathcal{F}^0. \tag{3.4}$$

We know from Theorem 2.1 and Corollary 2.1 that both $(\eta, \mathcal{F}^0)$ and $(\eta^*, \mathcal{F}^0)$ are regular lower bounded semi-Dirichlet forms. In order to get a similar property for the dual form $\widehat{\eta}$, we need to impose on the kernel k stronger conditions than (2.1)–(2.4) making the additional term on the right-hand side of (3.4) controllable.

In the rest of this section, we assume that the kernel k satisfies the condition

$$M_s \in L^2_{\mathrm{loc}}(E; m) \qquad \text{for } M_s(x) = \int_{y\neq x} \bigl(1 \wedge d(x,y)\bigr) k_s(x,y) m(dy), \quad x \in E, \tag{3.5}$$

in place of (2.1), and further satisfies condition (2.2) as well as (2.3) for $\gamma = 1$ so that

$$\begin{aligned} \frac{\beta_1}{2} &:= \sup_{x\in E} \int_{x\neq y} |k_a(x,y)| m(dy) \\ &= \sup_{x\in E} \frac{1}{2} \int_{x\neq y} |k(x,y) - k(y,x)| m(dy) < \infty. \end{aligned} \tag{3.6}$$

Notice that condition (2.4) for $\gamma = 1$ is always satisfied with $C_3 = 1$.

Then the integrals

$$\begin{aligned} \mathcal{L}u(x) &= \int_{y\neq x} \bigl(u(y) - u(x)\bigr) k(x,y) m(dy) \quad \text{and} \\ \mathcal{L}^* u(x) &= \int_{y\neq x} \bigl(u(y) - u(x)\bigr) k^*(x,y) m(dy), \end{aligned} \tag{3.7}$$

converge for $u \in C_0^{\mathrm{lip}}(E)$, $x \in E$, and we get from Proposition 2.1 the identities

$$\eta(u,v) = -(\mathcal{L}u, v), \qquad \eta^*(u,v) = -(\mathcal{L}^* u, v), \qquad u, v \in C_0^{\mathrm{lip}}(E). \tag{3.8}$$

Furthermore,

$$\begin{aligned} K(x) &:= 2 \int_{y\neq y} k_a(x,y) m(dy) \\ &= \int_{y\neq x} \bigl(k(x,y) - k(y,x)\bigr) m(dy), \qquad x \in E, \end{aligned} \tag{3.9}$$

defines a bounded function on E and (3.4) readily leads us to

$$\widehat{\eta}(u,v) = \eta^*(u,v) + (u, Kv), \qquad u, v \in \mathcal{F}^0,$$

which combined with (3.7) means that $\widehat{\mathcal{L}} = \mathcal{L}^* - K$ is the formal adjoint of $\mathcal{L}$. $\widehat{\eta}$ does not necessarily satisfy the contraction property (1.2), but the form

$$\widehat{\eta}_\beta(u,v) = \eta^*(u,v) + \bigl(u, (K+\beta)v\bigr), \qquad \beta \geq \beta_1,$$

does because so does the form η^* by Corollary 2.1 and $K + \beta \geq 0$ if $\beta \geq \beta_1$. So we have the following proposition.

PROPOSITION 3.1. *Assume that* (3.5) *and* (3.6) *hold. Then* $(\widehat{\eta}_\beta, \mathcal{F}^0)$, *which is the dual of* $(\eta_\beta, \mathcal{F}^0)$, *is a regular lower bounded semi-Dirichlet form on* $L^2(E;m)$ *provided that* $\beta \geq \beta_1$.

This proposition means that, under conditions (3.5) and (3.6), $\{e^{-\beta t}\widehat{T}_t; t > 0\}$ is Markovian for the dual semigroup $\{\widehat{T}_t; t > 0\}$ associated with η when $\beta \geq \beta_1$. If (3.6) fails, the dual semigroup of $\{e^{-\beta t}T_t; t > 0\}$ may not be Markovian no matter how large β is.

A nonnegative Borel function k on $E \times E$ is said to be a *probability kernel* if $\int_E k(x,y)m(dy) = 1, x \in E$. A probability kernel k with the additional property

$$\sup_{x \in E} \int_D k(y,x)m(dy) < \infty \tag{3.10}$$

satisfies conditions (3.5) and (3.6) and η defined by (3.8) yields a regular lower bounded semi-Dirichlet form on $L^2(E;m)$. We now give an example of a such a kernel on $\mathbb{R}^1$ with m being the Lebesgue measure for which the associated semi-Dirichlet form η is *not* nonnegative definite so that, according to Lemma 3.1, the associated dual semigroup $\{\widehat{T}_t, t > 0\}$ is *not* Markovian although $\{e^{-\beta t}\widehat{T}_t; t > 0\}$ is Markovian for a large $\beta > 0$ in view of Proposition 3.1. A transition probability density function with respect to the Lebesgue measure of the one-dimensional Brownian motion with a mildly localized drift serves to be an example of such a kernel k.

Consider a diffusion Y on $\mathbb{R}^1$ with generator $\mathcal{G}u = \frac{1}{2}u'' + \lambda b(x)u'$ where λ is a positive constant and b is a function in $C_0^1(\mathbb{R}^1)$ not identically 0. Then $\mathcal{G} = \frac{d}{dm} \cdot \frac{d}{ds}$ for

$$dm(x) = m(x)\,dx, \qquad ds(x) = 2m(x)^{-1}\,dx,$$

where

$$m(x) = 2\exp\left\{2\lambda \int_0^x b(y)\,dy\right\},$$

namely, Y is a diffusion with canonical scale s and canonical (speed) measure dm.

The following facts about Y are taken from [12]. Since $m(x)$ is bounded from above and from below by positive constants, both $\pm\infty$ are nonapproachable in the sense that $s(\pm\infty) = \pm\infty$. Therefore, Y is recurrent and consequently conservative: $q_t(x,E) = 1, x \in E$, where $\{q_t; t > 0\}$ denotes the transition function of Y. Y is m-symmetric and its Dirichlet form $(\mathcal{E}^Y, \mathcal{F}^Y)$ on $L^2(\mathbb{R}^1, m)$ is given by

$$\begin{cases} \mathcal{E}^Y(u,v) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^1} u'(x)v'(x)m(x)\,dx, \\ \mathcal{F}^Y = \{u \in L^2(\mathbb{R}^1; m) : u \text{ is absolutely} \\ \qquad\qquad \text{continuous and } \mathcal{E}^Y(u,u) < \infty\} \ (= H^1(\mathbb{R}^1)). \end{cases}$$

For $u \in C_0^1(\mathbb{R}^1)$, $\mathcal{E}^Y(u, \frac{u}{m})$ is seen to be equal to $\frac{1}{2}\int_{\mathbb{R}^1}((u')^2 - 2\lambda b u' u)\,dx$ and so

$$\mathcal{E}^Y\left(u, \frac{u}{m}\right) = \frac{1}{2}\left(\int_{\mathbb{R}^1} (u')^2\,dx + \lambda \int_{\mathbb{R}^1} b' u^2\,dx\right).$$

There is a finite interval $I \subset \mathbb{R}^1$ where b' is strictly negative. Choose $u_0 \in C_0^1(\mathbb{R}^1)$ not identically zero and with support being contained in I. We can then make a choice of $\lambda > 0$ such that the right-hand side of the above equation is negative for $u = u_0$.

Since q_t maps $L^2(\mathbb{R}^1; m)$ into $\mathcal{F}^Y \subset C(\mathbb{R}^1)$, $q_t(x, \cdot)$ is absolutely continuous with respect to m and hence with respect to the Lebesgue measure for each $x \in \mathbb{R}^1$. Denote by $q_t(x, y)$ its density with respect to the Lebesgue measure so that $\int_{\mathbb{R}^1} q_t(x, y)\,dy = 1$, $x \in \mathbb{R}^1$, with

$$q_t(y, x) = m(x) q_t(x, y) \frac{1}{m(y)}. \tag{3.11}$$

We know that the left-hand side of the above equation equals

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^1} \bigl(u(x) - q_t u(x)\bigr) \frac{u(x)}{m(x)} m(x)\,dx = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^1} \bigl(u(x) - q_t u(x)\bigr) u(x)\,dx$$

and so, for $k(x, y) = q_{t_0}(x, y)$ with a sufficiently small $t_0 > 0$,

$$\eta(u_0, u_0) = -\int_{\mathbb{R}^1} \left[\int_{\mathbb{R}^1} \bigl(u_0(y) - u_0(x)\bigr) k(x, y)\,dy\right] u_0(x)\,dx < 0.$$

Equality (3.10) follows from (3.11).

