The Annals of Probability
2011, Vol. 39, No. 3, 779–856
DOI: 10.1214/10-AOP571

AN OPTIMAL VARIANCE ESTIMATE IN STOCHASTIC HOMOGENIZATION OF DISCRETE ELLIPTIC EQUATIONS

BY ANTOINE GLORIA AND FELIX OTTO

INRIA and Max Planck Institute

We consider a discrete elliptic equation on the d-dimensional lattice $\mathbb{Z}^d$ with random coefficients A of the simplest type: they are identically distributed and independent from edge to edge. On scales large w.r.t. the lattice spacing (i.e., unity), the solution operator is known to behave like the solution operator of a (continuous) elliptic equation with constant deterministic coefficients. This symmetric "homogenized" matrix $A_{\text{hom}} = a_{\text{hom}}\,\text{Id}$ is characterized by $\xi \cdot A_{\text{hom}}\xi = \langle(\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)\rangle$ for any direction $\xi \in \mathbb{R}^d$, where the random field ϕ (the "corrector") is the unique solution of $-\nabla^* \cdot A(\xi + \nabla\phi) = 0$ such that $\phi(0) = 0$, $\nabla\phi$ is stationary and $\langle\nabla\phi\rangle = 0$, $\langle\cdot\rangle$ denoting the ensemble average (or expectation).

It is known ("by ergodicity") that the above ensemble average of the energy density $\mathcal{E} = (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)$, which is a stationary random field, can be recovered by a system average. We quantify this by proving that the variance of a spatial average of $\mathcal{E}$ on length scales L satisfies the optimal estimate, that is, $\text{var}[\sum \mathcal{E}\eta_L] \lesssim L^{-d}$, where the averaging function [i.e., $\sum \eta_L = 1$, $\text{supp}(\eta_L) \subset \{|x| \le L\}$] has to be smooth in the sense that $|\nabla\eta_L| \lesssim L^{-1-d}$. In two space dimensions (i.e., $d = 2$), there is a logarithmic correction. This estimate is optimal since it shows that smooth averages of the energy density $\mathcal{E}$ decay in L as if $\mathcal{E}$ would be independent from edge to edge (which it is not for $d > 1$).

This result is of practical significance, since it allows to estimate the dominant error when numerically computing a_{hom}.

1. Introduction.

1.1. *Motivation, informal statement and optimality of the result.* We study discrete elliptic equations. More precisely, we consider real functions u of the sites x in a d-dimensional Cartesian lattice $\mathbb{Z}^d$. Every edge e of the lattice is endowed with a "conductivity" $a(e) > 0$. This defines a discrete elliptic differential operator $-\nabla^* \cdot A\nabla$ via

$$-\nabla^* \cdot (A\nabla u)(x) := \sum_{y\in\mathbb{Z}^d, |x-y|=1} a(e)\big(u(x) - u(y)\big),$$

where the sum is over the $2d$ sites y which are connected by an edge $e = [x, y]$ to the site x. It is sometimes more convenient to think in terms of the associated

Received May 2009; revised May 2010.
MSC2010 subject classifications. 35B27, 39A70, 60H25, 60F99.
Key words and phrases. Stochastic homogenization, variance estimate, difference operator.

Dirichlet form, that is,

$$\sum_{x\in\mathbb{Z}^d}(\nabla v\cdot A\nabla u)(x) := \sum_{x\in\mathbb{Z}^d} v(x)\big(-\nabla^*\cdot(A\nabla u)(x)\big)$$
$$= \sum_e \big(v(x)-v(y)\big)a(e)\big(u(x)-u(y)\big),$$

where the last sum is over all edges e, and (x, y) denotes the two sites connected by e, that is, $e=[x,y]=[y,x]$ (with the convention that an edge is not oriented). We assume the conductivities a to be uniformly elliptic in the sense of

$$\alpha\le a(e)\le\beta \qquad \text{for all edges } e$$

for some fixed constants $0<\alpha\le\beta<\infty$.

We are interested in random coefficients. To fix ideas, we consider the simplest situation possible:

$$\{a(e)\}_e \qquad \text{are independently and identically distributed (i.i.d.).}$$

Hence, the statistics are described by a distribution on the finite interval $[\alpha,\beta]$. We would like to see this discrete elliptic operator with random coefficients as a good model problem for continuum elliptic operators with random coefficients of correlation length unity.

The first results in stochastic homogenization of linear elliptic equations in the continuous setting are due to Kozlov [11] and Papanicolaou and Varadhan [18], essentially using compensated compactness. The adaptation of these results to discrete elliptic equations in quite more general situations than the one considered above (i.e., under general ergodic assumptions) is due to Künnemann [13] following the approach by Papanicolaou and Varadhan for the continuous case, and also to Kozlov [12] (where more general discrete elliptic operators are considered). Note that the discrete elliptic operator $-\nabla^*\cdot A\nabla$ is the infinitesimal generator of a random walk in a random environment, whence the rephrasing of the homogenization result in [13] as the diffusion limit for reversible jump processes in $\mathbb{Z}^d$ with random bond conductivities. With the same point of view, it is also worth mentioning the seminal paper by Kipnis and Varadhan [9] using central limit theorems for martingales.

The general homogenization result proved in these articles states that there exist *homogeneous and deterministic* coefficients A_{hom} such that the solution operator of the continuum differential operator $-\nabla\cdot A_{\mathrm{hom}}\nabla$ describes the large scale behavior of the solution operator of the discrete differential operator $-\nabla^*\cdot A\nabla$. As a by product of this homogenization result, one obtains a characterization of the homogenized coefficients A_{hom}: it is shown that for every direction $\xi\in\mathbb{R}^d$, there exists a unique scalar field ϕ such that $\nabla\phi$ is stationary [stationarity means that the fields $\nabla\phi(\cdot)$ and $\nabla\phi(\cdot+z)$ have the same statistics for all shifts $z\in\mathbb{Z}^d$] and $\langle\nabla\phi\rangle=0$, solving the equation

$$-\nabla^*\cdot\big(A(\xi+\nabla\phi)\big)=0 \qquad \text{in } \mathbb{Z}^d, \tag{1.1}$$

and normalized by $\phi(0) = 0$. As in periodic homogenization, the function $\mathbb{Z}^d \ni x \mapsto \xi \cdot x + \phi(x)$ can be seen as the A-harmonic function which macroscopically behaves as the affine function $\mathbb{Z}^d \ni x \mapsto \xi \cdot x$. With this "corrector" ϕ, the homogenized coefficients A_{hom} (which in general form a symmetric matrix and for our simple statistics in fact a multiple of the identity: $A_{\mathrm{hom}} = a_{\mathrm{hom}}\,\mathrm{Id}$) can be characterized as follows:

$$\xi \cdot A_{\mathrm{hom}}\xi = \langle (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi) \rangle. \tag{1.2}$$

Since the scalar field $(\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)$ is stationary, it does not matter (in terms of the distribution) at which site x it is evaluated in the formula (1.2), so that we suppress the argument x in our notation.

The representation (1.2) is of no immediate practical use, since the equation (1.1) has to be solved:

- for *every realization* of the coefficients $\{a(e)\}_e$ and
- in the *whole space* $\mathbb{Z}^d$.

In order to overcome the first difficulty, it is natural to appeal to ergodicity (in the sense that ensemble averages are equal to system averages), which suggests to replace (1.2) by

$$\xi \cdot A_{\mathrm{hom}}\xi \rightsquigarrow \sum (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)\eta_L, \tag{1.3}$$

where η_L is a suitable averaging function of length scale $L \gg 1$, that is,

$$\mathrm{supp}(\eta_L) \subset \{|x| \le L\}, \qquad |\eta_L| \lesssim L^{-d}, \qquad \sum \eta_L = 1. \tag{1.4}$$

In fact, on expects the energy density $(\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)$, which is a stationary random field, to display a decay of correlations over large distances, so that (1.3) seems a good approximation for $L \gg 1$.

However, one still has to solve (1.1) on the whole space $\mathbb{Z}^d$, albeit for a single realization of the coefficients. In order to overcome this second difficulty, we start with the following observation: since ϕ on the ball $\{|x| \le L\}$ is expected to be little correlated to ϕ outside the ball $\{|x| \ge R\}$ for $R - L \gg 1$, it seems natural to replace ϕ in (1.3) by ϕ_R:

$$\sum (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)\eta_L \rightsquigarrow \sum (\xi + \nabla\phi_R) \cdot A(\xi + \nabla\phi_R)\eta_L, \tag{1.5}$$

where ϕ_R is the solution of an equation on a domain (say, a ball) of size R with homogeneous boundary conditions (say, Dirichlet):

$$\begin{aligned} -\nabla^* \cdot \big(A(\xi + \nabla\phi_R)\big) &= 0 \qquad \text{in } \mathbb{Z}^d \cap \{|x| < R\}, \\ \phi_R &= 0 \qquad \text{in } \mathbb{Z}^d \cap \{|x| \ge R\}, \end{aligned} \tag{1.6}$$

so that the right-hand side of (1.5) is indeed computable.

However, $\nabla\phi_R$ defined by (1.6) is not statistically stationary, which is a handicap for the error analysis. It is therefore common in the analysis of the error from spatial cut-off to introduce an intermediate step which consists in replacing equation (1.1) by

$$T^{-1}\phi_T - \nabla^* \cdot \big(A(\xi + \nabla\phi_T)\big) = 0 \qquad \text{in } \mathbb{Z}^d. \tag{1.7}$$

Clearly, the zero order term in (1.7) introduces a characteristic length scale $\sqrt{T}$ (the notation T that alludes to time is used because T^{-1} corresponds to the death rate in the random walker interpretation of the operator $T^{-1} - \nabla^* \cdot A\nabla$). In a second step, (1.7) is then replaced by

$$\begin{aligned} T^{-1}\phi_T - \nabla^* \cdot \big(A(\xi + \nabla\phi_{T,R})\big) &= 0 \qquad \text{in } \mathbb{Z}^d \cap \{|x| < R\}, \\ \phi_{T,R} &= 0 \qquad \text{in } \mathbb{Z}^d \cap \{|x| \ge R\}. \end{aligned}$$

The Green's function $G_T(x, y)$ of the operator $T^{-1} - \nabla^* \cdot A\nabla$ is known to decay faster than any power in $\frac{\sqrt{T}}{|x-y|} \ll 1$ uniformly in the realization of the coefficients [see, in particular, Lemma 2.8(iii)]. Therefore, one expects that ϕ_T and $\phi_{T,R}$ agree on the ball $\{|x| \le L\}$ up to an error which is of *infinite* order in $\varepsilon = \frac{\sqrt{T}}{R-L}$ (ε is the inverse of the distance of the ball $\{|x| \le L\}$ to the Dirichlet boundary $\{|x| = R\}$ measured in units of $\sqrt{T}$, see, e.g., [2], Section 3, for related arguments). Hence, we shall consider $\sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L$ as a very good proxy to the practically computable $\sum(\xi + \nabla\phi_{T,R}) \cdot A(\xi + \nabla\phi_{T,R})\eta_L$:

$$\sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L \approx \sum(\xi + \nabla\phi_{T,R}) \cdot A(\xi + \nabla\phi_{T,R})\eta_L.$$

In view of this remark, we restrict our attention to the error we make when replacing

$$\xi \cdot A_{\mathrm{hom}}\xi \rightsquigarrow \sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L.$$

It is natural to measure this error in terms of the expected value of its square. This error splits into two parts, the first arising from the finiteness of the averaging length scale L and the other arising from the finiteness of the cut-off length scale $\sqrt{T}$:

$$\begin{aligned} &\Big\langle \Big| \sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L - \xi \cdot A_{\mathrm{hom}}\xi \Big|^2 \Big\rangle \\ &\quad \overset{(1.2)}{=} \Big\langle \Big| \sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L - \langle(\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)\rangle \Big|^2 \Big\rangle \\ &\quad = \mathrm{var}\Big[\sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L\Big] \qquad (1.8) \\ &\qquad + \Big| \Big\langle \sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L \Big\rangle - \langle(\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)\rangle \Big|^2. \qquad (1.9) \end{aligned}$$

In view of the stationarity of $(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)$, of (1.4) and of (1.1), the second part (1.9) of the error can be rewritten as

$$\begin{aligned} &\Big| \Big\langle \sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L \Big\rangle - \langle(\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)\rangle \Big|^2 \\ &\quad = |\langle(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T) - (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)\rangle|^2 \qquad (1.10) \\ &\quad = \langle(\nabla\phi_T - \nabla\phi) \cdot A(\nabla\phi_T - \nabla\phi)\rangle^2. \end{aligned}$$

What scaling can we expect for the two error terms (1.8) and (1.10)? A heuristic prediction can be easily inferred from the regime of small ellipticity contrast, that is, $1-\frac{\alpha}{\beta} \ll 1$ (and $\alpha = 1$ w.l.o.g.). In this regime, to leading order, the two error terms (1.8) and (1.10) behave like

$$\operatorname{var}\Big[\sum(\xi \cdot (A - \langle A\rangle)\xi + 2\xi \cdot \nabla\bar{\phi})\eta_L\Big] \quad \text{and} \quad \langle|\nabla\bar{\phi}_T - \nabla\bar{\phi}|^2\rangle^2,$$

where $\bar{\phi}$ and $\bar{\phi}_T$ are defined via

$$-\triangle\bar{\phi} = \nabla^* \cdot \big((A - \langle A\rangle)\xi\big), \tag{1.11}$$

$$T^{-1}\bar{\phi}_T - \triangle\bar{\phi}_T = \nabla^* \cdot \big((A - \langle A\rangle)\xi\big), \tag{1.12}$$

respectively. In the first error term, we have replaced $\bar{\phi}_T$ by $\bar{\phi}$ for simplicity of the exposition.

These error terms can be computed in a straightforward manner. Indeed, as shown in the Appendix, they scale for any direction $|\xi| = 1$ as:

$$\operatorname{var}\Big[\sum(\xi \cdot (A - \langle A\rangle)\xi + 2\xi \cdot \nabla\bar{\phi})\eta_L\Big] \sim L^{-d} \tag{1.13}$$

$$\langle|\nabla\bar{\phi}_T - \nabla\bar{\phi}|^2\rangle^2 \sim \begin{cases} T^{-d}, & \text{for } d < 4,\\ T^{-4}\ln^2 T, & \text{for } d = 4,\\ T^{-4}, & \text{for } d > 4.\end{cases} \tag{1.14}$$

We now argue that the first error term (1.13) is the dominant one (in dimensions $d < 8$). In order to do so, we argue that the choice of $L \sim \sqrt{T}$ is natural [for which (1.13) dominates (1.14) in dimensions $d < 8$]. Indeed, we recall that in the ball $\{|x| \le L\}$, ϕ_T is a proxy for the computable $\phi_{T,R}$ (defined on the larger ball $\{|x| \le R\}$). The error is of *infinite* order in the distance between the two balls, measured in the length scale $\sqrt{T}$, that is, in $\varepsilon := \sqrt{T}/(R - L) \ll 1$. Hence, for the sake of discussing rates, we may indeed think of $L \sim \sqrt{T} \sim R$.

In this paper, we therefore focus on the error term (1.8) coming from the finite range L of the spatial average. In Theorem 2.1 (see also Remark 2.1), we shall establish that (1.13) holds as an estimate also for its nonlinear counterpart (1.8), that is,

$$\operatorname{var}\Big[\sum(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)\eta_L\Big] \lesssim L^{-d}, \tag{1.15}$$

with two minor restrictions:

- In dimension $d = 2$, the prefactor depends logarithmically on T (whereas for $d \neq 2$, the prefactor depends only on the ellipticity constants).
- The spatial averaging function η_L has to be smooth in the sense that $|\nabla\eta_L| \lesssim L^{-d-1}$ in addition to (1.4).

The estimate for the higher order term (1.9) will be the object of a subsequent work.

1.2. *Discussion of the works of Yurinskii and of Naddaf and Spencer.* In this subsection, we comment on two papers on error estimates (in the sense of the previous subsection) which from our perspective are the essential ones. We also explain how our work relates to these two papers.

Still unsurpassed is the first quantitative paper, the inspiring 1986 work by Yurinskii [21]. He essentially deals with the error (1.9) arising from the spatial cut-off T. In our discrete setting of i.i.d. coefficients $a(e)$ and for dimension $d > 2$, his result translates into

$$\langle |\nabla\phi_T - \nabla\phi|^2\rangle \lesssim T^{(2-d)/(4+d)+\delta}, \tag{1.16}$$

for $T \gg 1$ and some arbitrarily small $\delta > 0$, see [21], Theorem 2.1 (and [5], Lemma A.5, for this rephrasing of Yurinskii's result).

Yurinskii derives estimate (1.16) by fairly elementary arguments from the following crucial *variance estimate* of the spatial averages $\sum \phi_T \eta_L$ of ϕ_T on length scales L:

$$\operatorname{var}\Big[\sum \phi_T \eta_L\Big] \lesssim T\left(\frac{T}{L^d}\right)^{1/2-\delta} \tag{1.17}$$

for $1 \ll T \ll L^d$ and some arbitrarily small $\delta > 0$, see [21], Lemma 2.4. Let us comment a bit on the proof of (1.17): by stationarity of ϕ_T, the variance can be reformulated as a covariance, that is,

$$\operatorname{var}\Big[\sum \phi_T \eta_L\Big] = \operatorname{cov}\Big[\sum \phi_T \tilde{\eta}_L; \phi_T(0)\Big],$$

with a modified averaging function $\tilde{\eta}_L$. The starting point for (1.17) is to control the covariance by:

(i) An additive decomposition of $\phi_T(0)$ over all finite subsets S of the lattice $\mathbb{Z}^d$, that is, $\phi_T(0) = \sum_{S\subset\mathbb{Z}^d} \phi_{T,S}(0)$, where $\phi_{T,S}(0)$ only depends on $a_{|S}$, that is, the coefficients a restricted to the subset S.

(ii) An estimate on how sensitively $\sum \phi_T \tilde{\eta}_L$ depends on $a_{|S}$.

The decomposition in (i) is based on the probability measure on path space $[0,\infty) \ni t \mapsto \eta(t) \in \mathbb{Z}^d$ describing the random walk generated by the operator $-\nabla^* \cdot A\nabla$ (for a fixed realization of a). Indeed, this probability measure on path space allows for a well-known representation of $\phi_T(0)$ in terms of paths starting in 0 (via the expected value). Hence, the splitting can be obtained from restricting the expected value to all paths η with image S (up to some exit time larger than T), see [21], Lemma 2.3.

The sensitivity estimate (ii) comes in form of the deterministic energy-type estimate

$$\Big|\sum \phi_T \tilde{\eta}_L - \sum \tilde{\phi}_T \tilde{\eta}_L\Big|^2 \lesssim \frac{T}{L^d} \sum_{\text{edges } e \text{ s.t. } e\cap S\neq\varnothing} \big(1 + |\nabla\phi_T(e)|^2\big),$$

where $\tilde{\phi}_T$ is the solution of $T^{-1}\tilde{\phi}_T - \nabla^* \cdot \tilde{A}(\xi + \nabla\tilde{\phi}_T) = 0$ with coefficients $\tilde{A}$ which differ from A only on the subset S, see [21], (1.17).

The third ingredient for (1.17) is an estimate of the probability that a path η starting in 0 crosses a given edge e. This probability can be estimated in terms of the *Green's function* $G_T(x, 0)$ of the operator $T^{-1} - \nabla^* \cdot A\nabla$ (where x is one of the two sites on the edge e). Yurinskii then appeals to estimates on $G_T(x, y)$ that only depend on the ellipticity bounds $\alpha \le a \le \beta$ of A (and therefore do not depend on the realization of a) see [21], Lemma 2.1. As is well known, these type of estimates rely on the *Harnack inequality*.

Our variance estimate (1.15) also relies on these deterministic estimates of the Green's function $G_T(x, y)$, see Lemma 2.8. However, our strategy to estimate a variance differs substantially from Yurinskii's strategy of (i) and (ii). As a matter of fact, with our methods, we could derive the optimal variance estimate

$$\text{var}\Big[\sum \phi_T \eta_L\Big] \lesssim L^{2-d} \tag{1.18}$$

for $L \gg 1$. Estimate (1.18) is optimal in the sense that we obtain the above scaling in the regime of "vanishing ellipticity ratio" $1 - \frac{\alpha}{\beta} \ll 1$ by the arguments in the previous subsection. Still, the optimal estimate (1.18) would not yield the optimal estimate (1.14) by Yurinskii's argument to pass from (1.17) to (1.16).

Our strategy of estimating a variance is inspired by an unpublished paper by Naddaf and Spencer [17]. They use a *spectral gap* estimate to control the variance of some function X of the coefficients $\{a(e)\}_{\text{edges } e}$ (i.e., a random variable):

$$\text{var}[X] \lesssim \left\langle \sum_{\text{edges } e} \left(\frac{\partial X}{\partial a(e)}\right)^2 \right\rangle, \tag{1.19}$$

see [17], page 4. This type of estimate can be seen as a Poincaré estimate with mean value zero w.r.t. the infinite product measure that describes the distribution of the coefficients (and the optimal constant in this estimate is given by the smallest nonzero eigenvalue of the corresponding elliptic operator, whence "spectral gap"). Naddaf and Spencer derive (1.19) via the Brascamp–Lieb inequality for a large class of statistics for $\{a(e)\}_{\text{edges } e}$, which however does not include all i.i.d. statistics of $\{a(e)\}_{\text{edges } e}$ considered by us. We therefore rely on a slight modification of (1.19), see Lemma 2.3.

We also follow Naddaf and Spencer in the sense that we treat the variance of an *energy density*. However, they express their result not in terms of the energy density of ϕ_T but of a generic solution u with a *compactly supported, deterministic* right-hand side f, that is,

$$-\nabla^* \cdot A\nabla u = \nabla^* \cdot f. \tag{1.20}$$

Using (1.20), they obtain the formula $\frac{\partial}{\partial a(e)} \sum \nabla u \cdot A\nabla u = -|\nabla u(e)|^2$ so that an application of (1.19) yields the following estimate on the energy density

$X = \sum \nabla u \cdot A \nabla u$:

$$\operatorname{var}\Big[\sum \nabla u \cdot A\nabla u\Big] \lesssim \Big\langle \sum |\nabla u|^4\Big\rangle, \tag{1.21}$$

see [17], Proposition 1.

Naddaf and Spencer also remark that provided the ellipticity contrast $1-\frac{\alpha}{\beta}$ is small enough, *Meyer's estimate* holds which states that

$$\sum |\nabla u|^4 \lesssim \sum |f|^4, \tag{1.22}$$

with a constant that only depends on α, β. The combination of (1.21) and (1.22) yields the a priori estimate

$$\operatorname{var}\Big[\sum \nabla u \cdot A\nabla u\Big] \lesssim \sum |f|^4, \tag{1.23}$$

see [17], Theorem 1. Since the left-hand side of (1.23) scales as (volume)^2, while the right-hand side only scales as volume, this estimate reveals the optimal decay of fluctuations on the macroscopic level, very much like (1.15).—There is a somewhat theatrical convention in the homogenization literature to call the lattice spacing ε instead of 1 which highlights this scaling. Following Naddaf and Spencer, we use Meyer's estimate, albeit applied on the Green's function $G_T(x, y)$, see Lemma 2.9.

We will make use of the following notation:

- $d \geq 2$ is the dimension;
- $\int_{\mathbb{Z}^d} dx$ denotes the sum over $x \in \mathbb{Z}^d$, and $\int_D dx$ denotes the sum over $x \in \mathbb{Z}^d$ such that $x \in D$, D open subset of $\mathbb{R}^d$;
- $\langle \cdot \rangle$ is the ensemble average, or equivalently the expectation in the underlying probability space;
- $\operatorname{var}[\cdot]$ is the variance associated with the ensemble average;
- $\lesssim$ and $\gtrsim$ stand for $\leq$ and $\geq$ up to a multiplicative constant which only depends on the dimension d and the constants α, β (see Definition 2.1 below) if not otherwise stated;
- when both $\lesssim$ and $\gtrsim$ hold, we simply write $\sim$;
- we use $\gg$ instead of $\gtrsim$ when the multiplicative constant is (much) larger than 1;
- $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ denotes the canonical basis of $\mathbb{Z}^d$.

2. Main results.

2.1. *General framework.*

DEFINITION 2.1. We say that $a : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}^+$, $(x, y) \mapsto a(x, y)$ is a conductivity function on $\mathbb{Z}^d$ if there exist $0 < \alpha \leq \beta < \infty$ such that:

- $a(x, y) = 0$ if $|x - y| \neq 1$,

- $a(x, y) = a(y, x) \in [\alpha, \beta]$ if $|x - y| = 1$.

We denote by $\mathcal{A}_{\alpha\beta}$ the set of such conductivity functions.

DEFINITION 2.2. The elliptic operator $L : L^2_{\mathrm{loc}}(\mathbb{Z}^d) \to L^2_{\mathrm{loc}}(\mathbb{Z}^d)$, $u \mapsto Lu$ associated with a conductivity function $a \in \mathcal{A}_{\alpha\beta}$ is defined for all $x \in \mathbb{Z}^d$ by

$$(Lu)(x) = -\nabla^* \cdot A(x)\nabla u(x), \tag{2.1}$$

where

$$\nabla u(x) := \begin{bmatrix} u(x+\mathbf{e}_1) - u(x) \\ \vdots \\ u(x+\mathbf{e}_d) - u(x) \end{bmatrix}, \qquad \nabla^* u(x) := \begin{bmatrix} u(x) - u(x-\mathbf{e}_1) \\ \vdots \\ u(x) - u(x-\mathbf{e}_d) \end{bmatrix}$$

and

$$A(x) := \operatorname{diag}[a(x, x+\mathbf{e}_1), \dots, a(x, x+\mathbf{e}_d)].$$

In particular, it holds that

$$(Lu)(x) = \sum_{y, |x-y|=1} a(x, y)\big(u(x) - u(y)\big).$$

If $a(x, y) = 1$ for $|x - y| = 1$, then the associated elliptic operator L is the discrete Laplace operator, and is denoted by $-\triangle$.

DEFINITION 2.3 (Discrete integration by parts). Let $d \geq 2$, $h \in L^2(\mathbb{Z}^d)$ and $g \in L^2(\mathbb{Z}^d, \mathbb{R}^d)$. Then the discrete integration by parts reads

$$\int_{\mathbb{Z}^d} h(x)\nabla^* \cdot g(x)\,dx = -\int_{\mathbb{Z}^d} \nabla h(x) \cdot g(x)\,dx.$$

We now turn to the definition of the statistics of the conductivity function.

DEFINITION 2.4. A conductivity function is said to be independent and identically distributed (i.i.d.) if the coefficients $a(x, y)$ for $|x - y| = 1$ are i.i.d. random variables.

DEFINITION 2.5. The conductivity matrix A is obviously stationary in the sense that for all $z \in \mathbb{Z}^d$, $A(\cdot + z)$ and $A(\cdot)$ have the same statistics; and for all $x, z \in \mathbb{Z}^d$,

$$\langle A(x+z)\rangle = \langle A(x)\rangle.$$

Therefore, any translation invariant function of A, such as the modified corrector ϕ_T (see Lemma 2.2), is jointly stationary with A. In particular, not only are ϕ_T and its gradient $\nabla\phi_T$ stationary, but also any function of A, ϕ_T and $\nabla\phi_T$. A useful such example is the energy density $(\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T)$, which is stationary by joint stationarity of A and $\nabla\phi_T$.

Another translation invariant function of A is the Green functions G_T of Definition 2.7. In this case, stationarity means that $G_T(\cdot + z, \cdot + z)$ has the same statistics as $G_T(\cdot, \cdot)$ for all $z \in \mathbb{Z}^d$, so that in particular, for all $x, y, z \in \mathbb{Z}^d$,

$$\langle G_T(x+z, y+z)\rangle = \langle G_T(x, y)\rangle.$$

LEMMA 2.1 (Corrector ([13], Theorem 3)). *Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function, then for all $\xi \in \mathbb{R}^d$, there exists a unique random function $\phi : \mathbb{Z}^d \to \mathbb{R}$ which satisfies the corrector equation*

$$-\nabla^* \cdot A(x)\big(\nabla\phi(x) + \xi\big) = 0 \qquad \text{in } \mathbb{Z}^d, \tag{2.2}$$

and such that $\phi(0) = 0$, $\nabla\phi$ is stationary and $\langle\nabla\phi\rangle = 0$. In addition, $\langle|\nabla\phi|^2\rangle \lesssim |\xi|^2$.

We also define an "approximation" of the corrector as follows.

LEMMA 2.2 (Approximate corrector ([13], proof of Theorem 3)). *Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function, then for all $T > 0$ and $\xi \in \mathbb{R}^d$, there exists a unique stationary random function $\phi_T : \mathbb{Z}^d \to \mathbb{R}$ which satisfies the "approximate" corrector equation*

$$T^{-1}\phi_T(x) - \nabla^* \cdot A(x)\big(\nabla\phi_T(x) + \xi\big) = 0 \qquad \text{in } \mathbb{Z}^d, \tag{2.3}$$

and such that $\langle\phi_T\rangle = 0$. In addition, $T^{-1}\langle\phi_T^2\rangle + \langle|\nabla\phi_T|^2\rangle \lesssim |\xi|^2$.

Note that ϕ_T is stationary, whereas ϕ is not.

DEFINITION 2.6 (Homogenized coefficients). Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function and let $\xi \in \mathbb{R}^d$ and ϕ be as in Lemma 2.1. We define the homogenized $d \times d$-matrix A_{hom} as

$$\xi \cdot A_{\text{hom}}\xi = \langle(\xi + \nabla\phi) \cdot A(\xi + \nabla\phi)(0)\rangle. \tag{2.4}$$

Note that (2.4) fully characterizes A_{hom} since A_{hom} is a symmetric matrix (it is in particular of the form $a_{\text{hom}}\,\mathrm{Id}$ for an i.i.d. conductivity function).

