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HARNACK INEQUALITY FOR SDE WITH MULTIPLICATIVE NOISE AND EXTENSION TO NEUMANN SEMIGROUP ON NONCONVEX MANIFOLDS[1]

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By constructing a coupling with unbounded time-dependent drift, dimension-free Harnack inequalities are established for a large class of stochastic differential equations with multiplicative noise. These inequalities are applied to the study of heat kernel upper bound and contractivity properties of the semigroup. The main results are also extended to reflecting diffusion processes on Riemannian manifolds with nonconvex boundary.

1. Introduction.

Consider the following SDE on $\mathbb{R}^d$:

$$dX_t = \sigma(t, X_t)\,dB_t + b(t, X_t)\,dt, \tag{1.1}$$

where B_t is the d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\ge 0}, \mathbb{P})$, and

$$\sigma : [0,\infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d, \qquad b : [0,\infty) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$$

are progressively measurable and continuous in the second variable. Throughout the paper, we assume that for any $X_0 \in \mathbb{R}^d$ the equation (1.1) has a unique strong solution which is nonexplosive and continuous in t.

Let X_t^x be the solution to (1.1) for $X_0 = x$. We aim to establish the Harnack inequality for the operator P_t:

$$P_t f(x) := \mathbb{E} f(X_t^x), \qquad t \ge 0, x \in \mathbb{R}^d, f \in \mathscr{B}_b^+(\mathbb{R}^d),$$

where $\mathscr{B}_b^+(\mathbb{R}^d)$ is the class of all bounded nonnegative measurable functions on $\mathbb{R}^d$. To this end, we shall make use of the following assumptions.

(A1) There exists an increasing function $K : [0,\infty) \to \mathbb{R}$ such that almost surely

$$\begin{aligned}&\|\sigma(t,x) - \sigma(t,y)\|_{\mathrm{HS}}^2 + 2\langle b(t,x) - b(t,y), x - y\rangle \\ &\quad \le K_t |x-y|^2, \qquad x, y \in \mathbb{R}^d, t \ge 0.\end{aligned}$$

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(A2) There exists a decreasing function $\lambda:[0,\infty)\to(0,\infty)$ such that almost surely

$$\sigma(t,x)^*\sigma(t,x)\geq\lambda_t^2 I,\qquad x\in\mathbb{R}^d,t\geq 0.$$

(A3) There exists an increasing function $\delta:[0,\infty)\to(0,\infty)$ such that almost surely

$$\big|(\sigma(t,x)-\sigma(t,y))(x-y)\big|\leq\delta_t|x-y|,\qquad x,y\in\mathbb{R}^d,t\geq 0.$$

(A4) For $n\geq 1$, there exists a constant $c_n>0$ such that almost surely

$$\|\sigma(t,x)-\sigma(t,y)\|_{\mathrm{HS}}+|b(t,x)-b(t,y)|\leq c_n|x-y|,\qquad |x|,|y|,t\leq n.$$

It is well known that (A1) ensures the uniqueness of the solution to (1.1) while (A4) implies the existence and the uniqueness of the strong solution (see, e.g., [11] and references within for weaker conditions). On the other hand, if b and σ depend only on the variable $x\in\mathbb{R}^d$, then their continuity in x implies the existence of weak solutions (see [13], Theorem 2.3), so that by the Yamada–Watanabe principle [27], the uniqueness ensured by (A1) implies the existence and uniqueness of the strong solution.

Note that if $\sigma(t,x)$ and $b(t,x)$ are deterministic and independent of t, then the solution is a time-homogeneous Markov process generated by

$$L:=\frac{1}{2}\sum_{i,j=1}^d a_{ij}\,\partial_i\,\partial_j+\sum_{i=1}^d b_i\,\partial_i,$$

where $a:=\sigma\sigma^*$. If further more σ and b are smooth, we may consider the Bakry–Emery curvature condition [5]:

$$\Gamma_2(f,f)\geq -K\Gamma(f,f),\qquad f\in C^\infty(\mathbb{R}^d),\tag{1.2}$$

for some constant $K\in\mathbb{R}$, where

$$\Gamma(f,g):=\frac{1}{2}\sum_{i,j=1}^d a_{ij}(\partial_i f)(\partial_j g),\qquad f,g\in C^1(\mathbb{R}^d),$$

$$\Gamma_2(f,f):=\tfrac{1}{2}L\Gamma(f,f)-\Gamma(f,Lf),\qquad f\in C^\infty(\mathbb{R}^d).$$

According to [22], Lemma 2.2, and [23], Theorem 1.2, the curvature condition (1.2) is equivalent to the dimension-free Harnack inequality

$$(P_t f(x))^p\leq(P_t f^p(y))\exp\biggl[\frac{p\rho_a(x,y)^2}{2(p-1)(1-e^{-2Kt})}\biggr],$$
$$t\geq 0,p>1,f\in\mathscr{B}_b^+(\mathbb{R}^d),x,y\in\mathbb{R}^d,$$

where

$$\rho_a(x,y):=\sup\{|f(x)-f(y)|:f\in C^1(\mathbb{R}^d),\Gamma(f,f)\leq 1\},\qquad x,y\in\mathbb{R}^d.$$

This type of inequality has been extended and applied to the study of heat kernel (or transition probability) and contractivity properties for diffusion semigroups, see [1, 4, 18] for diffusions on manifolds with possibly unbounded below curvature, [15, 25] for stochastic generalized porous media and fast diffusion equations, and [2, 3, 8, 10, 14, 16, 17, 28] for the study of some other SPDEs with additive noise.

If σ depends on x, however, it is normally very hard to verify the curvature condition (1.2), which depends on second order derivatives of a^{-1}, the inverse matrix of a. This is the main reason why existing results on the dimension-free Harnack inequality for SPDEs are only proved for the additive noise case.

In this paper, we shall use the coupling argument developed in [4], which will allow us to establish Harnack inequalities for $\sigma(t,x)$ depending on x. This method has also been applied to the study of SPDEs in the above mentioned references. To see the difficulty in the study for $\sigma(t,x)$ depending on x, let us briefly recall the main idea of this argument.