4. Associated Hunt process and martingale problem. Let $(\eta, \mathcal{F})$ be a regular lower bounded semi-Dirichlet form on $L^2(E; m)$ as is defined in Section 1. For the symmetrization $\tilde{\eta}$, $(\tilde{\eta}_{\beta_0}, \mathcal{F})$ is then a closed symmetric form on $L^2(E; m)$ but not necessarily a symmetric Dirichlet form. A symmetric Dirichlet form $\mathcal{E}$ on $L^2(E; m)$ with domain $\mathcal{F}$ will be called a *reference* (*symmetric Dirichlet*) *form* of η if, for each fixed $\alpha > \beta_0$,

$$c_1 \mathcal{E}_1(u, u) \le \eta_\alpha(u, u) \le c_2 \mathcal{E}_1(u, u), \qquad u \in \mathcal{F}, \tag{4.1}$$

for some positive c_1, c_2 independent of $u \in \mathcal{F}$. $\mathcal{E}$ is then a regular Dirichlet form. In what follows, we assume that η admits a reference form $\mathcal{E}$. This assumption is really unnecessary (cf. [16, 19]) but convenient to simplify some arguments. The regular lower bounded semi-Diriclet form $(\eta, \mathcal{F}^0)$ constructed in Section 2 from a kernel k satisfying (2.1)–(2.4) has a reference form $(\mathcal{E}, \mathcal{F}^0)$ defined right after (1.4).

In formulating an association of a Hunt process with η, Carrillo Menendez adopted a functional capacity theorem due to Ancona [2]. More specifically, denote by $\mathcal{O}$ the family of all open sets $A \subset E$ with $\mathcal{L}_A = \{u \in \mathcal{F} : u \ge 1\ m\text{-a.e. on } A\} \neq$

$\emptyset$. Fix $\alpha > \beta_0$ and, for $A \in \mathcal{O}$, let e_A be the η_α-projection of 0 on $\mathcal{L}_A$ in Stampacchia's sense [21] (cf. [17], Theorem 2.6):

$$e_A \in \mathcal{L}_A, \qquad \eta_\alpha(e_A, w) \geq \eta_\alpha(e_A, e_A) \qquad \text{for any } w \in \mathcal{L}_A. \tag{4.2}$$

A set $N \subset E$ is called *η-polar* if there exist decreasing $A_n \in \mathcal{O}$ containing N such that e_{A_n} is η_α-convergent to 0 as $n \to \infty$. A numerical function u on E is called *η-quasi-continuous* if there exist decreasing $A_n \in \mathcal{O}$ such that e_{A_n} is η_α-convergent to 0 as $n \to \infty$ and $u|_{E \setminus A_n}$ is continuous for each n.

The capacity Cap for the reference form $\mathcal{E}$ is defined by

$$\operatorname{Cap}(A) = \inf\{\mathcal{E}_1(u, u) : u \in \mathcal{L}_A\}, \qquad A \in \mathcal{O}.$$

It then follows from (4.1) that

$$\begin{gathered} c_1 \operatorname{Cap}(A) \leq \eta_\alpha(e_A, e_A) \leq c_2 K_\alpha^2 \operatorname{Cap}(A), \qquad A \in \mathcal{O}, \\ K_\alpha = K + \frac{\alpha}{\alpha - \beta_0}, \end{gathered} \tag{4.3}$$

because (4.2) and (B.2) imply $\eta_\alpha(e_A, e_A) \leq K_\alpha^2 \eta_\alpha(w, w)$, $w \in \mathcal{L}_A$. Equation (4.3) means that a set N is η-polar iff it is $\mathcal{E}$-polar in the sense that $\operatorname{Cap}(N) = 0$, and a function u is η-quasi-continuous iff it is $\mathcal{E}$-quasi-continuous in the sense that there exist decreasing $A_n \in \mathcal{O}$ with $\operatorname{Cap}(A_n) \downarrow 0$ as $n \to \infty$ and $u|_{E \setminus A_n}$ is continuous for each n. Every element of $\mathcal{F}$ admits its η-quasi-continuous m-version. If $\{u_n\} \subset \mathcal{F}$ is η_α-convergent to $u \in \mathcal{F}$ and if each u_n is η-quasi-continuous, then (4.1) implies that a subsequence of $\{u_n\}$ converges η-q.e., namely, outside some η-polar set, to an η-quasi-continuous version of u. We shall occasionally drop η from the terms η-polar, η-q.e. and η-quasi-continuity for simplicity.

Recall that the L^2-resolvent $\{G_\alpha; \alpha > \beta_0\}$ associated with η determines the resolvent $\{G_\alpha; \alpha > 0\}$ on $L^\infty(E; m)$ with $\|G_\alpha f\|_\infty \leq \frac{1}{\alpha}\|f\|_\infty$, $\alpha > 0$, $f \in L^\infty(E; m)$.

LEMMA 4.1. *Suppose $G_\beta f$ admits a quasi-continuous m-version $R_\beta f$ for a fixed $\beta > \beta_0$ and for every bounded Borel $f \in L^2(E; m)$. Then, for any α with $0 < \alpha \leq \beta_0$ and for any bounded Borel $f \in L^2(E; m)$,*

$$R_\alpha f(x) = \sum_{k=1}^{\infty} (\beta - \alpha)^{k-1} R_\beta^k f(x)$$

converges q.e. and defines a quasi-continuous m-version of $G_\alpha f$. Further the resolvent equation

$$R_\alpha f - R_\beta f + (\alpha - \beta) R_\alpha R_\beta f = 0$$

holds q.e. for any bounded Borel $f \in L^2(E; m)$.

PROOF. Choose a regular nest $\{F_\ell\}$ so that $R_\beta^k f \in C(\{F_\ell\})$ for $k \geq 1$. Define $v_n(x) = \sum_{k=1}^n (\beta - \alpha)^{k-1} R_\beta^k f(x)$. By the resolvent equation for $\{G_\alpha; \alpha > 0\}$, we have

$$G_\alpha f = v_n + (\beta - \alpha)^n G_\beta^n G_\alpha f.$$

The L^∞-norm of the second term of the right-hand side is dominated by $\frac{1}{\alpha}(\frac{\beta-\alpha}{\beta})^n \|f\|_\infty$, which tends to 0 as $n \to \infty$. Therefore, $\{v_n\}$ is convergent uniformly on each set F_ℓ to a quasi-continuous version of $G_\alpha f$. The resolvent equation is clear. $\square$

THEOREM 4.1. *There exist a Borel η-polar set $N_0 \subset E$ and a Hunt process $X = (X_t, P_x)$ on $E \setminus N_0$ which is properly associated with $(\eta, \mathcal{F})$ in the sense that $R_\alpha f$ is a quasi continuous version of $G_\alpha f$ for any $\alpha > 0$ and any bounded Borel $f \in L^2(E; m)$. Here R_α is the resolvent of X and G_α is the resolvent associated with η.*

This theorem was proved in [8] first by assuming that $\beta_0 = 0$ and then reducing the situation to this case. Actually the proof can be carried out without such a reduction. Indeed, after constructing the kernel $\widetilde{V}_\lambda$ of [8], Proposition II.2.1, for every rational $\lambda > \beta_0$ ([8], Proposition II.2.2) can be shown first for every rational $\lambda > \beta_0$, and then for every $0 < \lambda \leq \beta_0$ by using Lemma 4.1. The rest of the arguments in [8] then works in getting to Theorem 4.1.

Our next concern will be exceptional sets and fine continuity for the Hunt process $X = (X_t, P_x)$ appearing in Theorem 4.1. Denote by $\mathcal{B}(E)$ the family of all Borel sets of E. For $B \in \mathcal{B}(E)$, we let

$$\sigma_B = \inf\{t > 0 : X_t \in B\}, \qquad \widehat{\sigma}_B = \inf\{t > 0 : X_{t-} \in B\}, \qquad \inf \varnothing = \infty.$$

$A \in \mathcal{B}(E)$ is called *X-invariant* if

$$P_x(\sigma_{E\setminus A} \wedge \widehat{\sigma}_{E\setminus A} < \infty) = 0 \qquad \forall x \in A.$$

$N \in \mathcal{B}(E)$ is called *properly exceptional* (with respect to X) if $m(N) = 0$ and $E \setminus N$ is X-invariant.