2.2. *Statement of the main result.* Our main result shows that the energy density $\mathcal{E} := T^{-1}\phi_T^2 + (\nabla\phi_T + \xi) \cdot A(\nabla\phi_T + \xi)$ of the approximate corrector ϕ_T, which is a stationary scalar field, decorrelates sufficiently rapidly so that smooth spatial averages (defined with help of η_L) fluctuate as they would if $\mathcal{E}$ would be independent from site to site (as is the case for the tensor field A of the coefficients). The strength of fluctuation is expressed in terms of the variance. In more than two space dimensions (i.e., $d > 2$), the estimate does *not* depend on the cut-off scale

$\sqrt{T}$ and thus carries over to the energy density of the corrector ϕ. In two space dimensions, we are not able to rule out a weak (i.e., logarithmic) dependence on the cut-off scale $\sqrt{T}$:

THEOREM 2.1. *Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function, and let ϕ and ϕ_T denote the corrector and approximate correctors associated with the conductivity function a and direction $\xi \in \mathbb{R}^d$, $|\xi| = 1$. We then define for all $L > 0$ and $T \gg 1$ the symmetric matrix $A_{L,T}$ characterized by*

$$\xi \cdot A_{L,T}\xi := \int_{\mathbb{Z}^d} \bigl(T^{-1}\phi_T(x)^2 + \bigl(\nabla\phi_T(x) + \xi\bigr) \cdot A(x)\bigl(\nabla\phi_T(x) + \xi\bigr)\bigr)\eta_L(x)\,dx,$$

where $x \mapsto \eta_L(x)$ is an averaging function on $(-L, L)^d$ such that $\int_{\mathbb{Z}^d} \eta_L(x)\,dx = 1$ and $\|\nabla\eta_L\|_{L^\infty} \lesssim L^{-d-1}$. Then, there exists an exponent $q > 0$ depending only on α, β such that

$$\begin{aligned} &\text{for } d = 2 \qquad \operatorname{var}[\xi \cdot A_{L,T}\xi] \lesssim L^{-2}(\ln T)^q, \\ &\text{for } d > 2 \qquad \operatorname{var}[\xi \cdot A_{L,T}\xi] \lesssim L^{-d}. \end{aligned} \tag{2.5}$$

In particular, for $d > 2$, the variance estimate (2.5) holds for the energy density of the corrector ϕ itself.

REMARK 2.1. While it is natural to include the zero-order term $T^{-1}\langle\phi_T^2\rangle$ into the definition of the energy density, it is not essential for our result. Here comes the reason: by a simplified version of the string of arguments which lead to Theorem 2.1 we can show that the variance of the zero-order term is estimated as

$$\operatorname{var}\left[\int_{\mathbb{Z}^d} \phi_T(x)^2\eta_L(x)\,dx\right] \lesssim \begin{cases} (\ln T)^q, & \text{for } d = 2, \\ L^{2-d}, & \text{for } d > 2. \end{cases}$$

Hence, this term is of lower order in the regime (of interest) $L \lesssim T$.

The main ingredient to the proof of Theorem 2.1 is of independent interest. It states that all finite stochastic moments of the approximate corrector ϕ_T are bounded independently of T for $d > 2$ and grow at most logarithmically in T for $d = 2$.

PROPOSITION 2.1. *Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function, $\xi \in \mathbb{R}^d$ with $|\xi| = 1$ and let ϕ_T denote the approximate corrector associated with the conductivity function a, and ξ. Then there exists a continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $q \in \mathbb{R}^+$, there exists a constant C_q such that for all $T > 0$,*

$$\begin{aligned} &\text{for } d = 2 \qquad \langle|\phi_T(0)|^q\rangle \le C_q(\ln T)^{\gamma(q)}, \\ &\text{for } d > 2 \qquad \langle|\phi_T(0)|^q\rangle \le C_q. \end{aligned} \tag{2.6}$$

In addition, $\gamma(2n) = n(n + 1)$ for all $n = 2^l$, $l \in \mathbb{N}$ large enough.

Let us give a heuristic argument for the behavior of $\langle|\phi_T(0)|^q\rangle$ for $d=1$. In this case, for $T=\infty$, the gradient of the corrector associated with $\xi=1$ is explicitly given by

$$\nabla\phi=\frac{1}{a\langle a^{-1}\rangle}-1.$$

Hence, $\phi(x)\in\mathbb{R}$ behaves as a discrete Brownian motion in $x\in\mathbb{Z}$ once we have fixed its value at 0. Usually, one imposes $\phi(0)=0$ almost surely, so that for $|x|\sim\sqrt{T}$,

$$\langle|\phi(x)|^q|\phi(0)=0\rangle\sim(\sqrt{T})^{q/2}.$$

Yet, one may choose a nontrivial initial value. In particular, one may also consider $\phi(0)=\phi_T(0)$ (which yields a corrector field different from the one in Definition 2.1). With ϕ defined this way, $\phi_T(x)$ and $\phi(x)$ are expected not to differ much provided $|x|\ll\sqrt{T}$. On the one hand, from this we deduce that $\phi_T(x)$ behaves locally as a discrete Brownian motion starting at $\phi_T(0)$, so that we have as above

$$\langle|\phi_T(x)-\phi_T(0)|^q\rangle\sim|x|^{q/2}$$

for all $q>0$ and $|x|\ll\sqrt{T}$. On the other hand, since ϕ_T is stationary,

$$\langle|\phi_T(x)-\phi_T(0)|^q\rangle\lesssim\langle|\phi_T(x)|^q\rangle+\langle|\phi_T(0)|^q\rangle=2\langle|\phi_T(0)|^q\rangle.$$

These two estimates indeed suggest that

$$\langle|\phi_T(0)|^q\rangle\gtrsim\sqrt{T}^{q/2-},$$

where the minus sign accounts for the fact that the argument only holds for $|x|\ll\sqrt{T}$—we may for instance miss logarithmic corrections. Hence, there is a transition between unboundedness and boundedness in T for some $d\in(1,3)$. The linearization of the problem in the regime of vanishing ellipticity contrast, that is, $1-\frac{\alpha}{\beta}\ll 1$, suggests that $d=2$ is indeed the critical dimension for Proposition 2.1, that is, the dimension where a logarithmic behavior is to be expected. However, there is no reason why $d=2$ should be critical for Theorem 2.1. Indeed, in the case of $d=1$, the statement of Theorem 2.1 holds without a logarithm.

In view of our discussion of the case $d=1$ and the observations in case of vanishing ellipticity contrast, it is not surprising that the statement of bounded stochastic moments is harder to prove the closer we are to $d=2$. For the experts in homogenization, let us give a quick sketch of the strategy of the proof of this result. Independent of the dimension, the proof always starts from the variance estimate (Lemma 2.3) applied to $\phi_T(0)^q$ and makes use of the representation of $\frac{\partial\phi_T(0)}{\partial a(e)}$ with help of the gradient $\nabla_x G_T(x,0)$ (Lemma 2.4).

- In the case of $d>4$, the uniform pointwise, but suboptimal, decay $|\nabla_x G_T(x,y)|\lesssim|x-y|^{d-2}$, which can be easily obtained from the same pointwise decay of the Green's function itself, is sufficient.

- In case $d = 4$, it would be enough to appeal to the Hölder estimate (with exponent γ only depending on the ellipticity contrast) in order to get the somewhat better pointwise decay $|\nabla_x G_T(x, y)| \lesssim |x - y|^{d-2-\gamma}$.
- In $d = 3$, we need (in addition) the *optimal* decay $|\nabla_x G_T(x, y)| \lesssim |x - y|^{d-1}$, which cannot be a pointwise control, but only an average control on dyadic annuli. In fact, we need the control of the *square* average, which we easily obtain from the Cacciopoli estimate.
- For $d = 2$, the square average is not sufficient anymore, we need the average to some power $p > 2$, as provided by Meyers' estimate (Lemma 2.9). This forces us—somewhat counterintuitively—to first estimate high moments of ϕ_T, so that the exponent we put on the gradient of the Green's function can be chosen close to 2 (and thus below Meyers' exponent).

In this presentation, we only display the last strategy (although it is an overkill for dimensions $d > 2$).

As a corollary of Proposition 2.1, we obtain the following existence and uniqueness result of stationary solutions to the corrector equation (1.1) for $d > 2$, which settles a long-standing open question.

COROLLARY 2.1. *Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function. Then, for $d > 2$ and for all $\xi \in \mathbb{R}^d$, there exists a unique stationary random field ϕ such that $\langle\phi\rangle = 0$ and*

$$-\nabla^* \cdot \big(A(\xi + \nabla\phi)\big) = 0 \qquad in\ \mathbb{Z}^d.$$

In addition, $\langle \phi^2 + |\nabla\phi|^2\rangle \lesssim |\xi|^2$.

We will not prove Corollary 2.1 in detail. Here comes the argument. Proposition 2.1 yields the a priori estimate $\langle\phi_T^2\rangle < C$ which is uniform in T. This additional estimate allows us to pass to the limit in the probability space for ϕ_T, as it is done for $\nabla\phi_T$ in [13], proof of Theorem 3. Note that the corrector fields of Lemma 2.1 and Theorem 2.1 do not coincide (only their gradients coincide). Uniqueness further requires the argument by Papanicolaou and Varadhan in [18], which does not appear in [13].

Let us point out that Proposition 2.1, Theorem 2.1 and Corollary 2.1 hold true for more general distributions, provided the variance estimate of Lemma 2.3 below holds. In particular, the law of $a(x, x + \mathbf{e}_i)$ may depend on the direction $\mathbf{e}_i$, which would give a general diagonal homogenized matrix (not necessarily a multiple of the identity matrix). More generally, $a(x, x')$ and $a(y, y')$ may also be slightly correlated. We do not pursue this direction in this article.

2.3. *Structure of the proof and statement of the auxiliary results.* Not surprisingly, in order to control the variance of some function X of the coefficients a (like the spatial average of the energy density of the approximate corrector ϕ_T), one needs to control the gradient of X w.r.t. a. As in [17], this is quantified by the following general variance estimate:

LEMMA 2.3 (Variance estimate). *Let $a = \{a_i\}_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with range $[\alpha, \beta]$. Let X be a Borel measurable function of $a \in \mathbb{R}^{\mathbb{N}}$ (i.e., measurable w.r.t. the smallest σ-algebra on $\mathbb{R}^{\mathbb{N}}$ for which all coordinate functions $\mathbb{R}^{\mathbb{N}} \ni a \mapsto a_i \in \mathbb{R}$ are Borel measurable, cf.* [10], *Definition* 14.4).

Then we have

$$\mathrm{var}[X] \le \left\langle \sum_{i=1}^{\infty} \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle \mathrm{var}[a_1], \tag{2.7}$$

where $\sup_{a_i} |\frac{\partial X}{\partial a_i}|$ denotes the supremum of the modulus of the ith partial derivative

$$\frac{\partial X}{\partial a_i}(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots)$$

of X with respect to the variable $a_i \in [\alpha, \beta]$.

REMARK 2.2. Let us comment a bit on Lemma 2.3. Estimate (2.7) is a weakened version of a spectral gap estimate

$$\mathrm{var}[X] \lesssim \left\langle \sum_{i=1}^{\infty} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle, \tag{2.8}$$

which already played a central role in Naddaf and Spencer's analysis of stochastic homogenization [17], Section 2. We note that for i.i.d. random variables, such a spectral gap estimate (2.8) follows "by tensorization" from the one-dimensional spectral gap estimate

$$\langle X(a_1)^2 \rangle - \langle X(a_1) \rangle^2 \lesssim \left\langle \left| \frac{\partial X}{\partial a_1} \right|^2 \right\rangle, \tag{2.9}$$

see, for instance, [14], Lemma 1.1. The one-dimensional spectral gap estimate (2.9) holds under mild assumptions on the distribution of a_1. Yet, (2.9) does not hold for atomic measures like $\langle X(a_1) \rangle = \frac{1}{2}(X(1) + X(2))$. Since Lemma 2.3 covers the case of atomic measures, we only obtain the weaker form (2.7) of (2.8). Despite this technical detail, the proof of Lemma 2.3 is very similar to the one in [14], Lemma 1.1.

As in [17], in the proof of Theorem 2.1, we will make use of the fact that $T^{-1}\phi_T^2 + (\nabla\phi_T + \xi) \cdot A(\nabla\phi_T + \xi)$ is an energy density, which yields the following elementary formula for the partial derivative w.r.t. the value $a(e)$ of the coefficient in the edge $e = [z, z + \mathbf{e}_i]$:

$$\begin{aligned} &\frac{\partial}{\partial a(e)} \int \big(T^{-1}\phi_T^2 + (\nabla\phi_T + \xi) \cdot A(\nabla\phi_T + \xi)\big)(x)\eta_L(x)\,dx \\ &\quad = -2 \int \left(\frac{\partial \phi_T}{\partial a(e)} \nabla\eta_L \cdot A(\nabla\phi_T + \xi) \right)(x)\,dx \\ &\qquad + \big(\eta_L(\nabla_i\phi_T + \xi_i)\big)^2(z), \end{aligned} \tag{2.10}$$

up to minor modifications coming from the discrete Leibniz rule, see Step 1 of the proof of Theorem 2.1.

This formula makes the gradient of the averaging function η_L appear; in order to benefit from this, we assume that the averaging function is smooth so that we get an extra power of L^{-1}. The merit of (2.10) is that we need to control the partial derivative $\frac{\partial \phi_T(x)}{\partial a(e)}$ of the approximate corrector $\phi_T(x)$ (and not of its spatial derivatives). Not surprisingly, this partial derivative involves the Green's function $G_T(x,\cdot)$. More precisely, it involves the gradient $\nabla_{z_i} G_T(x,z)$ of the Green's function with singularity in z [and not its second gradient $\nabla_{z_i}\nabla_x G_T(x,z)$, for which we would *not* have the optimal decay rate uniformly in a].

We define discrete Green's functions as follows.

DEFINITION 2.7 (Discrete Green's function). Let $d \geq 2$. For all $T > 0$, the Green's function $G_T : \mathcal{A}_{\alpha\beta} \times \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{Z}^d$, $(a,x,y) \mapsto G_T(x,y;a)$ associated with the conductivity function a is defined for all $y \in \mathbb{Z}^d$ and $a \in \mathcal{A}_{\alpha\beta}$ as the unique solution in $L^2_x(\mathbb{Z}^d)$ to

$$\begin{aligned}\int_{\mathbb{Z}^d} & T^{-1} G_T(x,y;a) v(x)\,dx \\ & + \int_{\mathbb{Z}^d} \nabla v(x) \cdot A(x) \nabla_x G_T(x,y;a)\,dx = v(y) \qquad \forall v \in L^2(\mathbb{Z}^d),\end{aligned} \tag{2.11}$$

where A is as in (2.1).

Note that the existence and uniqueness of discrete Green's functions is a consequence of Riesz' representation theorem. Throughout this paper, when no confusion occurs, we use the short-hand notation $G_T(x,y)$ for $G_T(x,y;a)$.

The following lemma provides the elementary formula relating the "susceptibility" $\frac{\partial \phi_T(x)}{\partial a(e)}$ of $\phi_T(x)$ to the Green's function $G_T(x,y)$.

LEMMA 2.4. *Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function, and let G_T and ϕ_T be the associated Green's function and approximate corrector for $T > 0$ and $\xi \in \mathbb{R}^d$, $|\xi| = 1$. Then, for all $e = [z, z + \mathbf{e}_i]$ and $x \in \mathbb{Z}^d$,*

$$\frac{\partial \phi_T(x;a)}{\partial a(e)} = -\big(\nabla_i \phi_T(z;a) + \xi_i\big) \nabla_{z_i} G_T(z,x;a), \tag{2.12}$$

and for all $n \in \mathbb{N}$,

$$\begin{aligned}\sup_{a(e)} & \left| \frac{\partial}{\partial a(e)} [\phi_T(x;a)^{n+1}] \right| \\ & \lesssim |\phi_T(x;a)|^n \big(|\nabla_i \phi_T(z;a)| + 1\big) |\nabla_{z_i} G_T(z,x;a)| \\ & \quad + \big(|\nabla_i \phi_T(z;a)| + 1\big)^{n+1} |\nabla_{z_i} G_T(z,x;a)|^{n+1}.\end{aligned} \tag{2.13}$$

In addition, it holds that

$$\sup_{a(e)}|\nabla_i\phi_T(z;a)| \lesssim |\nabla_i\phi_T(z;a)|+1. \tag{2.14}$$

Note that the multiplicative constant in (2.13) *depends on n next to* α, β *and* d.

In addition, Lemma 2.4 provides uniform estimates on $\frac{\partial[\phi_T(x)^n]}{\partial a(e)}$ in $a(e)$ (the case $n>1$ is needed in Proposition 2.1). In order to obtain this uniform control in $a(e)$, we need to control $\nabla_z G(z,x;a)$ uniformly in $a(e)$. Again, this comes from considering $\frac{\partial\nabla_z G(z,x;a)}{\partial a(e)}$. The following lemma provides the elementary formula for $\frac{\partial\nabla_z G(z,x;a)}{\partial a(e)}$ and a uniform estimate in $a(e)$.

LEMMA 2.5. *Let* $G_T:\mathcal{A}_{\alpha\beta}\times\mathbb{Z}^d\times\mathbb{Z}^d\to\mathbb{R}$, $(a,x,y)\mapsto G_T(x,y;a)$ *be the Green's function associated with the conductivity function a for* $T>0$. *For all* $e=[z,z+\mathbf{e}_i]$ *and for all* $x,y\in\mathbb{Z}^d$, *it holds that*

$$\frac{\partial}{\partial a(e)}G_T(x,y;a) = -\nabla_{z_i}G_T(x,z;a)\nabla_{z_i}G_T(z,y;a). \tag{2.15}$$

As a by-product, we also have: for all $x\in\mathbb{Z}^d$

$$\sup_{a(e)}|\nabla_{z_i}G_T(z,x;a)| \lesssim |\nabla_{z_i}G_T(z,x;a)|. \tag{2.16}$$

There is a technical difficulty arising from the fact that a has infinitely many components. In Lemma 2.3, this technical difficulty is handled by the strong measurability assumptions on X. The following lemma establishes these measurability properties for ϕ_T, so that we can apply Lemma 2.3.

LEMMA 2.6. *Let* $a\in\mathcal{A}_{\alpha\beta}$ *be an i.i.d. conductivity function, and let* $G_T(\cdot,\cdot;a)$ *and* $\phi_T(\cdot;a)$ *be the associated Green's function and approximate corrector for* $\xi\in\mathbb{R}^d$, $d\geq 2$, *and* $T>0$. *Then for fixed* $x,y\in\mathbb{Z}^d$, $G_T(x,y,\cdot)$ *and* $\phi_T(x;\cdot)$ *are continuous w.r.t. the product topology of* $\mathcal{A}_{\alpha\beta}$ (*i.e., the smallest/coarsest topology on* $\mathbb{R}^E$, *where* E *denotes the set of edges, such that the coordinate functions* $\mathbb{R}^E\ni a\mapsto a_e\in\mathbb{R}$ *are continuous for all edges* $e\in E$).

In particular, $G_T(x,y;\cdot)$ *and* $\phi_T(x;\cdot)$ *are Borel measurable functions of* $a\in\mathcal{A}_{\alpha\beta}$, *so that one may apply Lemma* 2.3 *to* $\phi_T(x;\cdot)$ *and nonlinear funtions thereof.*

The proof of Theorem 2.1 crucially relies on the fact that ϕ_T is almost bounded independently of T (in $d>2$). More precisely, it relies on the fact that any moment $\langle\phi_T(0)^n\rangle$ is bounded independently of T as stated in Proposition 2.1. Starting point for Proposition 2.1 is again Lemma 2.3, which is iteratively applied to $\phi_T(0)^m$ where m increases dyadically. This is how Lemma 2.4 comes in again. However, the crucial gain in stochastic integrability is provided by the following lemma. It can be interpreted as a Cacciopoli estimate in probability and relies on the stationarity of ϕ_T.

LEMMA 2.7. *Let $a \in \mathcal{A}_{\alpha\beta}$ be an i.i.d. conductivity function, and let ϕ_T be the approximate corrector associated with the coefficients a for $\xi \in \mathbb{R}^d$, $|\xi| = 1$. Then for $d \geq 2$ and for all $n \in 2\mathbb{N}$,*

$$\big\langle |\phi_T(0)|^n \big(|\nabla \phi_T(0)|^2 + |\nabla^* \phi_T(0)|^2\big)\big\rangle \lesssim \langle |\phi_T|^n(0)\rangle, \tag{2.17}$$

where the multiplicative constant does depend on n next to α, β, and d, but not on $T > 0$.

In order to prove Proposition 2.1 via Lemma 2.3 [applied to $\phi_T(0)^n$] and Lemma 2.4, we need some *weak version* of the optimal decay of the gradient $\nabla_z G_T(x,z)$ of Green's function in $|x-z|$, that is,

$$|\nabla_z G_T(x,z;a)| \lesssim |x-z|^{1-d} \qquad \text{uniformly in } a \text{ and } T. \tag{2.18}$$

This decay is the best we can hope as can be checked on the Green function for the Laplace equation. The same decay property is needed to prove Theorem 2.1 via Lemma 2.3 [applied to (2.10)] and Lemma 2.4. Yet it is well known from the continuum case that there are no *pointwise* in z bounds of the type (2.18) which would hold uniformly in the ellipticity constants α, β. (An elementary argument shows that any bound on $\nabla_x G(x,y)$ which would be uniform in a and in $1/2 \leq |x-y| \leq 1$ would yield that a bounded a-harmonic function has bounded gradient. However, for $d = 2$ and for any $\gamma > 0$, there are examples of a-harmonic functions from the theory of quasi-conformal mappings that are not Hölder continuous with exponent γ, see [6], Section 12.1.) Nevertheless, (2.18) holds in the *square averaged* sense on dyadic annuli, as can be seen by a standard Cacciopoli argument based on the optimal decay of the Green's function itself, that is,

$$G_T(x,z) \lesssim |x-z|^{2-d} \qquad \text{uniformly in } a \text{ and } T, \tag{2.19}$$

in the case $d > 2$. The pointwise estimate (2.19) in x and z is a classical result [7], Theorem 1.1, that relies on Harnack's inequality. It has been partially extended to discrete settings, see in particular the Harnack inequality on graphs [3]. However, we did not find a suitable reference for the BMO-type estimate in the case of $d = 2$. On the other hand, we do not require the *pointwise* version of (2.19), but just an averaged version on dyadic annuli. The statements we need are collected in the following lemma.

LEMMA 2.8. *Let $a \in \mathcal{A}_{\alpha\beta}$, $T > 0$ and G_T be the associated Green's function. For all $d \geq 2$ and $q \geq 1, r \geq 0$,*

(i) *BMO and L^q estimate*: *for all $R \gg 1$,*

$$\text{for } d = 2 \qquad \int_{|x-y|\leq R} \big|G_T(x,y) - \bar{G}_T(\cdot,y)_{\{|x-y|\leq R\}}\big|^q \, dx \lesssim R^2, \tag{2.20}$$

$$\text{for } d > 2 \qquad \int_{R\leq|x-y|\leq 2R} G_T(x,y)^q \, dx \lesssim R^d (R^{2-d})^q, \tag{2.21}$$

where $\bar{G}_T(\cdot, y)_{\{|y-x|\le R\}}$ *denotes the average of* $G_T(\cdot, y)$ *over the ball* $\{x \in \mathbb{Z}^d, |x-y| \le R\}$.

(ii) *Behavior for* $R \sim \sqrt{T}$ *and* $d = 2$:

$$R^{-2} \int_{|x-y|\le R} G_T(x, y)^2\, dx \lesssim 1. \tag{2.22}$$

(iii) *Decay at infinity*: *for all* $R \ge \sqrt{T}$,

$$\int_{R\le|x-y|\le 2R} G_T(x, y)^q\, dx \lesssim R^d (R^{2-d})^q \big(\sqrt{T} R^{-1}\big)^r. \tag{2.23}$$

The multiplicative constants in (2.20), (2.21) *and* (2.23) *depend on* q, r *next to* α, β *and* d.

We present a proof of Lemma 2.8 which for $d = 2$ is a direct version of the indirect argument developed in [4], Lemma 2.5, in case of a nonlinear, continuum equation. For the convenience of the reader, we also include the proof for $d > 2$—anyway, it has the same building blocks as the argument for $d = 2$. This makes our paper self-contained w.r.t. the properties of G_T.

However, it is not quite enough to know (2.18) in the *square*-averaged sense on dyadic annuli. In order to compensate for the fact that we only control *finite* stochastic moments of $\nabla\phi_T(0)$ via Proposition 2.1, we need to control a pth power of the gradient $\nabla_z G_T(x, z)$ of Green's function in the optimal way for some $p > 2$. This slight increase in integrability is provided by Meyers' estimate, which yields such a $p > 2$ as a function of the ellipticity bounds α, β only. Meyers' estimate has already been crucially used in [17], however in a somewhat different spirit. There it is used that for sufficiently small ellipticity contrast, $1 - \frac{\alpha}{\beta} \ll 1$, one has $p \ge 4$. The following lemma is the version of Meyers' estimate we need and will prove.

LEMMA 2.9 (Higher integrability of gradients). *Let* $a \in \mathcal{A}_{\alpha\beta}$ *be a conductivity function*, *and* G_T *be its associated Green's function. Then*, *for* $d \ge 2$, *there exists* $p > 2$ *depending only on* α, β, *and* d *such that for all* $T > 0$, $p \ge q \ge 2$, $k > 0$ *and* $R \gg 1$,

$$\int_{R\le|z|\le 2R} |\nabla_z G_T(z, 0)|^q\, dz \lesssim R^d (R^{1-d})^q \min\{1, \sqrt{T} R^{-1}\}^k. \tag{2.24}$$

For technical reasons, we need a *pointwise* decay of $G_T(x, y; a)$ in $|x-y|$ uniformly in a (but not in T). The decay we obtain is suboptimal and easily follows from Lemmas 2.8 and 2.9 using the discreteness.

COROLLARY 2.2. *For all* $d \ge 2$ *and* $T > 0$, *there exists a bounded radially symmetric function* $h_T \in L^1(\mathbb{Z}^d)$ *depending only on* d, α, β, *and* T *such that*

$$G_T(x, y; a) \le h_T(x - y)$$

for all $x, y \in \mathbb{Z}^d$ *and* $a \in \mathcal{A}_{\alpha\beta}$.

Lemmas 2.8 and 2.9 only treat G_T away from the diagonal $x = y$—which is a consequence of the fact that the scaling symmetry is broken by the discreteness. Using the discreteness, the following corollary establishes a bound independent of T and a.

COROLLARY 2.3. *For all $a \in \mathcal{A}_{\alpha\beta}$, $T > 0$ and $x, y \in \mathbb{Z}^d$,*

$$|\nabla G_T(x, y; a)| \lesssim 1.$$

Finally, for the proof of Theorem 2.1, we need to know that also the *convolution* of the gradients of the Green's functions decays at the optimal rate, that is,

$$\int_{\mathbb{Z}^d} |\nabla_z G_T(x, z)||\nabla_z G_T(x', z)|\, dz \lesssim |x - x'|^{2-d} \qquad \text{uniformly in } a \text{ and } T. \tag{2.25}$$

As for (2.18), it is not necessary to know (2.25) *pointwise* in (x, x'), but only in an averaged sense on dyadic annuli. The following lemma shows that (2.25) for linear averages can be inferred from (2.18) for quadratic averages.

LEMMA 2.10. *Let $h_T \in L^2_{\mathrm{loc}}(\mathbb{Z}^d)$ be such that for all $R \gg 1$ and $T > 0$,*

$$\textit{for } d = 2 \qquad \int_{R<|z|\le 2R} h_T^2(z)\, dz \lesssim \min\{1, \sqrt{T} R^{-1}\}^2, \tag{2.26}$$

$$\textit{for } d > 2 \qquad \int_{R<|z|\le 2R} h_T^2(z)\, dz \lesssim R^{2-d}, \tag{2.27}$$

and for $R \sim 1$

$$\textit{for } d \ge 2 \qquad \int_{|z|\le R} h_T^2(z)\, dz \lesssim 1. \tag{2.28}$$

Then for $R \gg 1$

$$\textit{for } d = 2 \qquad \int_{|x|\le R} \int_{\mathbb{Z}^d} h_T(z) h_T(z - x)\, dz\, dx \lesssim R^2 \max\{1, \ln(\sqrt{T} R^{-1})\}, \tag{2.29}$$

$$\textit{for } d > 2 \qquad \int_{|x|\le R} \int_{\mathbb{Z}^d} h_T(z) h_T(z - x)\, dz\, dx \lesssim R^2. \tag{2.30}$$

We present the proof of Proposition 2.1 and Theorem 2.1 in Section 3. We gather in Section 4 the proofs of the decay estimates for the discrete Green functions (i.e., Lemmas 2.8 and 2.9, and Corollaries 2.2 and 2.3) since they are needed at several places in the paper, and may be of independent interest. The proofs of the remaining auxiliary lemmas are the object of Section 5.