To explain the main idea of the coupling, we first consider the easy case where σ and b are independent of the second variable. For $x \neq y$ and $T > 0$, let X_t solve (1.1) with $X_0 = x$ and Y_t solve

$$dY_t = \sigma(t)\,dB_t + b(t)\,dt + \frac{|x-y|(X_t - Y_t)}{T|X_t - Y_t|}\,dt, \qquad Y_0 = y.$$

Then Y_t is well defined up to the coupling time

$$\tau := \inf\{t \geq 0 : X_t = Y_t\}.$$

Let $X_t = Y_t$ for $t \geq \tau$. We have

$$d|X_t - Y_t| = -\frac{|x-y|}{T}\,dt, \qquad t \leq \tau.$$

This implies $\tau = T$ and hence, $X_T = Y_T$. On the other hand, by the Girsanov theorem we have

$$P_T f(y) = \mathbb{E}[R f(Y_T)]$$

for

$$R := \exp\biggl[-\frac{|x-y|}{T}\int_0^T \frac{\langle \sigma(t)^{-1}(X_t - Y_t), dB_t\rangle}{|X_t - Y_t|} - \frac{|x-y|^2}{2T^2}\int_0^T \frac{|\sigma(t)^{-1}(X_t - Y_t)|^2}{|X_t - Y_t|^2}\,dt\biggr].$$

Therefore,

$$(P_T f(y))^p = (\mathbb{E}[R f(X_T)])^p \leq (P_T f^p(x))\bigl(\mathbb{E}R^{p/(p-1)}\bigr)^{p-1}.$$

Since by (A1) and (A2) it is easy to estimate moments of R, the desired Harnack inequality follows immediately.

In general, if $\sigma(t,x)$ depends on x, then the process $X_t - Y_t$ contains a nontrivial martingale term, which cannot be dominated by and bounded drift. So, in this case, any additional bounded drift put in the equation for Y_t is not enough to make the coupling successful before a fixed time T. This is the main difficulty to establish the Harnack inequality for diffusion semigroups with nonconstant diffusion coefficient.

In this paper, under assumptions (A1) and (A2), we are able to constructed a coupling with a drift which is unbounded around a fixed time T, such that the coupling is successful before T. In this case, the corresponding exponential martingale has finite entropy such that the log-Harnack inequality holds; if further more (A3) holds then the exponential martingale is L^p-integrable for some $p > 1$ such that the Harnack inequality with power holds. More precisely, we have the following result.

THEOREM 1.1. *Let $\sigma(t,x)$ and $b(t,x)$ either be deterministic and independent of t, or satisfy* (A4).

(1) *If* (A1) *and* (A2) *hold, then*

$$P_T \log f(y) \le \log P_T f(x) + \frac{K_T|x-y|^2}{2\lambda_T^2(1-e^{-K_T T})}, \qquad f \ge 1, x, y \in \mathbb{R}^d, T > 0.$$

(2) *If* (A1), (A2) *and* (A3) *hold, then for $p > (1 + \frac{\delta_T}{\lambda_T})^2$ and $\delta_{p,T} := \max\{\delta_T, \frac{\lambda_T}{2}(\sqrt{p}-1)\}$, the Harnack inequality*

$$(P_T f(y))^p \le (P_T f^p(x)) \exp\left[\frac{K_T\sqrt{p}(\sqrt{p}-1)|x-y|^2}{4\delta_{p,T}[(\sqrt{p}-1)\lambda_T - \delta_{p,T}](1-e^{-K_T T})}\right]$$

holds for all $T > 0, x, y \in \mathbb{R}^d$ and $f \in \mathscr{B}_b^+(\mathbb{R}^d)$.

Theorem 1.1(1) generalizes a recent result in [19] on the log-Harnack inequality by using the gradient estimate on P_t.

Let $p_t(x,y)$ be the density of P_t w.r.t. a Radon measure μ. Then according to [26], Proposition 2.4, the above log-Harnack inequality and Harnack inequality are equivalent to the following heat kernel inequalities, respectively:

$$\int_{\mathbb{R}^d} p_T(x,z) \log \frac{p_T(x,z)}{p_T(y,z)} \mu(dz) \le \frac{K|x-y|^2}{2\lambda_T^2(1-e^{-K_T T})}, \qquad x, y \in \mathbb{R}^d, T > 0, \tag{1.3}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^d} p_T(x,z)\left(\frac{p_t(x,z)}{p_t(y,z)}\right)^{1/(p-1)}\mu(dz)\\ \text{(1.4)}\qquad &\le \exp\left[\frac{K_T\sqrt{p}|x-y|^2}{4\delta_{p,T}(\sqrt{p}+1)[(\sqrt{p}-1)\lambda_T-\delta_{p,T}](1-e^{-K_T T})}\right],\\ &\qquad\qquad x,y\in\mathbb{R}^d, T>0.\end{aligned}$$

So, the following is a direct consequence of Theorem 1.1.

COROLLARY 1.2. *Let $\sigma(t,x)$ and $b(t,x)$ either be deterministic and independent of t, or satisfy* (A4). *Let P_t have a strictly positive density $p_t(x,y)$ w.r.t. a Radon measure μ. Then* (A1) *and* (A2) *imply* (1.3), *while* (A1)–(A3) *imply* (1.4).

Next, by standard applications of the Harnack inequality with power, we have the following consequence of Theorem 1.1 on contractivity properties of P_t.

COROLLARY 1.3. *Let $\sigma(t,x)$ and $b(t,x)$ be deterministic and independent of t, such that* (A1)–(A3) *hold for constant K,λ and δ. Let P_t have an invariant probability measure μ.*

(1) *If there exists $r>K^+/\lambda^2$ such that $\mu(e^{r|\cdot|^2})<\infty$, then P_t is hypercontractive, that is, $\|P_t\|_{L^2(\mu)\to L^4(\mu)}=1$ holds for some $t>0$.*

(2) *If $\mu(e^{r|\cdot|^2})<\infty$ holds for all $r>0$, then P_t is supercontractive, that is, $\|P_t\|_{L^2(\mu)\to L^4(\mu)}<\infty$ holds for all $t>0$.*

(3) *If $P_t e^{r|\cdot|^2}$ is bounded for any $t,r>0$, then P_t is ultracontractive, that is, $\|P_t\|_{L^2(\mu)\to L^\infty(\mu)}<\infty$ for any $t>0$.*

REMARK 1.1. To see that results in Corollary 1.3 are sharp, let P_t be symmetric w.r.t. μ. Then the hypercontractivity is equivalent to the validity of the log-Sobolev inequality

$$\mu(f^2\log f^2)\le C\mu(\Gamma(f,f)),\qquad f\in C_b^\infty(\mathbb{R}^d), \mu(f^2)=1,$$

for some constant $C>0$. Moreover, if there exists a constant $R>0$ such that

$$\Gamma(f,f)\le R^2|\nabla f|^2,\qquad f\in C^\infty(\mathbb{R}^d), \tag{1.5}$$

we have $\rho_a(x,y)\ge R^{-1}|x-y|$. So, by the concentration of measure for the log-Sobolev inequality, the hypercontractivity implies $\mu(e^{r|\cdot|^2})<\infty$ for some $r>0$, while the supercontractivity implies $\mu(e^{r|\cdot|^2})<\infty$ for all $r>0$. Combining this with Corollary 1.3, we have the following assertions under conditions (A1)–(A3) and (1.5):

(i) Let $K \leq 0$. Then P_t is hypercontractive if and only if $\mu(e^{r|\cdot|^2}) < \infty$ holds for some $r > 0$;

(ii) P_t is supercontractive if and only if $\mu(e^{r|\cdot|^2}) < \infty$ holds for all $r > 0$;

(iii) P_t is ultracontractive if and only if $P_t e^{r|\cdot|^2}$ is bounded for any $t, r > 0$.

