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DIMENSION RESULT AND KPZ FORMULA FOR TWO-DIMENSIONAL MULTIPLICATIVE CASCADE PROCESSES

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We prove a Hausdorff dimension result for the image of two-dimensional multiplicative cascade processes, and we obtain from this result a KPZ-type formula which normally has one point of phase transition.

1. Introduction. The famous Knizhnik–Polyakov–Zamolodchikov formula in quantum gravity relates the conformal dimension Δ^0 of any field operator of a two-dimensional conformal field theory to the analogous dimension Δ of the same operator when the theory is coupled to a two-dimensional quantum gravity,

$$\Delta^0 = \Delta + \frac{\gamma^2}{4}\Delta(\Delta - 1) \qquad \text{for } \gamma = \sqrt{\frac{25-c}{6}} - \sqrt{\frac{1-c}{6}}, \tag{1}$$

where c is the central charge of the conformal field theory. This formula was first derived by Knizhnik, Polyakov and Zamolodchikov [12] in 1988 via Liouville quantum gravity in a light cone gauge, building on a earlier work of Polyakov [18] in 1987. Shortly after, David [6] provided an alternative heuristic derivation of the KPZ formula by using Liouville field theory in the so-called conformal gauge. The KPZ formula has a great influence on string theory and conformal field theory, and it plays a core role in studying the connections of two-dimensional quantum gravity to random planar maps, two-dimensional lattice models, random matrix theory and Schramm–Loewner evolution.

In a recent inspiring paper [8] Duplantier and Sheffield provide (in a mathematically rigorous way) a geometrical KPZ formula under a similar framework as used in [6]. They relate the Euclidean scaling exponent x of a fractal subset of the domain D (with respect to the Lebesgue measure) to the quantum scaling exponent Δ of the same set (with respect to the Liouville quantum gravity measure, that is, roughly speaking, the measure $e^{\gamma h}\,dz$ with h being the Gaussian free field on D). By using large deviation estimates they prove that x and Δ satisfy the same formula as (1) (replacing Δ^0 by x) for $\gamma \in [0, 2)$.

Inspired by Duplantier and Sheffield's work, Benjamini and Schramm [5] prove a Hausdorff dimension version of the geometrical KPZ formula for random metrics built from Mandelbrot measures constructed in [11]. Adapting Benjamini

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and Schramm's proof, Rhodes and Vargas [19] prove a similar result for one-dimensional log-infinite divisible multifractal measures constructed in [1] and two-dimensional Gaussian multiplicative chaos constructed in [9, 20] (like the Liouville quantum gravity measure constructed in [8]). It is also worth mentioning that following [8] there is the paper of David and Bauer [7] that gives a physicist's derivation of the geometrical KPZ formula via heat kernel methods.

A common feature of the random measures mentioned above (Liouville quantum gravity measure, Mandelbrot measure, log-infinite divisible multifractal measure, etc.) is that they are all obtained through a limiting procedure, and along the procedure the random densities that are used to construct these measures can always be locally written as a product of independent weights. These random measures nowadays are mentioned as multiplicative chaos. The first work on this subject could be traced back to Kolmogorov [13] in 1941 regarding the local structure of turbulence in probabiliy interpretation. The study was developed by Yaglom [23] in 1966 (introducing the cascade structure) and Mandelbrot [16, 17] in the early 70s (refining the cascade structure and pointing out the necessity of using limiting procedures). Then in 1976 Kahane and Peyrière [11] completed the work in [17] regarding Mandelbrot measures, and introduced several fundamental ideas for the study of multiplicative chaos. Later in 1985 Kahane [9] defined rigorously the Gaussian multiplicative chaos suggested by Mandelbrot in [16]; in particular his theory gives a rigorous definition of the measure $e^{\gamma h}\,dz$ where h is the Gaussian free field. For more details on this subject one can see, for example, the survey paper [4].

Of special interest to this paper, we would like to present here more precisely Benjamini and Schramm's result on the geometrical KPZ formula for Mandelbrot measures: let W be a positive random variable of expectation $1/2$, and let $\{W(w) : w \in \bigcup_{n\geq 1}\{0,1\}^n\}$ be a sequence of independent copies of W encoded by the dyadic words. The Mandelbrot measure μ on $[0,1]$ generated by W is defined as the weak limit of

$$\left(d\mu_n(x) = 2^n \cdot W(x|_1)W(x|_2)\cdots W(x|_n)\,dx\right)_{n\geq 1},$$

where for $i = 1, 2, \dots,$ and $x \in [0,1]$, $x|_i$ stands for the first i letters of the dyadic expansion of x. From [11, 10] one knows that if $\mathbb{E}(W \log W) < 0$, then μ is almost surely nondegenerate and without atom, so it induces a random metric ρ_μ on $[0,1]$ given by $\rho_\mu(x,y) = \mu([x,y])$ for $0 \leq x < y \leq 1$ (such a metric was previously considered in [2]). Denote by $\dim_H$ the Hausdorff dimension with respect to the Euclidean metric and by $\dim_H^{\rho_\mu}$ the Hausdorff dimension with respect to ρ_μ, it is shown in [5] that if $\mathbb{E}(W^{-s}) < \infty$ for all $s \in [0,1)$, then for any Borel set $K \subset [0,1]$ with $\dim_H K = \xi_0$, almost surely $\dim_H^{\rho_\mu} K$ is equal to a constant $\xi \in [0,1]$ satisfying

$$2^{-\xi_0} = \mathbb{E}(W^{\xi}). \tag{2}$$

In the special case when $W = 2^{-1} \cdot 2^{-\gamma^2/4} e^{\gamma H/2}$, where $H = \mathcal{N}(0, 2\ln 2)$ is a normal random variable and $\gamma \in [0, 2)$ [notice that $\mathbb{E}(W \log W) < 0$ is equivalent to $\gamma < 2$], they obtain a geometrical KPZ formula for ρ_μ,

$$\xi_0 = \xi + \frac{\gamma^2}{4} \cdot \xi(1 - \xi). \tag{3}$$

To recover (1) one may let $\Delta^0 = 1 - \xi_0$ and $\Delta = 1 - \xi$. Notice that if we consider the indefinite integral of μ, that is the function $F_\mu(x) = \mu([0, x])$ for $x \in [0, 1]$, then by definition one directly gets

$$\dim_H^{\rho_\mu} K = \dim_H F_\mu(K).$$

So Benjamini and Schramm's result can be also understood as a Hausdorff dimension result for the image of the increasing process F_μ.

The main goal of this paper is to extend Benjamini and Schramm's result to signed multiplicative cascade processes, a class of random multifractal functions recently constructed in [3] as a natural generalization of F_μ. These processes are no longer increasing functions, and their graphs normally have Hausdorff dimension greater than 1, so one would naturally expect a formula that relates sets with dimension smaller than 1 to sets with dimension larger than 1. This remark led us to directly consider the case of two signed multiplicative cascades simultaneously. Before stating in more detail the result we need to recall the definition of two-dimensional multiplicative cascade processes. Let us begin with some notations on the coding space.