3. Proofs of the main results.

3.1. *Proof of Proposition* 2.1. Starting point are Lemmas 2.3 and 2.6, which yield

$$\operatorname{var}[\phi_T(0)^m] \lesssim \sum_e \left\langle \sup_{a(e)} \left| \frac{\partial \phi_T(0)^m}{\partial a(e)} \right|^2 \right\rangle,$$

where $\sum_e$ denotes the sum over the edges. Using now (2.13) in Lemma 2.4, this inequality turns into

$$\begin{aligned}\operatorname{var}[\phi_T(0)^m] \lesssim \int_{\mathbb{Z}^d} \sum_{i=1}^d \langle \phi_T(0)^{2(m-1)} \big(|\nabla_i \phi_T(z)| + 1\big)^2 |\nabla_{z_i} G_T(z,0)|^2 \\ + \big(|\nabla_i \phi_T(z)| + 1\big)^{2m} |\nabla_{z_i} G_T(z,0)|^{2m} \rangle \, dz,\end{aligned}$$

where we have replaced the sum over edges e by the sum over sites $z \in \mathbb{Z}^d$ and directions $\mathbf{e}_i$ for $i \in \{1, \dots, d\}$ according to $e = [z, z + \mathbf{e}_i]$. Simplifying further, we obtain

$$\begin{aligned}\operatorname{var}[\phi_T(0)^m] \lesssim \int_{\mathbb{Z}^d} \langle \phi_T(0)^{2(m-1)} \big(|\nabla \phi_T(z)| + 1\big)^2 |\nabla_z G_T(z,0)|^2 \\ + \big(|\nabla \phi_T(z)| + 1\big)^{2m} |\nabla_z G_T(z,0)|^{2m} \rangle \, dz.\end{aligned} \tag{3.1}$$

We proceed in four steps. Assuming first that for n big enough and for all $m \le n$ it holds that

$$\begin{aligned}&\int_{\mathbb{Z}^d} \langle \phi_T(0)^{2(m-1)} \big(|\nabla \phi_T(z)| + 1\big)^2 |\nabla_z G_T(z,0)|^2 \\ &\qquad + \big(|\nabla \phi_T(z)| + 1\big)^{2m} |\nabla_z G_T(z,0)|^{2m} \rangle \, dz \\ &\quad \lesssim \big(\langle \phi_T(0)^{2n} \rangle^{m/n - 1/(n(n+1))} + 1\big) \begin{cases} \ln T, & \text{for } d = 2, \\ 1, & \text{for } d > 2, \end{cases}\end{aligned} \tag{3.2}$$

we prove the claim in the first step. The last three steps are dedicated to the proof of (3.2) for n large enough.

Step 1. Proof that (3.1) and (3.2) imply (2.6).

For notational convenience, we set $\mu_d(T) = 1$ for $d > 2$ and $\mu_d(T) = \ln T$ for $d = 2$. Let $n = 2^l$, $l \in \mathbb{N}^*$. Using the elementary fact that

$$\langle \phi_T(0)^{2m} \rangle \le \langle \phi_T(0)^m \rangle^2 + \operatorname{var}[\phi_T(0)^m],$$

from the cascade of inequalities (3.1) and (3.2) for $m = 2^{l-q}$, $q \in \{0, \dots, l\}$, we deduce

$$\langle \phi_T(0)^{2 \cdot 2^l} \rangle \lesssim \langle \phi_T(0)^{2^l} \rangle^2 + \mu_d(T) \big(\langle \phi_T(0)^{2n} \rangle^{1 - 1/(n(n+1))} + 1\big) \tag{estimate 0}$$

$$\vdots$$

$$\langle \phi_T(0)^{2\cdot 2^{l-q}} \rangle \lesssim \langle \phi_T(0)^{2^{l-q}} \rangle^2 + \mu_d(T)\big(\langle \phi_T(0)^{2n} \rangle^{1/2^q - 1/(n(n+1))} + 1\big) \qquad \text{(estimate } q\text{)}$$

$$\vdots$$

$$\langle \phi_T(0)^{2\cdot 2^0} \rangle \lesssim \underbrace{\langle \phi_T(0) \rangle^2}_{\overset{\text{Lemma 2.2}}{=} 0} + \mu_d(T)\big(\langle \phi_T(0)^{2n} \rangle^{1/n - 1/(n(n+1))} + 1\big) \qquad \text{(estimate } l\text{)}.$$

We then take the power 2^q of each (estimate q) and obtain using Young's inequality:

$$(3.3) \qquad \begin{aligned} \langle \phi_T(0)^{2n} \rangle &\lesssim \langle \phi_T(0)^n \rangle^2 + \mu_d(T)\big(\langle \phi_T(0)^{2n} \rangle^{1-1/(n(n+1))} + 1\big), \\ &\vdots \\ \langle \phi_T(0)^{2\cdot 2^{l-q}} \rangle^{2^q} &\lesssim \langle \phi_T(0)^{2^{l-q}} \rangle^{2^{q+1}} \\ &\quad + \mu_d(T)^{2^q}\big(\langle \phi_T(0)^{2n} \rangle^{1-2^q/(n(n+1))} + 1\big), \\ \langle \phi_T(0)^{2^{l-q}} \rangle^{2^{q+1}} &\lesssim \langle \phi_T(0)^{2^{l-q-1}} \rangle^{2^{q+2}} \\ &\quad + \mu_d(T)^{2^{q+1}}\big(\langle \phi_T(0)^{2n} \rangle^{1-2^{q+1}/(n(n+1))} + 1\big), \\ &\vdots \\ \langle \phi_T(0)^2 \rangle^n &\lesssim \mu_d(T)^n\big(\langle \phi_T(0)^{2n} \rangle^{1-1/(n+1)} + 1\big). \end{aligned}$$

Since the multiplicative constants in each line of (3.3) only depend on α, β, d, n and q, a linear combination of these $l+1$ inequalities with suitable positive coefficients allows us to cancel the respective terms both on the left- and right-hand sides, which yields

$$(3.4) \qquad \langle \phi_T(0)^{2n} \rangle \lesssim \sum_{q=0}^{l} \mu_d(T)^{2^q}\big(\langle \phi_T(0)^{2n} \rangle^{1-2^q/(n(n+1))} + 1\big).$$

Using Young's inequality, each term gives the same contribution and (3.4) turns into

$$(3.5) \qquad \langle \phi_T(0)^{2n} \rangle \lesssim \mu_d(T)^{n(n+1)}.$$

Formula (2.6) is then proved for all $q \le 2n$ using Hölder's inequality in probability.

Step 2. Estimate for the Green's function.

Let $p > 2$ be as in Lemma 2.9. We shall prove that for all $q \ge 1$ and $R \gg 1$ the following holds

$$\text{for } d=2 \qquad \int_{R<|z|\le 2R} |\nabla_z G_T(z,0)|^q \, dz \lesssim R^{2\max\{1,q/p\}} R^{-q} \min\{1, \sqrt{T}R^{-1}\}^q, \tag{3.6}$$

$$\text{for } d>2 \qquad \int_{R<|z|\le 2R} |\nabla_z G_T(z,0)|^q \, dz \lesssim R^{d\max\{1,q/p\}} (R^{1-d})^q. \tag{3.7}$$

We split the argument into two parts to treat $q \ge p$ and $q < p$, respectively. For $q \ge p$, we use the discrete $L^p - L^q$ estimate:

$$\left(\int_{R<|z|\le 2R} |\nabla_z G_T(z,0)|^q \, dz\right)^{1/q} \le \left(\int_{R<|z|\le 2R} |\nabla_z G_T(z,0)|^p \, dz\right)^{1/p}.$$

Combined with (2.24) in Lemma 2.9, it proves (3.6) and (3.7).

For $q < p$, we simply use Hölder's inequality with exponents $(\frac{p}{q}, \frac{p}{p-q})$ in the form

$$\left(R^{-d}\int_{R<|z|\le 2R} |\nabla_z G_T(z,0)|^q \, dz\right)^{1/q} \lesssim \left(R^{-d}\int_{R<|z|\le 2R} |\nabla_z G_T(z,0)|^p \, dz\right)^{1/p},$$

that we also combine with (2.24).

Step 3. General estimate.

Let $\chi \ge 0$ be a random variable. In order to prove (3.2), we will need to estimate terms of the form

$$\int_{\mathbb{Z}^d} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r} \, dz$$

for $q, r > 1$. Relying on (3.6) and (3.7), we show that

$$\int_{\mathbb{Z}^d} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r} \, dz \lesssim \langle \chi \rangle^{1/r} \begin{cases} 1, & \text{if } d\max\left\{1, 1-\dfrac{1}{r}+\dfrac{q}{rp}\right\} + (1-d)\dfrac{q}{r} < 0, \\ & d \ge 2, \\ \ln T, & \text{if } 2\max\left\{1, 1-\dfrac{1}{r}+\dfrac{q}{rp}\right\} - \dfrac{q}{r} = 0, \\ & d=2. \end{cases} \tag{3.8}$$

Note that there is no overlap in (3.8). For $d > 2$, we will only make use of the estimate with $d\max\{1, 1-\frac{1}{r}+\frac{q}{rp}\} + (1-d)\frac{q}{r} < 0$. For $d = 2$, we will use the estimate both with $2\max\{1, 1-\frac{1}{r}+\frac{q}{rp}\} - \frac{q}{r} < 0$, and with $2\max\{1, 1-\frac{1}{r}+\frac{q}{rp}\} - \frac{q}{r} = 0$, which requires a specific argument.

Let $i_{\min} \in \mathbb{N}, i_{\min} \sim 1$ be such that Lemma 2.9 holds for $R \geq 2^{i_{\min}}$. To prove (3.8), we use a dyadic decomposition of $\mathbb{Z}^d$ in annuli of radii $R_i = 2^i$:

$$
\begin{aligned}
&\int_{\mathbb{Z}^d} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r}\, dz \\
&\qquad = \int_{|z|\leq 2^{i_{\min}}} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r}\, dz \\
&\qquad\quad + \sum_{i=i_{\min}}^{\infty} \int_{R_i<|z|\leq R_{i+1}} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r}\, dz.
\end{aligned}
\tag{3.9}
$$

Using Corollary 2.3, we bound the first term of the right-hand side by

$$
\int_{|z|\leq 2^{i_{\min}}} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r}\, dz \lesssim \langle \chi \rangle^{1/r}.
$$

For the second term of the right-hand side, we appeal to Hölder's inequality with $(r, \frac{r}{r-1})$:

$$
\begin{aligned}
&\sum_{i=i_{\min}}^{\infty} \int_{R_i<|z|\leq R_{i+1}} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r}\, dz \\
&\qquad \lesssim \sum_{i=i_{\min}}^{\infty} (R_i^d)^{1-1/r} \biggl(\int_{R_i<|z|\leq R_{i+1}} \langle \chi |\nabla_z G_T(z,0)|^q \rangle\, dz \biggr)^{1/r},
\end{aligned}
$$

so that (3.9) turns into

$$
\begin{aligned}
&\int_{\mathbb{Z}^d} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r}\, dz \\
&\qquad = \langle \chi \rangle^{1/r} + \sum_{i=i_{\min}}^{\infty} (R_i^d)^{1-1/r} \biggl\langle \chi \int_{R_i<|z|\leq R_{i+1}} |\nabla_z G_T(z,0)|^q\, dz \biggr\rangle^{1/r}.
\end{aligned}
$$

Using then (3.6) and (3.7), we get

$$
\begin{aligned}
&\biggl\langle \chi \int_{R_i<|z|\leq R_{i+1}} |\nabla_z G_T(z,0)|^q\, dz \biggr\rangle \\
&\qquad \lesssim \begin{cases} \langle \chi \rangle R_i^{2\max\{1,q/p\}} R_i^{-q} \min\{1, \sqrt{T} R_i^{-1}\}^q, & d=2, \\ \langle \chi \rangle R_i^{d\max\{1,q/p\}} (R_i^{1-d})^q, & d>2. \end{cases}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_{\mathbb{Z}^d} \langle \chi |\nabla_z G_T(z,0)|^q \rangle^{1/r}\, dz \\
&\lesssim \begin{cases} \langle \chi \rangle^{1/r} \displaystyle\sum_{i=0}^{\infty} R_i^{2\max\{1,1-1/r+q/(rp)\}-q/r} \min\{1, \sqrt{T} R_i^{-1}\}^{q/r}, & d=2, \\ \langle \chi \rangle^{1/r} \displaystyle\sum_{i=0}^{\infty} R_i^{d\max\{1,1-1/r+q/(rp)\}+(1-d)q/r}, & d>2. \end{cases}
\end{aligned}
$$

We distinguish two cases. If $d\max\{1, 1-\frac{1}{r}+\frac{q}{rp}\}+(1-d)\frac{q}{r}<0$, then

$$\int_{\mathbb{Z}^d}\langle\chi|\nabla_z G_T(z,0)|^q\rangle^{1/r}\,dz\lesssim\langle\chi\rangle^{1/r}\sum_{i=0}^{\infty}R_i^{d\max\{1,1-1/r+q/(rp)\}+(1-d)q/r}\lesssim\langle\chi\rangle^{1/r}.$$

This proves the first estimate of (3.8). For $d=2$, and $2\max\{1, 1-\frac{1}{r}+\frac{q}{rp}\}-\frac{q}{r}=0$, then we obtain

$$\begin{aligned}\int_{\mathbb{Z}^2}\langle\chi|\nabla_z G_T(z,0)|^q\rangle^{1/r}\,dz&\lesssim\langle\chi\rangle^{1/r}\sum_{i=0}^{\infty}\min\{1,\sqrt{T}R_i^{-1}\}^{q/r}\\&\lesssim\langle\chi\rangle^{1/r}\left(\ln T+\sum_{i=0}^{\infty}R_i^{-q/r}\right)\\&\lesssim\langle\chi\rangle^{1/r}(1+\ln T).\end{aligned}$$

This proves the second estimate of (3.8).

Step 4. Proof of (3.2).

Let $n\ge 1$ and $n\ge m\ge 1$. We first treat the first term of the left-hand side of (3.2). In that case Hölder's inequality in probability with exponents $(n+1,\frac{n+1}{n})$ and the stationarity of $\nabla\phi_T$ show

$$\begin{aligned}&\int_{\mathbb{Z}^d}\langle\phi_T(0)^{2(m-1)}(|\nabla\phi_T(z)|+1)^2|\nabla_z G_T(z,0)|^2\rangle\,dz\\&\quad\lesssim\int_{\mathbb{Z}^d}\left(\langle|\nabla\phi_T(z)|^{2(n+1)}\rangle^{1/(n+1)}+1\right)\\&\qquad\qquad\times\langle|\phi_T(0)|^{2(m-1)(n+1)/n}|\nabla_z G_T(z,0)|^{2(n+1)/n}\rangle^{n/(n+1)}\,dz\\&\quad=\left(\langle|\nabla\phi_T(0)|^{2(n+1)}\rangle^{1/(n+1)}+1\right)\\&\qquad\times\int_{\mathbb{Z}^d}\langle|\phi_T(0)|^{2(m-1)(n+1)/n}|\nabla_z G_T(z,0)|^{2(n+1)/n}\rangle^{n/(n+1)}\,dz.\end{aligned}\tag{3.10}$$

We apply Lemma 2.7 to bound the first ensemble average in (3.10):

$$\begin{aligned}&\langle|\nabla\phi_T(0)|^{2(n+1)}\rangle\\&\quad\lesssim\left\langle\sum_{i=1}^{d}|\nabla\phi_T(0)|^2\left(\phi_T(0)^{2n}+\phi_T(\mathbf{e}_i)^{2n}\right)\right\rangle\\&\quad\overset{\text{stationarity}}{=}2\left\langle\sum_{i=1}^{d}|\nabla\phi_T(0)|^2\phi_T(0)^{2n}\right\rangle\\&\quad\overset{(2.17)}{\lesssim}\langle\phi_T(0)^{2n}\rangle.\end{aligned}\tag{3.11}$$

We now want to apply Step 3 to the right-hand side integral of (3.10), that is, setting $q = \frac{2(n+1)}{n}$, $r = \frac{n+1}{n}$ and $\chi = |\phi_T(0)|^{2(m-1)(n+1)/n}$. Estimate (3.8) involves the number

$$\begin{aligned} &d \max\left\{1, 1 - \frac{1}{r} + \frac{q}{rp}\right\} + (1-d)\frac{q}{r} \\ &\quad = d \max\left\{1, \frac{1}{n+1} + \frac{2}{p}\right\} + 2(1-d). \end{aligned} \tag{3.12}$$

We distinguish the cases $d > 2$ and $d = 2$. For $d > 2$, we have that the number (3.12) is equal to $d + 2(1-d) = 2 - d$ and thus negative for n sufficiently large since $p > 2$. Hence, (3.8) yields

$$\begin{aligned} &\int_{\mathbb{Z}^d} \langle |\phi_T(0)|^{2(m-1)(n+1)/n} |\nabla_z G_T(z,0)|^{2(n+1)/n} \rangle^{n/(n+1)}\, dz \\ &\quad \lesssim \langle |\phi_T(0)|^{2(m-1)(n+1)/n} \rangle^{n/(n+1)} \le \langle |\phi_T(0)|^{2n} \rangle^{(m-1)/n}, \end{aligned}$$

where in the last inequality we appealed to Jensen in probability using

$$\frac{2(m-1)(n+1)}{n} \le \frac{2(n-1)(n+1)}{n} \le 2n.$$

The combination of this with (3.10) and (3.11) yields as desired

$$\begin{aligned} &\int_{\mathbb{Z}^d} \langle \phi_T(0)^{2(m-1)} (|\nabla \phi_T(z)| + 1)^2 |\nabla_z G_T(z,0)|^2 \rangle\, dz \\ &\quad \lesssim \langle \phi(0)^{2n} \rangle^{1/(n+1) + (m-1)/n} + 1 = \langle \phi(0)^{2n} \rangle^{m/n - 1/(n(n+1))} + 1. \end{aligned}$$

We turn to the case $d = 2$. We note that the number (3.12) is zero for n large enough since $p > 2$. Thus, from (3.8), we infer as we did above that

$$\begin{aligned} &\int_{\mathbb{Z}^d} \langle \phi_T(0)^{2(m-1)} (|\nabla \phi_T(z)| + 1)^2 |\nabla_z G_T(z,0)|^2 \rangle\, dz \\ &\quad \lesssim (\ln T)\big(\langle \phi(0)^{2n} \rangle^{m/n - 1/(n(n+1))} + 1\big). \end{aligned}$$

Let us now treat the second term of the left-hand side of (3.2), which differs from the first term only when $m \ge 2$. As for the first term, Hölder's inequality in probability with $(\frac{n+1}{m}, \frac{n+1}{n-m+1})$, the stationarity of $\nabla \phi_T$ and Lemma 2.7 imply

$$\begin{aligned} &\int_{\mathbb{Z}^d} \langle (|\nabla \phi_T(z)| + 1)^{2m} |\nabla_{z_i} G_T(z,0)|^{2m} \rangle\, dz \\ &\quad \lesssim \int_{\mathbb{Z}^d} \big(\langle |\nabla \phi_T(z)|^{2(n+1)} \rangle^{m/(n+1)} + 1\big) \\ &\qquad\qquad \times \langle |\nabla_z G_T(z,0)|^{2(n+1)m/(n-m+1)} \rangle^{(n-m+1)/(n+1)}\, dz \\ &\quad \lesssim \big(\langle \phi_T(0)^{2n} \rangle^{m/(n+1)} + 1\big) \\ &\qquad\quad \times \int_{\mathbb{Z}^d} \langle |\nabla_z G_T(z,0)|^{2(n+1)m/(n-m+1)} \rangle^{(n-m+1)/(n+1)}\, dz. \end{aligned} \tag{3.13}$$

We use (3.8) with $\chi \equiv 1$, $q = \frac{2(n+1)m}{n-m+1}$ and $r = \frac{n+1}{n-m+1}$, in which case we have

$$\begin{aligned}
&d \max\left\{1, 1 - \frac{1}{r} + \frac{q}{rp}\right\} + (1-d)\frac{q}{r} \\
&\quad = d \max\left\{1, \frac{m}{n+1} + \frac{2m}{p}\right\} + (1-d)2m.
\end{aligned} \tag{3.14}$$

We claim that this number is negative for n sufficiently large. Indeed, if $\max\{1, \frac{m}{n+1} + \frac{2m}{p}\} = 1$, then

$$\begin{aligned}
d \max\left\{1, \frac{m}{n+1} + \frac{2m}{p}\right\} + (1-d)2m &= d + 2m(1-d) \\
&= (2m-1)(1-d) + 1 < 0
\end{aligned}$$

since $d \geq 2$ and $m \geq 2$. Otherwise, $\max\{1, \frac{m}{n+1} + \frac{2m}{p}\} = \frac{m}{n+1} + \frac{2m}{p}$, and

$$\begin{aligned}
d \max\left\{1, \frac{m}{n+1} + \frac{2m}{p}\right\} + (1-d)2m &= 2m\left(d\left(\frac{1}{2(n+1)} + \frac{1}{p}\right) + 1 - d\right) \\
&< 2m\left(1 - \frac{d}{2}\right) \leq 0
\end{aligned}$$

for $d \geq 2$ and n large enough since $\frac{1}{p} < \frac{1}{2}$. This shows that (3.14) is negative so that we obtain by (3.8)

$$\int_{\mathbb{Z}^d} \left\langle |\nabla_z G_T(z,0)|^{2(n+1)m/(n-m+1)} \right\rangle^{(n-m+1)/(n+1)} dz \lesssim 1.$$

Combining this with (3.13) yields

$$\begin{aligned}
\int_{\mathbb{Z}^2} \left\langle \left(|\nabla \phi_T(z)| + 1\right)^{2m} |\nabla_z G_T(z,0)|^{2m} \right\rangle dz &\lesssim \langle \phi_T(0)^{2n} \rangle^{m/(n+1)} + 1 \\
&= \langle \phi_T(0)^{2n} \rangle^{m/n - m/(n(n+1))} + 1 \\
&\leq \langle \phi_T(0)^{2n} \rangle^{m/n - 1/(n(n+1))} + 1.
\end{aligned}$$

This concludes the proof of the proposition.

3.2. *Proof of Theorem* 2.1. Let us define the spatial average of a function $h : \mathbb{Z}^d \to \mathbb{R}$ with the mask η_L by

$$\langle\!\langle h \rangle\!\rangle_L := \int_{\mathbb{Z}^d} h(x) \eta_L(x)\, dx,$$

where η_L satisfies

$$\begin{aligned}
&\eta_L : \mathbb{Z}^d \to [0,1] \qquad \operatorname{supp}(\eta_L) \subset (-L, L)^d, \\
&\qquad \int_{\mathbb{Z}^d} \eta_L(x)\, dx = 1, \qquad |\nabla \eta_L| \lesssim L^{-d-1}.
\end{aligned} \tag{3.15}$$

The claim of the theorem is that there exists q depending only on α, β, and d such that

$$\operatorname{var}[\langle\langle T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T + \xi)\rangle\rangle_L] \lesssim L^{-d}\mu_d(T)^q,$$

where $\mu_d(T) = 1$ for $d > 2$ and $\mu_d(T) = \ln T$ for $d = 2$. Since we are not interested in the precise value of q, we adopt the convention that q is a nonnegative exponent which only depends on α, β, and d but which may vary from line to line in the proof.

Starting point is the estimate provided by Lemmas 2.3 and 2.6

$$\begin{aligned}&\operatorname{var}[\langle\langle T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T + \xi)\rangle\rangle_L] \\ &\quad\lesssim \left\langle \sum_e \sup_{a(e)} \left| \frac{\partial}{\partial a(e)} \langle\langle T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T + \xi)\rangle\rangle_L \right|^2 \right\rangle.\end{aligned} \tag{3.16}$$

Step 1. In this step, using the notation $e = [z, z + \mathbf{e}_i]$, we establish the formula

$$\begin{aligned}&\frac{\partial}{\partial a(e)} \langle\langle T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T + \xi)\rangle\rangle_L \\ &\qquad = 2\int_{\mathbb{Z}^d} (\nabla_i\phi_T(z) + \xi_i)\nabla_{z_i} G_T(z, x) \\ &\qquad\qquad \times \left(\sum_{j=1}^d a(x - \mathbf{e}_j, x)\nabla_j^*\eta_L(x)\big(\nabla_j^*\phi_T(x) + \xi_j\big)\right) dx \\ &\qquad\quad + \eta_L(z)(\nabla_i\phi_T + \xi_i)^2(z).\end{aligned} \tag{3.17}$$

Indeed, by definition of $\langle\langle\cdot\rangle\rangle_L$ we have

$$\begin{aligned}&\frac{\partial}{\partial a(e)} \langle\langle T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T + \xi)\rangle\rangle_L \\ &\quad = \int_{\mathbb{Z}^d} \eta_L(x) \frac{\partial}{\partial a(e)} \big(T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T + \xi)\big)(x)\, dx.\end{aligned}$$

We note

$$\begin{aligned}&\frac{\partial}{\partial a(e)} \big(T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T + \xi)\big)(x) \\ &\quad = \bigg(2T^{-1}\phi_T \frac{\partial\phi_T}{\partial a(e)} + 2\nabla \frac{\partial\phi_T}{\partial a(e)} \cdot A(\nabla\phi_T + \xi) \\ &\qquad\qquad + (\nabla\phi_T + \xi)\cdot \frac{\partial A}{\partial a(e)}(\nabla\phi_T + \xi)\bigg)(x) \\ &\quad = 2T^{-1}\bigg(\phi_T \frac{\partial\phi_T}{\partial a(e)}\bigg)(x) + 2\bigg(\nabla \frac{\partial\phi_T}{\partial a(e)} \cdot A(\nabla\phi_T + \xi)\bigg)(x) \\ &\qquad + (\nabla_i\phi_T + \xi_i)^2(z)\delta(x - z),\end{aligned}$$

so that

$$
\begin{aligned}
&\frac{\partial}{\partial a(e)}\langle\!\langle T^{-1}\phi_T^2+(\nabla\phi_T+\xi)\cdot A(\nabla\phi_T+\xi)\rangle\!\rangle_L \\
&\quad=2\int_{\mathbb{Z}^d}\biggl(\eta_L\biggl(T^{-1}\phi_T\,\frac{\partial\phi_T}{\partial a(e)}+\nabla\,\frac{\partial\phi_T}{\partial a(e)}\cdot A(\nabla\phi_T+\xi)\biggr)\biggr)(x)\,dx \\
&\qquad+\eta_L(z)(\nabla_i\phi_T+\xi_i)^2(z).
\end{aligned}
\tag{3.18}
$$

Using the discrete integration by parts formula of Definition 2.3, the first term of the right-hand side of (3.18) turns into

$$
\begin{aligned}
&\int_{\mathbb{Z}^d}\biggl(\eta_L\biggl(T^{-1}\phi_T\,\frac{\partial\phi_T}{\partial a(e)}+\nabla\,\frac{\partial\phi_T}{\partial a(e)}\cdot A(\nabla\phi_T+\xi)\biggr)\biggr)(x)\,dx \\
&\quad=-\int_{\mathbb{Z}^d}\frac{\partial\phi_T}{\partial a(e)}(x)\nabla^*\cdot\bigl(\eta_L A(\nabla\phi_T+\xi)\bigr)(x)\,dx \\
&\qquad+\int_{\mathbb{Z}^d}\biggl(\eta_L T^{-1}\phi_T\,\frac{\partial\phi_T}{\partial a(e)}\biggr)(x)\,dx.
\end{aligned}
\tag{3.19}
$$

We now use the following discrete Leibniz rule:

$$
\begin{aligned}
\nabla^*\cdot\bigl(\eta_L A(\nabla\phi_T+\xi)\bigr)(x)&=\eta_L(x)\bigl(\nabla^*\cdot A(\nabla\phi_T+\xi)\bigr)(x) \\
&\quad+\sum_{j=1}^d\nabla_j^*\eta_L(x)[A(\nabla\phi_T+\xi)]_j(x-\mathbf{e}_j),
\end{aligned}
$$

where $[A(\nabla\phi_T+\xi)]_j$ denotes the jth coordinate of the vector $A(\nabla\phi_T+\xi)$. For notational convenience, we take advantage of the diagonal structure of A (although this is not crucial) to rewrite the latter equality in the form

$$
\begin{aligned}
&\nabla^*\cdot\bigl(\eta_L A(\nabla\phi_T+\xi)\bigr)(x) \\
&\quad=\eta_L(x)\bigl(\nabla^*\cdot A(\nabla\phi_T+\xi)\bigr)(x) \\
&\qquad+\sum_{j=1}^d a(x-\mathbf{e}_j,x)\nabla_j^*\eta_L(x)\bigl(\nabla_j^*\phi_T(x)+\xi_j\bigr).
\end{aligned}
\tag{3.20}
$$

The combination of (3.20) with (3.19) and the use of the equation satisfied by ϕ_T,

$$
T^{-1}\phi_T-\nabla^*\cdot A(\nabla\phi_T+\xi)=0,
$$

yield

$$
\begin{aligned}
&\int_{\mathbb{Z}^d}\biggl(\eta_L\biggl(T^{-1}\phi_T\,\frac{\partial\phi_T}{\partial a(e)}+\nabla\,\frac{\partial\phi_T}{\partial a(e)}\cdot A(\nabla\phi_T+\xi)\biggr)\biggr)(x)\,dx \\
&\quad=-\int_{\mathbb{Z}^d}\frac{\partial\phi_T}{\partial a(e)}(x)\Biggl(\sum_{j=1}^d a(x-\mathbf{e}_j,x)\nabla_j^*\eta_L(x)\bigl(\nabla_j^*\phi_T(x)+\xi_j\bigr)\Biggr)\,dx.
\end{aligned}
$$

Using now Lemma 2.4, this turns into

$$
\begin{aligned}
&\int_{\mathbb{Z}^d} \left(\eta_L\left(T^{-1}\phi_T \frac{\partial \phi_T}{\partial a(e)} + \nabla \frac{\partial \phi_T}{\partial a(e)} \cdot A(\nabla\phi_T + \xi)\right)\right)(x)\,dx \\
&\quad \overset{(2.12)}{=} \int_{\mathbb{Z}^d} (\nabla_i \phi_T(z) + \xi_i) \nabla_{z_i} G_T(z,x) \\
&\qquad\qquad \times \left(\sum_{j=1}^d a(x-\mathbf{e}_j, x) \nabla_j^* \eta_L(x) \big(\nabla_j^*\phi_T(x) + \xi_j\big)\right) dx.
\end{aligned} \tag{3.21}
$$

Inserting (3.21) into (3.18) proves (3.17).