Therefore, conditions in Corollaries 1.3(2) and 1.3(3) are sharp for the supercontractivity and ultracontractivity of P_t. Moreover, as shown in [7] that when σ is constant, the sufficient condition $\mu(e^{r|\cdot|^2}) < \infty$ for some $r > K^+/\lambda^2$ is optimal for the hypercontractivity of P_t. So, Corollary 1.3(1) also provides a sharp sufficient condition for the hypercontractivity of P_t.

We will prove Theorem 1.1 and Corollary 1.3 in the next section. In Section 3, we extend these results to SDEs on Riemannian manifolds possibly with a convex boundary. Finally, combining results in Section 3 with a conformal change method introduced in [25], we are able to establish Harnack inequalities in Section 4 for the Neumann semigroup on a class of nonconvex manifolds.

2. Proofs of Theorem 1.1 and Corollary 1.3.

Let $x, y \in \mathbb{R}^d$, $T > 0$ and $p > (1 + \delta_T/\lambda_T)^2$ be fixed such that $x \neq y$. We have

$$\theta_T := \frac{2\delta_T}{(\sqrt{p} - 1)\lambda_T} \in (0, 2). \tag{2.1}$$

For $\theta \in (0, 2)$, let

$$\xi_t = \frac{2-\theta}{K_T}\bigl(1 - e^{K_T(t-T)}\bigr), \qquad t \in [0, T].$$

Then ξ is smooth and strictly positive on $[0, T)$ such that

$$2 - K_T\xi_t + \xi_t' = \theta, \qquad t \in [0, T]. \tag{2.2}$$

Consider the coupling

$$\begin{aligned} dX_t &= \sigma(t, X_t)\,dB_t + b(t, X_t)\,dt, \qquad X_0 = x,\\ dY_t &= \sigma(t, Y_t)\,dB_t + b(t, Y_t)\,dt\\ &\quad + \frac{1}{\xi_t}\sigma(t, Y_t)\sigma(t, X_t)^{-1}(X_t - Y_t)\,dt, \qquad Y_0 = y. \end{aligned} \tag{2.3}$$

Since the additional drift term $\xi_t^{-1}\sigma(t, y)\sigma(t, x)^{-1}(x - y)$ is locally Lipschitzian in y if (A4) holds, and continuous in y when σ and b are deterministic and time independent, the coupling (X_t, Y_t) is a well-defined continuous process for $t < T \wedge \zeta$, where ζ is the explosion time of Y_t; namely, $\zeta = \lim_{n\to\infty} \zeta_n$ for

$$\zeta_n := \inf\{t \in [0, T) : |Y_t| \geq n\},$$

where we set $\inf\varnothing = T$. Let

$$d\tilde{B}_t = dB_t + \frac{1}{\xi_t}\sigma(t, X_t)^{-1}(X_t - Y_t)\,dt, \qquad t < T \wedge \zeta.$$

If $\zeta = T$ and

$$\begin{aligned} R_s := \exp\biggl[& -\int_0^s \xi_t^{-1}\langle \sigma(t, X_t)^{-1}(X_t - Y_t), dB_t\rangle \\ & -\frac{1}{2}\int_0^s \xi_t^{-2}|\sigma(t, X_t)^{-1}(X_t - Y_t)|^2\,dt\biggr] \end{aligned}$$

is a uniformly integrable martingale for $s \in [0, T)$, then by the martingale convergence theorem, $R_T := \lim_{t\uparrow T} R_t$ exists and $\{R_t\}_{t\in[0,T]}$ is a martingale. In this case, by the Girsanov theorem $\{\tilde{B}_t\}_{t\in[0,T)}$ is a d-dimensional Brownian motion under the probability $R_T\mathbb{P}$. Rewrite (2.3) as

$$\begin{aligned} dX_t &= \sigma(t, X_t)\,d\tilde{B}_t + b(t, X_t)\,dt - \frac{X_t - Y_t}{\xi_t}\,dt, \qquad X_0 = x, \\ dY_t &= \sigma(t, Y_t)\,d\tilde{B}_t + b(t, Y_t)\,dt, \qquad Y_0 = y. \end{aligned} \tag{2.4}$$

Since $\int_0^T \xi_t^{-1}\,dt = \infty$, we will see that the additional drift $-\frac{X_t - Y_t}{\xi_t}\,dt$ is strong enough to force the coupling to be successful up to time T. So, we first prove the uniform integrability of $\{R_{s\wedge\zeta}\}_{s\in[0,T)}$ w.r.t. $\mathbb{P}$ so that $R_{T\wedge\zeta} := \lim_{s\uparrow T} R_{s\wedge\zeta}$ exists, then prove that $\zeta = T$ $\mathbb{Q}$-a.s. for $\mathbb{Q} := R_{T\wedge\zeta}\mathbb{P}$ so that $\mathbb{Q} = R_T\mathbb{P}$.

Let

$$\tau_n = \inf\{t \in [0, T) : |X_t| + |Y_t| \geq n\}.$$

Since X_t is nonexplosive as assumed, we have $\tau_n \uparrow \zeta$ as $n \uparrow \infty$.

LEMMA 2.1. *Assume* (A1) *and* (A2). *Let* $\theta \in (0, 2), x, y \in \mathbb{R}^d$ *and* $T > 0$ *be fixed.*

(1) *There holds*

$$\sup_{s\in[0,T),n\geq 1} \mathbb{E}R_{s\wedge\tau_n}\log R_{s\wedge\tau_n} \leq \frac{K_T|x - y|^2}{2\lambda_T^2\theta(2 - \theta)(1 - e^{-K_T T})|}.$$

Consequently,

$$R_{s\wedge\zeta} := \lim_{n\uparrow\infty} R_{s\wedge\tau_n\wedge(T-1/n)}, \qquad s \in [0, T], \qquad R_{T\wedge\zeta} := \lim_{s\uparrow T} R_{s\wedge\zeta}$$

exist such that $\{R_{s\wedge\zeta}\}_{s\in[0,T]}$ *is a uniformly integrable martingale.*

(2) *Let* $\mathbb{Q} = R_{T\wedge\zeta}\mathbb{P}$. *Then* $\mathbb{Q}(\zeta = T) = 1$ *so that* $\mathbb{Q} = R_T\mathbb{P}$.