Coding space. Let $b \geq 2$ be an integer, and let $\mathscr{A} = \{0, \ldots, b-1\}$ be the alphabet. Let $\mathscr{A}^* = \bigcup_{n \geq 0} \mathscr{A}^n$ (by convention $\mathscr{A}^0 = \{\varnothing\}$ the set of empty word) and $\mathscr{A}^{\mathbb{N}_+} = \{0, \ldots, b-1\}^{\mathbb{N}_+}$.

The word obtained by concatenation of $u \in \mathscr{A}^*$ and $v \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ is denoted by $u \cdot v$ and sometimes uv. If $n \geq 1$ and $u = u_1 \cdots u_n \in \mathscr{A}^n$, then for every $1 \leq i \leq n$, the word $u_1 \cdots u_i$ is denoted by $u|_i$, and if $i = 0$ then $u|_0$ stands for $\varnothing$. Also, for any infinite word $v = v_1 v_2 \cdots \in \mathscr{A}^{\mathbb{N}_+}$ and $n \geq 1$, $v|_n$ denotes the word $v_1 \cdots v_n$ and $v|_0$ the empty word.

The length of a word w is denoted by $|w| = n$ if $w \in \mathscr{A}^n$ and $|w| = \infty$ if $w \in \mathscr{A}^{\mathbb{N}_+}$. Let $\pi : w \in \mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+} \mapsto \sum_{i=1}^{|w|} w_i \cdot b^{-i}$ be the canonical projection from $\mathscr{A}^* \cup \mathscr{A}^{\mathbb{N}_+}$ onto the interval $[0, 1]$. For $w \in \mathscr{A}^*$ denote by $I_w = [\pi(w), \pi(w) + b^{-|w|})$ the b-adic interval encoded by w.

For $x \in [0, 1)$ and $n \geq 1$, let $x|_n = x_1 \cdots x_n$ be the unique element of $\mathscr{A}^n$ such that $x \in I_{x_1 \cdots x_n}$, as well as $1|_n = b - 1 \cdots b - 1$.

Two-dimensional multiplicative cascade processes. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the probability space, and let $W = (W_1, W_2)$ be a random vector satisfying:

(A0) $\mathbb{E}(W_1) = \mathbb{E}(W_2) = b^{-1}$;

(A1) $\exists q \in (1, 2]$ such that $\mathbb{E}(|W_1|^q) \vee \mathbb{E}(|W_2|^q) < b^{-1}$;
(A2) $\exists s > 2$ such that $\mathbb{E}(|W_1|^{-s}) \vee \mathbb{E}(|W_2|^{-s}) < \infty$.

Let $\{W(w) : w \in \mathscr{A}^*\}$ be a sequence of independent copies of W.
For $k \in \{1, 2\}$, $x \in [0, 1]$ and $n \geq 1$ define the product

$$Q_k(x|_n) = Q_k(I_{x|_n}) = W_k(x|_1) \cdot W_k(x|_2) \cdots W_k(x|_n).$$

For $k \in \{1, 2\}$ and $n \geq 1$ define the random piecewise linear function

$$F_{k,n} : t \in [0, 1] \mapsto \int_0^t b^n \cdot Q_k(x|_n)\, dx.$$

From [3] one has almost surely $F_{k,n}$ converges uniformly to a limit F_k. The two-dimensional multiplicative cascade process considered in this paper is defined as

$$F : t \in [0, 1] \mapsto (F_1(t), F_2(t)) \in \mathbb{R}^2.$$

Notice that if $\mathbb{P}(W_1 = W_2) = 1$, then almost surely $F_1 = F_2$, thus F degenerates to a one-dimensional multiplicative cascade process.

Main result. Given $\xi_0 \in [0, 1]$, denote by ξ the smallest solution of the equation

$$b^{-\xi_0} = \mathbb{E}(|W_1|^{\xi}) \vee \mathbb{E}(|W_2|^{\xi}) \tag{4}$$

and ζ the smallest solution of the equation

$$b^{-\xi_0} = \mathbb{E}(|W_1|^{\zeta-1} \cdot |W_2|) \vee \mathbb{E}(|W_1| \cdot |W_2|^{\zeta-1}). \tag{5}$$

Also denote by

$$\xi_* = -\log_b\big(\mathbb{E}(|W_1|) \vee \mathbb{E}(|W_2|)\big).$$

From assumptions (A0) and (A1) one can easily deduce that $\xi_* \in (1/2, 1]$.

THEOREM 1. *Let $K \subset [0, 1]$ be any Borel set with* $\dim_H K = \xi_0$.

(i) *If* $\mathbb{P}(W_1 = W_2) < 1$, *then almost surely*

$$\dim_H F(K) = \xi \wedge \zeta = \begin{cases} \xi, & \text{if } \xi_0 \in [0, \xi_*]; \\ \zeta, & \text{if } \xi_0 \in (\xi_*, 1]. \end{cases}$$

(ii) *If* $\mathbb{P}(W_1 = W_2) = 1$, *then almost surely*

$$\dim_H F(K) = \xi \wedge 1 = \begin{cases} \xi, & \text{if } \xi_0 \in [0, \xi_*]; \\ 1, & \text{if } \xi_0 \in (\xi_*, 1]. \end{cases}$$

Let us give two examples to help understand the result.

EXAMPLE 1. Let X_1 and X_2 be two random variables both taking values $b^{-\alpha}$ and $-b^{-\alpha}$ with respective probabilities $(1+b^{\alpha-1})/2$ and $(1-b^{\alpha-1})/2$ for some $\alpha \le 1$. Suppose that $\mathbb{P}(X_1 = X_2) < 1$. Let $\gamma \ge 0$ and let $H = \mathcal{N}(0, 2\ln b)$ be a normal random variable independent of X_1 and X_2. Define

$$W_1 = X_1 \cdot b^{-\gamma^2/4} e^{\gamma H/2} \quad \text{and} \quad W_2 = X_2 \cdot b^{-\gamma^2/4} e^{\gamma H/2}.$$

By simple calculation one has for $\{k, l\} = \{1, 2\}$,

$$\mathbb{E}(|W_k|^{\xi}) = \mathbb{E}(|W_l|^{\xi}) = \mathbb{E}(|W_k|^{\xi-1} \cdot |W_l|) = b^{-\xi\alpha} \cdot b^{-\xi(1-\xi)\gamma^2/4}.$$

Then assumption (A1) [(A0) and (A2) are automatically satisfied] is equivalent to requiring

$$\gamma < 2 \quad \text{and} \quad \begin{cases} \gamma - \gamma^2/4 < \alpha \le 1, & \text{if } \gamma \ge 1; \\ \gamma^2/4 + 1/2 < \alpha \le 1, & \text{if } \gamma < 1. \end{cases} \tag{6}$$

In such a case, Theorem 1 says that for any Borel set $K \subset [0, 1]$ with $\dim_H K = \xi_0$, almost surely $\dim_H F(K)$ is equal to a constant $\xi \in [0, 2)$ satisfying

$$\xi_0 = \alpha \cdot \xi + \frac{\gamma^2}{4} \cdot \xi(1-\xi).$$

Comparing to (3), this formula has a new parameter α varying in the region given by (6), and when $\alpha < 1$, the maximal dimension $\dim_H F([0, 1])$ is equal to

$$\frac{\gamma^2 + 4\alpha - \sqrt{(\gamma^2 + 4\alpha)^2 - 16\gamma^2}}{2\gamma^2} \in (1, 2)$$

if $\gamma > 0$ and is equal to $1/\alpha \in (1, 2)$ if $\gamma = 0$.