Step 2. In this step, we provide the estimate

$$
\begin{aligned}
&\sup_{a(e)} \left| \frac{\partial}{\partial a(e)} \langle\langle T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T+\xi)\rangle\rangle_L \right| \\
&\quad \lesssim \int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)| |\nabla^* \eta_L(x)| \big(|\nabla^*\phi_T(x)|^2 + |\nabla\phi_T(z)|^2 + 1\big)\,dx \\
&\qquad + \eta_L(z)\big(|\nabla\phi_T(z)|^2 + 1\big).
\end{aligned} \tag{3.22}
$$

Indeed, from Step 1, the boundedness of a, and $|\xi| = 1$, we infer that

$$
\begin{aligned}
&\sup_{a(e)} \left| \frac{\partial}{\partial a(e)} \langle\langle T^{-1}\phi_T^2 + (\nabla\phi_T + \xi)\cdot A(\nabla\phi_T+\xi)\rangle\rangle_L \right| \\
&\quad \lesssim \int_{\mathbb{Z}^d} \Big(\sup_{a(e)} |\nabla_i \phi_T(z)| + 1\Big) \sup_{a(e)} |\nabla_{z_i} G_T(z,x)| |\nabla^*\eta_L(x)| \\
&\qquad\qquad \times \Big(\sup_{a(e)} |\nabla^*\phi_T(x)| + 1\Big)\,dx \\
&\qquad + \eta_L(z)\Big(\sup_{a(e)} |\nabla_i\phi_T(z)|^2 + 1\Big).
\end{aligned} \tag{3.23}
$$

Hence, in the remainder of this step, we have to deal with the suprema over $a(e)$. Recalling that $e = [z, z+\mathbf{e}_i]$, the two following inequalities are consequences of Lemmas 2.5 and 2.4:

$$
\sup_{a(e)} |\nabla_{z_i} G_T(z,x)| \overset{(2.16)}{\lesssim} |\nabla_{z_i} G_T(z,x)| \qquad \text{for all } x \in \mathbb{Z}^d,
$$

$$
\sup_{a(e)} |\nabla_i \phi_T(z)| \overset{(2.14)}{\lesssim} |\nabla_i\phi_T(z)| + 1.
$$

The last inequality we need is

$$
\sup_{a(e)} |\nabla^*\phi_T(x)| \lesssim |\nabla^*\phi_T(x)| + \sup_{a(e)} |\nabla_i\phi_T(z)| + 1 \overset{(2.14)}{\lesssim} |\nabla^*\phi_T(x)| + |\nabla_i\phi_T(z)| + 1.
$$

It is then proved combining the boundedness of a and the following bound on the derivative of $\nabla^*\phi_T(x)$ with respect to $a(e)$:

$$\begin{aligned}
\left|\frac{\partial}{\partial a(e)}\nabla^*\phi_T(x)\right| &= \left|\nabla_x^*\frac{\partial}{\partial a(e)}\phi_T(x)\right| \\
&\overset{(2.12)}{=} \left|\nabla_x^*\big((\nabla_i\phi_T(z)+\xi_i)\nabla_{z_i}G_T(z,x)\big)\right| \\
&= \left|(\nabla_i\phi_T(z)+\xi_i)\nabla_{z_i}\nabla_x^*G_T(z,x)\right| \\
&\le 2\big(|\nabla_i\phi_T(z)|+|\xi_i|\big)\sup_{\mathbb{Z}^d\times\mathbb{Z}^d}|\nabla G_T| \\
&\lesssim |\nabla_i\phi_T(z)|+1,
\end{aligned}$$

where we have used the uniform bound on ∇G_T provided by Corollary 2.3.

Combining these three inequalities with (3.23) yields

$$\begin{aligned}
&\sup_{a(e)}\left|\frac{\partial}{\partial a(e)}\langle\langle T^{-1}\phi_T^2+(\nabla\phi_T+\xi)\cdot A(\nabla\phi_T+\xi)\rangle\rangle_L\right| \\
&\quad\lesssim \int_{\mathbb{Z}^d}(|\nabla\phi_T(z)|+1)|\nabla_z G_T(z,x)||\nabla^*\eta_L(x)| \\
&\qquad\qquad \times\big(|\nabla^*\phi_T(x)|+|\nabla\phi_T(z)|+1\big)\,dx \\
&\qquad + \eta_L(z)\big(|\nabla\phi_T(z)|^2+1\big)
\end{aligned}$$

from which we deduce (3.22).

Step 3. In this step, we argue that

$$\begin{aligned}
&\operatorname{var}[\langle\langle T^{-1}\phi_T^2+(\nabla\phi_T+\xi)\cdot A(\nabla\phi_T+\xi)\rangle\rangle_L] \\
&\lesssim \left\langle\int_{\mathbb{Z}^d}\left(\int_{\mathbb{Z}^d}|\nabla_z G_T(z,x)||\nabla^*\eta_L(x)||\nabla^*\phi_T(x)|^2\,dx\right)^2dz\right\rangle \qquad (3.24)\\
&\quad+\left\langle\int_{\mathbb{Z}^d}\left(\int_{\mathbb{Z}^d}|\nabla_z G_T(z,x)||\nabla^*\eta_L(x)||\nabla\phi_T(z)|^2\,dx\right)^2dz\right\rangle \qquad (3.25)\\
&\quad+\left\langle\int_{\mathbb{Z}^d}\left(\int_{\mathbb{Z}^d}|\nabla_z G_T(z,x)||\nabla^*\eta_L(x)|\,dx\right)^2dz\right\rangle \qquad (3.26)\\
&\quad+\left\langle\int_{\mathbb{Z}^d}\eta_L(z)^2\big(|\nabla\phi_T(z)|^2+1\big)^2\,dz\right\rangle. \qquad (3.27)
\end{aligned}$$

Indeed, inserting (3.22) in (3.16) yields

$$\begin{aligned}
&\operatorname{var}[\langle\langle T^{-1}\phi_T^2+(\nabla\phi_T+\xi)\cdot A(\nabla\phi_T+\xi)\rangle\rangle_L] \\
&\quad\lesssim\left\langle\sum_e\left(\int_{\mathbb{Z}^d}|\nabla_z G_T(z,x)||\nabla^*\eta_L(x)|\big(|\nabla^*\phi_T(x)|^2+|\nabla\phi_T(z)|^2+1\big)\,dx\right)^2\right\rangle \\
&\qquad+\left\langle\sum_e\eta_L^2(z)\big(|\nabla\phi_T(z)|^2+1\big)^2\right\rangle.
\end{aligned}$$

We then use Young's inequality in the first term of the right-hand side of this inequality and we replace the sum $\sum_e$ over edges $[z, z + \mathbf{e}_i]$ by d times the sum over $z \in \mathbb{Z}^d$ to establish this step.

It now remains to estimate the terms (3.24), (3.25), (3.26) and (3.27) to conclude the proof of the theorem.

Step 4. Estimate of (3.27):

$$\left\langle \int_{\mathbb{Z}^d} \eta_L(z)^2 \big(|\nabla\phi_T(z)|^2 + 1\big)^2 \, dz \right\rangle \lesssim \mu_d(T)^q L^{-d}. \tag{3.28}$$

Indeed, by stationarity we have

$$\langle |\nabla\phi_T(z)|^4\rangle \lesssim \sum_{i=1}^{d} \langle |\phi_T(z+\mathbf{e}_i)|^4 + |\phi_T(z)|^4\rangle = 2d\langle \phi_T(0)^4\rangle,$$

so that

$$\begin{aligned} \left\langle \int_{\mathbb{Z}^d} \eta_L(z)^2 \big(|\nabla\phi_T(z)|^2 + 1\big)^2 \, dz \right\rangle &\lesssim \left\langle \int_{\mathbb{Z}^d} \eta_L(z)^2 \big(|\nabla\phi_T(z)|^4 + 1\big) \, dz \right\rangle \\ &= \int_{\mathbb{Z}^d} \eta_L(z)^2 \big(\langle |\nabla\phi_T(z)|^4\rangle + 1\big) \, dz \\ &\lesssim \big(\langle \phi_T(0)^4\rangle + 1\big) \int_{\mathbb{Z}^d} \eta_L(z)^2 \, dz. \end{aligned}$$

On the one hand, it follows from Proposition 2.1 that

$$\langle \phi_T(0)^4\rangle \lesssim \mu_d(T)^q,$$

with $q = \gamma(4)$. On the other hand, it follows from (3.15) that

$$\int_{\mathbb{Z}^d} \eta_L(z)^2 \, dz \lesssim L^{-d}.$$

This establishes Step 4.

Step 5. Estimate of (3.26):

$$\left\langle \int_{\mathbb{Z}^d} \left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^*\eta_L(x)| \, dx \right)^2 dz \right\rangle \lesssim \mu_d(T)^q L^{-d}. \tag{3.29}$$

We expand the square

$$\begin{aligned} &\left\langle \int_{\mathbb{Z}^d} \left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^*\eta_L(x)| \, dx \right)^2 dz \right\rangle \\ &\quad = \left\langle \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} |\nabla^*\eta_L(x)||\nabla^*\eta_L(x')||\nabla_z G_T(z,x)||\nabla_z G_T(z,x')| \, dx \, dx' \, dz \right\rangle \\ &\quad = \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} |\nabla^*\eta_L(x)||\nabla^*\eta_L(x')| \int_{\mathbb{Z}^d} \langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')|\rangle \, dz \, dx \, dx'. \end{aligned}$$

We then use Cauchy–Schwarz' inequality in probability and the stationarity of G_T:

$$\begin{aligned}
&\langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')|\rangle \\
&\quad \le \langle |\nabla_z G_T(z,x)|^2\rangle^{1/2}\langle |\nabla_z G_T(z,x')|^2\rangle^{1/2} \\
&\quad = \langle |\nabla_z G_T(z-x,0)|^2\rangle^{1/2}\langle |\nabla_z G_T(z-x',0)|^2\rangle^{1/2}.
\end{aligned}$$

Hence, with the notation

$$h(y) := \langle |\nabla_y G_T(y,0)|^2\rangle^{1/2},$$

we have by definition of η_L:

$$\begin{aligned}
&\left\langle \int_{\mathbb{Z}^d}\left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^*\eta_L(x)|\,dx\right)^2 dz\right\rangle \\
&\quad \le \int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} |\nabla^*\eta_L(x)||\nabla^*\eta_L(x')|\int_{\mathbb{Z}^d} h(z-x)h(z-x')\,dz\,dx\,dx' \\
&\quad \lesssim L^{-2(d+1)}\int_{|x|\le L}\int_{|x'|\le L}\int_{\mathbb{Z}^d} h(z-x)h(z-x')\,dz\,dx\,dx' \\
&\quad = L^{-2(d+1)}\int_{|x|\le L}\int_{|x'|\le L}\int_{\mathbb{Z}^d} h(z')h(z'+x-x')\,dz'\,dx\,dx' \\
&\quad \le L^{-d-2}\int_{|y|\le 2L}\int_{\mathbb{Z}^d} h(z')h(z'-y)\,dz'\,dy.
\end{aligned}$$

We note that

$$\int_{R<|y|\le 2R} h^2(y)\,dy = \left\langle \int_{R<|y|\le 2R} |\nabla_y G_T(y,0)|^2\,dy\right\rangle.$$

On the one hand, for $R \gg 1$ we have according to Lemma 2.9 (for $q=2$)

$$\begin{aligned}
\text{for } d=2 \qquad &\int_{R<|y|\le 2R} h^2(y)\,dy \lesssim R^{2-2}\min\{1,\sqrt{T}R^{-1}\}^2 \\
&\qquad\qquad = \min\{1,\sqrt{T}R^{-1}\}^2, \\
\text{for } d>2 \qquad &\int_{R<|y|\le 2R} h^2(y)\,dy \lesssim R^d(R^{1-d})^2 \\
&\qquad\qquad = R^{2-d}.
\end{aligned}$$

On the other hand, for $R\sim 1$, Corollary 2.3 implies

$$\text{for } d\ge 2 \qquad \int_{|y|\le R} h^2(y)\,dy \lesssim 1.$$

Hence, we are in position to apply Lemma 2.10, which yields as desired

$$\int_{|y|\le 2L}\int_{\mathbb{Z}^d} h(z')h(z'-y)\,dz'\,dy \lesssim L^2\mu_d(T).$$

Note that for $d = 2$, we have used the elementary fact that $\max\{1, \ln \sqrt{T} L^{-1}\} \lesssim \ln T$.

Step 6. Estimate of (3.25):

$$\left\langle \int_{\mathbb{Z}^d} \left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^* \eta_L(x)||\nabla \phi_T(z)|^2 \, dx \right)^2 dz \right\rangle \lesssim \mu_d(T)^q L^{-d}. \tag{3.30}$$

As in Step 5,

$$\begin{aligned}
&\left\langle \int_{\mathbb{Z}^d} \left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^* \eta_L(x)||\nabla \phi_T(z)|^2 \, dx \right)^2 dz \right\rangle \\
&\quad = \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} |\nabla^* \eta_L(x)||\nabla^* \eta_L(x')| \\
&\qquad \times \int_{\mathbb{Z}^d} \langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')||\nabla \phi_T(z)|^4 \rangle \, dz \, dx \, dx'.
\end{aligned}$$

This time, we use Hölder's inequality with $(p, p, \frac{p}{p-2})$ in probability (where $p > 2$ is the exponent in Lemma 2.9):

$$\begin{aligned}
&\langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')||\nabla \phi_T(z)|^4 \rangle \\
&\quad \le \langle |\nabla_z G_T(z,x)|^p \rangle^{1/p} \langle |\nabla_z G_T(z,x')|^p \rangle^{1/p} \big\langle |\nabla \phi_T(z)|^{4p/(p-2)} \big\rangle^{(p-2)/p}.
\end{aligned}$$

By stationarity of G_T and ϕ_T, we obtain with Proposition 2.1

$$\begin{aligned}
&\langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')||\nabla \phi_T(z)|^4 \rangle \\
&\quad \lesssim \mu_d(T)^q \langle |\nabla_z G_T(z-x,0)|^p \rangle^{1/p} \langle |\nabla_z G_T(z-x',0)|^p \rangle^{1/p}.
\end{aligned}$$

Hence, with the notation

$$h(y) := \langle |\nabla_y G_T(y,0)|^p \rangle^{1/p},$$

by definition of η_L:

$$\begin{aligned}
&\left\langle \int_{\mathbb{Z}^d} \left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^* \eta_L(x)||\nabla \phi_T(z)|^2 \, dx \right)^2 dz \right\rangle \\
&\quad \lesssim \mu_d(T)^q \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} |\nabla^* \eta_L(x)||\nabla^* \eta_L(x')| \int_{\mathbb{Z}^d} h(z-x) h(z-x') \, dz \, dx \, dx' \\
&\quad \lesssim \mu_d(T)^q L^{-d-2} \int_{|y| \le 2L} \int_{\mathbb{Z}^d} h(z') h(z'-y) \, dz' \, dy.
\end{aligned}$$

As in Step 5, we shall establish that for $R \gg 1$

$$\begin{aligned}
&\text{for } d = 2 \qquad \int_{R < |y| \le 2R} h^2(y) \, dy \lesssim \min\{1, \sqrt{T} R^{-1}\}^2, \\
&\text{for } d > 2 \qquad \int_{R < |y| \le 2R} h^2(y) \, dy \lesssim R^{2-d},
\end{aligned} \tag{3.31}$$

and for $R \sim 1$

$$\text{for } d \geq 2 \qquad \int_{|y| \leq R} h^2(y)\, dy \lesssim 1. \tag{3.32}$$

Once this is done, Lemma 2.10 implies as desired

$$\int_{|y| \leq 2L} \int_{\mathbb{Z}^d} h(z')h(z'-y)\, dz'\, dy \lesssim L^2 \mu_d(T),$$

using in addition that $\max\{1, \ln \sqrt{T} L^{-1}\} \lesssim \ln T$ for $d = 2$. As above, (3.32) is a direct consequence of Corollary 2.3. We now deal with (3.31). Note that according to Lemma 2.9, we have for $R \gg 1$

$$\begin{aligned} &\text{for } d = 2 \qquad \int_{R < |y| \leq 2R} h^p(y)\, dy \lesssim R^{2-p} \min\{1, \sqrt{T} R^{-1}\}^p, \\ &\text{for } d > 2 \qquad \int_{R < |y| \leq 2R} h^p(y)\, dy \lesssim R^d (R^{1-d})^p. \end{aligned} \tag{3.33}$$

We now argue that this yields (3.31). Indeed, by Jensen's inequality

$$\begin{aligned} &\left(R^{-d} \int_{R < |x| \leq 2R} h^2(x)\, dx\right)^{1/2} \\ &\quad \leq \left(R^{-d} \int_{R < |x| \leq 2R} h^p(x)\, dx\right)^{1/p} \\ &\quad \overset{(3.33)}{\lesssim} \begin{cases} (R^{-2} R^{2-p} \min\{1, \sqrt{T} R^{-1}\}^p)^{1/p}, & d = 2, \\ (R^{-d} R^d (R^{1-d})^p)^{1/p}, & d > 2, \end{cases} \\ &\quad = \begin{cases} R^{-1} \min\{1, \sqrt{T} R^{-1}\}, & d = 2, \\ R^{1-d}, & d > 2, \end{cases} \end{aligned}$$

which implies (3.31).

Step 7. Estimate of (3.24):

$$\left\langle \int_{\mathbb{Z}^d} \left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^* \eta_L(x)||\nabla^* \phi_T(x)|^2\, dx \right)^2 dz \right\rangle \lesssim \mu_d(T)^q L^{-d}. \tag{3.34}$$

As in Steps 5 and 6,

$$\begin{aligned} &\left\langle \int_{\mathbb{Z}^d} \left(\int_{\mathbb{Z}^d} |\nabla_z G_T(z,x)||\nabla^* \eta_L(x)||\nabla^* \phi_T(x)|^2\, dx \right)^2 dz \right\rangle \\ &\quad = \int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} |\nabla^* \eta_L(x)||\nabla^* \eta_L(x')| \\ &\qquad\quad \times \int_{\mathbb{Z}^d} \langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')| \\ &\qquad\qquad \times |\nabla^* \phi_T(x)|^2 |\nabla^* \phi_T(x')|^2 \rangle\, dz\, dx\, dx'. \end{aligned}$$

Hölder's inequality with $(p, p, \frac{2p}{p-2}, \frac{2p}{p-2})$ in probability (where $p > 2$ is the exponent in Lemma 2.9) then yields

$$\begin{aligned}
&\langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')||\nabla^*\phi_T(x)|^2|\nabla^*\phi_T(x')|^2\rangle \\
&\quad\le \langle |\nabla_z G_T(z,x)|^p\rangle^{1/p}\langle |\nabla_z G_T(z,x')|^p\rangle^{1/p} \\
&\qquad \times \big\langle |\nabla^*\phi_T(x)|^{4p/(p-2)}\big\rangle^{(p-2)/(2p)}\big\langle |\nabla^*\phi_T(x')|^{4p/(p-2)}\big\rangle^{(p-2)/(2p)}.
\end{aligned}$$

The stationarity of G_T and ϕ_T, and Proposition 2.1 show

$$\begin{aligned}
&\langle |\nabla_z G_T(z,x)||\nabla_z G_T(z,x')||\nabla^*\phi_T(x)|^2|\nabla^*\phi_T(x')|^2\rangle \\
&\quad\lesssim \mu_d(T)^q\langle |\nabla_z G_T(z-x,0)|^p\rangle^{1/p}\langle |\nabla_z G_T(z-x',0)|^p\rangle^{1/p}.
\end{aligned}$$

We may now conclude as in Step 6.

The theorem follows from the combination of Step 3 with (3.28), (3.29), (3.30) and (3.34).

Step 8. Extension to the energy density of the corrector field for $d > 2$.

Let $A_{L,\infty}$ be defined by

$$\xi\cdot A_{L,\infty}\xi := \int_{\mathbb{Z}^d}\big(\nabla\phi(x)+\xi\big)\cdot A(x)\big(\nabla\phi(x)+\xi\big)\mu_L(x)\,dx,$$

for all $L \gg 1$. The claim is

$$\mathrm{var}[\xi\cdot A_{L,\infty}\xi] \lesssim L^{-d},$$

for $d > 2$. It is proved using (2.5) provided we show

$$\mathrm{var}[\xi\cdot A_{L,\infty}\xi] \le \liminf_{T\to\infty}\mathrm{var}[\xi\cdot A_{L,T}\xi]. \tag{3.35}$$

As we shall prove, the following two convergences hold:

$$\begin{aligned}
&\Big\langle\int_{\mathbb{Z}^d}\big(\xi+\nabla\phi_T(x)\big)\cdot A(x)\big(\xi+\nabla\phi_T(x)\big)\mu_L(x)\,dx\Big\rangle \\
&\quad\to \Big\langle\int_{\mathbb{Z}^d}\big(\xi+\nabla\phi(x)\big)\cdot A(x)\big(\xi+\nabla\phi(x)\big)\mu_L(x)\,dx\Big\rangle \\
&\quad= \xi\cdot A_{\mathrm{hom}}\xi,
\end{aligned}\tag{3.36}$$

which in fact amounts to $\langle(\xi+\nabla\phi_T)\cdot A(\xi+\nabla\phi_T)\rangle \to \langle(\xi+\nabla\phi)\cdot A(\xi+\nabla\phi)\rangle$ by stationarity, and

$$\begin{aligned}
&\int_{\mathbb{Z}^d}\big(\xi+\nabla\phi_T(x)\big)\cdot A(x)\big(\xi+\nabla\phi_T(x)\big)\mu_L(x)\,dx \\
&\quad\rightharpoonup \int_{\mathbb{Z}^d}\big(\xi+\nabla\phi(x)\big)\cdot A(x)\big(\xi+\nabla\phi(x)\big)\mu_L(x)\,dx
\end{aligned}\tag{3.37}$$

weakly in probability.

We may now conclude the proof of (3.35). Expanding the variance, one has

$$\operatorname{var}[\xi \cdot A_{L,T}\xi] = \left\langle \left(\int_{\mathbb{Z}^d} (\xi + \nabla\phi_T(x)) \cdot A(x)(\xi + \nabla\phi_T(x))\mu_L(x)\,dx \right)^2 \right\rangle$$
$$- \left\langle \int_{\mathbb{Z}^d} (\xi + \nabla\phi_T(x)) \cdot A(x)(\xi + \nabla\phi_T(x))\mu_L(x)\,dx \right\rangle^2.$$

By (3.36), the second term of the right-hand side converges to $(\xi \cdot A_{\mathrm{hom}}\xi)^2$ as $T \to \infty$, whereas by lower-semicontinuity of quadratic functionals, (3.37) implies that

$$\left\langle \left(\int_{\mathbb{Z}^d} (\xi + \nabla\phi(x)) \cdot A(x)(\xi + \nabla\phi(x))\mu_L(x)\,dx \right)^2 \right\rangle$$
$$\leq \liminf_{T\to\infty} \left\langle \left(\int_{\mathbb{Z}^d} (\xi + \nabla\phi_T(x)) \cdot A(x)(\xi + \nabla\phi_T(x))\mu_L(x)\,dx \right)^2 \right\rangle,$$

which shows (3.35).

It remains to prove (3.36) and (3.37). Note that by stationarity, (3.36) is a consequence of

$$\lim_{T\to\infty} |A_T - A_{\mathrm{hom}}| = 0, \tag{3.38}$$

for all $d \geq 2$, where $\xi \cdot A_T\xi := \langle (\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T) \rangle$. Starting point for (3.38) is the definition of A_T and A_{hom} from which we deduce

$$\begin{aligned} \xi \cdot (A_T - A_{\mathrm{hom}})\xi &= \langle (\xi + \nabla\phi_T) \cdot A(\xi + \nabla\phi_T) - (\xi + \nabla\phi) \cdot A(\xi + \nabla\phi) \rangle \\ &= \langle \xi \cdot A(\nabla\phi_T - \nabla\phi) \rangle + \langle \nabla\phi_T \cdot A(\xi + \nabla\phi_T) \rangle \\ &\quad - \langle \nabla\phi \cdot A(\xi + \nabla\phi) \rangle. \end{aligned} \tag{3.39}$$

Let us treat each term separately. For the second term, we shall argue that (2.3) yields: for every stationary field $\zeta : \mathbb{Z}^d \to \mathbb{R}$ such that $\langle \zeta^2 \rangle < \infty$, one has

$$T^{-1}\langle \phi_T \zeta \rangle + \langle \nabla\zeta \cdot A(\xi + \nabla\phi_T) \rangle = 0, \tag{3.40}$$

so that one may replace the second term of the right-hand side of (3.39) by $-T^{-1}\langle \phi_T^2 \rangle$. For the first term, we shall use the following weak convergence of $\nabla\phi_T(x)$ to $\nabla\phi(x)$ in probability: for every random variable χ taking values in $\mathbb{R}^d$ with $\langle |\chi|^2 \rangle < \infty$, one has for all $x \in \mathbb{Z}^d$,

$$\lim_{T\to\infty} \langle \chi \cdot (\nabla\phi_T(x) - \nabla\phi(x)) \rangle = 0, \tag{3.41}$$

so that taking $x = 0$ and $\chi \equiv A(0)\xi$ shows that the first term in the right-hand side of (3.39) vanishes as $T \uparrow \infty$. For the last term, combining (3.41) and (3.40), we will prove

$$\langle \nabla\phi \cdot A(\xi + \nabla\phi) \rangle = 0. \tag{3.42}$$

We directly draw the conclusion: the combination of (3.39), (3.41), (3.40) and (3.42) shows that

$$\limsup_{T\to\infty} |\xi \cdot (A_T - A_{\mathrm{hom}})\xi| = \limsup_{T\to\infty} T^{-1}\langle \phi_T^2\rangle,$$

which implies (3.38) by Proposition 2.1.

We give the arguments for (3.40), (3.41) and (3.42) for the reader's convenience (we could also directly appeal to [13]). Multiplying the defining equation for ϕ_T by ζ yields

$$T^{-1}(\phi_T\zeta)(z) - \big(\nabla^* \cdot A(\xi + \nabla\phi_T)\big)(z)\zeta(z) = 0. \tag{3.43}$$

We then use the discrete Leibniz rule in the form

$$\begin{aligned}\nabla^* \cdot \big(\zeta A(\xi + \nabla\phi_T)\big)(z) &= \big(\nabla^* \cdot A(\xi + \nabla\phi_T)\big)(z)\zeta(z)\\ &\quad + \sum_{j=1}^{d} \nabla_j^*\zeta(z)[A(\xi + \nabla\phi_T)(z - \mathbf{e}_j)]_j.\end{aligned} \tag{3.44}$$

Since $\nabla\phi_T$, ζ and A are jointly stationary random fields, the expectation of the left-hand side of (3.44) vanishes, and

$$\begin{aligned}\big\langle\big(\nabla^* \cdot A(\xi + \nabla\phi_T)\big)(z)\zeta(z)\big\rangle &= -\left\langle \sum_{j=1}^{d} \nabla_j^*\zeta(z)\big[A\big(\xi + \nabla\phi_T(z - \mathbf{e}_j)\big)\big]_j\right\rangle\\ &= -\langle \nabla\zeta \cdot A(\xi + \nabla\phi_T)\rangle,\end{aligned} \tag{3.45}$$

noting that $\nabla_j^*\zeta(z) = \nabla_j\zeta(z - \mathbf{e}_j)$. We then take the expectation of (3.43) and use (3.45) to obtain (3.40).

We recall the standard a priori estimate which one derives from (3.40):

$$\langle T^{-1}\phi_T(x)^2 + |\nabla\phi_T(x)|^2\rangle \lesssim 1.$$

Since the left-hand side does not depend on x by stationarity, there exists $g : \mathbb{Z}^d \to \mathbb{R}^d$ such that up to extraction, $\nabla\phi_T(x)$ converges to $g(x)$ weakly in probability for all $x \in \mathbb{Z}^d$. By construction, g is a gradient field, and is jointly stationary with A. By the boundedness of $\langle T^{-1}\phi_T^2\rangle^{1/2}$, one may pass to the limit in (3.40), and obtain for every stationary field ζ

$$\langle \nabla\zeta \cdot A(\xi + \nabla\phi)\rangle = 0. \tag{3.46}$$

As noticed by Künnemann in [13], this characterizes the gradient of the corrector, so that $g \equiv \nabla\phi$. This proves (3.41) by definition of weak convergence in probability.

We then use (3.46) for $\zeta = \phi_T$ and pass to the limit $T \uparrow \infty$ in (3.46) by the weak convergence (3.41). This proves (3.42).