PROOF. (1) Let $s \in [0, T)$ be fixed. By (2.4), (A1) and the Itô formula,

$$\begin{aligned} d\|X_t - Y_t\|^2 &\le 2\langle (\sigma(t, X_t) - \sigma(t, Y_t))(X_t - Y_t), d\tilde{B}_t\rangle \\ &\quad + K_T|X_t - Y_t|^2\,dt - \frac{2}{\xi_t}|X_t - Y_t|^2\,dt \end{aligned}$$

holds for $t \le s \wedge \tau_n$. Combining this with (2.2) we obtain

$$\begin{aligned} d\frac{|X_t - Y_t|^2}{\xi_t} &\le \frac{2}{\xi_t}\langle (\sigma(t, X_t) - \sigma(t, Y_t))(X_t - Y_t), d\tilde{B}_t\rangle \\ &\quad - \frac{|X_t - Y_t|^2}{\xi_t^2}(2 - K_T\xi_t + \xi_t')\,dt \\ &= \frac{2}{\xi_t}\langle (\sigma(t, X_t) - \sigma(t, Y_t))(X_t - Y_t), d\tilde{B}_t\rangle \\ &\quad - \frac{\theta}{\xi_t^2}|X_t - Y_t|^2\,dt, \qquad t \le s \wedge \tau_n. \end{aligned} \tag{2.5}$$

Multiplying by $\frac{1}{\theta}$ and integrating from 0 to $s \wedge \tau_n$, we obtain

$$\begin{aligned} \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2}\,dt &\le \int_0^{s\wedge\tau_n} \frac{2}{\theta\xi_t}\langle (\sigma(t, X_t) - \sigma(t, Y_t))(X_t - Y_t), d\tilde{B}_t\rangle \\ &\quad - \frac{|X_t - Y_t|^2}{\theta\xi_t} + \frac{|x - y|^2}{\theta\xi_0}. \end{aligned}$$

By the Girsanov theorem, $\{\tilde{B}_t\}_{t \le \tau_n \wedge s}$ is the d-dimensional Brownian motion under the probability measure $R_{s\wedge\tau_n}\mathbb{P}$. So, taking expectation $\mathbb{E}_{s,n}$ with respect to $R_{s\wedge\tau_n}\mathbb{P}$, we arrive at

$$\mathbb{E}_{s,n}\int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2}\,dt \le \frac{|x - y|^2}{\theta\xi_0}, \qquad s \in [0, T), n \ge 1. \tag{2.6}$$

By (A2) and the definitions of R_t and $\tilde{B}_t$, we have

$$\begin{aligned} \log R_r &= -\int_0^r \frac{1}{\xi_t}\langle \sigma(t, X_t)^{-1}(X_t - Y_t), d\tilde{B}_t\rangle + \frac{1}{2}\int_0^r \frac{|\sigma(t, X_t)^{-1}(X_t, Y_t)|^2}{\xi_t^2}\,dt \\ &\le -\int_0^r \frac{1}{\xi_t}\langle \sigma(t, X_t)^{-1}(X_t - Y_t), d\tilde{B}_t\rangle + \frac{1}{2\lambda_T^2}\int_0^r \frac{|X_t - Y_t|^2}{\xi_t^2}\,dt, \\ &\qquad\qquad r \le s \wedge \tau_n. \end{aligned}$$

Since $\{\tilde{B}_t\}$ is the d-dimensional Brownian motion under $R_{s\wedge\tau_n}\mathbb{P}$ up to $s \wedge \tau_n$, combining this with (2.6), we obtain

$$\mathbb{E}R_{s\wedge\tau_n}\log R_{s\wedge\tau_n} = \mathbb{E}_{s,n}\log R_{s\wedge\tau_n} \le \frac{|x - y|^2}{2\lambda_T^2\theta\xi_0}, \qquad s \in [0, T), n \ge 1.$$

By the martingale convergence theorem and the Fatou lemma, $\{R_{s\wedge\zeta}: s \in [0,T]\}$ is a well-defined martingale with

$$\mathbb{E}R_{s\wedge\zeta}\log R_{s\wedge\zeta} \le \frac{|x-y|^2}{2\lambda_T^2\theta\xi_0} = \frac{K_T|x-y|^2}{2\lambda_T^2\theta(2-\theta)(1-e^{-K_TT})}, \qquad s \in [0,T].$$

To see that $\{R_{s\wedge\zeta}: s \in [0,T]\}$ is a martingale, let $0 \le s < t \le T$. By the dominated convergence theorem and the martingale property of $\{R_{s\wedge\tau_n}: s \in [0,T)\}$, we have

$$\begin{aligned}\mathbb{E}(R_{t\wedge\zeta}|\mathscr{F}_s) &= \mathbb{E}\Big(\lim_{n\to\infty} R_{t\wedge\tau_n\wedge(T-1/n)}|\mathscr{F}_s\Big) = \lim_{n\to\infty}\mathbb{E}\big(R_{t\wedge\tau_n\wedge(T-1/n)}|\mathscr{F}_s\big)\\ &= \lim_{n\to\infty} R_{s\wedge\tau_n} = R_{s\wedge\zeta}.\end{aligned}$$

(2) Let $\sigma_n = \inf\{t \ge 0: |X_t| \ge n\}$. We have $\sigma_n \uparrow \infty$ $\mathbb{P}$-a.s and hence, also $\mathbb{Q}$-a.s. Since $\{\tilde{B}_t\}$ is a $\mathbb{Q}$-Brownian motion up to $T \wedge \zeta$, it follows from (2.5) that

$$\frac{(n-m)^2}{\xi_0}\mathbb{Q}(\sigma_m > t, \zeta_n \le t) \le \mathbb{E}_{\mathbb{Q}}\frac{|X_{t\wedge\sigma_m\wedge\zeta_n} - X_{t\wedge\sigma_m\wedge\zeta_n}|^2}{\xi_{t\wedge\sigma_m\wedge\zeta_n}} \le \frac{|x-y|^2}{\xi_0}$$

holds for all $n > m > 0$ and $t \in [0,T)$. By letting first $n \uparrow \infty$ then $m \uparrow \infty$, we obtain $\mathbb{Q}(\zeta \le t) = 0$ for all $t \in [0,T)$. This is equivalent to $\mathbb{Q}(\zeta = T) = 1$ according to the definition of ζ. $\square$

Lemma 2.1 ensures that under $\mathbb{Q} := R_{T\wedge\zeta}\mathbb{P}$, $\{\tilde{B}_t\}_{t\in[0,T]}$ is a Brownian motion. Then by (2.4), the coupling (X_t, Y_t) is well-constructed under $\mathbb{Q}$ for $t \in [0,T]$. Since $\int_0^T \xi_t^{-1}\,dt = \infty$, we shall see that the coupling is successful up to time T, so that $X_T = Y_T$ holds $\mathbb{Q}$-a.s. (see the proof of Theorem 1.1 below). This will provide the desired Harnack inequality for P_t as explained in Section 1 as soon as $R_{T\wedge\zeta}$ has finite $p/(p-1)$-moment. The next lemma provides an explicit upper bound on moments of $R_{T\wedge\zeta}$.