A set $N \subset E$ is called *m-polar* if there exists $N_1 \supset N$, $N_1 \in \mathcal{B}(E)$ such that $P_m(\sigma_{N_1} < \infty) = 0$. Any properly exceptional set is m-polar.

THEOREM 4.2.

(i) *For $A \in \mathcal{O}$, the function p_A^α defined by $p_A^\alpha(x) = E_x[e^{-\alpha\sigma_A}]$, $x \in E \setminus N_0$, is a quasi-continuous version of e_A, $\alpha > \beta_0$.*

(ii) *For any η-polar set B, there exists a Borel properly exceptional set N containing $N_0 \cup B$.*

(iii) *If u is η-quasi-continuous, then there exists a Borel properly exceptional set $N \supset N_0$ such that, for any $x \in E \setminus N$,*

$$P_x\Big(\lim_{t'\downarrow t} u(X_{t'}) = u(X_t)\ \forall t \geq 0 \text{ and } \lim_{t'\uparrow t} u(X_{t'}) = u(X_{t-})\ \forall t \in (0,\zeta)\Big) = 1, \tag{4.4}$$

where ζ is the lifetime of X. In particular, u is finely continuous with respect to the restricted Hunt process $X|_{E\setminus N}$.

(iv) *Any X-semi-polar set is η-polar.*

(v) *A set $N \subset E$ is η-polar if and only if N is m-polar.*

PROOF. (i) A function $u \in L^2(E;m)$ is said to be α-excessive if $u \geq 0$, $\beta G_{\alpha+\beta} u \leq u$, $\beta > 0$. A function $u \in \mathcal{F}$ is α-excessive iff $\eta_\alpha(u, v) \geq 0$ for all nonnegative $v \in \mathcal{F}$ (cf. [16], Theorem 2.4). In particular, e_A is α-excessive and further $v = e_A \wedge p_A^\alpha$ is an α-excessive function in $\mathcal{F}$ (cf. [16], Theorem 2.6). Hence, $\eta_\alpha(v, e_A - v) \geq 0$. Since $v \in \mathcal{L}_A$, $\eta_\alpha(e_A, e_A - v) \leq 0$ so that $v = e_A$ and $e_A \leq p_A^\alpha$. The converse inequality can be obtained as in the proof of Theorem 6.1 below by using the optional sampling theorem for a supermartingale but with time parameter set being a finite set.

Since the quasi-continuous function $\beta R_{\alpha+\beta} p_A^\alpha$ converges to p_A^α as $\beta \to \infty$ pointwise and in η_α, we get the quasi-continuity of p_A^α.

(ii) Choose a decreasing sets $A_n \in \mathcal{O}$ with $A_n \supset B$, $\mathrm{Cap}(A_n) \to 0$, $n \to \infty$ and put $B_1 = \bigcap_n A_n$. By (4.1) and (i), $\lim_{n\to\infty} p_{A_n}^\alpha = 0$ q.e. so that

$$P_x(\sigma_{B_1} \wedge \widehat{\sigma}_{B_1} < \infty) = 0, \qquad x \in E \setminus N_1,$$

for some polar set N_1. Choose next a decreasing sets $A'_n \in \mathcal{O}$ containing $B_1 \cup N_1 \cup N_0$ with $\mathrm{Cap}(A'_n) \to 0$, $n \to \infty$ and put $B_2 = \bigcap_n A'_n$. Then the above identity holds for $x \in E \setminus B_2$. Moreover, the above identity holds true for B_2 in place of B_1 and for some polar set N_2 in place of N_1. Repeating this procedure, we get an increasing sequence $\{B_k\}$ of G_δ-sets which are polar sets such that

$$P_x(\sigma_{B_k} \wedge \widehat{\sigma}_{B_k} < \infty) = 0, \qquad x \in E \setminus B_{k+1}.$$

It then suffices to put $N = \bigcup_k B_k$.

(iii) Choose decreasing $A_n \in \mathcal{O}$ such that $\mathrm{Cap}(A_n) \to 0$, $n \to 0$, and $u|_{E\setminus A_n}$ is continuous for each n. Let N be a properly exceptional set constructed in (ii) starting with this sequence $\{A_n\}$. Then, for any $x \in E \setminus N$, $\lim_{n\to\infty} p_{A_n}^\alpha(x) = 0$ and consequently $P_x(\lim_{n\to\infty} \sigma_{A_n} = \infty) = 1$, which readily implies (4.4).

(iv) We reproduce a proof by Silverstein [20]. For $B \in \mathcal{B}(E)$, consider the entry time $\dot{\sigma}_B = \inf\{t \geq 0 : X_t \in B\}$ and the function $\dot{p}_B^\alpha(x) = E_x[e^{-\alpha\dot{\sigma}_B}]$, $x \in E$, $\alpha > \beta_0$. Let K be a compact thin set: K admits no regular point relative to X. It suffices to show that K is η-polar.

Choose relatively compact open sets $\{G_n\}$ such that $G_n \supset \overline{G}_{n+1}$ and $\bigcap_n G_n = K$. Due to the quasi-left continuity of X, $p_{G_n}^\alpha(x) = \dot{p}_{G_n}^\alpha(x)$ then decreases to

$\dot{p}_K^\alpha(x)$ as $n \to \infty$ for each $x \in E$. By (i) and (4.1) and (4.2), the sequence $\{\dot{p}_{G_n}^\alpha\}$ is $\mathcal{E}_1$-bounded so that the Cesàro mean sequence f_n of its suitable subsequence is $\mathcal{E}_1$-convergent. Since f_n are quasi-continuous and converges to $\dot{p}_K^\alpha$ pointwise as $n \to \infty$, we conclude that $\dot{p}_K^\alpha$ is a quasi-continuous element of $\mathcal{F}$. On the other hand, the quasi-continuous function $\beta R_{\alpha+\beta}\dot{p}_K^\alpha$ converges to p_K^α as $\beta \to \infty$ pointwise and in η_α so that p_K^α is also a quasi-continuous version of $\dot{p}_K^\alpha$. Therefore, $p_K^\alpha = \dot{p}_K^\alpha$ q.e. and in particular K is η-polar.

(v) "only if" part follows from (ii). To show "if" part, assume that K is a compact m-polar set. Then $p_K^\alpha = 0$ m-a.e. Choose for K relatively compact open sets $\{G_n\}$ as in the proof of (iv) so that the Cesàro mean f_ℓ of a certain subsequence $\{p_{G_{n_\ell}}^\alpha\}$ is $\mathcal{E}_1$-convergent to p_K^α as $\ell \to \infty$ which is now a zero element of $\mathcal{F}^0$. Since $f_\ell \geq 1$ m-a.e. on G_{n_ℓ}, we have $\mathrm{Cap}(K) \leq \mathrm{Cap}(G_{n_\ell}) \leq \mathcal{E}_1(f_\ell, f_\ell)$ and we get $\mathrm{Cap}(K) = 0$ by letting $\ell \to \infty$. For any Borel m-polar set N, we have $\mathrm{Cap}(N) = \sup\{\mathrm{Cap}(K) : K \subset N, K \text{is compact}\} = 0$. $\square$

Clearly, the restriction of X outside its properly exceptional set is again a Hunt process properly associated with η.

Our final task in this section is to relate the Hunt process of Theorem 4.1 to a martingale problem.

We consider the case where η admits the expression

$$\eta(f, g) = -(\mathcal{L}f, g), \qquad f \in \mathcal{D}(\mathcal{L}), g \in \mathcal{F}, \tag{4.5}$$

for a operator $\mathcal{L}$ with domain $\mathcal{D}(\mathcal{L})$ satisfying the following:

(L.1) $\mathcal{D}(\mathcal{L})$ is a linear subspace of $\mathcal{F} \cap C_0(E)$,

(L.2) $\mathcal{L}$ is a linear operator sending $\mathcal{D}(\mathcal{L})$ into $L^2(E; m) \cap C_b(E)$,

(L.3) there exists a countable subfamily $\mathcal{D}_0$ of $\mathcal{D}(\mathcal{L})$ such that each $f \in \mathcal{D}(\mathcal{L})$ admits $f_n \in \mathcal{D}_0$ such that $f_n, \mathcal{L}f_n$ are uniformly bounded and converge pointwise to $f, \mathcal{L}f$, respectively, as $n \to \infty$.

We also consider an additional condition that

(L.4) there exists $f_n \in \mathcal{D}(\mathcal{L})$ such that $f_n, \mathcal{L}f_n$ are uniformly bounded and converge to $1, 0$, respectively, as $n \to \infty$.