We finally turn to the proof of (3.37). By definition, (3.37) is proved if for all bounded random variables χ,

$$\begin{aligned}\lim_{T\to\infty}\Big\langle \chi \int_{\mathbb{Z}^d} \big(\xi+\nabla\phi_T(x)\big)\cdot A(x)\big(\xi+\nabla\phi_T(x)\big)\mu_L(x)\,dx\Big\rangle\\ =\Big\langle \chi \int_{\mathbb{Z}^d} \big(\xi+\nabla\phi(x)\big)\cdot A(x)\big(\xi+\nabla\phi(x)\big)\mu_L(x)\,dx\Big\rangle.\end{aligned} \tag{3.47}$$

W.l.o.g. we may assume that χ takes values in [0, 1]. By lower-semicontinuity of quadratic functionals in probability, and since $\chi \geq 0$, the weak convergence (3.41) of $\nabla\phi_T(x)$ to $\nabla\phi(x)$ in L^2 in probability for all $x \in \mathbb{Z}^d$ yields

$$\begin{aligned}&\liminf_{T\to\infty}\Big\langle \chi \int_{\mathbb{Z}^d} \big(\xi+\nabla\phi_T(x)\big)\cdot A(x)\big(\xi+\nabla\phi_T(x)\big)\mu_L(x)\,dx\Big\rangle\\ &\quad=\int_{\mathbb{Z}^d}\mu_L(x)\Big(\liminf_{T\to\infty}\big\langle\chi\big(\xi+\nabla\phi_T(x)\big)\cdot A(x)\big(\xi+\nabla\phi_T(x)\big)\big\rangle\Big)\,dx\\ &\quad\geq\int_{\mathbb{Z}^d}\mu_L(x)\big\langle\chi\big(\xi+\nabla\phi_T(x)\big)\cdot A(x)\big(\xi+\nabla\phi_T(x)\big)\big\rangle\,dx\\ &\quad=\Big\langle \chi \int_{\mathbb{Z}^d} \big(\xi+\nabla\phi(x)\big)\cdot A(x)\big(\xi+\nabla\phi(x)\big)\mu_L(x)\,dx\Big\rangle.\end{aligned}$$

Likewise,

$$\begin{aligned}&\liminf_{T\to\infty}\Big\langle (1-\chi) \int_{\mathbb{Z}^d} \big(\xi+\nabla\phi_T(x)\big)\cdot A(x)\big(\xi+\nabla\phi_T(x)\big)\mu_L(x)\,dx\Big\rangle\\ &\quad\geq\Big\langle (1-\chi) \int_{\mathbb{Z}^d} \big(\xi+\nabla\phi(x)\big)\cdot A(x)\big(\xi+\nabla\phi(x)\big)\mu_L(x)\,dx\Big\rangle\end{aligned}$$

since $1-\chi \geq 0$. Combined with the convergence of the expectation (3.36) and the trivial identity $1 = \chi + (1-\chi)$, these two inequalities imply (3.47) for χ taking values in [0, 1], and therefore (3.37) as desired.

4. Proofs of the estimates on the Green functions.

Before addressing the proofs proper, let us make a general comment. In what follows, we shall replace the classical Leibniz rule by its discrete counterpart. Although they are essentially the same, the expressions that appear are more intricate in the discrete case. In order to keep the proofs clear, we first present the arguments using the classical Leibniz rule (though it does not hold at the discrete level) and we later give a separate argument to show that the various results still hold with the true discrete version.

4.1. *Proof of Lemma* 2.8. Without loss of generality, we may assume $y = 0$ and suppress the y-dependance of G_T in our notation. We will first give the proof in the continuum case (i.e., using the classical Leibniz rule) and then sketch the modifications arising from the discreteness.

We first argue that for any d,

$$T^{-1}\int_{\mathbb{Z}^d} G_{T,M}^2\,dx + \int_{\mathbb{Z}^d} |\nabla G_{T,M}|^2\,dx \lesssim M, \tag{4.1}$$

where for $0 < M < \infty$, $G_{T,M}$ denotes the following truncated version of G_T

$$G_{T,M} = \min\{G_T, M\} \ge 0.$$

Indeed, we consider $T^{-1}G_T - \nabla^* \cdot A\nabla G_T = \delta$ in its weak form, that is,

$$T^{-1}\int_{\mathbb{Z}^d} \zeta G_T\,dx + \int_{\mathbb{Z}^d} \nabla\zeta \cdot A\nabla G_T\,dx = \zeta(0) \tag{4.2}$$

and select $\zeta = G_{T,M}$. Since $G_{T,M}G_T \ge G_{T,M}^2$ and provided that $\nabla G_{T,M} \cdot A\nabla G_T \ge \nabla G_{T,M} \cdot A\nabla G_{T,M}$, we obtain (4.1) by uniform ellipticity. Indeed, since A is diagonal,

$$\begin{aligned}
&\nabla G_{T,M} \cdot A\nabla G_T(x)\\
&\quad = \sum_{i=1}^d a(x+\mathbf{e}_i, x)\big(G_{T,M}(x+\mathbf{e}_i) - G_{T,M}(x)\big)\big(G_T(x+\mathbf{e}_i) - G_T(x)\big)\\
&\quad \ge \sum_{i=1}^d a(x+\mathbf{e}_i, x)\big(G_{T,M}(x+\mathbf{e}_i) - G_{T,M}(x)\big)^2\\
&\quad \ge \alpha|\nabla G_{T,M}(x)|^2.
\end{aligned}$$

Step 1. Proof of (i) for $d > 2$.

Following [7], Theorem 1.1, we argue that (4.1) implies a weak-$L^{d/(d-2)}$ estimate, that is,

$$\mathcal{L}_d(\{G_T \ge M\}) \lesssim M^{-d/(d-2)}. \tag{4.3}$$

For this purpose, we appeal to Sobolev's inequality in $\mathbb{Z}^d$, that is,

$$\left(\int_{\mathbb{Z}^d} G_{T,M}^{2d/(d-2)}\,dx\right)^{(d-2)/(2d)} \lesssim \left(\int_{\mathbb{Z}^d} |\nabla G_{T,M}|^2\,dx\right)^{1/2},$$

which is a consequence of [22], Lemma 2.1 (when "$n \to \infty$"), or [3], Theorem 4.4 (when "$r \to \infty$"). Via Chebyshev's inequality and (4.1), this yields

$$M\mathcal{L}_d(\{G_T \ge M\})^{(d-2)/(2d)} \lesssim M^{1/2},$$

which is (4.3).

We now argue that the weak-$L^{d/(d-2)}$ estimate (4.3) in $\mathbb{Z}^d$ yields a strong L^q-estimate on balls $\{|x| \le R\}$ for $1 \le q < \frac{d}{d-2}$. More precisely, we have

$$\int_{|x|\le R} G_T^q\,dx \lesssim R^d(R^{2-d})^q. \tag{4.4}$$

Indeed, we have on the one hand

$$
\begin{aligned}
\int_{G_T>M} G_T^q\,dx \;&=\; q\int_M^\infty \mathcal{L}_d(\{G_T>M'\}){M'}^{q-1}\,dM' \\
&\qquad + M^q\mathcal{L}_d(\{|G_T|>M\}) \\
&\overset{(4.3)}{\lesssim} M^{q-d/(d-2)},
\end{aligned}
\tag{4.5}
$$

where we have used $q<\frac{d}{d-2}$. On the other hand, we have trivially

$$
\int_{\{G_T\le M\}\cap\{|x|\le R\}} G_T^q\,dx \lesssim R^d M^q. \tag{4.6}
$$

With the choice of $M=R^{2-d}$, the combination of (4.5) and (4.6) yields (4.4).

In order to increase the exponent q in (4.4), one combines a Cacciopoli estimate[1] for monotone functions of G_T with a Poincaré–Sobolev estimate to obtain a "reverse Hölder" inequality (as in the proof of Harnack's inequality, see [8], Chapter 4, Method II). We start with the Cacciopoli estimate, that is,

$$
\int_{2R\le|x|\le 4R}|\nabla G_T^{q/2}|^2\,dx \lesssim R^{-2}\int_{R\le|x|\le 8R} G_T^q\,dx \tag{4.7}
$$

for all $1<q<\infty$. For that purpose, we test (4.2) with $\zeta=\eta^2 G_T^{q-1}$, where the spatial cut-off function η has the properties

$$
\begin{aligned}
&\eta\equiv 1 \qquad \text{in } \{2R\le|x|\le 4R\},\\
&\eta\equiv 0 \qquad \text{outside } \{R\le|x|\le 8R\}, \qquad |\nabla\eta|\lesssim R^{-1}, \qquad 0\le\eta\le 1.
\end{aligned}
\tag{4.8}
$$

This yields

$$
T^{-1}\int_{\mathbb{Z}^d}\eta^2 G_T^q\,dx + \int_{\mathbb{Z}^d}\nabla(\eta^2 G_T^{q-1})\cdot A\nabla G_T\,dx = 0. \tag{4.9}
$$

Since by the uniform ellipticity of A, there exists a generic constant $C<\infty$ (which only depends on q, α, β) such that

$$
\begin{aligned}
&\nabla(\eta^2 G_T^{q-1})\cdot A\nabla G_T \\
&\quad = (q-1)\eta^2 G_T^{q-2}\nabla G_T\cdot A\nabla G_T + 2\eta G_T^{q-1}\nabla\eta\cdot A\nabla G_T \\
&\quad \overset{\text{Young}}{\ge} C^{-1}\eta^2 G_T^{q-2}|\nabla G_T|^2 - C G_T^q|\nabla\eta|^2 \\
&\quad \gtrsim C^{-1}\eta^2|\nabla G_T^{q/2}|^2 - C G_T^q|\nabla\eta|^2,
\end{aligned}
$$

we obtain

$$
\int_{\mathbb{Z}^d}\eta^2|\nabla G_T^{q/2}|^2\,dx \lesssim \int_{\mathbb{Z}^d} G_T^q|\nabla\eta|^2\,dx.
$$

[1] This is the only place where we use the Leibniz rule.

In view of the properties (4.8) of η, this yields (4.7) for $d > 2$.

We now derive the "reverse Hölder" inequality

$$\begin{aligned}(4.10)\qquad &\left(R^{-d}\int_{2R\le|x|\le 4R} G_T^{qd/(d-2)}\,dx\right)^{(d-2)/(qd)}\\ &\qquad\lesssim \left(R^{-d}\int_{R\le|x|\le 8R} G_T^{q}\,dx\right)^{1/q}.\end{aligned}$$

For that purpose, we appeal to the Poincaré–Sobolev estimate (see [22], Lemma 2.1, or [3], Theorem 4.4) on the annulus $\{2R \le |x| \le 4R\}$:

$$\begin{aligned}\left(R^{-d}\int_{2R\le|x|\le 4R} |u|^{2d/(d-2)}\,dx\right)^{(d-2)/(2d)} &\lesssim \left(R^{2-d}\int_{2R\le|x|\le 4R} |\nabla u|^2\,dx\right)^{1/2}\\ &\quad+\left(R^{-d}\int_{2R\le|x|\le 4R} |u|^2\,dx\right)^{1/2}.\end{aligned}$$

We apply the latter to $u = G_T^{q/2}$:

$$\begin{aligned}\left(R^{-d}\int_{2R\le|x|\le 4R} G_T^{qd/(d-2)}\,dx\right)^{(d-2)/(qd)} &\lesssim \left(R^{2-d}\int_{2R\le|x|\le 4R} |\nabla G_T^{q/2}|^2\,dx\right)^{1/q}\\ &\quad+\left(R^{-d}\int_{2R\le|x|\le 4R} G_T^{q}\,dx\right)^{1/q}.\end{aligned}$$

The combination of this with (4.7) yields (4.10).

We now may conclude in the case of $d > 2$: indeed, (4.10) allows us to iteratively increase the integrability q in multiplicative increments of $\frac{d}{d-2}$ in the estimate (4.4). Since any $p < \infty$ can be reached in finite multiplicative increments starting from a $1 < q < \frac{d}{d-2}$, the side effect that the annuli get dyadically larger at every step does not matter qualitatively (in this sense, the above argument is much less subtle than the proof of the Harnack inequality). This proves (2.21).

Step 2. Proof of (i) for $d = 2$.

We now tackle the case of $d = 2$, which in fact amounts to the L^1-BMO estimate

$$(4.11)\qquad \left(R^{-2}\int_{|x|\le R} |u-\bar u_{\{|x|\le R\}}|^q\,dx\right)^{1/q} \lesssim \int_{\mathbb{Z}^2} |f|\,dx$$

for

$$(4.12)\qquad T^{-1}u - \nabla^*\cdot A\nabla u = f,$$

where $\bar u_{\{|x|\le R\}}$ denotes the average of u on the ball of radius R. We fix an exponent $q < \infty$ and a radius $1 \ll R < \infty$ and assume w.l.o.g.

$$(4.13)\qquad \bar u_{\{|x|\le R\}} = 0.$$

As in (4.1), we have

$$\int_{|x|\le R} |\nabla u_M|^2\,dx \lesssim M\int_{\mathbb{Z}^2} |f|\,dx. \tag{4.14}$$

As opposed to the case of $d > 2$, this is the only time we use the equation (4.12).

Estimate (4.14) is used in connection with the Poincaré–Sobolev inequality with mean value zero, that is,

$$\left(R^{-2}\int_{|x|\le R} \big|u_M - (\overline{u}_M)_{\{|x|\le R\}}\big|^s\,dx\right)^{1/s} \lesssim \left(\int_{|x|\le R} |\nabla u_M|^2\,dx\right)^{1/2},$$

for any $s < \infty$, which we use once for $s = q$, that is,

$$\begin{aligned}\left(R^{-2}\int_{|x|\le R} \big|u_M - (\overline{u}_M)_{\{|x|\le R\}}\big|^q\,dx\right)^{1/q} \quad &\lesssim \quad \left(\int_{|x|\le R} |\nabla u_M|^2\,dx\right)^{1/2} \\ &\overset{(4.14)}{\lesssim} \left(M\int_{\mathbb{Z}^2} |f|\,dx\right)^{1/2},\end{aligned} \tag{4.15}$$

and once for arbitrary s (which we think of being larger than q) in the form

$$\begin{aligned}&\left(R^{-2}\int_{|x|\le R} |u_M|^s\,dx\right)^{1/s} \\ &\quad\lesssim \left(\int_{|x|\le R} |\nabla u_M|^2\,dx\right)^{1/2} + \big|(\overline{u}_M)_{\{|x|\le R\}}\big| \\ &\quad\overset{(4.14)}{\lesssim} \left(M\int_{\mathbb{Z}^2} |f|\,dx\right)^{1/2} + \left(R^{-2}\int_{|x|\le R} |u|^q\,dx\right)^{1/q}.\end{aligned} \tag{4.16}$$

We use (4.16) to estimate the peaks of u. More precisely, we claim that for $s > 2q$,

$$\begin{aligned}&\left(R^{-2}\int_{\{|x|\le R\}\cap\{|u|>M\}} |u|^q\,dx\right)^{1/q} \\ &\quad\lesssim M^{1-s/(2q)}\left(\int_{\mathbb{Z}^2} |f|\,dx\right)^{s/(2q)} + M^{1-s/q}\left(R^{-2}\int_{|x|\le R} |u|^q\,dx\right)^{s/q^2}.\end{aligned} \tag{4.17}$$

The argument for (4.17) is similar to the case of $d > 2$: estimate (4.16) yields the weak estimate

$$\begin{aligned}&M\big(R^{-2}\mathcal{L}_2(\{|x|\le R\}\cap\{|u|>M\})\big)^{1/s} \\ &\quad\lesssim \left(M\int_{\mathbb{Z}^2} |f|\,dx\right)^{1/2} + \left(R^{-2}\int_{|x|\le R} |u|^q\,dx\right)^{1/q},\end{aligned}$$

which we rewrite as

$$
\begin{aligned}
&R^{-2}\mathcal{L}_2\bigl(\{|x|\le R\}\cap\{|u|>M\}\bigr)\\
&\qquad\lesssim M^{-s/2}\biggl(\int_{\mathbb{Z}^2}|f|\,dx\biggr)^{s/2}+M^{-s}\biggl(R^{-2}\int_{|x|\le R}|u|^q\,dx\biggr)^{s/q}.
\end{aligned}
\tag{4.18}
$$

On the other hand, we have

$$
\begin{aligned}
\int_{\{|x|\le R\}\cap\{|u|>M\}}|u|^q\,dx&=q\int_M^\infty \mathcal{L}_2(\{|x|\le R\}\cap\{|u|>M'\})M'^{q-1}\,dM'\\
&\quad+M^q\mathcal{L}_2(\{|x|\le 1\}\cap\{|u|>M\}).
\end{aligned}
\tag{4.19}
$$

Since $s>2q$, the combination of (4.18) and (4.19) yields

$$
\begin{aligned}
&R^{-2}\int_{\{|x|\le R\}\cap\{|u|>M\}}|u|^q\,dx\\
&\qquad\lesssim M^{q-s/2}\biggl(\int_{\mathbb{Z}^2}|f|\,dx\biggr)^{s/2}+M^{q-s}\biggl(R^{-2}\int_{|x|\le R}|u|^q\,dx\biggr)^{s/q},
\end{aligned}
$$

which is (4.17).

We now combine (4.15) and (4.17) as follows

$$
\begin{aligned}
&\biggl(R^{-2}\int_{|x|\le R}|u|^q\,dx\biggr)^{1/q}\\
&\quad\overset{(4.13)}{\le}\biggl(R^{-2}\int_{|x|\le R}\bigl|u-(\overline{u}_M)_{\{|x|\le R\}}\bigr|^q\,dx\biggr)^{1/q}\\
&\quad\le\biggl(R^{-2}\int_{|x|\le R}\bigl|u_M-(\overline{u}_M)_{\{|x|\le R\}}\bigr|^q\,dx\biggr)^{1/q}\\
&\qquad+\biggl(R^{-2}\int_{\{|x|\le R\}\cap\{|u|>M\}}|u|^q\,dx\biggr)^{1/q}\\
&\overset{(4.15)\text{ and }(4.17)}{\lesssim} M^{1/2}\biggl(\int_{\mathbb{Z}^2}|f|\,dx\biggr)^{1/2}+M^{1-s/(2q)}\biggl(\int_{\mathbb{Z}^2}|f|\,dx\biggr)^{s/(2q)}\\
&\qquad+M^{1-s/q}\biggl(R^{-2}\int_{|x|\le R}|u|^q\,dx\biggr)^{s/q^2}.
\end{aligned}
$$

We claim that this estimate contains the desired estimate. Indeed, using the abbreviations

$$
U:=\biggl(R^{-2}\int_{|x|\le R}|u|^q\,dx\biggr)^{1/q}\quad\text{and}\quad F:=\int_{\mathbb{Z}^2}|f|\,dx,
$$

we rewrite the above as

$$
U\lesssim M^{1/2}F^{1/2}+M^{1-s/(2q)}F^{s/(2q)}+M^{1-s/q}U^{s/q}.
\tag{4.20}
$$

Since $s > q$, choosing $M \sim U$ sufficiently large, we may absorb the last term of (4.20) into the left-hand side. This yields

$$U \lesssim U^{1/2} F^{1/2} + U^{1-s/(2q)} F^{s/(2q)}.$$

Using Young's inequality twice in the right-hand side since $s > 2q$, we obtain as desired $U \lesssim F$, which shows

$$\left(R^{-2} \int_{|x|\le R} |G_T - \overline{G}_{T\,|x|\le R}|^q \, dx\right)^{1/q} \lesssim 1.$$

Step 3. Proof of (ii).

We first derive a weak L^4-estimate on $\{|x| \le R\}$:

$$R^{-2}\mathcal{L}_2(\{G_T > M\} \cap \{|x| \le R\}) \lesssim M^{-4}. \tag{4.21}$$

For that purpose, we combine (4.1), which for $R \sim \sqrt{T}$ turns into

$$R^{-2} \int_{\mathbb{Z}^2} G_{T,M}^2 \, dx + \int_{\mathbb{Z}^2} |\nabla G_{T,M}|^2 \, dx \lesssim M, \tag{4.22}$$

with the Poincaré–Sobolev estimate

$$\left(R^{-2} \int_{|x|\le R} |G_{T,M} - \overline{G}_{T,M\{|x|\le R\}}|^8 \, dx\right)^{1/8} \lesssim \left(\int_{|x|\le R} |\nabla G_{T,M}|^2 \, dx\right)^{1/2}$$

in form of

$$\left(R^{-2} \int_{|x|\le R} G_{T,M}^8 \, dx\right)^{1/8} \lesssim \left(\int_{|x|\le R} |\nabla G_{T,M}|^2 \, dx\right)^{1/2} + \left(R^{-2} \int_{|x|\le R} G_{T,M}^2 \, dx\right)^{1/2}.$$

This yields (4.21):

$$\big(R^{-2} M^8 \mathcal{L}_2(\{G_T > M\} \cap \{|x| \le R\})\big)^{1/8} \le \left(R^{-2} \int_{|x|\le R} G_{T,M}^8 \, dx\right)^{1/8} \lesssim M^{1/2}.$$

We now argue that (4.21) yields (2.22). Indeed, combining

$$\begin{aligned} R^{-2} \int_{\{G_T > M\}\cap\{|x|\le R\}} G_T^2 \, dx &= qR^{-2} \int_M^\infty \mathcal{L}_2(\{G_T > M'\} \cap \{|x| \le R\}) M' \, dM' \\ &\quad + R^{-2} M^2 \mathcal{L}_2(\{G_T > M\} \cap \{|x| \le R\}) \\ &\overset{(4.21)}{\lesssim} M^{-2} \end{aligned}$$

with the trivial inequality

$$R^{-2} \int_{|x|\le R} G_{T,M}^2 \, dx \lesssim M^2$$

for $M = 1$ yields property (ii) of the lemma.

Step 4. Proof of (iii).

We establish for all $q > 1$ and $R \gg 1$

$$(2R)^{-d} \int_{|x|\geq 2R} G_T^q \, dx \lesssim \frac{T}{R^2} R^{-d} \int_{R\leq|x|\leq 2R} G_T^q \, dx. \tag{4.23}$$

Indeed, we test (4.2) with $\eta^2 G_T^{q-1}$ where the cut-off function η is chosen as follows

$$\eta \equiv 1 \qquad \text{in } \{|x| \geq 2R\},$$
$$\eta \equiv 0 \qquad \text{in } \{|x| \leq R\}, \qquad |\nabla \eta| \lesssim R^{-1}, \qquad 0 \leq \eta \leq 1,$$

yielding

$$T^{-1} \int_{\mathbb{Z}^d} \eta^2 G_T^q \, dx + \int_{\mathbb{Z}^d} \nabla(\eta^2 G_T^{q-1}) \cdot A \nabla G_T \, dx = 0.$$

Arguing as for (4.9), this yields

$$T^{-1} \int_{|x|\geq 2R} G_T^q \, dx + \int_{|x|\geq 2R} |\nabla G_T^{q/2}|^2 \, dx \lesssim R^{-2} \int_{R\leq|x|\leq 2R} G_T^q \, dx,$$

so in particular (4.23).

We now turn to (2.23). We introduce the abbreviations

$$R_k := 2^k \sqrt{T},$$
$$\Lambda_k := R_k^{-d} \int_{R_k\leq|x|\leq R_{k+1}} G_T^q \, dx,$$

so that (4.23) turns into

$$\Lambda_{k+1} \leq C \frac{T}{R_k^2} \Lambda_k = C 4^{-k} \Lambda_k,$$

where C denotes a constant depending only on α, β, and d. This yields by iteration

$$\Lambda_k \leq \Lambda_0 C^k \prod_{i=0}^{k-1} 4^{-i} = \Lambda_0 C^k 4^{-(k-1)k/2} = \Lambda_0 C^k 2^{-(k-1)k}.$$

Thus, for all $r > 0$,

$$\ln\left(\left(\frac{R_k}{\sqrt{T}}\right)^r \frac{\Lambda_k}{\Lambda_0}\right) \leq kr \ln 2 + (k+1) \ln C - k^2 \ln 2$$
$$\lesssim 1$$

for k large enough. Hence,

$$\int_{R_k\leq|x|\leq R_{k+1}} G_T^q \, dx \lesssim \Lambda_0 R_k^d \left(\frac{R_k}{\sqrt{T}}\right)^{-r}.$$

To conclude the proof of (iii), it remains to argue that

$$\Lambda_0 \lesssim \begin{cases} d = 2, & 1\,, \\ d > 2, & (\sqrt{T}^{2-d})^q. \end{cases} \tag{4.24}$$

For $d > 2$, this a consequence of (2.21), whereas for $d = 2$ we combine (2.20) with (2.22) as follows:

$$
\begin{aligned}
\Lambda_0 &\le T^{-1} \int_{|x|\le 2\sqrt{T}} G_T(x)^q \, dx \\
&\le T^{-1}\Bigg(\underbrace{\bigg(\int_{|x|\le 2\sqrt{T}} \big|G_T(x) - \overline{G}_{T\,|x|\le 2\sqrt{T}}\big|^q \, dx\bigg)^{1/q}}_{\overset{(2.20)}{\lesssim}\ \sqrt{T}^2} \\
&\qquad + T^{-1}\bigg(\int_{|x|\le 2\sqrt{T}} (\underbrace{\overline{G}_{T\,|x|\le 2\sqrt{T}}}_{\overset{(2.22)}{\lesssim}\ \sqrt{T}^2})^q \, dx\bigg)^{1/q}\Bigg)^q \\
&\lesssim 1.
\end{aligned}
$$

Step 5. Modifications due to the discreteness.

The only place where we have used the Leibniz rule is the proof of the Caccioppoli inequality (4.7). At the discrete level, we have for $i \in \{1, \dots, d\}$

$$
\begin{aligned}
&\nabla_i(\eta^2 G_T^{q-1})(x) \\
&\quad = \eta^2(x+\mathbf{e}_i)G_T^{q-1}(x+\mathbf{e}_i) - \eta^2(x)G_T^{q-1}(x) \\
&\quad = \frac{\eta^2(x+\mathbf{e}_i)+\eta^2(x)}{2}\big(G_T^{q-1}(x+\mathbf{e}_i) - G_T^{q-1}(x)\big) \\
&\qquad + \frac{\eta^2(x+\mathbf{e}_i)-\eta^2(x)}{2}\big(G_T^{q-1}(x+\mathbf{e}_i) + G_T^{q-1}(x)\big).
\end{aligned} \tag{4.25}
$$

Taking advantage of the diagonal structure of A (although this is not essential), we obtain

$$
\begin{aligned}
\nabla(\eta^2 G_T^{q-1}) \cdot A\nabla G_T(x) &= \sum_{i=1}^d \nabla_i(\eta^2 G_T^{q-1})(x) a(x, x+\mathbf{e}_i)\nabla_i G_T(x) \\
&\overset{(4.25)}{=} \sum_{i=1}^d a(x, x+\mathbf{e}_i)\frac{\eta^2(x+\mathbf{e}_i)+\eta^2(x)}{2} \\
&\qquad \times \underbrace{\big(G_T^{q-1}(x+\mathbf{e}_i) - G_T^{q-1}(x)\big)\nabla_i G_T(x)}_{\ge 0} \\
&\quad + \sum_{i=1}^d a(x, x+\mathbf{e}_i)\frac{\eta^2(x+\mathbf{e}_i)-\eta^2(x)}{2} \\
&\qquad \times \big(G_T^{q-1}(x+\mathbf{e}_i) + G_T^{q-1}(x)\big)\nabla_i G_T(x).
\end{aligned}
$$

Since the underbraced term is nonnegative, the lower and upper bounds on a yield

$$
\begin{aligned}
&\nabla(\eta^2 G_T^{q-1}) \cdot A\nabla G_T(x) \\
&\quad \geq \alpha \sum_{i=1}^{d} \frac{\eta^2(x+\mathbf{e}_i)+\eta^2(x)}{2}\big(G_T^{q-1}(x+\mathbf{e}_i) - G_T^{q-1}(x)\big)\nabla_i G_T(x) \\
&\qquad - \beta \sum_{i=1}^{d} |\nabla_i \eta(x)| \frac{\eta(x+\mathbf{e}_i)+\eta(x)}{2}\big(G_T^{q-1}(x+\mathbf{e}_i) + G_T^{q-1}(x)\big)|\nabla_i G_T(x)| \\
&\quad \overset{\text{Young}}{\geq} \alpha \sum_{i=1}^{d} \frac{\eta^2(x+\mathbf{e}_i)+\eta^2(x)}{2}\big(G_T^{q-1}(x+\mathbf{e}_i) - G_T^{q-1}(x)\big)\nabla_i G_T(x) \\
&\qquad - \beta C \sum_{i=1}^{d} \big(G_T(x+\mathbf{e}_i)^q + G_T(x)^q\big)|\nabla_i \eta(x)|^2 \\
&\qquad - \beta C^{-1} \sum_{i=1}^{d} \underbrace{\left(\frac{\eta(x+\mathbf{e}_i)+\eta(x)}{2}\right)^2}_{\leq(\eta^2(x+\mathbf{e}_i)+\eta^2(x))/2} (\nabla_i G_T(x))^2 \\
&\qquad\qquad \times \big(G_T^{q-2}(x+\mathbf{e}_i) + G_T^{q-2}(x)\big).
\end{aligned}
$$

Using the inequality (proved at the end of the step)

$$
\begin{aligned}
&2(b^{q-1} - c^{q-1})(b-c) \geq (b-c)^2(b^{q-2}+c^{q-2}) \\
&\qquad\qquad \text{for } b, c \geq 0, q \geq 2,
\end{aligned}
\tag{4.26}
$$

we may absorb the last term of the right-hand side of the latter inequality into the first term for C large enough, so that it turns into

$$
\begin{aligned}
&\nabla(\eta^2 G_T^{q-1}) \cdot A\nabla G_T(x) \\
&\quad \geq (\alpha - 2\beta C^{-1}) \sum_{i=1}^{d} \frac{\eta^2(x+\mathbf{e}_i)+\eta^2(x)}{2} \\
&\qquad\qquad \times \big(G_T^{q-1}(x+\mathbf{e}_i) - G_T^{q-1}(x)\big)\nabla_i G_T(x) \\
&\qquad - \beta C \sum_{i=1}^{d} \big(G_T(x+\mathbf{e}_i)^q + G_T(x)^q\big)|\nabla_i \eta(x)|^2.
\end{aligned}
\tag{4.27}
$$

Using now the following inequality

$$
(b^{q-1} - c^{q-1})(b-c) \gtrsim (b^{q/2} - c^{q/2})^2 \qquad \text{for } b, c \geq 0, q > 1, \tag{4.28}
$$

(4.27) finally turns into

$$\begin{aligned}
&\nabla(\eta^2 G_T^{q-1}) \cdot A\nabla G_T(x) \\
&\quad \gtrsim \sum_{i=1}^{d} \frac{\eta^2(x+\mathbf{e}_i)+\eta^2(x)}{2}\big(G_T^{q/2}(x+\mathbf{e}_i)-G_T^{q/2}(x)\big)^2 \\
&\qquad - C\sum_{i=1}^{d}\big(G_T(x+\mathbf{e}_i)^q + G_T(x)^q\big)|\nabla_i\eta(x)|^2.
\end{aligned}$$

Combining this with (4.9) yields

$$\begin{aligned}
&\int_{\mathbb{Z}^d} \eta^2(x)|\nabla G_T^{q/2}(x)|^2\,dx \\
&\qquad \lesssim \int_{\mathbb{Z}^d} \big(G_T(x+\mathbf{e}_i)^q + G_T(x)^q\big)|\nabla_i\eta(x)|^2\,dx,
\end{aligned} \tag{4.29}$$

which implies as desired

$$\int_{2R\le|x|<4R} |\nabla G_T^{q/2}(x)|^2\,dx \lesssim R^{-2}\int_{R\le|x|<8R} G_T(x)^q\,dx,$$

provided that η satisfies in addition

$$\eta(x)=0 \qquad \text{for } x\notin\{y : R+1\le|y|\le 8R-1\},$$

which is no restriction since $R\gg 1$.