EXAMPLE 2. Now let

$$W_1 = X_1 \cdot b^{-\gamma^2/4} e^{\gamma H/2} \quad \text{and} \quad W_2 = b^{-1} \cdot b^{-\gamma^2/4} e^{\gamma H/2},$$

so W_2 is almost surely positive. For $\xi \ge 0$ one has

$$\mathbb{E}(|W_1|^{\xi}) \vee \mathbb{E}(|W_2|^{\xi}) = b^{-\xi\alpha} \cdot b^{-\xi(1-\xi)\gamma^2/4},$$

and for $\zeta \ge 1$ one has

$$\mathbb{E}(|W_1|^{\zeta-1}|W_2|) \vee \mathbb{E}(|W_2|^{\zeta-1}|W_1|) = b^{-(\zeta-1+\alpha)} \cdot b^{-\zeta(1-\zeta)\gamma^2/4}$$

as well as $\xi_* = \alpha$. We need the same condition as in (6). In this case, since F_2 is almost surely increasing, one can deduce a random metric ρ_F from F on $[0, 1]$ given by $\rho_F(x, y) = |F(x) - F(y)|$ for $x, y \in [0, 1]$. Then Theorem 1 says that for any Borel set $K \subset [0,1]$ with $\dim_H K = \xi_0$, almost surely $\dim_H^{\rho_F} K$ is equal to a constant $\xi \in [0,2)$ satisfying

$$\begin{cases} \xi_0 = \alpha \cdot \xi + \dfrac{\gamma^2}{4} \cdot \xi(1-\xi), & \text{if } \xi_0 \in [0,\alpha]; \\ \xi_0 = \xi - 1 + \alpha + \dfrac{\gamma^2}{4} \cdot \xi(1-\xi), & \text{if } \xi_0 \in (\alpha,1]. \end{cases}$$

If $\alpha = 1$, then we go back to (3). If $\alpha < 1$, then this KPZ-type formula has a phase transition at α, and the maximal dimension $\dim_H^{\rho_F}[0,1]$ is equal to

$$\frac{\gamma^2 + 4 - \sqrt{(\gamma^2+4)^2 - 16\gamma^2(2-\alpha)}}{2\gamma^2} \in (1,2)$$

if $\gamma > 0$ and is equal to $2 - \alpha \in (1, 3/2)$ if $\gamma = 0$.

REMARK 1. The reason why we consider the two-dimensional case can be easily seen from Theorem 1 and Examples 1, 2. If we only consider the one-dimensional case, as already shown in Theorem 1(ii), the formula will also have a phase transition at ξ_*, but such a phase transition is indeed caused by the limitation of the image space.

REMARK 2. Examples 1 and 2 are special cases of Theorem 1. In general, the theorem could provide us with more colorful formulas. In principle, the formula can have as many points of phase transition as we want.

REMARK 3. Finding the Hausdorff dimension of the image of a stochastic process restricted to any Borel set is a classical problem in probability theory. The first work on this subject could be traced back to Lévy [14] and Taylor [21] in 1953, regarding the Hausdorff dimension and Hausdorff measure of the image of Brownian motion. Since then much progress has been made for fractional Brownian motion, stable Lévy process and many other processes. We refer to the survey paper [22] and the references therein for more information on this subject.

The proof of Theorem 1 will be given in the next section. We end this section with some preliminaries.

Hausdorff dimension. If (X,ρ) is a locally compact metric space, for $d \geq 0$, $\delta > 0$ and $K \subset X$ let

$$\mathcal{H}_\delta^{\rho,d}(K) = \inf\left\{\sum_{i\in I} |U_i|_\rho^d\right\},$$

where the infimum is taken over the set of all the at most countable coverings $\{U_i\}_{i\in I}$ of K such that $0 \leq |U_i|_\rho \leq \delta$, where $|U_i|_\rho$ stands for the diameter of U_i

with respect to ρ. Define

$$\mathcal{H}^{\rho,d}(K) = \lim_{\delta \searrow 0} \mathcal{H}^{\rho,d}_{\delta}(K).$$

Then $\mathcal{H}^{\rho,d}(K)$ is called the d-dimensional Hausdorff measure of K with respect to ρ, and the Hausdorff dimension of K with respect to ρ is the number

$$\dim_H^{\rho} K = \inf\{d : \mathcal{H}^{\rho,d}(K) < \infty\}.$$

When ρ is the standard Euclidean metric, we often omit the index ρ.

Stationary self-similarity of multiplicative cascade processes. For $k \in \{1, 2\}$, $w \in \mathscr{A}^*$ and $n \geq 1$ define

$$F^{[w]}_{k,n} : t \in [0, 1] \mapsto \int_0^t b^n \cdot W_k(w \cdot x|_1) \cdots W_k(w \cdot x|_n)\, dx.$$

Since $\mathscr{A}^*$ is countable, we have almost surely for all $w \in \mathscr{A}^*$, $F^{[w]}_{k,n}$ converges uniformly to a limit $F^{[w]}_k$ and $F^{[w]}_k$ has the same law as F_k.

By construction for any $w \in \mathscr{A}^*$ and $t \in [0, 1]$ one has

$$F_k\big(\pi(w) + t \cdot b^{-|w|}\big) - F_k(\pi(w)) = Q_k(w) \cdot F^{[w]}_k(t). \tag{7}$$

For $w \in \mathscr{A}^*$ define

$$Z_k(w) = F^{[w]}_k(1)$$

and

$$X_k(w) = \sup_{s,t \in [0,1]} \big|F^{[w]}_k(s) - F^{[w]}_k(t)\big|.$$

Then from (7) one has

$$F_k\big(\pi(w) + b^{-|w|}\big) - F_k(\pi(w)) = Q_k(w) \cdot Z_k(w)$$

and

$$O_k(w) = O_k(I_w) := \sup_{s,t \in I_w} |F_k(s) - F_k(t)| = |Q_k(w)| \cdot X_k(w),$$

where $Q_k(w)$ is independent of $Z_k(w)$ and $X_k(w)$.

We will use the convention that $Z_k = Z_k(\varnothing)$ and $X_k = X_k(\varnothing)$.

By direct calculation, for any $q_1, q_2 \in \mathbb{R}$ and $w \in \mathscr{A}_*$ one has

$$\mathbb{E}(O_1(w)^{q_1} O_2(w)^{q_2}) = \mathbb{E}(|W_1|^{q_1} |W_2|^{q_2})^{|w|} \cdot \mathbb{E}(X_1^{q_1} X_2^{q_2}),$$

whenever the expectation exists.