THEOREM 4.3. *Assume that η admits the expression* (4.5) *with $\mathcal{L}$ satisfying conditions* (L.1), (L.2), (L.3).

(i) *There exists then a Borel properly exceptional set N containing N_0 such that, for every $f \in \mathcal{D}(\mathcal{L})$,*

$$M_t^{[f]} = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)\,ds, \qquad t \geq 0, \tag{4.6}$$

is a P_x-martingale for each $x \in E \setminus N$.

(ii) *If the additional condition* (L.4) *is satisfied, then the Hunt process $X|_{E\setminus N}$ is conservative.*

PROOF. (i) Take $f \in \mathcal{D}(\mathcal{L})$ and $g \in L^2(E; m)$. By (4.5) and (3.2), we have, for $\alpha > \beta_0$,

$$\begin{aligned}(G_\alpha \mathcal{L} f, g) &= (\mathcal{L} f, \widehat{G}_\alpha g) = -\eta(f, \widehat{G}_\alpha g) \\ &= -\eta_\alpha(f, \widehat{G}_\alpha g) + \alpha(f, \widehat{G}_\alpha g) \\ &= -(f, g) + \alpha(G_\alpha f, g).\end{aligned}$$

Thus, $(G_\alpha \mathcal{L} f, g) = (\alpha G_\alpha f - f, g)$ holds for any $g \in \mathcal{F}$ and

$$\frac{1}{\alpha} G_\alpha(\mathcal{L} f)(x) = G_\alpha f(x) - \frac{f(x)}{\alpha}, \qquad m\text{-a.e.}$$

We denote by $\{P_t; t \geq 0\}$ and $\{R_\alpha; \alpha > 0\}$ the transition function and the resolvent of X, respectively:

$$P_t h(x) = \mathbb{E}_x[h(X_t)], \qquad R_\alpha h(x) = \int_0^\infty e^{-\alpha t} P_t h(x)\, dt.$$

Since X is properly associated with η by Theorem 4.1, we get

$$\frac{1}{\alpha} R_\alpha(\mathcal{L} f)(x) = R_\alpha f(x) - \frac{f(x)}{\alpha}, \qquad \text{q.e.}$$

Hence, by virtue of Theorem 4.2(ii), there exists a Borel properly exceptional set N such that

$$\int_0^\infty e^{-\alpha t} \left(\int_0^t P_s(\mathcal{L} f)(x)\, ds \right) dt = \int_0^\infty e^{-\alpha t} \big(P_t f(x) - f(x)\big)\, dt, \qquad x \in E \setminus N,$$

holds for any $\alpha \in \mathbb{Q}_+$ with $\alpha > \beta_0$ and for any $f \in \mathcal{D}_0$.

Since $P_t h(x)$ is a right continuous in $t \geq 0$ for any $h \in C_b(E)$, we get

$$P_t f(x) - f(x) = \int_0^t P_s(\mathcal{L} f)(x)\, ds, \qquad t \geq 0, x \in E \setminus N, \tag{4.7}$$

holding for any $f \in \mathcal{D}_0$. By virtue of condition (L.3), we conclude that the equation (4.7) holds true for any $f \in \mathcal{D}(\mathcal{L})$. Equation (4.7) implies that, for any $f \in \mathcal{D}(\mathcal{L})$, the functional $M_t^{[f]}, t \geq 0$, defined by (4.6) is a mean zero, square integrable additive functional of the Hunt process $X|_{E \setminus N}$ so that it is a P_x-martingale for each $x \in E \setminus N$.

(ii) Under the additional condition (L.4), we let $n \to \infty$ in equation (4.7) with f_n in place of f arriving at $P_t 1 = 1, t \geq 0$. $\square$

Theorem 4.3 will enable us in the next section to relate our Hunt process to the solution of a martingale problem in a specific case.

5. Stable-like process. In this section, we consider the case that $E = \mathbb{R}^d$ and $m(dx) = dx$ is the Lebesgue measure on $\mathbb{R}^d$. For a positive measurable function $\alpha(x)$ defined on $\mathbb{R}^d$, Bass introduced the following integro-differential operator in [5] (see also [4, 6]): for $u \in C_b^2(\mathbb{R}^d)$,

$$\mathcal{L}u(x) = w(x)\int_{h\neq 0}\big(u(x+h) - u(x) - \nabla u(x)\cdot h\mathbf{1}_{B(1)}(h)\big)|h|^{-d-\alpha(x)}\,dh, \qquad x \in \mathbb{R}^d,$$

where $w(x)$ is a function chosen so that $\mathcal{L}e^{iux} = -|u|^{\alpha(x)}e^{iux}$ and $C_b^2(\mathbb{R}^d)$ denotes the set of twicely differentiable bounded functions. If α is Lipschitz continuous, bounded below by a constant which is greater than 0, and bounded above by a constant which is less than 2, then he constructed a unique strong Markov process associated with $\mathcal{L}$ by solving the $\mathcal{L}$-martingale problem for every starting point $x \in \mathbb{R}^d$. Using the theory of stochastic differential equation with jumps, Tsuchiya [22] also succeeded in constructing the Markov process associated with $\mathcal{L}$ (see also [18]). Note that the weight function $w(x)$ is given by

$$w(x) = \frac{\Gamma((1+\alpha(x))/2)\Gamma((\alpha(x)+d)/2)\sin(\pi\alpha(x)/2)}{2^{1-\alpha(x)}\pi^{d/2+1}}, \qquad x \in \mathbb{R}^d \tag{5.1}$$

(see, e.g., [3]).

Put $k(x,y) = w(x)|x-y|^{-d-\alpha(x)}$, $x, y \in \mathbb{R}^d$ with $x \neq y$. Then this falls into our case when we consider the following conditions: there exist positive constants $\underline{\alpha}, \overline{\alpha}, M$ and δ so that for $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &0 < \underline{\alpha} \le \alpha(x) \le \overline{\alpha} < 2, \overline{\alpha} < 1 + \frac{\underline{\alpha}}{2} \quad \text{and} \\ &|\alpha(x) - \alpha(y)| \le M|x-y|^{\delta} \qquad \text{for } \delta \text{ with } 0 < \frac{1}{2}(2\overline{\alpha} - \underline{\alpha}) < \delta \le 1. \end{aligned} \tag{5.2}$$

PROPOSITION 5.1. *Assume* (5.2) *holds. Then conditions* (2.1)–(2.4) *are satisfied by the function*

$$k(x,y) = w(x)|x-y|^{-d-\alpha(x)}, \qquad x, y \in \mathbb{R}^d, x \neq y. \tag{5.3}$$

PROOF. Note first that, from equation (5.1) defining the weight $w(x)$, we easily see that there exist constants c_i $(i = 1, 2, 3)$ so that for $x, y \in \mathbb{R}^d$,

$$c_1 \le w(x) \le c_2, \qquad |w(x) - w(y)| \le c_3|\alpha(x) - \alpha(y)|.$$

Then

$$\begin{aligned} k_s(x,y) &= \tfrac{1}{2}\big(w(x)|x-y|^{-d-\alpha(x)} + w(y)|x-y|^{-d-\alpha(y)}\big) \\ &\le \begin{cases} M|x-y|^{-d-\overline{\alpha}}, & |x-y| \le 1, \\ M|x-y|^{-d-\underline{\alpha}}, & |x-y| > 1. \end{cases} \end{aligned}$$

This and the condition $0 < \underline{\alpha} \le \overline{\alpha} < 2$ imply that condition (2.1) is fulfilled because the function M_s in it is bounded. Condition (2.2) is also valid as $|k_a(x, y)| \le k_s(x, y)$.