We quickly sketch the proofs of (4.26) and (4.28) to conclude. Inequality (4.26) follows by symmetry from

$$\begin{aligned}
(b^{q-1}-c^{q-1})(b-c)-(b-c)^2c^{q-2} &= (b-c)(b^{q-1}-bc^{q-2}) \\
&= b(b-c)(b^{q-2}-c^{q-2}) \\
&= b|b-c||b^{q-2}-c^{q-2}| \ge 0.
\end{aligned}$$

To prove (4.28) we first note that by homogeneity and nonnegativity of b and c, it is enough to consider $c=1$ and $b\ge 0$. We introduce the function $h=\mathbb{R}^+\to\mathbb{R}^+$ defined by

$$h(b)=\begin{cases} \dfrac{(b^{q/2}-1)^2}{(b^{q-1}-1)(b-1)}, & b\ne 1, \\[2ex] \dfrac{q^2}{4(q-1)}, & b=1. \end{cases}$$

Since $h\ge 0$, the claim is proved if h is bounded on $\mathbb{R}^+$. As $h(0)=1$ and $\lim_{b\to\infty} h(b)=1$, it is enough to prove that h is continuous on $\mathbb{R}^+$. A Taylor expansion around $b = 1$ yields

$$(b^{q/2} - 1)^2 = \frac{q^2}{4}(b-1)^2 + o\big((b-1)^2\big),$$
$$(b^{q-1} - 1)(b-1) = (q-1)(b-1)^2 + o\big((b-1)^2\big).$$

Hence, $\lim_{b \to 1} h(b) = h(1)$, h is continuous and therefore bounded on $\mathbb{R}^+$, as desired.

4.2. *Proof of Lemma* 2.9. The proof relies on three ingredients: a Meyers' estimate based on the L^q theory for the constant-coefficients Helmholtz projection, a Cacciopoli estimate and the estimates of Lemma 2.8.

We begin with Meyers' estimates. Let $u : \mathbb{Z}^d \to \mathbb{R}$, $f : \mathbb{Z}^d \to \mathbb{R}$, and $g : \mathbb{Z}^d \to \mathbb{R}^d$ have support in $\{|x| < R\}$, and let u satisfy the equation

$$-\nabla^* \cdot A(x)\nabla u(x) = \nabla^* \cdot g(x) + f(x) \qquad \text{in } \mathbb{Z}^d. \tag{4.30}$$

We claim that there exists $p > 2$ depending only on α, β, and d such that for all $R \gg 1$, the following L^p-estimate holds

$$\begin{aligned}\left(\int_{\mathbb{Z}^d} |\nabla u(x)|^p \, dx\right)^{1/p} &\lesssim \left(\int_{\mathbb{Z}^d} |g(x)|^p \, dx\right)^{1/p} \\ &\quad + R^{1-d(1/2-1/p)}\left(\int_{\mathbb{Z}^d} |f(x)|^2 \, dx\right)^{1/2}.\end{aligned} \tag{4.31}$$

As in the original paper [16] by Meyers, the proof of (4.31) relies on a perturbation argument and on the L^q regularity theory for the Helmholtz projection.

Step 1. L^q regularity for the Helmholtz projection.

Let $\mathcal{H} : L^2(\mathbb{Z}^d, \mathbb{R}^d) \to L^2(\mathbb{Z}^d, \mathbb{R}^d)$ denote the Helmholtz projection, that is, the orthogonal projection onto gradient fields for the inner product of $L^2(\mathbb{Z}^d, \mathbb{R}^d)$. By definition, $\mathcal{H}$ is continuous on $L^2(\mathbb{Z}^d, \mathbb{R}^d)$ and satisfies

$$\|\mathcal{H}g\|_{L^2(\mathbb{Z}^d,\mathbb{R}^d)} \le \|g\|_{L^2(\mathbb{Z}^d,\mathbb{R}^d)}. \tag{4.32}$$

Let us show that $\mathcal{H}$ can be extended to a continuous operator from $L^q(\mathbb{Z}^d, \mathbb{R}^d)$ to $L^q(\mathbb{Z}^d, \mathbb{R}^d)$ for all $1 < q < \infty$. The proof is standard, appealing to Calderón–Zygmund singular integral theory and to Marcinkiewicz interpolation theorem (such theorems apply to the discrete case under investigation since the associated measure has the so-called "doubling" property). Since $\mathcal{H}$ commutes with translations, it is a convolution operator: there exists a matrix-valued kernel K such that

$$\mathcal{H}g(x) = \int_{\mathbb{Z}^d} K(x-y)g(y)\, dy. \tag{4.33}$$

From an elementary Fourier series analysis (see [15] for related arguments), we infer that the symbol of K coincides with the symbol of the second derivative of the Green's function of the Laplace equation studied in [15]. In particular, from the analysis of [15], we learn that

$$|\nabla K(x)| \lesssim \frac{1}{1+|x|^{d+1}}. \tag{4.34}$$

We are therefore in position to apply Calderón–Zygmund's theory (see [20], Theorem 2, page 17), which shows that $\mathcal{H}$ is of weak type (1, 1) (see the proof of [20], Theorem 3, page 19). Appealing to Marcinkiewicz' interpolation theorem (see [1], Theorem 1.3.1, page 9) then shows that $\mathcal{H}$ can be extended to a continuous operator from $L^q(\mathbb{Z}^d, \mathbb{R}^d)$ to $L^q(\mathbb{Z}^d, \mathbb{R}^d)$ for all $1 < q < 2$. A standard duality argument (see, [19], 2.5(c), page 33, e.g.) implies that $\mathcal{H}$ can also be extended to a continuous operator from $L^q(\mathbb{Z}^d, \mathbb{R}^d)$ to $L^q(\mathbb{Z}^d, \mathbb{R}^d)$ for all $2 < q < \infty$. Let $r > 2$ be fixed, and for all $q > 1$ let denote by C_q the norm of $\mathcal{H}$ in $\mathcal{L}(L^q(\mathbb{Z}^d, \mathbb{R}^d), L^q(\mathbb{Z}^d, \mathbb{R}^d))$. Then Riesz–Thorin interpolation theorem (see [1], Theorem 1.1.1, page 2) shows that for all $\theta \in (0, 1)$, $C_{2\theta+r(1-\theta)} \le C_2^{\theta} C_r^{1-\theta}$, so that

$$\limsup_{q\to 2} C_q \le 1 \tag{4.35}$$

since $C_2 \le 1$ by (4.32).

We now turn to the proof of (4.31) proper and proceed with the perturbation argument.

Step 2. Proof of (4.31) for $f \equiv 0$.

We first assume that $f \equiv 0$, and rewrite the left-hand side of (4.30) as a perturbation of the operator $-\frac{\alpha+\beta}{2}\triangle$:

$$-\frac{\alpha+\beta}{2}\triangle u = \nabla^* \cdot \left(g + \left(A - \frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u\right)$$

or equivalently in the form

$$-\triangle u = \nabla^* \cdot \left(\frac{2}{\alpha+\beta}\left(g + \left(A - \frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u\right)\right). \tag{4.36}$$

In order to apply the L^q theory for the Helmholtz projection, we need to show that

$$\nabla u \equiv \mathcal{H}\left(\frac{2}{\alpha+\beta}\left(g + \left(A - \frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u\right)\right). \tag{4.37}$$

Since ∇u is obviously a gradient, it remains to show that for all $\zeta : \mathbb{Z}^d \to \mathbb{R}$ such that $\nabla\zeta \in L^2(\mathbb{Z}^d, \mathbb{R}^d)$ one has

$$\begin{aligned}&\int_{\mathbb{Z}^d} \nabla u(x)\cdot\nabla\zeta(x)\,dx\\ &\quad= \int_{\mathbb{Z}^d}\left(\frac{2}{\alpha+\beta}\left(g + \left(A - \frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u\right)\right)(x)\cdot\nabla\zeta(x)\,dx.\end{aligned} \tag{4.38}$$

To this aim, we multiply (4.36) by ζ and integrate by parts using that u, ∇u and g have compact supports. This yields (4.38) and proves therefore (4.37). The continuity of $\mathcal{H}$ from $L^q(\mathbb{Z}^d,\mathbb{R}^d)$ to $L^q(\mathbb{Z}^d,\mathbb{R}^d)$ proved in Step 1 then implies that

$$\begin{aligned}(4.39)\qquad &\left(\int_{|x|\le R}|\nabla u(x)|^q\,dx\right)^{1/q}\\ &\quad\le C_q\frac{2}{\alpha+\beta}\left(\int_{|x|\le R}\left|g(x)+\left(A(x)-\frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u(x)\right|^q dx\right)^{1/q}.\end{aligned}$$

Using the triangle inequality, (4.39) turns into

$$\begin{aligned}&\left(\int_{|x|\le R}|\nabla u(x)|^q\,dx\right)^{1/q}\\ (4.40)\qquad &\quad\le C_q\frac{2}{\alpha+\beta}\left(\int_{|x|\le R}|g(x)|^q\,dx\right)^{1/q}\\ &\qquad+C_q\frac{2}{\alpha+\beta}\left(\int_{|x|\le R}\left|\left(A(x)-\frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u(x)\right|^q dx\right)^{1/q}.\end{aligned}$$

Since $a\in\mathcal{A}_{\alpha\beta}$, $|(A(x)-\frac{\alpha+\beta}{2}\,\mathrm{Id})\nabla u(x)|\le\frac{\beta-\alpha}{2}|\nabla u(x)|$ and we may absorb the term

$$\begin{aligned}&C_q\frac{2}{\alpha+\beta}\left(\int_{|x|\le R}\left|\left(A(x)-\frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u(x)\right|^q dx\right)^{1/q}\\ &\qquad\le C_q\frac{\beta-\alpha}{\alpha+\beta}\left(\int_{|x|\le R}|\nabla u(x)|^q\,dx\right)^{1/q}\end{aligned}$$

into the left-hand side of (4.40) provided that

$$(4.41)\qquad C_q\underbrace{\frac{\beta-\alpha}{\alpha+\beta}}_{<1}<1.$$

The interpolation property (4.35) ensures there exists $p>2$ such that (4.41) holds for all $p\ge q\ge 2$. For such a q, we then have

$$(4.42)\qquad \left(\int_{|x|\le R}|\nabla u(x)|^q\,dx\right)^{1/q}\lesssim\left(\int_{|x|\le R}|g(x)|^q\,dx\right)^{1/q},$$

as desired.

Step 3. Proof of (4.31) for general f.

Note that since u and g have compact supports, equation (4.30) implies that $\int_{\mathbb{Z}^d}f(x)\,dx=0$. We first show that there exists $\nabla w\in L^2(\mathbb{Z}^d,\mathbb{R}^d)$ such that for all $\zeta:\mathbb{Z}^d\to\mathbb{R}$ with $\nabla\zeta\in L^2(\mathbb{Z}^d,\mathbb{R}^d)$, one has

$$(4.43)\qquad \int_{\mathbb{Z}^d}\nabla w(x)\cdot\nabla\zeta(x)\,dx=\int_{\mathbb{Z}^d}f(x)\zeta(x)\,dx,$$

so that (4.37) turns into

$$\nabla u \equiv \mathcal{H}\left(\frac{2}{\alpha+\beta}\left(g+\left(A-\frac{\alpha+\beta}{2}\,\mathrm{Id}\right)\nabla u\right)+\nabla w\right).$$

Provided

$$\left(\int_{\mathbb{Z}^d} |\nabla w(x)|^q\,dx\right)^{1/q} \lesssim R^{1-d(1/2-1/q)}\left(\int_{\mathbb{Z}^d} f(x)^2\,dx\right)^{1/2}, \tag{4.44}$$

for all $2 \le q \le \tilde{q}$ for some $\tilde{q} > 2$, we then conclude as in the case $f \equiv 0$ (with potentially a smaller p). To prove the existence of such a ∇w, we proceed by minimization and consider the problem

$$\begin{aligned}\inf\Big\{&\int_{\mathbb{Z}^d} |\nabla\zeta(x)|^2\,dx - \int_{\mathbb{Z}^d} f(x)\zeta(x)\,dx;\\ &\qquad\qquad \zeta:\mathbb{Z}^d \to \mathbb{R}, \nabla\zeta \in L^2(\mathbb{Z}^d,\mathbb{R}^d)\Big\}.\end{aligned} \tag{4.45}$$

The same argument as in the proof of Riesz' theorem yields the existence of a minimizer once one shows that the functional is coercive. Let R be large enough so that f has support in $\{|x| < R\}$, and denote by $\bar{\zeta}_{\{|x|<R\}}$ the average of ζ on $\{|x| < R\}$. Since f has zero average, one may subtract the average of ζ and obtain by Cauchy–Schwarz and Poincaré's inequalities

$$\begin{aligned}\left|\int_{\mathbb{Z}^d} f(x)\zeta(x)\,dx\right| &= \left|\int_{|x|<R} f(x)\big(\zeta(x)-\bar{\zeta}_{\{|x|<R\}}\big)\,dx\right|\\ &\lesssim R\left(\int_{\mathbb{Z}^d} f(x)^2\,dx\right)^{1/2}\left(\int_{|x|<R} |\nabla\zeta(x)|^2\,dx\right)^{1/2}\\ &\lesssim -2R^2\int_{\mathbb{Z}^d} f(x)^2\,dx + \frac{1}{2}\int_{\mathbb{Z}^d} |\nabla\zeta(x)|^2\,dx.\end{aligned}$$

This shows that for all test functions ζ

$$\begin{aligned}&\int_{\mathbb{Z}^d} |\nabla\zeta(x)|^2\,dx - \int_{\mathbb{Z}^d} f(x)\zeta(x)\,dx\\ &\qquad \ge -2R^2\int_{\mathbb{Z}^d} f(x)^2\,dx + \frac{1}{2}\int_{\mathbb{Z}^d} |\nabla\zeta(x)|^2\,dx,\end{aligned} \tag{4.46}$$

as desired. This proves the existence of a minimizer $w:\mathbb{Z}^d \to \mathbb{R}$ such that $\nabla w \in L^2(\mathbb{R}^d,\mathbb{Z}^d)$. In addition, it satisfies the estimate

$$\int_{\mathbb{Z}^d} |\nabla w(x)|^2\,dx \le 4R^2\int_{\mathbb{Z}^d} f(x)^2\,dx. \tag{4.47}$$

Since w is a minimizer of (4.45), the first variation of the energy at w vanishes, and w satisfies (4.43).

It remains to estimate the L^q norm of ∇w for some $q > 2$. To this aim, we argue that

$$\begin{aligned}\int_{\mathbb{Z}^d} |\nabla\nabla w(x)|^2\,dx &= \int_{\mathbb{Z}^d} (\triangle w(x))^2\,dx \\ &= \int_{\mathbb{Z}^d} f(x)^2\,dx.\end{aligned} \tag{4.48}$$

As in the continuum case, the first identity in (4.48) follows directly from two integrations by parts for w with compact support. For general w, the boundary term involves products of first and second derivatives of w on spheres of large radius R. In our discrete setting, these boundary terms can be estimated by the integral of $|\nabla w|^2$ outside the ball of radius R, which is finite since $|\nabla w|^2$ is integrable by construction. Hence, the boundary terms can be made to vanish in the limit $R \to \infty$. The second identity in (4.48) follows from the fact that w solves the equation

$$-\triangle w(x) = f(x) \qquad \text{in } \mathbb{Z}^d,$$

which is a consequence of (4.43). We are in position to conclude. For $d > 2$, we appeal to Poincaré–Sobolev inequality on ∇w to turn (4.48) into

$$\int_{\mathbb{Z}^d} |\nabla w(x)|^{2d/(d-2)}\,dx \lesssim \int_{\mathbb{Z}^d} f(x)^2\,dx. \tag{4.49}$$

Combined with (4.47), (4.49) implies (4.44) for all $2 \le q \le \frac{2d}{d-2}$ by Hölder's inequality. For $d = 2$, we appeal to Poincaré–Sobolev inequality on $(\nabla_i w)^2$ for $i \in \{1, \dots, d\}$ to turn (4.47) and (4.48) into

$$\begin{aligned}\int_{\mathbb{Z}^2} (\nabla_i w(x))^4\,dx &\lesssim \left(\int_{\mathbb{Z}^2} |\nabla_i(\nabla_i w(x))^2|\,dx\right)^2 \\ &\lesssim \int_{\mathbb{Z}^2} |\nabla\nabla w(x)|^2\,dx \int_{\mathbb{Z}^2} |\nabla w(x)|^2\,dx \\ &\lesssim R^2\left(\int_{\mathbb{Z}^2} f(x)^2\,dx\right)^2.\end{aligned} \tag{4.50}$$

Combined with (4.47), (4.50) implies (4.44) for all $2 \le q \le 4$ by Hölder's inequality.

Step 4. Cacciopoli estimate.

We need the following finer version of (4.7): For all $\kappa \in \mathbb{R}$,

$$\begin{aligned}&\int_{2R\le|x|\le 16R} |\nabla G_T(x)|^2\,dx \\ &\quad\lesssim R^{-2}\int_{R\le|x|\le 32R} \big(G_T(x) - \kappa\big)^2\,dx \\ &\qquad + T^{-1}|\kappa| \int_{R\le|x|\le 32R} |G_T(x) - \kappa|\,dx.\end{aligned} \tag{4.51}$$

This variant of Caccioppoli's estimate can be proved along the lines of (4.7), multiplying the equation by $\eta^2(G_T(x)-\kappa)$ instead of $\eta^2 G_T(x)$. The zero order term then brings the new term in the right-hand side of (4.51). By Young and Cauchy–Schwarz' inequalities, the second term of the right-hand side is controlled by

$$\begin{aligned}
&T^{-1}|\kappa|\int_{R\le|x|\le 32R}|G_T(x)-\kappa|\,dx\\
&\quad= T^{-1}|\kappa|R\int_{R\le|x|\le 32R}R^{-1}|G_T(x)-\kappa|\,dx\\
&\quad\lesssim T^{-2}|\kappa|^2R^2+R^{-2}\int_{R\le|x|\le 32R}|G_T(x)-\kappa|^2\,dx.
\end{aligned}$$

Hence, it only remains to estimate the first term of the right-hand side of (4.51). To this aim, we appeal to Hölder's inequality with exponents $(p/2, p/(p-2))$ for $p\ge 2$:

$$\begin{aligned}
&\int_{R\le|x|\le 32R}|G_T(x)-\kappa|^2\,dx\\
&\quad\le\left(\int_{R\le|x|\le 32R}|G_T(x)-\kappa|^p\,dx\right)^{2/p}(R^d)^{(p-2)/p}\\
&\quad= R^{d-2d/p}\left(\int_{R\le|x|\le 32R}|G_T(x)-\kappa|^p\,dx\right)^{2/p}.
\end{aligned}$$

Using these last two estimates and the elementary inequality $(a^2+b^2)^{1/2}\lesssim a+b$, (4.51) turns into

$$\begin{aligned}
&\left(\int_{2R\le|x|\le 16R}|\nabla G_T(x)|^2\,dx\right)^{1/2}\\
&\quad\lesssim R^{-1}R^{d/2-d/p}\left(\int_{R\le|x|\le 32R}|G_T(x)-\kappa|^p\,dx\right)^{1/p}\\
&\qquad+T^{-1}R^{d/2+1}|\kappa|,
\end{aligned}\tag{4.52}$$

that we will use with $\kappa=\overline{G}_{T\{R\le|x|\le 32R\}}$.

Step 5. In this step, we use Steps 1 and 4 to argue that

$$\begin{aligned}
&\left(\int_{4R<|x|\le 8R}|\nabla G_T(x)|^p\,dx\right)^{1/p}\\
&\quad\lesssim(R^{-1}+RT^{-1})\left(\int_{2R\le|x|\le 16R}|G_T(x)-\overline{G}_{T\{R\le|x|\le 32R\}}|^p\,dx\right)^{1/p}\\
&\qquad+T^{-1}R^{d/p+1}\overline{G}_{T\{R\le|x|\le 32R\}}.
\end{aligned}\tag{4.53}$$

We apply Meyers' estimate (4.31) to the function $u=\eta(G_T-\overline{G}_{T\{R\le|x|\le 32R\}})$, where the cut-off function $\eta:\mathbb{Z}^d\to[0,1]$ is such that

$$\eta(x)=1 \qquad \text{for } 4R\le|x|\le 8R,$$
$$\eta(x)=0 \qquad \text{for } \begin{cases}|x|\le 2R+1,\\ |x|\ge 16R-1,\end{cases} \qquad |\nabla\eta|\lesssim R^{-1}. \tag{4.54}$$

For all $i\in\{1,\dots,d\}$, the discrete Leibniz rule yields

$$\nabla_i u(x)=\eta(x)\nabla_i G_T(x)+\big(G_T(x+\mathbf{e}_i)-\overline{G}_{T\{R\le|x|\le 32R\}}\big)\nabla_i\eta(x).$$

Based on this, a direct calculation shows

$$\begin{aligned}
&-\nabla^*\cdot A\nabla u(x)\\
&\quad=-\underbrace{\eta(x)\nabla^*\cdot A\nabla G_T(x)}_{\overset{(4.54)\text{ and }(2.11)}{=}\eta(x)T^{-1}G_T(x)}-\sum_{i=1}^d\nabla_i^*\eta(x)a(x-\mathbf{e}_i,x)\nabla_i^*G_T(x)\\
&\qquad-\sum_{i=1}^d\nabla_i^*\big(\big(G_T(x+\mathbf{e}_i)-\overline{G}_{T\{R\le|x|\le 32R\}}\big)a(x,x+\mathbf{e}_i)\nabla_i\eta(x)\big)\\
&\quad=\nabla^*\cdot\left(-\sum_{i=1}^d\big(G_T(x+\mathbf{e}_i)-\overline{G}_{T\{R\le|x|\le 32R\}}\big)a(x,x+\mathbf{e}_i)\nabla_i\eta(x)\mathbf{e}_i\right)\\
&\qquad-\sum_{i=1}^d\nabla_i^*\eta(x)a(x-\mathbf{e}_i,x)\nabla_i^*G_T(x)-\eta(x)T^{-1}G_T(x).
\end{aligned}$$

This identity has the form of (4.30) provided we define the functions f and g by

$$f(x)=-\sum_{i=1}^d\nabla_i^*\eta(x)a(x-\mathbf{e}_i,x)\nabla_i^*G_T(x)-\eta(x)T^{-1}G_T(x),$$

$$g(x)=-\sum_{i=1}^d\big(G_T(x+\mathbf{e}_i)-\overline{G}_{T\{R\le|x|\le 32R\}}\big)a(x,x+\mathbf{e}_i)\nabla_i\eta(x)\mathbf{e}_i.$$

Since u, f and g have support in $\{|x|\le 16R\}$, we may apply estimate (4.31) which yields

$$\begin{aligned}
&\left(\int_{|x|\le 16R}|\nabla u(x)|^p\,dx\right)^{1/p}\\
&\quad\lesssim\left(\sum_{i=1}^d\int_{\mathbb{Z}^d}|\nabla_i\eta(x)|^p\big|G_T(x+\mathbf{e}_i)-\overline{G}_{T\{R\le|x|\le 32R\}}\big|^p\,dx\right)^{1/p}\\
&\qquad+R^{1-d(1/2-1/p)}\\
&\qquad\quad\times\left(\int_{\mathbb{Z}^d}\big(|\nabla^*\eta(x)|^2|\nabla^*G_T(x)|^2+T^{-2}\eta(x)^2G_T(x)^2\big)\,dx\right)^{1/2}.
\end{aligned}$$

Using the property (4.54) of η, and the triangle inequality, we are left with

$$
\begin{aligned}
&\left(\int_{|x|\le 16R} |\nabla u(x)|^p \, dx\right)^{1/p} \\
&\quad \lesssim R^{-1}\left(\int_{2R\le|x|\le 16R} \left|G_T(x)-\overline{G}_{T\{R\le|x|\le 32R\}}\right|^p dx\right)^{1/p} \\
&\qquad + R^{d/p-d/2}\left(\int_{2R\le|x|\le 16R} |\nabla G_T(x)|^2 \, dx\right)^{1/2} \\
&\qquad + R^{d/p-d/2+1}\left(\int_{2R\le|x|\le 16R} T^{-2} G_T(x)^2 \, dx\right)^{1/2}.
\end{aligned}
\tag{4.55}
$$

Let us rearrange the terms. For the third term, the triangle inequality and Hölder's inequality with exponents $(p/2, p/(p-2))$ show that

$$
\begin{aligned}
&\left(\int_{2R\le|x|\le 16R} G_T(x)^2 \, dx\right)^{1/2} \\
&\quad \lesssim R^{d/2}\overline{G}_{T\{R\le|x|\le 32R\}} \\
&\qquad + R^{d/2-d/p}\left(\int_{2R\le|x|\le 16R} \left|G_T(x)-\overline{G}_{T\{R\le|x|\le 32R\}}\right|^p dx\right)^{1/p},
\end{aligned}
$$

whereas for the second term we appeal to the Cacciopoli estimate (4.52) with $\kappa = \overline{G}_{T\{R\le|x|\le 32R\}}$. Hence, (4.55) finally turns into

$$
\begin{aligned}
&\left(\int_{|x|\le 16R} |\nabla u(x)|^p \, dx\right)^{1/p} \\
&\quad \lesssim (R^{-1}+RT^{-1})\left(\int_{2R\le|x|\le 16R} \left|G_T(x)-\overline{G}_{T\{R\le|x|\le 32R\}}\right|^p dx\right)^{1/p} \\
&\qquad + T^{-1}R^{d/p+1}\overline{G}_{T\{R\le|x|\le 32R\}}.
\end{aligned}
\tag{4.56}
$$

We are in position to conclude the proof of this step. For all $i \in \{1,\dots,d\}$, the discrete Leibniz rule yields $\nabla_i u(x) = \eta(x)\nabla_i G_T(x) + G_T(x+\mathbf{e}_i)\nabla_i\eta(x)$. Hence, (4.54) implies that $\nabla u(x) = \nabla G_T(x)$ for $4R \le |x| \le 8R$, so that (4.56) yields (4.53).

Step 6. Proof of (2.24).

We claim that (2.24) follows from (4.53) and the estimates of Lemma 2.8.

We distinguish two regimes: $R \le \sqrt{T}$ and $R \ge \sqrt{T}$. We begin with $R \le \sqrt{T}$. For the first term of the right-hand side of (4.53), we appeal to the BMO estimate (2.20) of Lemma 2.8 for $d = 2$ and to the decay estimate (2.21) with "$q = p$" for

$d > 2$, so that

$$(4.57)\quad \begin{aligned} &(R^{-1}+RT^{-1})\Big(\int_{2R\le|x|\le 16R}\big|G_T(x)-\overline{G}_{T\{R\le|x|\le 32R\}}\big|^p\,dx\Big)^{1/p} \\ &\quad\lesssim R^{-1}\big(R^d R^{(2-d)p}\big)^{1/p} = R^{d/p-d+1}. \end{aligned}$$

For the second term, we estimate the average using (2.22) for $d = 2$

$$\overline{G}_{T\{R\le|x|\le 32R\}} \lesssim R^{-2}\sqrt{T}^2\overline{G}_{T\{|x|\le 32\sqrt{T}\}} \overset{(2.22)}{\lesssim} R^{-2}T,$$

and using (2.21) with "$q = 1$" for $d > 2$

$$\overline{G}_{T\{R\le|x|\le 32R\}} \overset{(2.21)}{\lesssim} R^{2-d} \lesssim R^{-d}T,$$

since $R \le \sqrt{T}$. Hence, in both cases,

$$(4.58)\quad T^{-1}R^{d/p+1}\overline{G}_{T\{R\le|x|\le 32R\}} \lesssim R^{d/p-d+1}.$$

From (4.57) and (4.58), we then conclude that (2.24) holds for $R \le \sqrt{T}$.