Moments control. It is proved in [3] that for $k \in \{1, 2\}$:

$$\begin{aligned} &\text{(i)} \quad \text{If } \mathbb{E}(|W_k|^q) < b^{-1} \text{ for some } q > 1, \text{ then } \mathbb{E}(X_k^q) < \infty; \\ &\text{(ii)} \quad \text{If } \mathbb{E}(|W_k|^{-s}) < \infty \text{ for some } s > 0, \text{ then } \mathbb{E}(X_k^{-s}) < \infty. \end{aligned} \tag{8}$$

2. Proof of Theorem 1.

2.1. *Upper bound estimate.* For $p \geq 0$ let

$$\phi(p) = \mathbb{E}(|W_1|^p) \vee \mathbb{E}(|W_2|^p)$$

and

$$\widetilde{\phi}(p) = \mathbb{E}(|W_1|^{p-1} \cdot |W_2|) \vee \mathbb{E}(|W_1| \cdot |W_2|^{p-1}).$$

We have the following lemma:

LEMMA 1. *One has $\phi(p) \leq \widetilde{\phi}(p)$ if $p \in [0, 1]$ and $\phi(p) \geq \widetilde{\phi}(p)$ if $p \geq 1$.*

PROOF. Obviously $\phi(1) = \widetilde{\phi}(1)$.
Since $\frac{|W_1|}{|W_2|} + \frac{|W_2|}{|W_1|} \geq 2$, we get $\widetilde{\phi}(0) \geq 1 = \phi(0)$.
Let $\{k, l\} = \{1, 2\}$. For $p > 1$ from Hölder's inequality one gets

$$\begin{aligned}
\mathbb{E}(|W_k|^{p-1}|W_l|) &\leq \mathbb{E}\big(|W_k|^{(p-1)\cdot p/(p-1)}\big)^{(p-1)/p} \cdot \mathbb{E}(|W_l|^p)^{1/p} \\
&= \mathbb{E}(|W_k|^p)^{(p-1)/p} \cdot \mathbb{E}(|W_l|^p)^{1/p} \\
&\leq \phi(p).
\end{aligned}$$

This implies $\widetilde{\phi}(p) \leq \phi(p)$. For $p \in (0, 1)$ from Hölder's inequality one gets

$$\begin{aligned}
\mathbb{E}(|W_k|^p) &= \mathbb{E}\big(|W_k|^p \cdot |W_l|^{-p(1-p)} \cdot |W_l|^{p(1-p)}\big) \\
&\leq \mathbb{E}\big(\big(|W_k|^p \cdot |W_l|^{-p(1-p)}\big)^{1/p}\big)^p \cdot \mathbb{E}\big(\big(|W_l|^{p(1-p)}\big)^{1/(1-p)}\big)^{1-p} \\
&= \mathbb{E}(|W_k| \cdot |W_l|^{p-1})^p \cdot \mathbb{E}(|W_l|^p)^{1-p} \\
&\leq \widetilde{\phi}(p)^p \cdot \mathbb{E}(|W_l|^p)^{1-p}.
\end{aligned}$$

In an analogous way one can also obtain

$$\mathbb{E}(|W_l|^p) \leq \widetilde{\phi}(p)^p \cdot \mathbb{E}(|W_k|^p)^{1-p}.$$

Then

$$\mathbb{E}(|W_k|^p) \leq \widetilde{\phi}(p)^p \cdot \widetilde{\phi}(p)^{p(1-p)} \cdot \mathbb{E}(|W_k|^p)^{(1-p)(1-p)},$$

which implies $\mathbb{E}(|W_k|^p) \leq \widetilde{\phi}(p)$, thus $\phi(p) \leq \widetilde{\phi}(p)$. $\square$

Given $\xi_0 \in [0, 1]$ recall the definition of ξ and ζ in (4) and (5).

Notice that under assumptions (A0) and (A1), Lemma 1 ensures that, by the convexity of ϕ and $\widetilde{\phi}$, ϕ and $\widetilde{\phi}$ are non increasing on $\widetilde{\phi}^{-1}([0, 1])$. This implies that $\xi \leq \zeta \leq 1$ if $\xi_0 \in [0, \xi_*]$ and $\xi \geq \zeta > 1$ if $\xi_0 \in (\xi_*, 1]$, as well as $\phi'(\xi+) \leq 0$ and $\widetilde{\phi}'(\zeta+) \leq 0$. Thus given any $\varepsilon > 0$ small enough, one can find an $\eta > 0$ such that

$$\phi(\xi + \eta) \leq b^{-(\xi_0+\varepsilon)} \quad \text{and} \quad \widetilde{\phi}(\zeta + \eta) \leq b^{-(\xi_0+\varepsilon)}.$$

From the moments control (8) it is easy to deduce that

$$\mathbb{E}(X_1^{\xi+\eta}) \vee \mathbb{E}(X_2^{\xi+\eta}) < \infty$$

as well as for $\{k, l\} = \{1, 2\}$ and $\zeta > 1$,

$$\mathbb{E}(X_k^{\zeta+\eta-1} X_l) \leq \mathbb{E}(X_k^{\zeta+\eta})^{(\zeta+\eta-1)/(\zeta+\eta)} \cdot \mathbb{E}(X_l^{\zeta+\eta})^{1/(\zeta+\eta)} < \infty.$$

From the definition of Hausdorff dimension, for each $n \geq 1$ one can find a sequence $\mathcal{I}_n$ of b-adic intervals such that

$$K \subset \bigcup_{I \in \mathcal{I}_n} I \quad \text{and} \quad \sum_{I \in \mathcal{I}_n} |I|^{\xi_0+\varepsilon} \leq 2^{-n}.$$

Let $\delta_n = \sup_{I \in \mathcal{I}_n} |F(I)|$. Since F is almost surely continuous, $\delta_n \to 0$ almost surely. For any interval $I \in \mathcal{I}_n$ denote by

$$O_*(I) = O_1(I) \wedge O_2(I) \quad \text{and} \quad O^*(I) = O_1(I) \vee O_2(I).$$

Then we can obtain the desired upper bounds from the following two facts:

(i) If $\xi_0 \in [0, \xi_*]$: for each $I \subset \mathcal{I}_n$ one can use a single square of side length $2O^*(I)$ to cover $F(I)$, thus

$$\begin{aligned}
\mathbb{E}(\mathcal{H}_{\delta_n}^{\xi+\eta}(F(K))) &\leq 2^{\xi+\eta}\mathbb{E}\Big(\sum_{I \in \mathcal{I}_n} O_1(I)^{\xi+\eta} \vee O_2(I)^{\xi+\eta}\Big) \\
&\leq 2^{\xi+\eta} \sum_{I \in \mathcal{I}_n} \mathbb{E}\big(O_1(I)^{\xi+\eta} + O_2(I)^{\xi+\eta}\big) \\
&\leq C \cdot \sum_{I \in \mathcal{I}_n} |I|^{\xi_0+\varepsilon} \\
&\leq C \cdot 2^{-n},
\end{aligned}$$

where $C = 2^{\xi+\eta+1}\mathbb{E}(X_1^{\xi+\eta}) \vee \mathbb{E}(X_2^{\xi+\eta})$.