On the other hand, since

$$\begin{aligned} k_a(x, y) &= w(x)|x-y|^{-d-\alpha(x)} - w(y)|x-y|^{-d-\alpha(y)} \\ &= \big(w(x) - w(y)\big)|x-y|^{-d-\alpha(x)} \\ &\quad + w(y)|x-y|^{-d}\big(|x-y|^{-\alpha(x)} - |x-y|^{-\alpha(y)}\big) \end{aligned}$$

and

$$|x-y|^{-\alpha(x)} - |x-y|^{-\alpha(y)} = \int_{\alpha(y)}^{\alpha(x)} |x-y|^{-u} \frac{1}{\ln|x-y|^{-1}} \, du,$$

we see that for $|x - y| < 1$,

$$\begin{aligned} |k_a(x, y)| &\le |w(x) - w(y)| \cdot |x-y|^{-d-\alpha(x)} \\ &\quad + w(y)|x-y|^{-d}|\alpha(x) - \alpha(y)| \cdot |x-y|^{-(\alpha(x)\vee\alpha(y))} \frac{1}{\ln|x-y|^{-1}} \\ &\le M\left(|x-y|^{-d-\overline{\alpha}+\delta} + |x-y|^{-d-\overline{\alpha}+\delta} \frac{1}{\ln|x-y|^{-1}}\right) \\ &\le M'|x-y|^{-d-\overline{\alpha}+\delta} \frac{1}{\ln|x-y|^{-1}}. \end{aligned}$$

So if γ satisfies

$$\gamma(d + \overline{\alpha} - \delta) - (d - 1) < 1,$$

then condition (2.3) holds. As for condition (2.4), note that

$$k_s(x, y) \ge M'|x-y|^{-d-\underline{\alpha}}, \qquad |x - y| < 1.$$

So, (2.4) is valid when

$$(d + \overline{\alpha} - \delta)(2 - \gamma) < d + \underline{\alpha}.$$

Therefore, conditions (2.3) and (2.4) hold provided that γ satisfies

$$\frac{d + 2\overline{\alpha} - 2\delta - \underline{\alpha}}{d + \overline{\alpha} - \delta} < \gamma < \frac{d}{d + \overline{\alpha} - \delta}. \qquad \square$$

Let $(\eta, \mathcal{F}^0)$ be the regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$ associated with the kernel (5.3) satisfying (5.2) according to Theorem 2.1. Let $X = (X_t, P_x)$ be the Hunt process on $\mathbb{R}^d$ properly associated with $(\eta, \mathcal{F})$ by Theorem 4.1.

Define a linear operator $\mathcal{L}$ by

$$(5.4)\qquad \begin{cases} \mathcal{D}(\mathcal{L}) = C_0^2(\mathbb{R}^d), \\ \mathcal{L}u(x) = \displaystyle\int_{h\neq 0} \big(u(x+h) - u(x) - \nabla u(x)\cdot h\mathbf{1}_{B_1(0)}(h)\big)\frac{w(x)\,dh}{|h|^{d+\alpha(x)}}, \\ \qquad x \in \mathbb{R}^d. \end{cases}$$

$C_0^2(\mathbb{R}^d)$ is a linear subspace of $\mathcal{F}^0 \cap C_0(\mathbb{R}^d)$ and, by condition (5.2), we can see that $\mathcal{L}$ maps $C_0^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. As any continuously differentiable function and its derivatives can be simultaneously approximated by polynomials and their derivatives uniformly on each rectangles (cf. [9], Chapter II), conditions (L.1), (L.2), (L.3) in the preceding section on $\mathcal{L}$ are fulfilled. We can easily verify that the present $\mathcal{L}$ satisfies condition (L.4) as well.

Since the vector valued function $hw(x)\mathbf{1}_{B_1(0)}(h)|h|^{-d-\alpha(x)}$ is odd with respect to the variable h for each $x \in \mathbb{R}^d$, we get for $u \in C_0^2(\mathbb{R}^d)$,

$$\begin{aligned} \eta^n(u,v) &= -\iint_{|x-y|>1/n} \big(u(y)-u(x)\big)v(x)\frac{w(x)}{|x-y|^{d+\alpha(x)}}\,dx\,dy \\ &= -\iint_{|h|>1/n} \big(u(x+h)-u(x)\big)v(x)\frac{w(x)}{|h|^{d+\alpha(x)}}\,dx\,dh \\ &= -\iint_{|h|>1/n} \big(u(x+h)-u(x)-\nabla u(x)\cdot h\mathbf{1}_{B_1(0)}(h)\big)v(x) \\ &\qquad\qquad \times \frac{w(x)}{|h|^{d+\alpha(x)}}\,dx\,dh. \end{aligned}$$

By letting $n \to \infty$, we have

$$\eta(u,v) = -(\mathcal{L}u, v),$$

that is, η is related to $\mathcal{L}$ by (4.5).

By virtue of Theorem 4.3, there exists a Borel properly exceptional set $N \subset \mathbb{R}^d$ so that $X|_{\mathbb{R}^d\setminus N}$ is conservative and, for each $x \in \mathbb{R}^d \setminus N$,

$$M_t^{[f]} = f(X_t) - f(X_0) - \int_0^t (\mathcal{L}f)(X_s)\,ds, \qquad t \geq 0,$$

is a martingale under P_x for every $f \in C_0^2(\mathbb{R}^d)$. Approximating $f \in C_b^2(\mathbb{R}^d)$ by a uniformly bounded sequence $\{f_n\} \subset C_0^2(\mathbb{R}^d)$ such that $\{\mathcal{L}f_n\}$ is uniformly bounded and convergent to $\mathcal{L}f$, we see that (4.6) remains valid for $f \in C_b^2(\mathbb{R}^d)$ and $M_t^{[f]}$ is still a martingale under $\mathbb{P}_x$ for $x \in \mathbb{R}^d \setminus N$. For each $x \in \mathbb{R}^d \setminus N$, the measure $\mathbb{P}_x$ is thus a solution to the martingale problem for the operator $\mathcal{L}$ of (5.4) starting at x so that $\mathbb{P}_x$ coincides with the law constructed by Bass [5] because of the uniqueness also due to [5].

REMARK 5.1. Let

$$k^*(x,y)=\frac{w(y)}{|x-y|^{d+\alpha(y)}},\qquad x,\in\mathbb{R}^d, x\neq y. \tag{5.5}$$

Under condition (5.2), the form η^* corresponding to the kernel k^* is a regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$ by virtue of Proposition 5.1 and Corollary 2.1. By Theorem 4.1, η^* admits a properly associated Hunt process X^* on $\mathbb{R}^d$. Furthermore, we can have an explicit expression $\eta^*(u,v)=-(\mathcal{L}^*u,v)$ for $u\in C_0^2(\mathbb{R}^d)$ and $v\in\mathcal{F}^0$ with

$$\begin{aligned}\mathcal{L}^*u(x)=&\int_{h\neq 0}\big(u(x+h)-u(x)-\nabla u(x)\cdot h\mathbf{1}_{B_1(0)}(h)\big)\frac{w(x+h)\,dh}{|h|^{d+\alpha(x+h)}}\\&+\frac{1}{2}\int_{0<|h|<1}\nabla u(x)\cdot h\bigg(\frac{w(x+h)}{|h|^{d+\alpha(x+h)}}-\frac{w(x-h)}{|h|^{d+\alpha(x-h)}}\bigg)\,dh,\qquad x\in\mathbb{R}^d.\end{aligned}$$

In a lower order case as is considered in Section 3, both $\mathcal{L}$ and $\mathcal{L}^*$ admit simpler expressions (3.7) and $\mathcal{L}^*-K$ is a formal adjoint of $\mathcal{L}$ for a function K defined by (3.9).

6. Associated Hunt processes on open subsets and on their closures. We make the same assumptions on E,m,k as in Section 2. Let D be an arbitrary open subset of E and $\overline{D}$ be the closure of D, m_D is defined to be $m_D(B)=m(B\cap D)$, $B\in\mathcal{B}(E)$ and $(u,v)_D$ denotes the inner product of $L^2(D,m_D)$ ($=L^2(\overline{D},m_D)$). Consider the related function spaces $C_0^{\mathrm{lip}}(\overline{D})$ and $C_0^{\mathrm{lip}}(D)$ introduced in Section 1. Define

$$\begin{cases}\mathcal{E}_D(u,v):=\displaystyle\iint_{D\times D\setminus\mathrm{diag}}\big(u(y)-u(x)\big)\big(v(y)-v(x)\big)\\ \qquad\qquad\qquad\qquad\times k_s(x,y)m_D(dx)m_D(dy),\\ \mathcal{F}_D^r=\{u\in L^2(D;m_D): u \text{ is Borel measurable and } \mathcal{E}_D(u,u)<\infty\},\end{cases} \tag{6.1}$$

and let $\mathcal{F}_{\bar{D}}$ and $\mathcal{F}_D^0$ be the $\mathcal{E}_{D,1}$-closures of $C_0^{\mathrm{lip}}(\overline{D})$ and $C_0^{\mathrm{lip}}(D)$ in $\mathcal{F}_D^r$, respectively. $(\mathcal{E}_D,\mathcal{F}_{\bar{D}})$ [resp., $(\mathcal{E}_D^0,\mathcal{F}_D^0)$] is a regular symmetric Dirichlet form on $L^2(\overline{D};m_D)$ [resp., $L^2(D;m_D)$] where $\mathcal{E}_D^0$ denotes the restriction of $\mathcal{E}_D$ to $\mathcal{F}_D^0\times\mathcal{F}_D^0$. Furthermore, in view of [13], Theorem 4.4.3, we have the identity

$$\mathcal{F}_D^0=\{u\in\mathcal{F}_{\bar{D}}:\tilde{u}=0, \mathcal{E}_D\text{-q.e. on }\partial D\}, \tag{6.2}$$

where $\tilde{u}$ denotes an $\mathcal{E}_D$-quasi continuous version of $u\in\mathcal{F}_{\bar{D}}$. We keep in mind that a subset of D is polar for $(\mathcal{E}_D,\mathcal{F}_D^0)$ iff so it is for $(\mathcal{E}_D,\mathcal{F}_{\bar{D}})$, and the restriction to D of a quasi continuous function with respect to the latter is quasi-continuous with respect to the former.