LEMMA 2.2. *Assume* (A1)–(A3). *Let R_t and ξ_t be fixed for $\theta = \theta_T$. We have*

$$\tag{2.7}\begin{aligned}&\sup_{s\in[0,T]}\mathbb{E}\bigg\{R_{s\wedge\zeta}\exp\bigg[\frac{\theta_T^2}{8\delta_T^2}\int_0^{s\wedge\zeta}\frac{|X_t-Y_t|^2}{\xi_t^2}\,dt\bigg]\bigg\}\\ &\qquad \le \exp\bigg[\frac{\theta_TK_T|x-y|^2}{4\delta_T^2(2-\theta_T)(1-e^{-K_TT})}\bigg].\end{aligned}$$

Consequently,

$$\tag{2.8}\sup_{s\in[0,T]}\mathbb{E}R_{s\wedge\zeta}^{1+r_T} \le \exp\bigg[\frac{\theta_TK_T(2\delta_T+\theta_T\lambda_T)|x-y|^2}{8\delta_T^2(2-\theta_T)(\delta_T+\theta_T\lambda_T)(1-e^{-K_TT})}\bigg]$$

holds for

$$r_T = \frac{\lambda_T^2\theta_T^2}{4\delta_T^2 + 4\theta_T\lambda_T\delta_T}.$$

PROOF. Let $\theta = \theta_T$. By (2.5), for any $r > 0$ we have

$$\begin{aligned}
&\mathbb{E}_{s,n} \exp\biggl[r \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2}\,dt\biggr] \\
&\quad\le \exp\biggl[\frac{r|x-y|^2}{\theta_T \xi_0}\biggr] \\
&\qquad \times \mathbb{E}_{s,n} \exp\biggl[\frac{2r}{\theta_T} \int_0^{s\wedge\tau_n} \frac{1}{\xi_t} \langle (\sigma(t, X_t) - \sigma(t, Y_t))(X_t - Y_t), d\tilde{B}_t\rangle\biggr] \\
&\quad\le \exp\biggl[\frac{r K_T |x-y|^2}{\theta_T (2-\theta_T)(1-e^{-K_T T})}\biggr] \\
&\qquad \times \biggl(\mathbb{E}_{s,n} \exp\biggl[\frac{8r^2\delta_T^2}{\theta_T^2} \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2}\,dt\biggr]\biggr)^{1/2},
\end{aligned}$$

where the last step is due to (A3) and the fact that

$$\mathbb{E} e^{M_t} \le (\mathbb{E} e^{2\langle M\rangle_t})^{1/2}$$

for a continuous exponential integrable martingale M_t. Taking $r = \theta_T^2/(8\delta_T^2)$, we arrive at

$$\mathbb{E}_{s,n} \exp\biggl[\frac{\theta_T^2}{8\delta_T^2} \int_0^{s\wedge\tau_n} \frac{|X_t - Y_t|^2}{\xi_t^2}\,dt\biggr] \le \biggl[\frac{\theta_T K_T |x-y|^2}{4\delta_T^2 (2-\theta_T)(1-e^{-K_T T})}\biggr], \qquad n \ge 1.$$

This implies (2.7) by letting $n \to \infty$.

Next, by (A2) and the definition of R_s, we have

$$\begin{aligned}
\mathbb{E} R_{s\wedge\tau_n}^{1+r_T} &= \mathbb{E}_{s,n} R_{s\wedge\tau_n}^{r_T} \\
&= \mathbb{E}_{s,n} \exp\biggl[-r_T \int_0^{s\wedge\tau_n} \frac{1}{\xi_t} \langle \sigma(t, X_t)^{-1}(X_t - Y_t), d\tilde{B}_t\rangle \\
&\qquad + \frac{r_T}{2} \int_0^{s\wedge\tau_n} \frac{|\sigma(t, X_t)^{-1}(X_t - Y_t)|^2}{\xi_t^2}\,dt\biggr].
\end{aligned} \tag{2.9}$$

Noting that for any exponential integrable martingale M_t w.r.t. $R_{s\wedge\tau_n}\mathbb{P}$, one has

$$\begin{aligned}
&\mathbb{E}_{s,n} \exp[r_T M_t + r_T \langle M\rangle_t/2] \\
&\quad= \mathbb{E}_{s,n} \exp[r_T M_t - r_T^2 q \langle M\rangle_t/2 + r_T(q r_T + 1)\langle M\rangle_t/2] \\
&\quad\le (\mathbb{E}_{s,n} \exp[r_T q M_t - r_T^2 q^2 \langle M\rangle_t/2])^{1/q} \\
&\qquad \times \biggl(\mathbb{E}_{s,n} \exp\biggl[\frac{r_T q(r_T q + 1)}{2(q-1)} \langle M\rangle_t\biggr]\biggr)^{(q-1)/q} \\
&\quad= \biggl(\mathbb{E}_{s,n} \exp\biggl[\frac{r_T q(r_T q + 1)}{2(q-1)} \langle M\rangle_t\biggr]\biggr)^{(q-1)/q}, \qquad q > 1,
\end{aligned}$$

it follows from (2.9) that

$$\mathbb{E}R_{s\wedge\tau_n}^{1+r_T} \le \left(\mathbb{E}_{s,n}\exp\left[\frac{qr_T(qr_T+1)}{2(q-1)\lambda_T^2}\int_0^{s\wedge\tau_n}\frac{|X_t-Y_t|^2}{\xi_t^2}\,dt\right]\right)^{(q-1)/q}. \tag{2.10}$$

Take

$$q = 1+\sqrt{1+r_T^{-1}}, \tag{2.11}$$

which minimizes $q(qr_T+1)/(q-1)$ such that

$$\begin{aligned}\frac{qr_T(qr_T+1)}{2\lambda_T^2(q-1)} &= \frac{r_T+\sqrt{r_T(r_T+1)}}{2\lambda_T^2\sqrt{1+r_T^{-1}}}\left(r_T+1+\sqrt{r_T(r_T+1)}\right)\\ &= \frac{(r_T+\sqrt{r_T^2+r_T})^2}{2\lambda_T^2} = \frac{\theta_T^2}{8\delta_T^2}.\end{aligned} \tag{2.12}$$

Combining (2.10) with (2.7) and (2.12), and noting that due to (2.11) and the definition of r_T

$$\frac{q-1}{q} = \frac{\sqrt{1+r_T^{-1}}}{1+\sqrt{1+r_T^{-1}}} = \frac{2\delta_T+\theta_T\lambda_T}{2\delta_T+2\theta_T\lambda_T},$$

we obtain

$$\mathbb{E}R_{s\wedge\tau_n}^{1+r_T} \le \exp\left[\frac{\theta_T K_T(2\delta_T+\theta_T\lambda_T)|x-y|^2}{8\delta_T^2(2-\theta_T)(\delta_T+\theta_T\lambda_T)(1-e^{-K_T T})}\right].$$

According to the Fatou lemma, the proof is then completed by letting $n\to\infty$. □

PROOF OF THEOREM 1.1. Since (A3) also holds for $\delta_{p,T}$ in place of δ_T, it suffices to prove the desired Harnack inequality for δ_T in place of $\delta_{p,T}$.