(ii) If $\xi_0 \in (\xi_*, 1]$: for each $I \subset \mathcal{I}_n$ one can use no more than $\lfloor O^*(I)/O_*(I) \rfloor$-many squares of side length $2O_*(I)$ to cover $F(I)$, thus

$$\begin{aligned}
&\mathbb{E}(\mathcal{H}_{\delta_n}^{\zeta+\eta}(F(K))) \\
&\quad \leq 2^{\zeta+\eta}\mathbb{E}\Big(\sum_{I \in \mathcal{I}_n}\Big(\frac{O_2(I)}{O_1(I)} \cdot O_1(I)^{\zeta+\eta}\Big) \vee \Big(\frac{O_1(I)}{O_2(I)} \cdot O_2(I)^{\zeta+\eta}\Big)\Big) \\
&\quad \leq 2^{\zeta+\eta} \sum_{I \in \mathcal{I}_n} \mathbb{E}\big(O_2(I) O_1(I)^{\zeta+\eta-1} + O_1(I) O_2(I)^{\zeta+\eta-1}\big) \\
&\quad \leq C' \cdot \sum_{I \in \mathcal{I}_n} |I|^{\xi_0+\varepsilon} \\
&\quad \leq C' \cdot 2^{-n},
\end{aligned}$$

where $C' = 2^{\zeta+\eta+1}\mathbb{E}(X_1^{\zeta+\eta-1} X_2) \vee \mathbb{E}(X_2^{\zeta+\eta-1} X_1)$.

2.2. *Lower bound estimate.* We will use a similar method as in [5] to estimate the lower bound. First we consider the case $\mathbb{P}(W_1 = W_2) < 1$.

There is nothing to prove when $\dim_H K = 0$, since $F(K)$ is always nonempty. Let $\dim_H K = \xi_0 > 0$. Given any $\delta \in (0, \xi_0)$, due to Frostman's lemma there exists a Borel probability measure μ_0 carried by K such that

$$\iint_{s,t\in[0,1]} \frac{d\mu_0(s)\, d\mu_0(t)}{|s-t|^{\xi_0-\delta}} < \infty.$$

Let $\{k, l\} = \{1, 2\}$ and let $d \in (0, 2)$ be the unique number such that

$$\begin{cases} \mathbb{E}(|W_k|^d) = b^{-(\xi_0-\delta)}, & \text{if } \xi_0 \in (0, \xi_*]; \\ \mathbb{E}(|W_k| \cdot |W_l|^{d-1}) = b^{-(\xi_0-\delta)}, & \text{if } \xi_0 \in (\xi_*, 1]. \end{cases}$$

We may assume that δ is small enough such that $d > 1$ if $\xi_0 \in (\xi_*, 1]$, and $d \in (0, 1)$ if $\xi_0 \in (0, \xi_*]$.

For $w \in \mathscr{A}^*$ let

$$\widetilde{W}(w) = \begin{cases} b^{\xi_0-\delta} \cdot |W_k(w)|^d, & \text{if } \xi_0 \in (0, \xi_*]; \\ b^{\xi_0-\delta} \cdot |W_k(w)| \cdot |W_l(w)|^{d-1}, & \text{if } \xi_0 \in (\xi_*, 1] \end{cases}$$

and

$$Q(w) = \widetilde{W}(w|_1)\widetilde{W}(w|_2)\cdots \widetilde{W}(w).$$

For $n \geq 1$ define the random measure μ_n by

$$d\mu_n(x) = Q(x|_n)\, d\mu_0(x).$$

By construction, $(\mu_n)_{n\geq 1}$ is a measure-valued martingale thus yields a weak limit μ, and $\mu([0, 1] \setminus K) = 0$ almost surely.

For $s, t \in [0, 1]$ define

$$\mathcal{K}_n^d(s,t) = |F_k(s) - F_k(t)|^d \vee O_k(s|_n)^d \tag{9}$$

if $d \in (0, 1]$ and

$$\begin{aligned} \mathcal{K}_n^d(s,t) = {} & \big(|F_k(s) - F_k(t)|^2 + |F_l(s) - F_l(t)|^2\big)^{d/2} \\ & \vee \big(O_k(s|_n)^2 + O_l(s|_n)^2\big)^{d/2} \end{aligned} \tag{10}$$

if $d > 1$. Due to the continuity of F, one has almost surely $\mathcal{K}_n^d$ converges uniformly to

$$\mathcal{K}^d(s,t) = \begin{cases} |F_k(s) - F_k(t)|^d, & \text{if } d \in (0, 1]; \\ \big(|F_k(s) - F_k(t)|^2 + |F_l(s) - F_l(t)|^2\big)^{d/2}, & \text{if } d > 1. \end{cases}$$

We have the following proposition:

PROPOSITION 1. *There exists a constant C such that for any $0 \le s < t \le 1$ and $n \ge 1$,*

$$\mathbb{E}\biggl(\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\biggr) \le C \cdot \frac{d\mu_0(s)\,d\mu_0(t)}{|s-t|^{\xi_0-\delta}}.$$

By using Fubini's theorem, Proposition 1 yields that for any $n \ge 1$,

$$\mathbb{E}\biggl(\iint_{s,t\in[0,1]} \frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\biggr) \le 2C \iint_{s,t\in[0,1]} \frac{d\mu_0(s)\,d\mu_0(t)}{|s-t|^{\xi_0-\delta}} < \infty. \tag{11}$$

For any $s, t \in [0,1]$ one has

$$\mathcal{K}_n^d(s,t) \le \sup_{s,t\in[0,1]} |F_k(s) - F_k(t)|^d = X_k^d,$$

so (11) implies

$$\sup_{n\ge 1} \mathbb{E}\bigl(X_k^{-d} \cdot \mu_n([0,1])^2\bigr) < \infty. \tag{12}$$

Notice that for $d \in (0,1)$ we have

$$\begin{aligned}\mathbb{E}\bigl(\mu_n([0,1])^{2/(1+d)}\bigr) &= \mathbb{E}\bigl(X_k^{d/(1+d)} \cdot X_k^{-d/(1+d)} \cdot \mu_n([0,1])^{2/(1+d)}\bigr)\\ &\le \mathbb{E}(X_k)^{d/(1+d)} \cdot \mathbb{E}\bigl(X_k^{-d} \cdot \mu_n([0,1])^2\bigr)^{1/(1+d)},\end{aligned}$$

and for $d \in (1,2)$ we have for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}\bigl(\mu_n([0,1])^{1+\varepsilon}\bigr) &= \mathbb{E}\bigl(X_k^{d(1+\varepsilon)/2} \cdot X_k^{-d(1+\varepsilon)/2} \cdot \mu_n([0,1])^{1+\varepsilon}\bigr)\\ &\le \mathbb{E}\bigl(X_k^{d(1+\varepsilon)/(1-\varepsilon)}\bigr)^{(1-\varepsilon)/2} \cdot \mathbb{E}\bigl(X_k^{-d} \cdot \mu_n([0,1])^2\bigr)^{(1+\varepsilon)/2}.\end{aligned}$$

Thus by using the corresponding martingale convergence theorem we get from (12) that $\mathbb{E}(\mu([0,1])) = 1$. Then by using the same tail event argument as in [5] we can get $\mathbb{P}(\mu([0,1]) > 0) = 1$.