Now define for $u\in C_0^{\mathrm{lip}}(\overline{D})$ and $n\in\mathbb{N}$

$$\mathcal{L}_D^n u(x):=\int_{\{y\in D: d(x,y)>1/n\}}\big(u(y)-u(x)\big)k(x,y)m_D(dy),\qquad x\in D. \tag{6.3}$$

Then, just as in Proposition 2.1 and Theorem 2.1 of Section 2, we conclude that the finite limit

$$\eta_D(u,v) = -\lim_{n\to\infty}\int_D \mathcal{L}_D^n u(x)v(x)m_D(dx) \qquad \text{for } u,v \in C_0^{\mathrm{lip}}(\overline{D}) \tag{6.4}$$

exists, η_D extends to $\mathcal{F}_{\bar{D}} \times \mathcal{F}_{\bar{D}}$ and $(\eta_D, \mathcal{F}_{\bar{D}})$ becomes a regular lower bounded semi-Dirichlet form on $L^2(\overline{D}; m_D)$ possessing $(\mathcal{E}_D, \mathcal{F}_{\bar{D}})$ as its reference symmetric Dirichlet form. In parallel with $(\eta_D, \mathcal{F}_{\bar{D}})$, the space $(\eta_D^0, \mathcal{F}_D^0)$ becomes a regular lower bounded semi-Dirichlet form on $L^2(D; m_D)$ possessing $(\mathcal{E}_D^0, \mathcal{F}_D^0)$ as its reference symmetric Dirichlet form. Here η_D^0 is the restriction of η_D to $\mathcal{F}_D^0 \times \mathcal{F}_D^0$.

Let $X^{\bar{D}} = (X_t, P_x)$ be a Hunt process on $\overline{D}$ properly associated with the form $(\eta_D, \mathcal{F}_{\bar{D}})$ on $L^2(\overline{D}; m_D)$. Denote by $X^{D,0} = (X_t^{D,0}, P_x)$ the part process of $X^{\bar{D}}$ on D, namely, $X_t^{D,0}$ is obtained from X_t by killing upon hitting the boundary ∂D:

$$X_t^{D,0} = X_t, \qquad t < \sigma_{\partial D}; \qquad X_t^{D,0} = \Delta, \qquad t \geq \sigma_{\partial D},$$

$X^{D,0}$ is a Hunt process with state space D.

THEOREM 6.1. *The part process $X^{D,0}$ of $X^{\bar{D}}$ on D is properly associated with the regular lower bounded semi-Dirichlet form $(\eta_D^0, \mathcal{F}_D^0)$ on $L^2(D; m_D)$.*

PROOF. Let $\{R_\alpha; \alpha > 0\}$ be the resolvent of $X^{\bar{D}}$. σ will denote the hitting time of ∂D by $X^{\bar{D}}: \sigma = \sigma_{\partial D}$. Put, for $\alpha > 0$ and $x \in \overline{D}$,

$$R_\alpha^{D,0} f(x) = E_x\left[\int_0^\sigma e^{-\alpha t} f(X_t)\,dt\right],$$

$$H_\alpha^{\partial D} u(x) = E_x[e^{-\alpha\sigma} u(X_\sigma)], \qquad x \in \overline{D}.$$

$\{R_\alpha^{D,0}|_D; \alpha > 0\}$ is the resolvent of the part process $X^{D,0}$ of $X^{\bar{D}}$ on D.

We need to prove that, for any $\alpha > \beta_0$ and any $f \in \mathcal{B}(\overline{D}) \cap L^2(\overline{D}, m_D)$,

$$\begin{aligned} &R_\alpha^{D,0} f \text{ is } \eta_D^0\text{-quasi-continuous,} \\ &R_\alpha^{D,0} f \in \mathcal{F}_D^0, \qquad \eta_{D,\alpha}^0(R_\alpha^{D,0} f, v) = (f, v)_D \qquad \text{for any } v \in \mathcal{F}_D^0. \end{aligned} \tag{6.5}$$

We denote by $\mathcal{G}$ the space appearing in the right-hand side of (6.2). Notice that $\mathcal{E}_D$-q.e. (resp., $\mathcal{E}_D$-quasi-continuity) is now a synonym of η_D-q.e. (resp., η_D-quasi-continuity). As the set of points of ∂D that are irregular for ∂D is known to be semi-polar, we have $P_x(\sigma = 0) = 1$ and so $R_\alpha^{D,0} f(x) = 0$ for η_D-q.e. $x \in \partial D$ owing to Theorem 4.2(iv). Since

$$\begin{aligned} &R_\alpha f \text{ is } \eta_D\text{-quasi-continuous,} \\ &R_\alpha f \in \mathcal{F}_{\bar{D}}, \qquad \eta_{D,\alpha}(R_\alpha f, v) = (f, v)_D \qquad \text{for any } v \in \mathcal{F}_{\bar{D}} \end{aligned}$$

and

$$R_\alpha f(x) = R_\alpha^{D,0} f(x) + H_\alpha^{\partial D} R_\alpha f(x), \qquad x \in \overline{D}, \tag{6.6}$$

we see that, for the proof of (6.5), it is enough to show that

$$\begin{aligned} &H_\alpha^{\partial D} R_\alpha f \text{ is } \eta_D\text{-quasi-continuous,} \\ &H_\alpha^{\partial D} R_\alpha f \in \mathcal{F}_{\bar{D}}, \qquad \eta_{D,\alpha}(H_\alpha^{\partial D} R_\alpha f, v) = 0 \qquad \text{for any } v \in \mathcal{G}. \end{aligned} \tag{6.7}$$

To this end, we fix $\alpha > \beta_0$, $f \in \mathcal{B}_+(\overline{D}) \cap L^2(\overline{D}; m_D)$ and put $u = R_\alpha f$. Consider a closed convex subset of $\mathcal{F}_{\bar{D}}$ defined by

$$\mathcal{L}_{u,\partial D} = \{v \in \mathcal{F}_{\bar{D}}, \tilde{v} \geq \tilde{u} \text{ q.e. on } \partial D\}.$$

Let u_α be the $\eta_{D,\alpha}$-projection of 0 on $\mathcal{L}_{u,\partial D}$:

$$u_\alpha \in \mathcal{L}_{u,\partial D}, \qquad \eta_{D,\alpha}(u_\alpha, v - u_\alpha) \geq 0, \qquad \text{for any } v \in \mathcal{L}_{u,\partial D}.$$

Both u and u_α are α-excessive elements of $\mathcal{F}_{\bar{D}}$. By making use of the function $v = u_\alpha \wedge u$ as in the proof of Proposition 3.1(i), we readily get

$$\tilde{u}_\alpha = u \text{ q.e. on } \partial D, \qquad \eta_{D,\alpha}(u_\alpha, v) = 0 \qquad \text{for any } v \in \mathcal{G}. \tag{6.8}$$

Finally, we prove that

$$H_\alpha^{\partial D} u \text{ is } \eta_D\text{-quasi continuous}, \qquad H_\alpha^{\partial D} u = u_\alpha, \tag{6.9}$$

which leads us to the desired property (6.7). By (6.6), $H_\alpha^{\partial D} u$ is an α-excessive function dominated by $u \in \mathcal{F}_{\bar{D}}$ so that $H_\alpha^{\partial D} u$ is a quasi-continuous element of $\mathcal{F}_{\bar{D}}$. Further $H_\alpha^{\partial D} u = u$ q.e. on ∂D by (6.6) and an observation made preceding it. Let $v = H_\alpha^{\partial D} u \wedge u_\alpha$. Then $\tilde{v} = H_\alpha^{\partial D} u \wedge \tilde{u}_\alpha = u$ q.e. on ∂D so that $\eta_{D,\alpha}(u_\alpha, u_\alpha - v) = 0$ by (6.8). On the other hand, v is α-excessive and so $\eta_{D,\alpha}(v, u_\alpha - v) \geq 0$. Consequently, $\eta_\alpha(u_\alpha - v, u_\alpha - v) \leq 0$ and we get the inequality $u_\alpha \leq H_\alpha^{\partial D} u$.