We now deal with the case $R \ge \sqrt{T}$. For the first term of the right-hand side of (4.53), we use the decay estimate (2.23) with exponents "$q = p, r = k + 2p$," which yields

$$\begin{aligned} \int_{2R\le|x|\le 16R}\big|G_T(x)-\overline{G}_{T\{R\le|x|\le 32R\}}\big|^p\,dx &\lesssim \int_{R\le|x|\le 32R} G_T(x)^p\,dx \\ &\overset{(2.23)}{\lesssim} R^d R^{(2-d)p}\big(\sqrt{T}R^{-1}\big)^{k+2p}, \end{aligned}$$

and therefore

$$(4.59)\quad \begin{aligned} &(R^{-1}+RT^{-1})\Big(\int_{2R\le|x|\le 16R}\big|G_T(x)-\overline{G}_{T\{R\le|x|\le 32R\}}\big|^p\,dx\Big)^{1/p} \\ &\quad\lesssim R^{d/p-d+1}\big(\sqrt{T}R^{-1}\big)^{k/p}. \end{aligned}$$

For the second term, we proceed the same way, and appeal to (2.23) with exponents "$q = 1, r = k/p + 2$," which yields

$$\overline{G}_{T\{R\le|x|\le 32R\}} \lesssim R^{2-d}\big(\sqrt{T}R^{-1}\big)^{k/p+2} = TR^{-d}\big(\sqrt{T}R^{-1}\big)^{k/p},$$

and therefore

$$(4.60)\quad T^{-1}R^{d/p+1}\overline{G}_{T\{R\le|x|\le 32R\}} \lesssim R^{d/p-d+1}\big(\sqrt{T}R^{-1}\big)^{k/p}.$$

From (4.59) and (4.60), we then deduce that (2.24) holds for $R \ge \sqrt{T}$ as well.

4.3. *Proof of Corollaries* 2.2 *and* 2.3. These results are easy consequences of Lemmas 2.8 and 2.9. We include their proofs for convenience.

4.3.1. *Proof of Corollary* 2.2. W.l.o.g. we assume $y=0$ and skip the dependence on y in the notation. We distinguish two regimes: $|x|\le\sqrt{T}$ and $|x|\ge\sqrt{T}$.

In the first case, we use (2.22) and the intermediate results (4.4) in the proof of Lemma 2.8, which yield

$$\text{for } d=2 \qquad \int_{|x|\le\sqrt{T}} G_T^2(x)\,dx \lesssim T,$$
$$\text{for } d>2 \qquad \int_{|x|\le\sqrt{T}} G_T^q(x)\,dx \lesssim \sqrt{T}^d\bigl(\sqrt{T}^{2-d}\bigr)^q,$$

and imply for $q=\frac{d-1}{d-2}\in(1,\frac{d}{d-2})$ by the L^2-L^∞ estimate

$$G_T(x)\lesssim\sqrt{T} \qquad \text{for } |x|\le\sqrt{T}. \tag{4.61}$$

For $|x|\ge\sqrt{T}$, we use the decay estimate (2.23) of Lemma 2.8 with "$q=d, r=d(d+1)+1$"

$$\int_{R\le|x|\le 2R} G_T^d(x)\,dx \lesssim R^d\bigl(\sqrt{T}R^{-1}\bigr)^{d(d+1)+1}=\sqrt{T}^d\bigl(\sqrt{T}R^{-1}\bigr)^{d^2+1},$$

so that we may deduce

$$G_T(x)\lesssim\bigl(\sqrt{T}R^{-1}\bigr)^{d+1/d}\sqrt{T} \qquad \text{for } R\le|x|\le 2R. \tag{4.62}$$

We then define $h_T\in L^1(\mathbb{R}^d)$ by

$$h_T(x)\sim\begin{cases}\sqrt{T}2^{-k(d+1/d)}, & \sqrt{T}2^k\le|x|\le\sqrt{T}2^{k+1}, \qquad k\in\mathbb{N},\\ \sqrt{T}, & |x|\le\sqrt{T},\end{cases}$$

so that $G_T(x)\le h_T(x)$ for all $x\in\mathbb{Z}^d$. This concludes the proof since the factors in (4.61) and (4.62) only depend on α, β and d.

4.3.2. *Proof of Corollary* 2.3. We divide the proof in three steps. We first prove that the Green function $G_T(x,y)$ is symmetric so that $\nabla_x G_T(x,y)=\nabla_x G_T(y,x)$. In the second step, we show the uniform bound for $|x-y|\ge R$ sufficiently large, and in the third step for $|x-y|\le R$.

Step 1. Symmetry of G_T.

Let $y,\tilde{y}\in\mathbb{Z}^d$. Testing the defining equation (2.11) with $x\mapsto G_T(x,\tilde{y})$ yields

$$\int_{\mathbb{Z}^d} T^{-1}G_T(x,y)G_T(x,\tilde{y})\,dx+\int_{\mathbb{Z}^d}\nabla G_T(x,\tilde{y})\cdot A(x)\nabla G_T(x,y)\,dx=G_T(y,\tilde{y}).$$

Since A is symmetric, the left-hand side of this identity is symmetric in y and $\tilde{y}$. Hence, the right-hand side is also symmetric, that is, $G_T(y,\tilde{y})=G_T(\tilde{y},y)$.

Let $R\sim 1$ be sufficiently large so that Lemma 2.9 applies.

Step 2. Estimate for $|x-y|\ge R$.

For $q = 2$, formula (2.24) yields for all $k \in \mathbb{N}$

$$\int_{2^k R \le |x-y| \le 2^{k+1} R} |\nabla_x G_T(x,y)|^2 \, dx \lesssim (2^k R)^d ((2^k R)^{1-d})^2 = (2^k R)^{2-d} \overset{d \ge 2}{\lesssim} 1.$$

Hence, by the discrete $L^2 - L^\infty$ estimate, this shows

$$|\nabla_x G_T(x,y)| \lesssim 1 \qquad \text{for } |x-y| \ge R. \tag{4.63}$$

Step 3. Estimate for $|x-y| \le R$.

We now use an a priori estimate. Let $i \in \{1, \dots, d\}$ be fixed. We set $u(x) := G_T(x, y + \mathbf{e}_i) - G_T(x,y) = \nabla_{y_i} G_T(x,y)$. This function solves the equation

$$T^{-1} u - \nabla^* \cdot A \nabla u = f \qquad \text{in } \mathbb{Z}^d, \tag{4.64}$$

where $f(x) = \delta(y + \mathbf{e}_i - x) - \delta(y - x)$. Since f satisfies $\int_{\mathbb{Z}^d} f(x) = 0$, one has by integration by parts, ellipticity of A and Poincaré's inequality

$$\begin{aligned}
\int_{\mathbb{Z}^d} T^{-1} u(x)^2 \, dx + \alpha \int_{\mathbb{Z}^d} |\nabla u(x)|^2 \, dx &\le \int_{\mathbb{Z}^d} f(x) u(x) \, dx \\
&= \int_{|x-y| \le R} f(x) \big(u(x) - \bar{u}_{\{|x-y| \le R\}} \big) \, dx \\
&\lesssim R \bigg(\int_{\{|x-y| \le R\}} |\nabla u(x)|^2 \, dx \bigg)^{1/2}.
\end{aligned}$$

Hence,

$$\int_{\mathbb{Z}^d} |\nabla u(x)|^2 \, dx \lesssim R^2 \sim 1.$$

This shows that $\sup |\nabla u| \lesssim 1$. Therefore, for all x such that $|x-y| \le R$, we have using Step 2 and the fact that R is of order 1

$$|u(x)| \le R \sup |\nabla u| + \sup_{|z-y| \ge R} |u(z)| \lesssim 1.$$

Recalling that $u(x) = \nabla_{y_i} G_T(x,y)$, we conclude by Step 1 that this implies $|\nabla_{y_i} G_T(y,x)| \lesssim 1$, as desired.

5. Proofs of the other auxiliary lemmas.

5.1. *Proof of Lemma* 2.3. W.l.o.g. we may assume

$$\sum_{i=1}^{\infty} \bigg\langle \sup_{a_i} \bigg| \frac{\partial X}{\partial a_i} \bigg|^2 \bigg\rangle < \infty. \tag{5.1}$$

Let X_n denote the expected value of X conditioned on $a_1, \dots, a_n$, that is,

$$X_n(a_1, \dots, a_n) = \langle X | a_1, \dots, a_n \rangle.$$

We will establish the following two inequalities for $n < \tilde{n} \in \mathbb{N}$:

$$\langle X_n^2 \rangle - \langle X_n \rangle^2 \leq \sum_{i=1}^{n} \left\langle \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle \mathrm{var}[a_1], \tag{5.2}$$

$$\langle (X_{\tilde{n}} - X_n)^2 \rangle \leq \sum_{i=n+1}^{\tilde{n}} \left\langle \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle \mathrm{var}[a_1]. \tag{5.3}$$

Before proving (5.2) and (5.3), we draw the conclusion. There is a slight technical difficulty due to the fact that there are infinitely many random variables.

From (5.3) and (5.1), we learn that $\{X_n\}_{n\uparrow\infty}$ is a Cauchy sequence in L^2 w.r.t. probability. Hence, there exists a square integrable function $\tilde{X}$ of a such that

$$\lim_{n\uparrow\infty} \langle (\tilde{X} - X_n)^2 \rangle = 0. \tag{5.4}$$

By construction of X_n, (5.4) implies

$$\langle \tilde{X} | a_1, \dots, a_n \rangle = \langle X | a_1, \dots, a_n \rangle \qquad \text{for a. e. } (a_1, \dots, a_n) \text{ and all } n \in \mathbb{N}.$$

This means that the random variables X and $\tilde{X}$ agree on all measurable finite rectangular cylindrical sets, that is, measurable sets of the form $A_1 \times \cdots \times A_n \times \mathbb{R} \times \cdots$, where n is finite. Since these sets are stable under intersection and generate the entire σ-algebra of measurable sets, the random variables X and $\tilde{X}$ are uniquely determined by their value on these sets [10], Satz 14.12. Hence, the two random variables coincide, yielding

$$\tilde{X} = X \qquad \text{almost surely.} \tag{5.5}$$

From (5.2), (5.4) and (5.5), we obtain in the limit $n \uparrow \infty$ as desired

$$\mathrm{var}[X] = \langle X^2 \rangle - \langle X \rangle^2 \leq \sum_{i=1}^{\infty} \left\langle \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle \mathrm{var}[a_1].$$

We now turn to (5.2) and (5.3). Notice that we have the decomposition

$$\langle X_n^2 \rangle - \langle X_n \rangle^2 = \sum_{i=1}^{n} (\langle X_i^2 \rangle - \langle X_{i-1}^2 \rangle),$$

where we have set $X_0 :\equiv \langle X \rangle$ so that $\langle X_n \rangle^2 = \langle X_0^2 \rangle$. Hence, (5.2) reduces to

$$\langle X_i^2 \rangle - \langle X_{i-1}^2 \rangle \leq \left\langle \sup_{a_i} \left| \frac{\partial X}{\partial a_i} \right|^2 \right\rangle \mathrm{var}[a_1]. \tag{5.6}$$

Likewise,

$$\langle (X_{\tilde{n}} - X_n)^2 \rangle = \langle X_{\tilde{n}}^2 \rangle - \langle X_n^2 \rangle = \sum_{i=n+1}^{\tilde{n}} (\langle X_i^2 \rangle - \langle X_{i-1}^2 \rangle),$$

so that also (5.3) reduces to (5.6).

We finally turn to (5.6). We note that by our assumption that $\{a_i\}_{i\in\mathbb{N}}$ are i.i.d., we have

$$\langle X_i^2(a_1,\dots,a_i)\rangle = \left\langle \int X_i^2(a_1,\dots,a_{i-1},a_i')\beta(da_i')\right\rangle,$$

$$X_{i-1}(a_1,\dots,a_{i-1}) = \int X_i(a_1,\dots,a_{i-1},a_i'')\beta(da_i''),$$

where β denotes the distribution of a_1. Hence, we obtain

$$\begin{aligned}
&\langle X_i^2\rangle - \langle X_{i-1}^2\rangle \\
&\quad= \left\langle \int X_i^2(a_1,\dots,a_{i-1},a_i')\beta(da_i') - \left(\int X_i(a_1,\dots,a_{i-1},a_i'')\beta(da_i'')\right)^2\right\rangle \\
&\quad= \left\langle \int\int \frac{1}{2}\big(X_i(a_1,\dots,a_{i-1},a_i') - X_i(a_1,\dots,a_{i-1},a_i'')\big)^2\beta(da_i')\beta(da_i'')\right\rangle \\
&\quad\le \left\langle \int\int \sup_{a_i'''}\left|\frac{\partial X_i}{\partial a_i}(a_1,\dots,a_{i-1},a_i''')\right|^2 \frac{1}{2}(a_i'-a_i'')^2\beta(da_i')\beta(da_i'')\right\rangle \\
&\quad= \left\langle \sup_{a_i'''}\left|\frac{\partial X_i}{\partial a_i}(a_1,\dots,a_{i-1},a_i''')\right|^2\right\rangle \left(\int (a_i')^2\beta(da_i') - \left(\int a_i''\beta(da_i'')\right)^2\right) \\
&\quad= \left\langle \sup_{a_i'''}\left|\frac{\partial X_i}{\partial a_i}(a_1,\dots,a_{i-1},a_i''')\right|^2\right\rangle \operatorname{var}[a_1].
\end{aligned}$$

We conclude by noting that by the definition of X_i and Jensen's inequality

$$\left|\frac{\partial X_i}{\partial a_i}(a_1,\dots,a_i)\right|^2 = \left|\left\langle \frac{\partial X}{\partial a_i}\Big| a_1,\dots,a_i\right\rangle\right|^2 \le \left\langle \left|\frac{\partial X}{\partial a_i}\right|^2 \Big| a_1,\dots,a_i\right\rangle,$$

so that

$$\begin{aligned}
&\left\langle \sup_{a_i'}\left|\frac{\partial X_i}{\partial a_i}(a_1,\dots,a_{i-1},a_i')\right|^2\right\rangle \\
&\qquad\le \left\langle\left\langle \sup_{a_i'}\left|\frac{\partial X}{\partial a_i}(a_1,\dots,a_{i-1},a_i',a_{i+1},\dots)\right|^2 \Big| a_1,\dots,a_i\right\rangle\right\rangle \\
&\qquad= \left\langle \sup_{a_i'}\left|\frac{\partial X}{\partial a_i}(a_1,\dots,a_{i-1},a_i',a_{i+1},\dots)\right|^2\right\rangle.
\end{aligned}$$

5.2. *Proof of Lemma* 2.5. Let us divide the proof in four steps.

Step 1. Proof of (2.15).

We recall the definition of the operator

$$(Lu)(x) = \sum_{x', |x'-x|=1} a(x,x')\big(u(x) - u(x')\big).$$

For convenience, we set $e = [z, z']$, $z' = z + \mathbf{e}_i$. We recall that $G_T(\cdot, y)$, $y \in \mathbb{Z}^d$, is defined via

$$(T^{-1} + L)G_T(\cdot, y)(x) = \delta(x - y), \qquad x \in \mathbb{Z}^d. \tag{5.7}$$

Hence, we obtain by differentiating (5.7)

$$\begin{aligned}\Big((T^{-1} + L)\,\frac{\partial}{\partial a(e)} G_T(\cdot, y)\Big)(x) + \big(G_T(z, y) - G_T(z', y)\big)\delta(x - z)\\ + \big(G_T(z', y) - G_T(z, y)\big)\delta(x - z') = 0,\end{aligned}$$

which, in view of (5.7), can be rewritten as

$$\begin{aligned}(T^{-1} + L)\Big(\frac{\partial}{\partial a(e)} G_T(\cdot, y) + \big(G_T(z, y) - G_T(z', y)\big)G_T(\cdot, z)\\ + \big(G_T(z', y) - G_T(z, y)\big)G_T(\cdot, z')\Big) \equiv 0.\end{aligned} \tag{5.8}$$

From this, we would like to conclude

$$\begin{aligned}\frac{\partial}{\partial a(e)} G_T(\cdot, y) + \big(G_T(z, y) - G_T(z', y)\big)G_T(\cdot, z)\\ + \big(G_T(z', y) - G_T(z, y)\big)G_T(\cdot, z') \equiv 0,\end{aligned} \tag{5.9}$$

which is nothing but (2.15).

In order to draw this conclusion, we will appeal to the following uniqueness result in $L^2(\mathbb{Z}^d)$: any $u \in L^2(\mathbb{Z}^d)$ which satisfies $((T^{-1} + L)u)(x) = 0$ for all $x \in \mathbb{Z}^d$ vanishes identically. However, we cannot apply this directly to u given by the left-hand side of (5.9), since we do not know a priori that $\frac{\partial}{\partial a(e)} G_T(\cdot, y)$ is in $L^2(\mathbb{Z}^d)$.

For that purpose, we replace the derivative $\frac{\partial}{\partial a(e)}$ by the difference quotient. We thus fix a step size $h \neq 0$ and introduce the abbreviations

$$G_T(x, y) := G_T(x, y; a) \quad \text{and} \quad G'_T(x, y) := G_T(x, y, a'),$$

where the coefficients a' are defined by modifying a only at edge e by the increment h, that is,

$$a'(e) = a(e) + h \quad \text{and} \quad a'(e') = a(e') \qquad \text{for all } e' \neq e.$$

We further denote by $L_T := T + L_a$ and $L'_T := T + L_{a'}$ the operators with coefficients a and a', respectively. We mimic the derivation of (5.8) on the discrete level: from (5.7), we obtain

$$
\begin{aligned}
0 &= \frac{1}{h}\big(L_T G_T(\cdot, y) - L'_T G'_T(\cdot, y)\big) \\
&= L_T \frac{1}{h}\big(G_T(\cdot, y) - G'_T(\cdot, y)\big) + \frac{1}{h}(L_T - L'_T)G'_T(\cdot, y) \\
&= L_T \frac{1}{h}\big(G_T(\cdot, y) - G'_T(\cdot, y)\big) + \big(G'_T(z, y) - G'_T(z', y)\big)\delta(\cdot - z) \\
&\quad + \big(G'_T(z', y) - G'_T(z, y)\big)\delta(\cdot - z') \\
&= L_T \bigg(\frac{1}{h}\big(G_T(\cdot, y) - G'_T(\cdot, y)\big) + \big(G'_T(z, y) - G'_T(z', y)\big)G_T(\cdot, z) \\
&\qquad\qquad + \big(G'_T(z', y) - G'_T(z, y)\big)G_T(\cdot, z')\bigg).
\end{aligned}
$$

Since for fixed $h \neq 0$,

$$
\begin{aligned}
u_h :\equiv{}& \frac{1}{h}\big(G_T(\cdot, y) - G'_T(\cdot, y)\big) + \big(G'_T(z, y) - G'_T(z', y)\big)G_T(\cdot, z) \\
&+ \big(G'_T(z', y) - G'_T(z, y)\big)G_T(\cdot, z')
\end{aligned}
$$

does inherit the integrability properties of $G_T(\cdot, y)$ and $G'_T(\cdot, y)$ from Corollary 2.2, we now may conclude that $u_h \in L^2(\mathbb{Z}^d)$, and therefore $u_h \equiv 0$, that is,

$$
\begin{aligned}
&\frac{1}{h}\big(G_T(x, y) - G'_T(x, y)\big) + \big(G'_T(z, y) - G'_T(z', y)\big)G_T(x, z) \\
&\qquad + \big(G'_T(z', y) - G'_T(z, y)\big)G_T(x, z') = 0
\end{aligned}
$$

for every $x \in \mathbb{Z}^d$. Since by Lemma 6, $G_T(x, y; \cdot)$ is continuous in $a(e)$, we learn that $G_T(x, y; \cdot)$ is continuously differentiable w.r.t. $a(e)$ and that (2.15) holds.

We set for abbreviation

$$
\begin{aligned}
&G_T(x, e) := G_T(x, z) - G_T(x, z'), \\
&G_T(e, y) := G_T(z, y) - G_T(z', y), \\
&G_T(e, e) := G_T(z, z) + G_T(z', z') - G_T(z, z') - G_T(z', z).
\end{aligned}
\tag{5.10}
$$

Step 2. Proof of

$$
\begin{aligned}
&\frac{\partial}{\partial a(e)} G_T(x, e) = -G_T(e, e)G_T(x, e), \\
&\frac{\partial}{\partial a(e)} G_T(e, y) = -G_T(e, e)G_T(e, y).
\end{aligned}
\tag{5.11}
$$

This is a consequence of (2.15) for $y = z, z'$:

$$\begin{aligned}
&\frac{\partial}{\partial a(e)}\big(G_T(x,z) - G_T(x,z')\big) \\
&\quad = \frac{\partial}{\partial a(e)} G_T(x,z) - \frac{\partial}{\partial a(e)} G_T(x,z') \\
&\quad \overset{(2.15)}{=} -\big(G_T(x,z) - G_T(x,z')\big)\big(G_T(z,z) - G_T(z',z)\big) \\
&\qquad + \big(G_T(x,z) - G_T(x,z')\big)\big(G_T(z,z') - G_T(z',z')\big) \\
&\quad = -\big(G_T(z,z) + G_T(z',z') - G_T(z,z') - G_T(z,z')\big) \\
&\qquad \times \big(G_T(x,z) - G_T(x,z')\big)
\end{aligned}$$

and for $x = z, z'$, respectively.

Step 3. Conclusion.

Note that Corollary 2.3 implies

$$|G_T(e,e)| \lesssim 1. \tag{5.12}$$

The combination of (5.11) with (5.12) yields

$$\left|\frac{\partial}{\partial a(e)} G_T(x,e)\right| \lesssim |G_T(x,e)|, \qquad \left|\frac{\partial}{\partial a(e)} G_T(e,y)\right| \lesssim |G_T(e,y)|.$$

Since $a(e)$ is bounded, this also yields

$$\sup_{a(e)} |G_T(x,e)| \sim |G_T(x,e)|, \qquad \sup_{a(e)} |G_T(e,y)| \sim |G_T(e,y)|,$$

which is nothing but (2.16).

5.3. *Proof of Lemma* 2.4. We recall that $e = [z, z']$, $z' = z + \mathbf{e}_i$.

Step 1. Proof of (2.12).

We first give a heuristic argument for (2.12) based on the defining equation

$$T^{-1}\phi_T(x) - \big(\nabla^* \cdot A\big(\nabla\phi_T(x) + \xi\big)\big)(x) = 0. \tag{5.13}$$

Differentiating (5.13) w.r.t. $a(e)$ yields as in Step 1 of the proof of Lemma 2.5

$$\begin{aligned}
&T^{-1}\frac{\partial \phi_T}{\partial a(e)}(x) - \left(\nabla^* \cdot A\nabla \frac{\partial \phi_T}{\partial a(e)}\right)(x) \\
&\qquad - \big(\nabla_i \phi_T(z) + \xi_i\big)\big(\delta(x-z) - \delta(x-z')\big) = 0.
\end{aligned} \tag{5.14}$$

Provided we have $\frac{\partial \phi_T}{\partial a(e)} \in L^2(\mathbb{Z}^d)$, this yields by definition of G_T

$$\frac{\partial \phi_T}{\partial a(e)}(x) = -\big(\nabla_i \phi_T(z) + \xi_i\big)\big(G_T(x,z') - G_T(x,z)\big),$$

which is (2.12).

In order to turn the above into a rigorous argument, we need to argue that $\phi_T(x)$ is differentiable w.r.t. $a(e)$ and that $\frac{\partial \phi_T}{\partial a(e)} \in L^2(\mathbb{Z}^d)$. Starting point is the representation formula from Step 2 of the proof of Lemma 2.6, that is,

$$\phi_T(x) = \int_{\mathbb{Z}^d} G_T(x,y)\nabla^* \cdot (A(y)\xi)\,dy. \tag{5.15}$$

Combined with Corollary 2.2, (5.15) and (2.15) in Lemma 2.5 show that $\phi_T(x)$ is differentiable w.r.t. $a(e)$. We may now switch the order of the differentiation and the sum as follows:

$$\begin{aligned}\frac{\partial \phi_T}{\partial a(e)}(x) &= -\nabla_{z_i} G_T(x,z)\xi_i \\ &\quad - \int_{\mathbb{Z}^d} \nabla_{z_i} G_T(x,z)\nabla_{z_i} G_T(z,y)\nabla^* \cdot (A(y)\xi)\,dy \\ &= -\underbrace{\nabla_{z_i} G_T(x,z)}_{\in L^2_x(\mathbb{Z}^d)}\biggl(\xi_i + \int_{\mathbb{Z}^d} \underbrace{\nabla_{z_i} G_T(z,y)}_{\in L^1_y(\mathbb{Z}^d)}\underbrace{\nabla^* \cdot (A(y)\xi)}_{\in L^\infty(\mathbb{Z}^d)}\,dy\biggr),\end{aligned} \tag{5.16}$$

since $G_T(\cdot,z) \in L^2(\mathbb{Z}^d)$ by definition of the Green's function, $G_T(z,\cdot) \in L^1(\mathbb{Z}^d)$ by Corollary 2.2 and A is bounded. This proves that $\frac{\partial \phi_T}{\partial a(e)} \in L^2(\mathbb{Z}^d)$.

Step 2. Proof of

$$\sup_{a(e)} |\phi_T(x)| \lesssim |\phi_T(x)| + \bigl(|\nabla_i \phi_T(z)| + 1\bigr)|\nabla_{z_i} G_T(z,x)|, \tag{5.17}$$

$$\sup_{a(e)} \left|\frac{\partial \phi_T(x)}{\partial a(e)}\right| \lesssim \bigl(|\nabla_i \phi_T(z)| + 1\bigr)|\nabla_{z_i} G_T(z,x)|. \tag{5.18}$$

We argue that it is enough to prove (2.14). Indeed, the combination of (2.12), (2.16) and (2.14) with the boundedness of a implies (5.17) and (5.18). In order to prove (2.14), we proceed as follows

$$\begin{aligned}-\left(\nabla_i \frac{\partial \phi_T}{\partial a(e)}\right)(z) &= \frac{\partial \phi_T}{\partial a(e)}(z) - \frac{\partial \phi_T}{\partial a(e)}(z') \\ &\overset{(2.12)}{=} \bigl(\nabla_i \phi_T(z) + \xi_i\bigr)\bigl(G_T(z,z) - G_T(z,z')\bigr) \\ &\quad - \bigl(\nabla_i \phi_T(z) + \xi_i\bigr)\bigl(G_T(z',z) - G_T(z',z')\bigr) \\ &= \bigl(\nabla_i \phi_T(z) + \xi_i\bigr) \\ &\quad \times \bigl(G_T(z,z) - G_T(z,z') - G_T(z',z) + G_T(z',z')\bigr) \\ &= \bigl(\nabla_i \phi_T(z) + \xi_i\bigr)G_T(e,e),\end{aligned} \tag{5.19}$$

where we used the abbreviation

$$G_T(e,e) = G_T(z,z) - G_T(z,z') - G_T(z',z) + G_T(z',z').$$

Recalling that Corollary 2.3 implies

$$G_T(e,e) \lesssim 1,$$

inequality (2.14) follows now from (5.19) and the boundedness of a.

Step 3. Proof of (2.13).

For $n \geq 0$, the chain rule yields

$$\frac{\partial[\phi_T(x)^{n+1}]}{\partial a(e)} = (n+1)\phi_T(x)^n \frac{\partial \phi_T(x)}{\partial a(e)}.$$

Using (5.17) and (5.18), this implies

$$\sup_{a(e)}\left|\frac{\partial[\phi_T(x)^{n+1}]}{\partial a(e)}\right| \lesssim \big(|\phi_T(x)| + \big(|\nabla_i \phi_T(z)|+1\big)|\nabla_{z_i} G_T(z,x)|\big)^n$$
$$\times \big(\big(|\nabla_i \phi_T(z)|+1\big)|\nabla_{z_i} G_T(z,x)|\big),$$

which turns into (2.13) using Young's inequality.

5.4. *Proof of Lemma* 2.6. We first prove the claim for G_T and deduce the result for ϕ_T appealing to an integral representation using the Green's function.

Step 1. Properties of G_T.

The product topology is the topology of componentwise convergence. Hence, we consider an arbitrary sequence $\{a_\nu\}_{\nu\uparrow\infty} \subset \mathcal{A}_{\alpha\beta}$ of coefficients such that

$$\lim_{\nu\uparrow\infty} a_\nu(e) = a(e) \qquad \text{for all edges } e. \tag{5.20}$$

Fix $y \in \mathbb{Z}^d$; by the uniform bounds on $G_T(\cdot, y; a_\nu)$ from Corollary 2.2, we can select a subsequence ν' such that

$$u_T(x) := \lim_{\nu'\uparrow\infty} G_T(x, y; a_{\nu'}) \qquad \text{exists for all } x \in \mathbb{Z}^d. \tag{5.21}$$

It remains to argue that $u_T(x) = G_T(x, y; a)$. Because of (5.20) and (5.21), we can pass to the limit in $(T^{-1}G_T(\cdot, y; a_{\nu'}) + L_{a_{\nu'}} G_T(\cdot, y; a_{\nu'}))(x) = \delta(x-y)$ to obtain

$$(T^{-1}u_T + L_a u_T)(x) = \delta(x-y) \qquad \text{for all } x \in \mathbb{Z}^d. \tag{5.22}$$

Moreover, the uniform decay of $G_T(\cdot, y; a_\nu)$ from Corollary 2.2 is preserved in the limit, so that $u_T \in L^1(\mathbb{Z}^d) \subset L^2(\mathbb{Z}^d)$. Note that Riesz's representation theorem on $L^2(\mathbb{Z}^d)$ yields uniqueness for the solution of (5.22) in $L^2(\mathbb{Z}^d)$. Hence, we conclude as desired that $u_T(\cdot) = G_T(\cdot, y; a)$. Borel measurability of $G_T(x, y; \cdot)$ in the sense of Lemma 2.3 follows from continuity w.r.t. the product topology, cf. [10], Satz 14.8.

Step 2. Properties of ϕ_T.