Due to the fact that almost surely μ_n converges weakly to μ and $\mathcal{K}_n^d$ converges uniformly to $\mathcal{K}^d$, we get from (11) that

$$\mathbb{E}\biggl(\iint_{s,t\in[0,1]} \frac{d\mu(s)\,d\mu(t)}{\mathcal{K}^d(s,t)}\biggr) \le \liminf_{n\to\infty} \mathbb{E}\biggl(\iint_{s,t\in[0,1]} \frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\biggr) < \infty.$$

Since almost surely μ is carried by K and $\mu(K) > 0$, by using the mass distribution principle we get the desired lower bound.

For the case $\mathbb{P}(W_1 = W_2) = 1$, it is the same proof as above when $\xi_0 \in (0, \xi_*]$. When $\xi_0 \in (\xi_*, 1]$, we may take $d \in (0,1)$ such that $\mathbb{E}(|W_k|^d) = b^{-(\xi_*-\delta)}$ and for $w \in \mathscr{A}_*$ let

$$\widetilde{W}(w) = b^{\xi_*-\delta} \cdot |W_k(w)|^d.$$

Then the same procedure as the case $\xi_0 \in (0, \xi_*]$ will yield a lower bound d, which can be arbitrarily close to 1, thus the conclusion.

2.3. *Proof of Proposition* 1. Recall that $Z_k = F_k(1)$. We will frequently use the following lemma, whose proof will be given in the Section 2.4.

LEMMA 2.
(i) *For any* $d \in (0,1)$ *there exists a constant* C_d *such that for any constants* $A, B \in \mathbb{R}$ *with* $A \neq 0$, *one has*

$$\mathbb{E}(|AZ_k + B|^{-d}) \leq C_d \cdot |A|^{-d}.$$

(ii) *If* $\mathbb{P}(W_1 = W_2) < 1$, *then for any* $d \in (1,2)$ *there exists a constant* C_d *such that for any constants* $A_1, A_2, B_1, B_2 \in \mathbb{R}$ *with* $A_1 A_2 \neq 0$, *one has*

$$\mathbb{E}\big((|A_1 Z_k + B_1|^2 + |A_2 Z_l + B_2|^2)^{-d/2}\big) \leq C_d \cdot |A_1|^{-1} \cdot |A_2|^{-d+1}.$$

For $n \geq 1$ and $w \in \mathscr{A}^n \setminus \{b-1 \cdots b-1\}$ denote by w^+ the unique word in $\mathscr{A}^n$ such that $\pi(w^+) = \pi(w) + b^{-n}$.

Since $s < t$, there exists a unique $j \geq 0$ such that $s|_j^+ = t|_j$ and $s|_{j+1}^+ \neq t|_{j+1}$. This implies $\pi(s|_{j+1}^+) + b^{-j-1} \leq t$ and

$$b^{-(j+1)} \leq |s-t| \leq 2b^{-j} \leq b^{-(j-1)}.$$

Notice that one has either $s_{j+1} \in \{0, \ldots, b-2\}$ or $s_{j+1} = b-1$. Without loss of generality we may assume $s_{j+1} \in \{0, \ldots, b-2\}$ thus $s|_{j+1}^+ = s|_j \cdot r$ for $r = s_{j+1} + 1 \in \{1, \ldots, b-1\}$.

Recall the definition of $\mathcal{K}_n^d$ in (9) and (10). We have the following two situations.

2.3.1. *When* $d < 1$.
(i) If $j \geq n$, then

$$\begin{aligned}\frac{d\mu_n(s)\, d\mu_n(t)}{\mathcal{K}_n^d(s,t)} &\leq O_k(s|_n)^{-d} \cdot Q(s|_n) \cdot Q(t|_n)\, d\mu_0(s)\, d\mu_0(t) \\ &= b^{n(\xi_0 - \delta)} \cdot X_k(s|_n)^{-d} \cdot |Q(t|_n)|\, d\mu_0(s)\, d\mu_0(t).\end{aligned}$$

Since $X_k(s|_n)$ and $Q(t|_n)$ are independent, we get

$$\begin{aligned}\mathbb{E}\left(\frac{d\mu_n(s)\, d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\right) &\leq b^{n(\xi_0-\delta)} \cdot \mathbb{E}(X_k^{-d})\, d\mu_0(s)\, d\mu_0(t) \\ &\leq b^{\xi_0-\delta} \cdot \mathbb{E}(X_k^{-d}) \cdot b^{(j-1)(\xi_0-\delta)}\, d\mu_0(s)\, d\mu_0(t) \\ &\leq b^{\xi_0-\delta} \cdot \mathbb{E}(X_k^{-d}) \cdot \frac{d\mu_0(s)\, d\mu_0(t)}{|s-t|^{\xi_0-\delta}}.\end{aligned}$$

(ii) If $j \leq n-1$, then

$$\begin{aligned}\mathcal{K}_n^d(s,t)^{-1} &\leq |F_k(s) - F_k(t)|^{-d} \\ &= \big|Q_k(s|_{j+1}^+) \cdot Z_k(s|_{j+1}^+) + \Delta_k\big|^{-d},\end{aligned}$$

where $\Delta_k = F_k(t) - F_k(\pi(s|_{j+1}^+) + b^{-j-1}) + F_k(s|_{j+1}^+) - F_k(s)$. Notice that $Z_k(s|_{j+1}^+)$ is independent of $Q(s|_n)$, $Q(t|_n)$, $Q_k(s|_{j+1}^+)$ and Δ_k. Let

$$\mathcal{A}(s|_{j+1}^+) = \sigma\big(W(w) : |w| \le j+1 \text{ or } w|_{j+1} \ne s|_{j+1}^+\big). \tag{13}$$

From Lemma 2(i) we get

$$\begin{aligned}
&\mathbb{E}\Big(\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)} \,\Big|\, \mathcal{A}(s|_{j+1}^+)\Big) \\
&\quad \le C_d \cdot \big|Q_k(s|_j \cdot r)\big|^{-d} \cdot Q(s|_n) \cdot Q(t|_n)\, d\mu_0(s)\, d\mu_0(t) \\
&\quad = C_d \cdot \big|W_k(s|_j \cdot r)\big|^{-d} \cdot b^{(j+1)(\xi_0-\delta)} \cdot \prod_{i=j+1}^{n} \widetilde{W}(s|_i) \cdot Q(t|_n)\, d\mu_0(s)\, d\mu_0(t).
\end{aligned}$$

Since all the random variables in the above products are independent, we get

$$\begin{aligned}
\mathbb{E}\Big(\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\Big) &\le C_d \cdot \mathbb{E}(|W_k|^{-d}) \cdot b^{(j+1)(\xi_0-\delta)}\, d\mu_0(s)\, d\mu_0(t) \\
&\le C_d \cdot b^{2(\xi_0-\delta)} \cdot \mathbb{E}(|W_k|^{-d}) \cdot \frac{d\mu_0(s)\,d\mu_0(t)}{|s-t|^{\xi_0-\delta}}.
\end{aligned}$$