To get the converse inequality, consider a bounded nonnegative Borel function h on D with $\int_D h\, dm = 1$. Denote by $\{p_t; t \geq 0\}$ the transition function of $X^{\bar{D}}$. We choose a Borel measurable quasi-continuous version $\tilde{u}_\alpha$ of $u_\alpha \in \mathcal{F}_{\bar{D}}$. We set $\tilde{u}_\alpha(\Delta) = 0$ for the cemetery Δ of $X^{\bar{D}}$. Since u_α is α-excessive, $e^{-\alpha t} p_t \tilde{u}_\alpha \leq \tilde{u}_\alpha$ m-a.e., and we can see that the process $\{Y_t = e^{-\alpha t} \tilde{u}_\alpha(X_t); t \geq 0\}$ is a right continuous positive supermartingale under $P_{h\cdot m}$ in view of Theorem 4.2(iii). For any compact set $K \subset \partial D$, we get from the optional sampling theorem and (6.8),

$$\begin{aligned} E_{h\cdot m}[Y_{\sigma_K}] &= E_{h\cdot m}[e^{-\alpha\sigma_K} \tilde{u}_\alpha(X_{\sigma_K})] \\ &= E_{h\cdot m}[e^{-\alpha\sigma_K} u(X_{\sigma_K})] \leq E_{h\cdot m}[Y_0] \\ &= (h, u_\alpha)_D. \end{aligned}$$

By choosing K such that $\sigma_K \downarrow \sigma$ $P_{h\cdot m}$-a.e., we obtain $(h, H_\alpha^{\partial D} u)_D \leq (h, u_\alpha)_D$ and $H_\alpha^{\partial D} u \leq u_\alpha$. $\square$

As a preparation for the next lemma, we take any open set $G \subset D$ and denote by m_G the restriction of m to G. Let $\mathcal{F}_G^0$ be the $\mathcal{E}_{D,1}$-closure of $C_0^{\mathrm{lip}}(G)$ in $\mathcal{F}_D^r$ and η_G^0 be the restriction of η_D to $\mathcal{F}_G^0 \times \mathcal{F}_G^0$. Then, just as above,

$$\mathcal{F}_G^0 = \{u \in \mathcal{F}_{\bar{D}} : \tilde{u} = 0\ \mathcal{E}_D \text{ q.e. on } \overline{D} \setminus G\}$$

and $(\eta_G^0, \mathcal{F}_G^0)$ becomes a regular lower bounded semi-Dirichlet form on $L^2(G; m_G)$ with which the part process $X^{G,0}$ of $X^{\bar{D}}$ on G is properly associated. The resolvent of $X^{G,0}$ will be denoted by $R_\alpha^{G,0}$.

Define

$$H_\alpha^{\bar{D}\setminus G} u(x) = E_x[e^{-\alpha\sigma_{\bar{D}\setminus G}} u(X_{\sigma_{\bar{D}\setminus G}})], \qquad x \in \overline{D}.$$

As (6.7), we have, for $u = R_\alpha f$, $f \in \mathcal{B}(\overline{D}) \cap L^2(\overline{D}; m_D)$, $\alpha > \beta_0$,

$$\begin{aligned} &H_\alpha^{\bar{D}\setminus G} u \text{ is } \eta_D\text{-quasi-continuous,} \\ &H_\alpha^{\bar{D}\setminus G} u \in \mathcal{F}_{\bar{D}}, \qquad \eta_{D,\alpha}(H_\alpha^{\bar{D}\setminus G} u, v) = 0 \qquad \text{for any } v \in \mathcal{F}_G^0, \end{aligned} \tag{6.10}$$

and the bound $\eta_{D,\alpha}(H_\alpha^{\bar{D}\setminus G} u, H_\alpha^{\bar{D}\setminus G} u) \le \eta_{D,\alpha}(u, u)$. We can easily see that (6.10) holds true for any $u \in \mathcal{F}^{\bar{D}} \cap C_0(\overline{D})$ where $C_0(\overline{D})$ denotes the restrictions to $\overline{D}$ of functions in $C_0(E)$. In fact, by the resolvent equation, (6.10) is true for $R_\beta u$, $\beta > \beta_0$, in place of u. Since $\{\beta_n R_{\beta_n} u\}$ converges to u pointwise as well as in $\eta_{D,\alpha}$-metric as $\beta_n \to \infty$, so does the sequence $\{\beta_n H_\alpha^{\bar{D}\setminus G} R_{\beta_n} u\}$, arriving at the validity of (6.10) for such u.

LEMMA 6.1. *Let G be a relatively compact open set with $\overline{G} \subset D$. Then for any $v \in \mathcal{F}^{\bar{D}} \cap C_0(\overline{D})$ with* $\mathrm{supp}[v] \subset \overline{D} \setminus \overline{G}$, *it follows for $\alpha > \beta_0$ that*

$$E_x[e^{-\alpha\tau_G} v(X_{\tau_G})] = R_\alpha^{G,0} g_v(x) \qquad \textit{for q.e. } x \in G, \tag{6.11}$$

where $\tau_G = \sigma_{\bar{D}\setminus G} \wedge \zeta$ is the first leaving time from G and g_v is a function given by

$$g_v(x) = 1_G(x) \int_{\bar{D}\setminus\bar{G}} k(x, y) v(y) m_D(dy), \qquad x \in \overline{D}. \tag{6.12}$$

PROOF. Take any $u \in \mathcal{F}^{\bar{D}} \cap C_0(\overline{D})$ such that $\mathrm{supp}[u] \subset G$. From (6.3) and (6.4), we then have

$$\eta_D(u, v) = -\int_{G\times(\bar{D}\setminus\bar{G})} u(y) v(x) k(x, y) m_D(dx) m_D(dy). \tag{6.13}$$

We can now proceed as in [13], page 163. The function g_v defined by (6.12) belongs to $L^2(G; m_G)$ on account of condition (2.1) on the kernel k. Therefore, we obtain from (6.13)

$$\begin{aligned}\eta^0_{G,\alpha}(R^{G,0}_\alpha g_v, u) &= \int_G g_v(x)u(x)m_G(dx)\\ &= \int_{G\times(\bar D\setminus\bar G)} u(x)v(y)k(x,y)m_D(dx)m_D(dy)\\ &= -\eta_D(v,u) = -\eta_{D,\alpha}(v,u)\\ &= -\eta^0_{G,\alpha}(v - H^{\bar D\setminus G}_\alpha v, u), \qquad \alpha > \beta_0,\end{aligned}$$

the last identity being a consequence of (6.10). Since $\mathcal{F}^{\bar D} \cap C_0(G)$ is $\eta^0_{G,\alpha}$-dense in $\mathcal{F}^0_G$, we get

$$H^{\bar D\setminus G}_\alpha v(x) = H^{\bar D\setminus G}_\alpha v(x) - v(x) = R^{G,0}_\alpha g_v(x) \qquad \text{for } m_G\text{-a.e. on } G.$$

We then obtain (6.11) because $H^{\bar D\setminus G}_\alpha v$ and $R^{G,0}_\alpha g_v$ are η^0_G-quasi-continuous by (6.10). □

THEOREM 6.2.