(1) By Lemma 2.1, $\{R_{s\wedge\zeta}\}_{s\in[0,T]}$ is an uniformly integrable martingale and $\{\tilde{B}_t\}_{t\le T}$ is a d-dimensional Brownian motion under the probability $\mathbb{Q}$. Thus, Y_t can be solved up to time T. Let

$$\tau = \inf\{t\in[0,T]: X_t = Y_t\}$$

and set $\inf\varnothing = \infty$ by convention. We claim that $\tau \le T$ and thus, $X_T = Y_T$, $\mathbb{Q}$-a.s. Indeed, if for some $\omega\in\Omega$ such that $\tau(\omega) > T$, by the continuity of the processes we have

$$\inf_{t\in[0,T]}|X_t-Y_t|^2(\omega) > 0.$$

So,

$$\int_0^T \frac{|X_t-Y_t|^2}{\xi_t^2}\,dt = \infty$$

holds on the set $\{\tau > T\}$. But according to Lemma 2.2, we have

$$\mathbb{E}_{\mathbb{Q}} \int_0^T \frac{|X_t - Y_t|^2}{\xi_t^2} \, dt < \infty,$$

we conclude that $\mathbb{Q}(\tau > T) = 0$. Therefore, $X_T = Y_T$ $\mathbb{Q}$-a.s.

Now, combining Lemma 2.1 with $X_T = Y_T$ and using the Young inequality, for $f \geq 1$ we have

$$\begin{aligned} P_T \log f(y) &= \mathbb{E}_{\mathbb{Q}}[\log f(Y_T)] = \mathbb{E}[R_{T\wedge\zeta} \log f(X_T)] \\ &\leq \mathbb{E} R_{T\wedge\zeta} \log R_{T\wedge\zeta} + \log \mathbb{R} f(X_T) \\ &\leq \log P_T f(x) + \frac{K_T |x-y|^2}{2\lambda_T^2 \theta(2-\theta)(1-e^{-K_T T})}. \end{aligned}$$

This completes the proof of (1) by taking $\theta = 1$.

(2) Let $\theta = \theta_T$. Since $X_T = Y_T$ and $\{\tilde{B}_t\}_{t\in[0,T]}$ is the d-dimensional Brownian motion under $\mathbb{Q}$, we have

$$\begin{aligned} (P_T f(y))^p &= (\mathbb{E}_{\mathbb{Q}}[f(Y_T)])^p = (\mathbb{E}[R_{T\wedge\zeta} f(X_T)])^p \\ &\leq (P_T f^p(x)) \bigl(\mathbb{E} R_{T\wedge\zeta}^{p/(p-1)}\bigr)^{p-1}. \end{aligned} \tag{2.13}$$

Due to (2.1), we see that

$$\frac{p}{p-1} = 1 + \frac{\lambda_T^2 \theta_T^2}{4\delta_T(\delta_T + \theta_T \lambda_T)}.$$

So, it follows from Lemma 2.2 and (2.1) that

$$\begin{aligned} \bigl(\mathbb{E} R_{T\wedge\zeta}^{p/(p-1)}\bigr)^{p-1} &= (\mathbb{E} R_{T\wedge\zeta}^{1+r_T})^{p-1} \leq \exp\biggl[\frac{(p-1)\theta_T K_T (2\delta_T + \theta_T \lambda_T)|x-y|^2}{8\delta_T^2 (2-\theta_T)(\delta_T + \theta_T\lambda_T)(1-e^{-K_T T})}\biggr] \\ &= \exp\biggl[\frac{K_T \sqrt{p}(\sqrt{p}-1)|x-y|^2}{4\delta_T[(\sqrt{p}-1)\lambda_T - \delta_T](1-e^{-K_T T})}\biggr]. \end{aligned}$$

Then the proof is finished by combining this with (2.13). $\square$

PROOF OF COROLLARY 1.3. Let $f \in \mathscr{B}_b^+(\mathbb{R}^d)$ be such that $\mu(f^p) \leq 1$. Let $p > (1 + \delta/\lambda)^2$. By Theorem 1.1(2), we have

$$(P_t f(y))^p \exp\biggl[-\frac{K\sqrt{p}(\sqrt{p}-1)|x-y|^2}{4\delta_p[(\sqrt{p}-1)\lambda - \delta_p](1-e^{-Kt})}\biggr] \leq P_t f^p(x), \qquad x, y \in \mathbb{R}^d,$$

where $\delta_p = \max\{\delta, \frac{\lambda}{2}(\sqrt{p}-1)\}$. Integrating w.r.t. $\mu(dx)$ and noting that μ is P_t-invariant, we obtain

$$(P_t f(y))^p \int_{\mathbb{R}^d} \exp\biggl[-\frac{K\sqrt{p}(\sqrt{p}-1)|x-y|^2}{4\delta[(\sqrt{p}-1)\lambda - \delta](1-e^{-Kt})}\biggr] \mu(dx) \leq 1. \tag{2.14}$$

Taking $f = n \wedge (p_t(y,\cdot))^{1/p}$ and letting $n \uparrow \infty$, we prove the first assertion.

Next, let $B(0,1) = \{x \in \mathbb{R}^d : |x| \le 1\}$. Since μ is an invariant measure, it has a strictly positive density w.r.t. the Lebesgue measure so that $\mu(B(0,1)) > 0$ (cf. [6]). Let $p \ge (1 + 2\delta/\lambda)^2$. We have $\delta_p = (\sqrt{p} - 1)\lambda/2$ and thus

$$\frac{\sqrt{p}(\sqrt{p}-1)}{4\delta_p[(\sqrt{p}-1)\lambda - \delta_p]} = \frac{\sqrt{p}}{\lambda^2(\sqrt{p}-1)}.$$