2.3.2. *When $d > 1$.*

(i) If $j \ge n$, then

$$\begin{aligned}
\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)} &\le \frac{Q(s|_n) \cdot Q(t|_n)\, d\mu_0(s)\, d\mu_0(t)}{(O_k(s|_n)^2 + O_l(s|_n)^2)^{d/2}} \\
&\le \frac{Q(s|_n) \cdot Q(t|_n)\, d\mu_0(s)\, d\mu_0(t)}{((Q_k(s|_n) \cdot Z_k(s|_n))^2 + (Q_l(s|_n) \cdot Z_l(s|_n))^2)^{d/2}}.
\end{aligned}$$

Let $\mathcal{A}_n = \sigma(W(w) : |w| \le n)$. From Lemma 2(ii) we get

$$\begin{aligned}
&\mathbb{E}\Big(\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)} \,\Big|\, \mathcal{A}_n\Big) \\
&\quad \le C_d \cdot \big(|Q_k(s|_n)| \cdot |Q_l(s|_n)|^{d-1}\big)^{-1} \cdot Q(s|_n) \cdot Q(t|_n)\, d\mu_0(s)\, d\mu_0(t) \\
&\quad = C_d \cdot b^{n(\xi_0-\delta)} \cdot Q(t|_n)\, d\mu_0(s)\, d\mu_0(t).
\end{aligned}$$

This implies

$$\begin{aligned}
\mathbb{E}\Big(\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\Big) &\le C_d \cdot b^{n(\xi_0-\delta)}\, d\mu_0(s)\, d\mu_0(t) \\
&\le C_d \cdot b^{\xi_0-\delta} \cdot \frac{d\mu_0(s)\,d\mu_0(t)}{|s-t|^{\xi_0-\delta}}.
\end{aligned}$$

(ii) If $j \le n-1$, like in Section 2.3.1(ii) one has

$$\begin{aligned}
&\mathcal{K}_n^d(s,t)^{-1} \\
&\quad \le \big(|F_k(s)-F_k(t)|^2 + |F_l(s)-F_l(t)|^2\big)^{-d/2} \\
&\quad = \big(\big|Q_k(s|_{j+1}^+)\cdot Z_k(s|_{j+1}^+) + \Delta_k\big|^2 + \big|Q_l(s|_{j+1}^+)\cdot Z_l(s|_{j+1}^+) + \Delta_l\big|^2\big)^{-d/2}.
\end{aligned}$$

By using Lemma 2(ii) we get

$$\begin{aligned}
&\mathbb{E}\bigg(\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\,\Big|\,\mathcal{A}(s|_{j+1}^+)\bigg) \\
&\quad \le C_d\cdot\big(\big|Q_k(s|_j\cdot r)\big|\cdot\big|Q_l(s|_j\cdot r)\big|^{d-1}\big)^{-1}\cdot Q(s|_n)\cdot Q(t|_n)\,d\mu_0(s)\,d\mu_0(t) \\
&\quad = C_d\cdot Q(s|_j\cdot r)^{-1}\cdot Q(s|_n)\cdot Q(t|_n)\,d\mu_0(s)\,d\mu_0(t) \\
&\quad = C_d\cdot \widetilde{W}(s|_j\cdot r)^{-1}\cdot b^{(j+1)(\xi_0-\delta)}\cdot\prod_{i=j+1}^{n}\widetilde{W}(s|_i)\cdot Q(t|_n)\,d\mu_0(s)\,d\mu_0(t).
\end{aligned}$$

All the random variables in the above products are independent, so

$$\begin{aligned}
\mathbb{E}\bigg(\frac{d\mu_n(s)\,d\mu_n(t)}{\mathcal{K}_n^d(s,t)}\bigg) &\le C_d\cdot\mathbb{E}(|W_k|^{-1}|W_l|^{1-d})\cdot b^{(j+1)(\xi_0-\delta)}\,d\mu_0(s)\,d\mu_0(t) \\
&\le C_d\cdot\mathbb{E}(|W_k|^{-1}|W_l|^{1-d})b^{2(\xi_0-\delta)}\cdot\frac{d\mu_0(s)\,d\mu_0(t)}{|s-t|^{\xi_0-\delta}}.
\end{aligned}$$

2.3.3. *Conclusion*. Let

$$C = \begin{cases} \max\{b^{\xi_0-\delta}\mathbb{E}(X_k^{-d}),\, C_d b^{2(\xi_0-\delta)}\mathbb{E}(|W_k|^{-d})\}, & \text{if } \xi_0\in(0,\xi_*]; \\ \max\{C_d b^{\xi_0-\delta},\, C_d b^{2(\xi_0-\delta)}\mathbb{E}(|W_k|^{-1}|W_l|^{1-d})\}, & \text{if } \xi_0\in(\xi_*,1]. \end{cases}$$

Then we get the conclusion from Section 2.3.1 and 2.3.2.

2.4. *Proof of Lemma 2*.

(i) Let $\varphi_k(x) = \mathbb{E}(e^{ixZ_k})$ be the characteristic function of Z_k. From (7) we have the following functional equation:

$$Z_k = \sum_{j=0}^{b-1} W_k(j)\cdot Z_k(j). \tag{14}$$

This implies

$$\varphi_k(x) = \mathbb{E}\bigg(\prod_{j=0}^{b-1}\varphi_k\big(x\cdot W_k(j)\big)\bigg).$$

Notice that given $x \in \mathbb{R}$ one has $|\varphi_k(x)| = |\varphi_k(-x)| = |\varphi_k(|x|)|$, so

$$|\varphi_k(x)| \le \mathbb{E}\left(\prod_{j=0}^{b-1} \big|\varphi_k\big(x \cdot W_k(j)\big)\big|\right)$$
$$= \mathbb{E}\left(\prod_{j=0}^{b-1} \big|\varphi_k\big(x \cdot |W_k(j)|\big)\big|\right),$$

which implies

$$|\varphi_k(x)| \le \mathbb{E}\big(\big|\varphi_k(x \cdot |W_k|)\big|\big)^b. \tag{15}$$

Starting from (15) and following the proof of Theorem 2.1 in [15] one can prove the following result:

If $\mathbb{E}(|W_k|^{-s}) < \infty$ for some $s > 0$, then $|\varphi_k(x)| = O(|x|^{-s})$ when $x \to \infty$.