(i) *$X^{\bar D} = (X_t, P_x)$ admits no jump from D to ∂D:*

$$P_x(X_{t-} \in D, X_t \in \partial D \text{ for some } t > 0) = 0 \qquad \textit{for q.e. } x \in D. \tag{6.14}$$

(ii) *If D is relatively compact, then $X^{\bar D}$ is conservative: denoting by ζ the lifetime of $X^{\bar D}$,*

$$P_x(\zeta = \infty) = 1 \qquad \textit{for q.e. } x \in \overline{D}. \tag{6.15}$$

(iii) *If D is relatively compact, then $X^{D,0} = (X^{D,0}_t, P_x)$ admits no killing inside D: denoting by ζ^0 the lifetime of $X^{D,0}$,*

$$P_x(X^{D,0}_{\zeta^0-} \in D, \zeta^0 < \infty) = 0 \qquad \textit{for q.e. } x \in D. \tag{6.16}$$

PROOF. (i) For any open set G as Lemma 6.1 and any compact subset F of ∂D, we can find a uniformly bounded sequence $\{v_n\} \subset \mathcal{F}^{\bar D} \cap C_0(\overline{D})$ with support being contained in a common compact subset of $\overline{D} \setminus \overline{G}$ and $\lim_{n\to\infty} v_n = 1_F$. Then $g_{v_n}(x)$ are uniformly bounded and converge to $g_{1_F}(x) = 0$ as $n \to \infty$. Therefore, by letting $n \to \infty$ in (6.11) with v_n in place of v, we get $P_x(X_{\tau_G} \in F) = 0$ for q.e. $x \in G$. Since G and F are arbitrary with the stated properties, we have (6.14).

(ii) When D is relatively compact, $1 \in C^{\mathrm{lip}}_0(\overline{D})$ so that we see from (6.3) and (6.4) that $1 \in \mathcal{F}^{\bar D}$ and $\eta_D(1, v) = 0$ for any $v \in \mathcal{F}^{\bar D}$. We have therefore, for any $\alpha > \beta_0$ and $f \in L^2(\overline{D}, m_D)$,

$$0 = \eta_D(1, \widehat{G}_\alpha f) = (1, f)_D - \alpha(1, \widehat{G}_\alpha f)_D = (1 - \alpha R_\alpha 1, f)_D,$$

where $\widehat{G}_\alpha$ is the dual resolvent. This implies that $\alpha R_\alpha 1 = 1$ m_D-a.e. for $\alpha > \beta_0$ and consequently q.e. on $\overline{D}$ because $R_\alpha 1$ is quasi-continuous. Equation (6.15) is proven.

(iii) This is an immediate consequence of (i), (ii) as $X^{D,0}$ is the part process of $X^{\bar{D}}$ on D. □

We conjecture that the property (6.16) for $X^{D,0}$ holds true without the assumption of the relative compactness of D and especially for the minimal process X^0 on E.

Finally, we consider the case where E is $\mathbb{R}^d$ and m is the Lebesgue measure on it. For $\alpha \in (0,2)$ and an arbitrary open set $D \subset \mathbb{R}^d$, we make use of the Lévy kernel

$$k^{[\alpha]}(x,y) = \frac{\alpha 2^{\alpha-1}\Gamma((\alpha+d)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)} \frac{1}{|x-y|^{d+\alpha}}, \qquad x, y \in \mathbb{R}^d,$$

of the symmetric α-stable process to introduce the Dirichlet form

$$\text{(6.17)} \quad \begin{cases} \mathcal{E}_D^{[\alpha]}(u,v) := \displaystyle\iint_{D\times D\setminus \mathrm{diag}} \big(u(y)-u(x)\big)\big(v(y)-v(x)\big) k^{[\alpha]}(x,y)\,dx\,dy, \\ \mathcal{F}_D^{[\alpha],r} = \big\{u \in L^2(D) : u \text{ is Borel measurable and } \mathcal{E}_D^{[\alpha]}(u,u) < \infty\big\}, \end{cases}$$

on $L^2(D)$ based on the Lebesgue measure on D. Denote by $\mathcal{F}_{\bar{D}}^{[\alpha]}$ the $\mathcal{E}_{D,1}^{[\alpha]}$-closure of $C_0^{\mathrm{lip}}(\overline{D})$ in $\mathcal{F}_D^{[\alpha],r}$. For $s \in (0,d]$, a Borel subset Γ of $\mathbb{R}^d$ is said to be an *s-set* if there exist positive constants c_1, c_2 such that for all $x \in \Gamma$ and $r \in (0,1]$, $c_1 r^s \le \mathcal{H}^s(\Gamma \cap B(x,r)) \le c_2 r^s$, where $\mathcal{H}^s$ denotes the s-dimensional Hausdorff measure on $\mathbb{R}^d$ and $B(x,r)$ is the ball of radius r centered at $x \in \mathbb{R}^d$.

If the open set D is a d-set, then, by making use of Jonsson–Wallin's trace theorem [14] as in [7], one can show that $\mathcal{F}_{\bar{D}}^{[\alpha]} = \mathcal{F}_D^{[\alpha],r}$ and moreover that a subset of $\overline{D}$ is $\mathcal{E}_D^{[\alpha]}$-polar iff it is polar with respect to the symmetric α-stable process on $\mathbb{R}^d$.

Let us consider the kernel $k^{(1)}$ of (1.9) for $w(x)$ given by (5.1) and $\alpha(x)$ satisfying condition (5.2). In particular, it is assumed that

$$0 < \underline{\alpha} \le \alpha(x) \le \overline{\alpha} < 2$$

for some constant $\underline{\alpha}, \overline{\alpha}$. $k^{(1)}$ satisfies conditions (2.1)–(2.4) by Proposition 5.1 and one can associate with it the regular lower bounded semi-Dirichlet form η_D (resp., η_D^0) on $L^2(\overline{D}; 1_D\,dx)$ [resp., $L^2(D)$] possessing as its reference form $\mathcal{E}_D$ (resp., $\mathcal{E}_D^0$) defined right after (6.1) for $k^{(1)}$ and the Lebesgue measure in place of k and m.

Suppose D is bounded, then there exist positive constants c_3, c_4 with

$$c_3 k^{[\underline{\alpha}]}(x, y) \le k_s^{(1)}(x, y) \le c_4 k^{[\overline{\alpha}]}(x, y), \qquad x, y \in \overline{D},$$

so that

$$c_3 \mathcal{E}_D^{[\underline{\alpha}]}(u, u) \le \mathcal{E}_D(u, u) \le c_4 \mathcal{E}_D^{[\overline{\alpha}]}(u, u), \qquad u \in C_0^{\mathrm{lip}}(\overline{D}). \tag{6.18}$$

For the kernel $k^{(1)}$, the Hunt process $X^{\bar{D}}$ on $\overline{D}$ associated with $(\eta_D, \mathcal{F}_{\bar{D}})$ is called a modified reflecting stable-like process, while its part process $X^{D,0}$ on D, which is associated with $(\eta_D^0, \mathcal{F}_D^0)$, is called a censored stable-like process.

PROPOSITION 6.1. *Assume that D is a bounded open d-set.*

(i) *If ∂D is polar with respect to the symmetric $\overline{\alpha}$-stable process on $\mathbb{R}^d$, then the censored stable-like process $X^{D,0} = (X_t^{D,0}, P_x, \zeta^0)$ is conservative and it does not approach to ∂D in finite time*:

$$P_x(\zeta^0 = \infty) = 1, \qquad P_x(X_{t-}^{D,0} \in \partial D \text{ for some } t > 0) = 0. \tag{6.19}$$

(ii) *If ∂D is nonpolar with respect to the symmetric $\underline{\alpha}$-stable process on $\mathbb{R}^d$, then the censored stable-like process $X^{D,0}$ satisfies*

$$\int_D P_x(X_{\zeta^0-}^{D,0} \in \partial D, \zeta^0 < \infty) h(x)\, dx = \int_D P_x(\zeta^0 < \infty) h(x)\, dx > 0 \tag{6.20}$$

for any strictly positive Borel function h on D with $\int_D h(x)\, dx = 1$.

PROOF. (i) Since $\mathcal{E}_D$ is a reference form of $(\eta_D, \mathcal{F}_{\bar{D}})$, we see that ∂D is η_D-polar by (6.18) and the stated observation in [7]. The assertions of (i) then follows from Theorem 4.2(ii) and Theorem 6(ii).

(ii) ∂D is not η_D-polar by (6.18) and accordingly not m-polar with respect to the process $X^{\bar{D}}$ by Theorem 4.2(v), where m is the Lebesgue measure on D. Taking Theorem 6.2(i), (iii) into account, we then get (6.20). □

The polarity of a set $N \subset \mathbb{R}^d$ with respect to the symmetric α-stable process is equivalent to $C^{\alpha/2,2}(N) = 0$ for the Bessel capacity $C^{\alpha/2,2}$ (cf. Section 2.4 of the second edition of [13]). The latter has been well studied in [1] in relation to the Hausdorff measure and the Hausdorff content. For instance, when $\alpha \le d$ and ∂D is a s-set, ∂D is polar in this sense if and only if $\alpha + s \le d$. Of course, we get the same results as above for the second kernel $k^{(1)*}$ in (1.9).

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