Combining this with (2.14) and noting that

$$\begin{aligned}\int_{\mathbb{R}^d} & \exp\biggl[-\frac{K\sqrt{p}(\sqrt{p}-1)|x-y|^2}{4\delta[(\sqrt{p}-1)\lambda - \delta](1-e^{-Kt})}\biggr]\mu(dx) \\ &\ge \mu(B(0,1))\exp\biggl[-\frac{K\sqrt{p}(\sqrt{p}-1)(1+|y|)^2}{4\delta[(\sqrt{p}-1)\lambda - \delta](1-e^{-Kt})}\biggr],\end{aligned}$$

we obtain

$$(P_t f(y))^p \le C_1 \exp\biggl[\frac{K\sqrt{p}(1+|y|)^2}{\lambda_T^2(\sqrt{p}-1)(1-e^{-Kt})}\biggr], \qquad t > 0, y \in \mathbb{R}^d, \tag{2.15}$$

for some constant $C_1 > 0$ and all $f \in \mathscr{B}_b^+(\mathbb{R}^d)$ with $\mu(f^p) \le 1$. Since

$$\lim_{p\to\infty}\lim_{t\to\infty}\frac{K\sqrt{p}}{\lambda^2(\sqrt{p}-1)(1-e^{-Kt})} = \frac{K^+}{\lambda^2},$$

for any $r > K^+/\lambda^2$ there exist $p > (1 + 2\delta_T/\lambda)^2$, $\beta > 1$ and $t_1 > 0$ such that

$$(P_{t_1} f(y))^{\beta p} \le C_2 e^{r|y|^2}, \qquad y \in \mathbb{R}^d, f \in \mathscr{B}_b^+(\mathbb{R}^d), \mu(f^p) \le 1,$$

holds for some constant $C_2 > 0$. Thus, $\mu(e^{r|\cdot|^2}) < \infty$ implies that

$$\|P_{t_1}\|_{L^p(\mu)\to L^{p\beta}(\mu)} < \infty.$$

Since $\|P_s\|_{L^q(\mu)} = 1$ holds for any $q \in [1,\infty]$, by the interpolation theorem and the semigroup property one may find $t_2 > t_1$ such that

$$\|P_{t_2}\|_{L^2(\mu)\to L^4(\mu)} < \infty. \tag{2.16}$$

Moreover, by [12], Theorem 3.6(ii), there exist some constants $\eta, C_3 > 0$ such that

$$\|P_t - \mu\|_{L^2(\mu)} \le C_3 e^{-\eta t}, \qquad t \ge 0.$$

Combining this with (2.16) we conclude that $\|P_t\|_{L^2(\mu)\to L^4(\mu)} \le 1$ holds for sufficiently large $t > 0$, that is, (2) holds.

Finally, (3) and (4) follow immediately from (2.15) and the interpolation theorem. □

3. Extension to manifolds with convex boundary. Let M be a d-dimensional complete, connected Riemannian manifold, possibly with a convex boundary ∂M. Let N be the inward unit normal vector filed of ∂M when $\partial M \neq \varnothing$. Let P_t be the (Neumann) semigroup generated by

$$L := \psi^2(\Delta + Z)$$

on M, where $\psi \in C^1(M)$ and Z is a C^1 vector field on M. Assume that ψ is bounded and

$$\mathrm{Ric} - \nabla Z \geq -K_0 \tag{3.1}$$

holds for some constant $K_0 \geq 0$. Then the (reflecting) diffusion process generated by L is nonexplosive.

To formulate P_t as the semigroup associated to a SDE like (1.1), we set

$$\sigma = \sqrt{2}\psi, \qquad b = \psi^2 Z. \tag{3.2}$$

Let d_I denote the Itô differential on M. In local coordinates the Itô differential for a continuous semi-martingale X_t on M is given by (see [4] or [9])

$$(d_I X_t)^k = dX_t^k + \frac{1}{2}\sum_{i,j=1}^{d} \Gamma_{ij}^k(X_t)\, d\langle X^i, X^j\rangle_t, \qquad 1 \leq k \leq d.$$

Then P_t is the semigroup for the solution to the SDE

$$d_I X_t = \sigma(X_t)\Phi_t\, dB_t + b(X_t)\, dt + N(X_t)\, dl_t, \tag{3.3}$$

where B_t is the d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, Φ_t is the horizontal lift of X_t onto the frame bundle $O(M)$, and l_t is the local time of X_t on ∂M. When $\partial M = \varnothing$, we simply set $l_t = 0$.

To derive the Harnack inequality as in Section 2, we assume that

$$\lambda := \inf \sigma > 0, \qquad \delta := \sup \sigma - \inf \sigma < \infty. \tag{3.4}$$

Now, let $x, y \in M$ and $T > 0$ be fixed. Let ρ be the Riemannian distance on M, that is, $\rho(x, y)$ is the length of the minimal geodesic on M linking x and y, which exits if ∂M is either convex or empty.

Let X_t solve (3.3) with $X_0 = x$. Next, any strictly positive function $\xi \in C([0, T))$, let Y_t solve

$$\begin{aligned} d_I Y_t = {} & \sigma(Y_t) P_{X_t, Y_t} \Phi_t\, dB_t + b(X_t)\, dt \\ & - \frac{\sigma(Y_t)\rho(X_t, Y_t)}{\sigma(X_t)\xi_t} \nabla\rho(X_t, \cdot)(Y_t)\, dt + N(Y_t)\, d\tilde{l}_t \end{aligned}$$

for $Y_0 = y$, where $\tilde{l}_t$ is the local time of Y_t on ∂M, and $P_{X_t, Y_t} : T_{X_t}M \to T_{Y_t}M$ is the parallel displacement along the minimal geodesic from X_t to Y_t, which exists since ∂M is convex or empty. As explained in [4], Section 3, we may and do assume that the cut-locus of M is empty such that the parallel displacement is smooth. Let

$$d\tilde{B}_t = dB_t + \frac{\rho(X_t, Y_t)}{\xi_t \sigma(X_t)} \Phi_t^{-1} \nabla \rho(\cdot, Y_t)(X_t)\, dt, \qquad t < T.$$

By the Girsanov theorem, for any $s \in (0, T)$ the process $\{\tilde{B}_t\}_{t\in[0,s]}$ is the d-dimensional Brownian motion under the weighted probability measure $R_s\mathbb{P}$, where

$$\begin{aligned} R_s := \exp\biggl[&-\int_0^s \frac{\rho(X_t, Y_t)}{\xi_t \sigma(X_t)} \langle \nabla \rho(\cdot, Y_t)(X_t), \Phi_t\, dB_t\rangle \\ &\qquad - \frac{1}{2}\int_0^s \frac{\rho(X_t, Y_t)^2}{\xi_t^2 \sigma(X_t)^2}\, dt\biggr]. \end{aligned} \tag{3.5}$$

Thus, by (3.2) we have

$$\begin{aligned} d_I X_t &= \sqrt{2}\psi(X_t)\Phi_t\, d\tilde{B}_t + (\psi^2 Z)(X_t)\, dt \\ &\quad - \frac{\rho(X_t, Y_t)}{\xi_t} \nabla\rho(\cdot, Y_t)(X_t)\, dt + N(X_t)\, dl_t, \\ d_I Y_t &= \sqrt{2}\psi(Y_t)\Phi_t\, d\tilde{B}_t + (\psi^2 Z)(Y_t)\, dt + N(Y_t)\, d\tilde{l}_t. \end{aligned}$$

Let $\xi \in C^1([0, T))$ be strictly positive and take

$$\beta_t = -\frac{\rho(X_t, Y_t)}{\sqrt{2}\xi_t \psi(X_t)} \Phi_t^{-1} \nabla\rho(\cdot, Y_t)(X_t).$$