Under assumption (A2) this result will imply that $\varphi_k \in L^1(\mathbb{R})$, thus Z_k has a bounded density function f_k with $\|f_k\|_\infty \le C_k := \int_{\mathbb{R}} |\varphi_k(x)|\,dx < \infty$. This gives us

$$\begin{aligned}
\mathbb{E}(|AZ_k + B|^{-d}) &= \int_{\mathbb{R}} \frac{f_k(x)}{|Ax + B|^d}\,dx \\
&= |A|^{-d} \int_{\mathbb{R}} \frac{f_k(x)}{|x + B/A|^d}\,dx \\
&= |A|^{-d} \int_{\mathbb{R}} \frac{f_k(u - B/A)}{|u|^d}\,du \\
&= |A|^{-d} \left(\int_{|u|>1} \frac{f_k(u - B/A)}{|u|^d}\,du + \int_{|u|\le 1} \frac{f_k(u - B/A)}{|u|^d}\,du \right) \\
&\le |A|^{-d} \cdot \left(1 + C_k \int_{|u|\le 1} \frac{1}{|u|^d}\,du\right).
\end{aligned}$$

(ii) First we assume that Z_k and Z_l have a bounded joint density function f with $\|f\|_\infty = C < \infty$, then

$$\begin{aligned}
&\mathbb{E}\big((|A_1 Z_k + B_1|^2 + |A_2 Z_l + B_2|^2)^{-d/2}\big) \\
&\quad = \iint \frac{f(x,y)}{(|A_1 x + B_1|^2 + |A_2 y + B_2|^2)^{d/2}}\,dx\,dy \\
&\quad = |A_2|^{-d} \iint \frac{f(x,y)}{(|(A_1/A_2)x + B_1/A_2|^2 + |y + B_2/A_2|^2)^{d/2}}\,dx\,dy \\
&\quad \le |A_2|^{-d} \frac{|A_2|}{|A_1|} \iint \frac{f((A_2/A_1)u - B_1/A_1, v - B_2/A_2)}{(u^2+v^2)^{d/2}}\,du\,dv \\
&\quad \le |A_1|^{-1} |A_2|^{-d+1} \left(1 + C \iint_{|u|^2+|v|^2\le 1} \frac{1}{(u^2+v^2)^{d/2}}\,du\,dv\right),
\end{aligned}$$

which gives us the conclusion. So it is enough to show that the characteristic function

$$\varphi : (x, y) \in \mathbb{R}^2 \mapsto \varphi(x, y) = \mathbb{E}\big(e^{i(xZ_k + yZ_l)}\big)$$

is in $L^1(\mathbb{R}^2)$. For we consider the polar coordinates: for $r \in \mathbb{R}_+$ and $\theta \in [0, 2\pi)$ define

$$\overline{\varphi}(r, \theta) = \varphi(r\cos\theta, r\sin\theta) = \mathbb{E}\big(e^{i(r\cos\theta Z_k + r\sin\theta Z_l)}\big). \tag{16}$$

Let $\psi(r) = \sup_{\theta\in[0,2\pi)} |\overline{\varphi}(r, \theta)|$. Clearly $\psi(r) \le 1$, so it is enough to show that $\psi(r) = O(r^{-s})$ for some $s > 2$ when $r \to \infty$. This can be done by using a similar argument as in (i): from (16) and (14) one has

$$\overline{\varphi}(r, \theta) = \mathbb{E}\left(\prod_{j=0}^{b-1} \overline{\varphi}\big(r \cdot \overline{W}(j), \theta + \overline{\theta}(j)\big)\right),$$

where $\overline{W}(j) = \sqrt{|W_k(j)|^2 + |W_l(j)|^2}$ and $\overline{\theta}_j = \arccos(W_k(j)/\overline{W}(j))$. This gives us

$$\psi(r) \le \mathbb{E}\big(\psi(r \cdot \overline{W})\big)^b \qquad \text{where } \overline{W} = \sqrt{|W_k|^2 + |W_l|^2}. \tag{17}$$

Again, starting from inequality (17) and following the proof of Theorem 2.1 in [15] (with a nontrivial modification which we will present later), one can prove the following result:

$$\text{If } \mathbb{E}(\overline{W}^{-s}) < \infty \text{ for some } s > 0, \text{ then } \psi(r) = O(r^{-s}) \text{ when } r \to \infty. \tag{18}$$

Then we can get the conclusion due to assumption (A2).

The nontrivial modification for proving (18) is the part that proves $\psi(r) < 1$ holds for all $r > 0$, the rest of the proof will follow easily from the proof of Theorem 2.1 in [15]. In order to prove that $\psi(r) < 1$ holds for all $r > 0$, first we show that $\psi(r) < 1$ holds for all r small enough.

Suppose that it is not the case. Then we can find sequences $r_n \to 0$ and $\theta_n \in [0, 2\pi)$ such that $|\overline{\varphi}(r_n, \theta_n)| = 1$, and thus there exists a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ and a sequence $\zeta_n \in [0, 2\pi)$ such that

$$r_n \cos\theta_n Z_k(\omega) + r_n \sin\theta_n Z_l(\omega) \in \zeta_n + 2\pi\mathbb{Z}$$

holds for all $n \ge 1$ and $\omega \in \Omega'$. In other words, for any $\omega, \omega' \in \Omega'$ one has

$$r_n \cos\theta_n\big(Z_k(\omega) - Z_k(\omega')\big) + r_n \sin\theta_n\big(Z_l(\omega) - Z_l(\omega')\big) \in 2\pi\mathbb{Z} \qquad \forall n \ge 1.$$

From $r_n \to 0$ one gets

$$\cos\theta_n\big(Z_k(\omega) - Z_k(\omega')\big) + \sin\theta_n\big(Z_l(\omega) - Z_l(\omega')\big) = 0$$

for all n large enough. Since $\cos\theta_n$ and $\sin\theta_n$ cannot be equal to 0 at the same time and Z_k, Z_l are not almost surely a constant, there exist a subset $\Omega'' \subset \Omega'$ with $\mathbb{P}(\Omega'') = 1$ and a constant $c \neq 0$ such that

$$Z_k(\omega) - Z_k(\omega') = c\big(Z_l(\omega) - Z_l(\omega')\big)$$

holds for all $\omega, \omega' \in \Omega''$. This implies that $Z_k - cZ_l$ is a constant on Ω''. In other words, $\sum_{j=0}^{b-1} W_k(j)Z_k(j) - cW_l(j)Z_l(j)$ is almost surely a constant. But this could happen only if $W_k(j)Z_k(j) - cW_l(j)Z_l(j)$ is almost surely equal to 0 for each $j = 0, \dots, b-1$ (since they are i.i.d. random variables). So we get $c = 1$ and $W_k = W_l$ almost surely, which is contradictory to the assumption $\mathbb{P}(W_1 = W_2) < 1$.

Now suppose that there exists an $h > 0$ such that $\psi(h) = 1$, and we may assume that $\psi(r) < 1$ holds for all $0 < r < h$. From (17) we get

$$1 = \psi(h) \le \mathbb{E}\big(\psi(h \cdot \overline{W})\big)^b \le 1.$$

This implies that almost surely $\psi(h \cdot \overline{W}) = 1$. Due to (A1) there exists $q \in (1, 2]$ such that $\mathbb{E}(|W_1|^q) \vee \mathbb{E}(|W_2|^q) < b^{-1}$. Since $q/2 < 1$, by using subadditivity of $x \to x^{q/2}$, we get that

$$\begin{aligned}
\mathbb{P}(\overline{W} \ge 1) &\le \mathbb{E}\big((|W_k|^2 + |W_l|^2)^{q/2}\big) \\
&\le \mathbb{E}(|W_k|^q + |W_l|^q) \\
&< 2b^{-1} \\
&\le 1.
\end{aligned}$$

Thus there exists $\delta < 1$ such that $\psi(h \cdot \delta) = 1$, which is a contradiction.

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