Repeating the proof of (4.10) in [21], we obtain

$$\begin{aligned} d\rho(X_t, Y_t) &\le \bigl(\sigma(X_t) - \sigma(Y_t)\bigr)\langle \nabla\rho(\cdot, Y_t)(X_t), \Phi_t\, d\tilde{B}_t\rangle \\ &\quad + K_1 \rho(X_t, Y_t)\, dt - \frac{\rho(X_t, Y_t)}{\xi_t}\, dt, \qquad t < T, \end{aligned}$$

where

$$K_1 = K_0\|\psi\|_\infty^2 + 2\|Z\|_\infty \|\nabla\psi\|_\infty \|\psi\|_\infty.$$

This implies that

$$\begin{aligned} d\frac{\rho(X_t, Y_t)^2}{\xi_t} &\le \frac{2}{\xi_t}\rho(X_t, Y_t)\bigl(\sigma(X_t) - \sigma(Y_t)\bigr)\langle \nabla\rho(\cdot, Y_t)(X_t), \Phi_t\, d\tilde{B}_t\rangle \\ &\quad - \frac{\rho(X_t, Y_t)^2}{\xi_t^2}(2 - K\xi_t + \xi_t')\, dt \end{aligned}$$

holds for $t < T$ and

$$\begin{aligned} K &:= 2K_1 + \|\nabla\sigma\|_\infty^2 \\ &= 2K_0\|\psi\|_\infty^2 + 4\|Z\|_\infty\|\nabla\psi\|_\infty\|\psi\|_\infty + 2\|\nabla\psi\|_\infty^2. \end{aligned} \tag{3.6}$$

In particular, letting

$$\xi_t = \frac{2-\theta}{K}\big(1-e^{K(t-T)}\big), \qquad t \in [0,T], \theta \in (0,2),$$

we have

$$2 - K\xi_t + \xi_t' = \theta.$$

Therefore, the following result follows immediately by repeating calculations in Section 2.

THEOREM 3.1. *Assume that ∂M is either empty or convex. Let* (4.1) *and Z, ϕ be bounded such that*

$$K := 2K_0\|\psi\|_\infty^2 + 4\|Z\|_\infty\|\nabla\psi\|_\infty\|\psi\|_\infty + 2\|\nabla\psi\|_\infty^2 < \infty.$$

Then all assertions in Theorem 1.1 *and Corollaries* 1.2, 1.3 *hold for P_t the* (*Neumann*) *semigroup generated by $L = \psi^2(\Delta + Z)$ on M with $\rho(x,y)$ replacing $|x-y|$, and for constant functions $K, \delta := \sup\psi - \inf\psi$ and $\lambda := \inf|\psi|$.*

4. Neumann semigroup on nonconvex manifolds. Following the line of [24], we are able to make the boundary from nonconvex to convex by using a conformal change of metric. This will enable us to extend our results to the Neumann semigroup on a class of nonconvex manifolds.

Let $\partial M \neq \varnothing$ with N the inward normal unit vector field. Then the second fundamental form of ∂M is a two-tensor on the tangent space of ∂M defined by

$$\mathbb{I}(X,Y) := -\langle \nabla_X N, Y\rangle, \qquad X, Y \in T\partial M.$$

Assume that there exists $\kappa > 0$ and $K_0 \in \mathbb{R}$ such that

$$\mathrm{Ric} - \nabla Z \geq -K_0, \qquad \mathbb{I} \geq -\kappa \tag{4.1}$$

holds for M and a C^1 vector field Z. We shall consider the Harnack inequality for the Neumann semigroup P_t generated by

$$L = \Delta + Z.$$

To make the boundary convex, let $f \in C_b^\infty(M)$ such that $f \geq 1$ and $N \log f|_{\partial M} \geq \kappa$. By [24], Lemma 2.1, ∂M is convex under the metric

$$\langle\cdot,\cdot\rangle' = f^{-2}\langle\cdot,\cdot\rangle.$$

Let Δ' and ∇' be the Laplacian and gradient induced by the new metric. We have (see (2.2) in [20])

$$L = f^{-2}(\Delta' + Z'), \qquad Z' = f^2 Z + \frac{d-2}{2}\nabla f^2.$$

Let Ric' be the Ricci curvature induced by the metric $\langle\cdot,\cdot\rangle'$. We have (see the proof of [21], Theorem 5.1)

$$\mathrm{Ric}' - \nabla' Z' \geq -K_f \langle\cdot,\cdot\rangle'$$

for

$$K_f = \sup\{Kf^2 - d\Delta f + (d-3)|\nabla f|^2 + 3|Z||f|\nabla f|\}. \tag{4.2}$$

Applying Theorem 3.1 to the convex manifold $(M, \langle\cdot,\cdot\rangle')$, $\psi = f^{-1}$ and

$$\begin{aligned} K &= 2K_f^+ \|f^{-1}\|_\infty + 4\|Z'\|'_\infty \|\nabla' f^{-1}\|'_\infty \|f^{-1}\|_\infty + 2\|\nabla' f^{-1}\|'^2_\infty \\ &\leq 2K_f^+ + 4\|fZ + (d-2)\nabla f\|_\infty \|\nabla f\|_\infty + 2\|\nabla f\|^2_\infty, \end{aligned} \tag{4.3}$$

where $\|\cdot\|'$ is the norm induced by $\langle\cdot,\cdot\rangle'$ and we have used that $f \geq 1$, we obtain the following result.

THEOREM 4.1. *Let* (4.1) *hold for some* $\kappa > 0$ *and* $K_0 \in \mathbb{R}$, *and let* P_t *be the Neumann semigroup generated by* $L = \Delta + Z$ *on* M. *Then for any* $f \in C_b^\infty(M)$ *such that* $\inf f = 1$, $N \log f|_{\partial M} \geq \kappa$ *and* $K < \infty$, *where* K *is fixed by* (4.2) *and* (4.3), *all assertions in Theorem* 1.1 *and Corollaries* 1.2 *and* 1.3 *hold with* $\rho(x,y)$ *replacing* $|x-y|$ *for constant functions* $K, \delta := \sup f^{-1} - \inf f^{-1}$ *and* $\lambda := \inf f^{-1}$.

REMARK 4.1. A simple choice of f in Theorem 4.1 is $f = \phi \circ \rho_\partial$, where ρ_∂ is the Riemannian distance to the boundary which is smooth on $\{\rho_\partial \leq r_T\}$ for some $r_T > 0$ provided the injectivity radius of the boundary is positive, and $f \in C_b^\infty([0,\infty))$ is such that $f(0) = 1$, $f'(0) = \kappa$ and $f(r) = f(r_T)$ for $r \geq r_T$. In general, f is taken according to r_T and bounds of the second fundamental form and sectional curvatures, see, for example, [21, 24] for details. With specific choices of f, Theorem 4.1 provides explicit Harnack type inequalities, heat kernels estimates and criteria on contractivity properties for the Neumann semigroup on manifolds with nonconvex boundary.

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