Corollary 2.2 ensures that $G_T(x, \cdot) \in L^1(\mathbb{Z}^d)$ for all $x \in \mathbb{Z}^d$ and one may then define a function $\tilde{\phi}_T$ by

$$\tilde{\phi}_T(x) = \int_{\mathbb{Z}^d} G_T(x,y)\nabla^* \cdot (A(y)\xi_i)\, dy. \tag{5.23}$$

Since $G_T(\cdot + z, \cdot + z)$ has the same law as $G_T(\cdot, \cdot)$ by uniqueness of the Green's function and joint stationarity of the coefficient A, $\tilde{\phi}_T(\cdot + z)$ has the same law as $\tilde{\phi}_T$. This shows that $\tilde{\phi}_T$ is stationary. In addition, $\tilde{\phi}_T$ is a solution of (2.3) by construction. Hence, by the uniqueness of stationary solutions of (2.3), $\tilde{\phi}_T = \phi_T$ almost surely, so that by the measurability properties we may assume $\tilde{\phi}_T \equiv \phi_T$.

Introducing for $R \geq 1$

$$\phi_{T,R}(x) := \int_{|y| \leq R} G_T(x, y) \nabla^* \cdot (A(y)\xi_i)\, dy,$$

one may rewrite (5.23) as

$$\phi_T(x) = \lim_{R \to \infty} \phi_{T,R}(x). \tag{5.24}$$

From Step 1, $\phi_{T,R}(x)$ is a continuous function of a since $G_T(x, y)$ is and the formula for $\phi_{T,R}(x)$ involves only a *finite* number of operations. Note that Corollary 2.2 implies that

$$\lim_{R \uparrow \infty} \sup_{a \in \mathcal{A}_{\alpha\beta}} \int_{|y| > R} G_T(x, y; a)\, dy = 0.$$

Hence, the convergence in (5.24) is uniform in a and the continuity of $\phi_{T,R}$ in a is preserved at the limit. Therefore, ϕ_T (and continuous functions thereof) are continuous with respect to the product topology, and hence Borel measurable.

5.5. *Proof of Lemma* 2.7. We first sketch the proof in the continuous case, that is, with $\mathbb{Z}^d$ replaced by $\mathbb{R}^d$.

Step 1. Continuous version.

Starting point is the defining equation (2.3) of the corrector ϕ_T in its continuous version, that is,

$$T^{-1}\phi_T - \nabla \cdot A(\nabla \phi_T + \xi) = 0 \qquad \text{in } \mathbb{R}^d. \tag{5.25}$$

We multiply (5.25) with ϕ_T^{n+1} and obtain by Leibniz' rule:

$$\begin{aligned} 0 &= T^{-1}\phi_T^{n+2} + \big(-\nabla \cdot A(\nabla \phi_T + \xi)\big)\phi_T^{n+1} \\ &= T^{-1}\phi_T^{n+2} - \nabla \cdot \big(\phi_T^{n+1} A(\nabla \phi_T + \xi)\big) + \nabla \phi_T^{n+1} \cdot A(\nabla \phi_T + \xi) \\ &= T^{-1}\phi_T^{n+2} - \nabla \cdot \big(\phi_T^{n+1} A(\nabla \phi_T + \xi)\big) \\ &\quad + (n+1)\phi_T^n \nabla \phi_T \cdot A(\nabla \phi_T + \xi). \end{aligned} \tag{5.26}$$

We then take the expected value. Since the random fields A and ϕ_T are jointly stationary, and thus also $\phi_T^{n+1} A(\nabla \phi_T + \xi)$, we obtain

$$\langle T^{-1}\phi_T^{n+2}\rangle + (n+1)\langle \phi_T^n \nabla \phi_T \cdot A(\nabla \phi_T + \xi)\rangle = 0,$$

and therefore

$$\langle \phi_T^n \nabla \phi_T \cdot A(\nabla \phi_T + \xi)\rangle \leq 0$$

since $n+2$ is even. By the uniform ellipticity of A and since $\phi_T^n \geq 0$ (n is even) and $|\xi|=1$, this yields the estimate

$$\langle \phi_T^n |\nabla\phi_T|^2\rangle \lesssim \langle \phi_T^n |\nabla\phi_T|\rangle.$$

Applying Cauchy–Schwarz' inequality in probability on the right-hand side of this inequality yields the continuum version of (2.17), that is,

$$\langle \phi_T^n |\nabla\phi_T|^2\rangle \lesssim \langle \phi_T^n \rangle.$$

We now turn to our discrete case.

Step 2. Discrete version.

We need a discrete version of the Leibniz rule $\nabla\cdot(fg) = f\nabla\cdot g + \nabla f\cdot g$ used in (5.26). Let $f\in L^2_{\mathrm{loc}}(\mathbb{Z}^d)$ and $g\in L^2_{\mathrm{loc}}(\mathbb{Z}^d,\mathbb{R}^d)$, then this formula is replaced by

$$\begin{aligned}\nabla^*\cdot(fg)(z) &= \sum_{j=1}^d \bigl(f(z)[g(z)]_j - f(z-\mathbf{e}_j)[g(z-\mathbf{e}_j)]_j\bigr)\\ &= f(z)\nabla^*\cdot g(z) + \sum_{j=1}^d \nabla_j^* f(z)[g(z-\mathbf{e}_j)]_j.\end{aligned} \tag{5.27}$$

We also need a substitute for the identity $\nabla\phi_T^{n+1} = (n+1)\phi_T^n\nabla\phi_T$ used in (5.26). This substitute is provided by the two calculus estimates

$$(\tilde\phi^{n+1}-\phi^{n+1})(\tilde\phi-\phi) \gtrsim (\tilde\phi^n+\phi^n)(\tilde\phi-\phi)^2, \tag{5.28}$$

$$|\tilde\phi^{n+1}-\phi^{n+1}| \lesssim (\tilde\phi^n+\phi^n)|\tilde\phi-\phi|. \tag{5.29}$$

For the convenience of the reader, we sketch their proof: by the well-known formula for $\tilde\phi^{n+1}-\phi^{n+1}$, they are equivalent to

$$\sum_{m=0}^n \phi^m\tilde\phi^{n-m} \sim \tilde\phi^n+\phi^n.$$

By homogeneity, we may assume $\tilde\phi = 1$, so that the above turns into

$$\sum_{m=0}^n \phi^m \sim 1+\phi^n.$$

The upper estimate is obvious by Hölder's inequality since n is even. Also for the lower bound, we use the evenness of n to rearrange the sum as follows:

$$\begin{aligned}\sum_{m=0}^n \phi^m &= \frac{1}{2}1 + \frac{1}{2}(1+2\phi+\phi^2) + \frac{1}{2}\phi^2(1+2\phi+\phi^2) + \cdots\\ &\quad + \frac{1}{2}\phi^{n-2}(1+2\phi+\phi^2) + \frac{1}{2}\phi^n\\ &\geq \frac{1}{2}(1+\phi^n).\end{aligned}$$

After these motivations and preparations, we turn to the proof of Lemma 2.7 proper. With $f(z) := \phi_T^{n+1}(z)$ and $g(z) := A(\nabla\phi_T + \xi)(z)$, (5.27) turns into

$$\begin{aligned}
&\nabla^* \cdot \big(\phi_T^{n+1}(z) A(\nabla\phi_T + \xi)(z)\big) \\
&\quad = \phi_T^{n+1}(z)\nabla^* \cdot A(\nabla\phi_T + \xi)(z) + \sum_{j=1}^{d} \nabla_j^*\phi_T^{n+1}(z) \underbrace{[A(\nabla\phi_T + \xi)(z - \mathbf{e}_j)]_j}_{\substack{= a(z-\mathbf{e}_j, z)(\nabla_j\phi_T(z-\mathbf{e}_j)+\xi_j) \\ = a(z-\mathbf{e}_j, z)(\nabla_j^*\phi_T(z)+\xi_j)}} .
\end{aligned}$$

Hence,

$$\begin{aligned}
&-\phi_T^{n+1}(z)\nabla^* \cdot A(\nabla\phi_T + \xi)(z) \\
&\quad = \sum_{j=1}^{d} \nabla_j^*\phi_T^{n+1}(z) a(z - \mathbf{e}_j, z)\big(\nabla_j^*\phi_T(z) + \xi_j\big) \\
&\qquad - \nabla^* \cdot \big(\phi_T^{n+1}(z) A(\nabla\phi_T + \xi)(z)\big).
\end{aligned} \tag{5.30}$$

Multiplying (2.3) with $\phi_T^{n+1}(z)$ and using (5.30) emulate (5.26) and yield

$$\begin{aligned}
0 &= T^{-1}\phi_T^{n+2}(z) - \nabla^* \cdot \big(\phi_T^{n+1}(z) A(\nabla\phi_T + \xi)(z)\big) \\
&\quad + \sum_{j=1}^{d} \nabla_j^*\phi_T^{n+1}(z) a(z - \mathbf{e}_j, z)\big(\nabla_j^*\phi_T(z) + \xi_j\big).
\end{aligned} \tag{5.31}$$

Taking the expectation of (5.31) and noting that $\phi_T^{n+2} \geq 0$, we obtain as for the continuous case

$$\begin{aligned}
&\left\langle \sum_{j=1}^{d} a(z - \mathbf{e}_j, z)\nabla_j^*\phi_T^{n+1}(z)\nabla_j^*\phi_T(z) \right\rangle \\
&\quad \lesssim \left\langle \sum_{j=1}^{d} a(z - \mathbf{e}_j, z)|\nabla_j^*\phi_T^{n+1}(z)| \right\rangle.
\end{aligned} \tag{5.32}$$

On the one hand, we have

$$\begin{aligned}
&\sum_{j=1}^{d} a(z - \mathbf{e}_j, z)\nabla_j^*\phi_T^{n+1}(z)\nabla_j^*\phi_T(z) \\
&\quad = \sum_{j=1}^{d} a(z - \mathbf{e}_j, z)\big(\phi_T^{n+1}(z) - \phi_T^{n+1}(z - \mathbf{e}_j)\big)\big(\phi_T(z) - \phi_T(z - \mathbf{e}_j)\big) \\
&\quad \overset{(5.28)}{\gtrsim} \sum_{j=1}^{d} \big(\phi_T^{n}(z) + \phi_T^{n}(z - \mathbf{e}_j)\big)\big(\phi_T(z) - \phi_T(z - \mathbf{e}_j)\big)^2.
\end{aligned} \tag{5.33}$$

On the other hand, we observe

$$
\begin{aligned}
&\sum_{j=1}^{d} a(z-\mathbf{e}_j,z)|\nabla_j^* \phi_T^{n+1}(z)| \\
&\quad = \sum_{j=1}^{d} a(z-\mathbf{e}_j,z)|\phi_T^{n+1}(z)-\phi_T^{n+1}(z-\mathbf{e}_j)| \qquad (5.34)\\
&\quad \overset{(5.29)}{\lesssim} \sum_{j=1}^{d} \bigl(\phi_T^n(z)+\phi_T^n(z-\mathbf{e}_j)\bigr)|\phi_T(z)-\phi_T(z-\mathbf{e}_j)|.
\end{aligned}
$$

Now (5.32), (5.33) and (5.34) combine to

$$
\begin{aligned}
&\left\langle \sum_{j=1}^{d} \bigl(\phi_T^n(z)+\phi_T^n(z-\mathbf{e}_j)\bigr)\bigl(\phi_T(z)-\phi_T(z-\mathbf{e}_j)\bigr)^2 \right\rangle \\
&\quad \lesssim \sum_{j=1}^{d} \bigl\langle \bigl(\phi_T^n(z)+\phi_T^n(z-\mathbf{e}_j)\bigr)|\phi_T(z)-\phi_T(z-\mathbf{e}_j)|\bigr\rangle.
\end{aligned}
$$

By stochastic homogeneity, this reduces to

$$
\begin{aligned}
&\left\langle \sum_{j=1}^{d} \bigl(\phi_T^n(\mathbf{e}_j)+\phi_T^n(0)\bigr)\bigl(\phi_T(\mathbf{e}_j)-\phi_T(0)\bigr)^2 \right\rangle \\
&\quad \lesssim \left\langle \sum_{j=1}^{d} \bigl(\phi_T^n(\mathbf{e}_j)+\phi_T^n(0)\bigr)|\phi_T(\mathbf{e}_j)-\phi_T(0)| \right\rangle.
\end{aligned}
$$

An application of Cauchy–Schwarz inequality yields

$$
\left\langle \sum_{j=1}^{d} \bigl(\phi_T^n(\mathbf{e}_j)+\phi_T^n(0)\bigr)\bigl(\phi_T(\mathbf{e}_j)-\phi_T(0)\bigr)^2 \right\rangle \lesssim \left\langle \sum_{j=1}^{d} \bigl(\phi_T^n(\mathbf{e}_j)+\phi_T^n(0)\bigr) \right\rangle.
$$

A last application of stochastic homogeneity gives as desired

$$
\left\langle \phi_T^n(0) \sum_{j=1}^{d} \Bigl(\bigl(\phi_T(\mathbf{e}_j)-\phi_T(0)\bigr)^2+\bigl(\phi_T(0)-\phi_T(-\mathbf{e}_j)\bigr)^2\Bigr) \right\rangle \lesssim \langle \phi_T^n(0)\rangle.
$$

5.6. *Proof of Lemma* 2.10. The proof relies on a doubly dyadic decomposition of space. First note that by symmetry,

$$
\begin{aligned}
\int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz &= \int_{|z|\ge|z-x|} h_T(z)h_T(z-x)\,dz \\
&\ge \frac{1}{2}\int_{\mathbb{Z}^d} h_T(z)h_T(z-x)\,dz.
\end{aligned}
$$

Hence, it is enough to consider

$$\int_{|x|\le R}\int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx.$$

In the three first steps, we treat the case $d>2$. We then sketch the modification for $d=2$ in the last step. Let $\tilde{R}\sim 1$ be such that (2.27) holds with a constant independent of R for all $R\ge \tilde{R}/2$.

Step 1. Proof of

$$\int_{R<|x|\le 2R}\int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx \lesssim R^2 \qquad \text{for } R\ge 2\tilde{R}. \tag{5.35}$$

Let $N\in\mathbb{N}$ be such that $\tilde{R}\le 2^{-N}R\le 2\tilde{R}$. We then decompose the sum over $|z|\le|z-x|$ into three contributions: $R/2<|z|$, a dyadic decomposition for $\tilde{R}<|z|\le R/2$ and a remainder on $|z|\le\tilde{R}$. More precisely,

$$\begin{aligned}
&\int_{R<|x|\le 2R}\int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx\\
&\quad=\int_{R<|x|\le 2R}\int_{R/2<|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx\\
&\qquad+\sum_{n=1}^{N}\int_{R<|x|\le 2R}\int_{\{2^{-(n+1)}R<|z|\le 2^{-n}R\}\cap\{|z|\le|z-x|\}} h_T(z)h_T(z-x)\,dz\,dx\\
&\qquad+\int_{R<|x|\le 2R}\int_{\{|z|\le 2^{-(N+1)}R\}\cap\{|z|\le|z-x|\}} h_T(z)h_T(z-x)\,dz\,dx\\
&\quad\le\int_{|x|\le 2R}\underbrace{\int_{R/2<|z|\le|z-x|} h_T(z)h_T(z-x)\,dz}_{=I_1}\,dx\\
&\qquad+\sum_{n=1}^{N}\underbrace{\int_{R<|x|\le 2R}\int_{2^{-(n+1)}R<|z|\le 2^{-n}R} h_T(z)h_T(z-x)\,dz\,dx}_{=I_2(n)}\\
&\qquad+\underbrace{\int_{R<|x|\le 2R}\int_{|z|\le\tilde{R}} h_T(z)h_T(z-x)\,dz\,dx}_{=I_3(N)}.
\end{aligned}$$

We use Young's inequality, a dyadic decomposition of $\{|z|>R/2\}$, and the assumption (2.27) to bound I_1:

$$\begin{aligned}
I_1&\le\frac{1}{2}\left(\int_{R/2<|z|} h_T(z)^2\,dz+\int_{R/2<|z-x|} h_T(z-x)^2\,dz\right)\\
&=\sum_{k=-1}^{\infty}\int_{2^kR<|z|\le 2^{k+1}R} h_T^2(z)\,dz \overset{(2.27)}{\lesssim}\sum_{k=-1}^{\infty}\left(\frac{1}{2^{d-2}}\right)^k R^{2-d}\lesssim R^{2-d}.
\end{aligned}$$

In order to bound $I_2(n)$, we will use the following fact

$$\big(|x| > R \text{ and } |z| \le \tfrac{1}{2}R\big) \Rightarrow \big(|z-x| > \tfrac{1}{2}R\big). \tag{5.36}$$

We have by Cauchy–Schwarz inequality

$$\begin{aligned} I_2(n) &\le \bigg(\int_{|x|\le 2R}\int_{2^{-(n+1)}R<|z|\le 2^{-n}R} h_T(z)^2\,dz\,dx \\ &\qquad \times \int_{R<|x|\le 2R}\int_{|z|\le 2^{-n}R} h_T(z-x)^2\,dz\,dx\bigg)^{1/2} \\ &\lesssim \bigg(R^d \underbrace{\int_{2^{-(n+1)}R<|z|\le 2^{-n}R} h_T(z)^2\,dz}_{\overset{(2.27)}{\lesssim}(2^{-n}R)^{2-d}} \\ &\qquad \times \underbrace{\int_{R<|x|\le 2R}\int_{|z|\le 2^{-n}R} h_T(z-x)^2\,dz\,dx}_{\overset{(5.36)}{\lesssim}\underbrace{\int_{|z|\le 2^{-n}R}\int_{R/2<|z-x|\le 5R/2} h_T(z-x)^2\,dx\,dz}_{\overset{(2.27)}{\lesssim}\int_{|z|\le 2^{-n}R} R^{2-d}\,dz=(2^{-n}R)^d R^{2-d}}}\bigg)^{1/2} \\ &\lesssim 2^{-n}R^2. \end{aligned}$$

We proceed the same way to bound $I_3(N)$. Recalling that $R \ge 2\tilde{R} \sim 1$, it holds that $|z| \le \tilde{R} \Rightarrow |z| \le R/2$. Hence, we are in position to use (5.36) and we obtain

$$\begin{aligned} I_3(N) &\le \bigg(\int_{|x|\le 2R}\int_{|z|\le\tilde{R}} h_T(z)^2\,dz\,dx \int_{R<|x|\le 2R}\int_{|z|\le\tilde{R}} h_T(z-x)^2\,dz\,dx\bigg)^{1/2} \\ &\lesssim \bigg(R^d \underbrace{\int_{|z|\le\tilde{R}} h_T(z)^2\,dz}_{\overset{(2.28)}{\lesssim}1} \underbrace{\int_{R<|x|\le 2R}\int_{|z|\le\tilde{R}} h_T(z-x)^2\,dz\,dx}_{\overset{(5.36)}{\lesssim}\underbrace{\int_{|z|\le\tilde{R}}\int_{R/2<|z-x|\le 5R/2} h_T(z-x)^2\,dx\,dz}_{\overset{(2.27)}{\lesssim}\int_{|z|\le\tilde{R}} R^{2-d}\,dz \sim R^{2-d}}}\bigg)^{1/2} \\ &\lesssim R. \end{aligned}$$

Since $\sum_{n=1}^{\infty} 2^{-n}R^2 \sim R^2$ and $|\{|x| \le 2R\}|R^{2-d} \sim R^2$, the bounds on I_1, $I_2(n)$ and $I_3(N)$ imply the claim (5.35).

Step 2. Proof of

$$\int_{|x|\le 4\tilde{R}}\int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx \lesssim 1. \tag{5.37}$$

This time, we decompose the sum over $|z| \le |z-x|$ in two contributions only: $|z| \le \tilde{R}$ and $\tilde{R} < |z|$. We then obtain

$$\begin{aligned}
&\int_{|x|\le 4\tilde{R}} \int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx \\
&\quad = \int_{|x|\le 4\tilde{R}} \underbrace{\int_{\tilde{R}<|z|\le|z-x|} h_T(z)h_T(z-x)\,dz}_{=I_1'}\,dx \\
&\qquad + \underbrace{\int_{|x|\le 4\tilde{R}} \int_{\{|z|\le\tilde{R}\}\cap\{|z|\le|z-x|\}} h_T(z)h_T(z-x)\,dz\,dx}_{=I_2'}\,.
\end{aligned}$$

Proceeding as for I_1 in Step 1 using (2.27) yields

$$I_1' \lesssim 1.$$

For I_2', we use Cauchy–Schwarz inequality, (2.28), and $\tilde{R} \sim 1$:

$$I_2' \le \left(\int_{|x|\le 4\tilde{R}} \int_{|z|\le\tilde{R}} h_T^2(z)\,dz\,dx\right)^{1/2} \left(\int_{|x|\le 4\tilde{R}} \int_{|z'|\le 5\tilde{R}} h_T^2(z')\,dz'\,dx\right)^{1/2} \lesssim 1.$$

This proves (5.37).

Step 3. Proof of (2.30).

It only remains to use a dyadic decomposition of the ball of radius R into the ball of radius $\tilde{R}$ and annuli of the form $2^{-k}R < |z| \le 2^{-k+1}R$, as follows. Taking M such that $2\tilde{R} \le 2^{-M}R \le 4\tilde{R}$, it holds that

$$\begin{aligned}
&\int_{|x|\le R} \int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx \\
&\quad = \underbrace{\int_{|x|\le 2^{-M}R} \int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx}_{\overset{(5.37)}{\lesssim} 1} \\
&\qquad + \sum_{n=1}^{M} \underbrace{\int_{2^{n-M-1}R<|x|\le 2^{n-M}R} \int_{|z|\le|z-x|} h_T(z)h_T(z-x)\,dz\,dx}_{\overset{(5.35)}{\lesssim} (2^{n-M}R)^2} \\
&\quad \lesssim 1 + R^2 \sum_{n=1}^{M} 4^{-n} \sim R^2,
\end{aligned}$$

which proves (2.30).

Step 4. Proof of (2.29).

For the case $d = 2$, we use the same strategy as for $d > 2$. The bounds on $I_2(n)$ and $I_3(N)$ are the same as for $d > 2$. However the estimate for I_1 is slightly worse. Indeed, we split the dyadic sums $2^k R < |z| \le 2^{k+1} R$ into two categories in order to take advantage of the fast decay in (2.26): the first class is for k such that $2^k R \le \sqrt{T}$ and the other class for k such that $2^k R > \sqrt{T}$. More precisely, setting $\mathcal{I}(R, T) := \{k \in \mathbb{N} : 2^{k-1} R \le \sqrt{T}\}$, we have

$$
\begin{aligned}
I_1 &= \int_{R/2<|z|\le|z-x|} h_T(z) h_T(z-x)\, dz \\
&\overset{\text{Young}}{\le} \int_{R/2<|z|} h_T(z)^2\, dz \\
&= \sum_{k=-1}^{\infty} \int_{2^k R<|z|\le 2^{k+1} R} h_T(z)^2\, dz \\
&= \underbrace{\sum_{k\in\mathcal{I}(R,T)} \int_{2^{k-1} R<|z|\le 2^k R} h_T^2(z)\, dz}_{\overset{(2.26)}{\lesssim} \max\{0, \ln(\sqrt{T} R^{-1})\}} + \underbrace{\sum_{k\in\mathbb{N}\setminus\mathcal{I}(R,T)} \int_{2^{k-1} R<|z|\le 2^k R} h_T^2(z)\, dz}_{\overset{(2.26)}{\lesssim} \sum_{k\in\mathbb{N}} 2^{-2k} \lesssim 1} \\
&\lesssim \max\{1, \ln(\sqrt{T} R^{-1})\},
\end{aligned}
$$

which gives the extra factor in (2.29).

APPENDIX: HEURISTICS FOR (1.13) AND (1.14)

Let $\bar{\phi}_i$ and $\bar{\phi}_{T,i}$ denote for $i \in \{1, \ldots, d\}$ the solutions of (1.11) and (1.12), respectively, with ξ replaced by the ith unit vector $\mathbf{e}_i$ of $\mathbb{R}^d$. We claim that

$$
\sum_{i=1}^{d} \sum_{j=1}^{d} \operatorname{var}\Big[\sum (\mathbf{e}_j \cdot (A - \langle A \rangle)\mathbf{e}_i + 2\mathbf{e}_j \cdot \nabla \bar{\phi}_i) \eta_L\Big] = d \operatorname{var}[a] \sum \eta_L^2, \tag{A.1}
$$

$$
\sum_{i=1}^{d} \langle |\nabla \bar{\phi}_{T,i} - \nabla \bar{\phi}_i|^2 \rangle = \operatorname{var}[a] T^{-2} \sum \bar{G}_T^2, \tag{A.2}
$$

where $\bar{G}_T$ denotes the fundamental solution of the constant coefficient operator $T^{-1} - \triangle$. We also denote by $\bar{G}$ the fundamental solution of the Laplacian. Since

$$
\sum \bar{G}_T^2 \sim \begin{cases} T^{2-d/2}, & \text{for } d < 4, \\ \ln T, & \text{for } d = 4, \\ 1, & \text{for } d > 4, \end{cases}
$$

and

$$
\sum \eta_L^2 \sim L^{-d},
$$

(1.13) and (1.14) follow from (A.1) and (A.2), that we prove now.

Step 1. Argument for (A.2).

Since

$$-\triangle(\bar\phi_T-\bar\phi)=-T^{-1}\bar\phi_T, \tag{A.3}$$

one has

$$\langle|\nabla(\bar\phi_T-\bar\phi)|^2\rangle=-T^{-1}\langle\bar\phi_T(\bar\phi_T-\bar\phi)\rangle. \tag{A.4}$$

Rewriting (A.3) in the form

$$T^{-1}(\bar\phi_T-\bar\phi)-\triangle(\bar\phi_T-\bar\phi)=-T^{-1}\bar\phi$$

yields the formula

$$(\bar\phi_T-\bar\phi)(0)=-T^{-1}\sum_x\bar G_T(x)\bar\phi(x). \tag{A.5}$$

Using (A.5), (A.4) turns into

$$\langle|\nabla(\bar\phi_T-\bar\phi)|^2\rangle=-T^{-2}\sum_x\bar G_T(x)\langle\bar\phi_T(0)\bar\phi(x)\rangle. \tag{A.6}$$

Expressing now $\bar\phi_{T,i}(0)$ and $\bar\phi_i(x)$ in terms the Green's functions[2] $\bar G_T$ and $\bar G$,

$$\begin{aligned}\bar\phi_{T,i}(0)&=\sum_{x'}\bar G_T(x')\nabla^*\cdot(A(x')\mathbf{e}_i)\\&=-\sum_{x'}\nabla_i\bar G_T(x')\big(a_i(x')-\langle a\rangle\big),\\\bar\phi_i(x)&=\sum_{x'}\bar G(x-x')\nabla^*\cdot(A(x')\mathbf{e}_i)\\&=-\sum_{x''}\nabla_i\bar G(x-x'')\big(a_i(x'')-\langle a\rangle\big),\end{aligned}$$

and using the independence of $a_i(x')$ and $a_i(x'')$ for $x'\neq x''$, we get

$$\langle\bar\phi_{T,i}(0)\bar\phi_i(x)\rangle=\sum_{x'}\nabla_i\bar G_T(x')\nabla_i\bar G(x-x')\big\langle\big(a_i(x')-\langle a\rangle\big)^2\big\rangle.$$

Hence,

$$\begin{aligned}\sum_{i=1}^d\langle\bar\phi_{T,i}(0)\bar\phi_i(x)\rangle&=\operatorname{var}[a]\sum_{x'}\nabla\bar G_T(x')\cdot\nabla\bar G(x-x')\\&=\operatorname{var}[a]\bar G_T(x),\end{aligned}$$

[2]Attention should be paid here to turn this into a rigorous argument since $\bar G$ is not in $L^1(\mathbb{Z}^d)$.

since $-\triangle\bar{G}(x)=\delta(x)$. Combined with (A.6), this proves (A.2).

Step 2. Argument for (A.1).

Using the Green's function, one has

$$\begin{aligned}\bar{\phi}_i(x)&=\sum_{x'}\bar{G}(x-x')\nabla^*\cdot\big((A-\langle A\rangle)\mathbf{e}_i\big)(x')\\&=-\sum_{x'}\nabla_i\bar{G}(x-x')\big(a_i(x')-\langle a\rangle\big),\end{aligned}$$

and therefore

$$\nabla\bar{\phi}_i(x)=-\sum_{x'}\nabla\nabla_i\bar{G}(x-x')\big(a_i(x')-\langle a\rangle\big).$$

Hence, denoting by $\mathcal{A}_{ij}$ the argument of the variance in (A.1), one has

$$\begin{aligned}\mathcal{A}_{ij}&:=\sum_{x}\big(\mathbf{e}_j\cdot\mathbf{e}_i\big(a_i(x)-\langle a\rangle\big)+2\mathbf{e}_j\cdot\nabla\bar{\phi}_i(x)\big)\eta_L(x)\\&=\sum_{x}\sum_{x'}\big(a_i(x')-\langle a\rangle\big)\mathbf{e}_j\cdot\big(\delta(x-x')\mathbf{e}_i-2\nabla\nabla_i\bar{G}(x-x')\big)\eta_L(x).\end{aligned}$$

Using the independence of the a_i, one obtains for the variance

$$\begin{aligned}\operatorname{var}[\mathcal{A}_{ij}]=\operatorname{var}[a]\sum_{x}\sum_{x'}\sum_{x''}&\mathbf{e}_j\cdot\big(\delta(x-x')\mathbf{e}_i-2\nabla\nabla_i\bar{G}(x-x')\big)\\&\times\mathbf{e}_j\cdot\big(\delta(x''-x')\mathbf{e}_i-2\nabla\nabla_i\bar{G}(x''-x')\big)\\&\times\eta_L(x)\eta_L(x'').\end{aligned}$$

Rearranging the terms yields

$$\begin{aligned}\operatorname{var}[\mathcal{A}_{ij}]&=\operatorname{var}[a]\sum_{x}\sum_{x'}\delta(j-i)\big(\delta(x-x')-4\nabla_i\nabla_i\bar{G}(x-x')\big)\eta_L(x)\eta_L(x')\\&\quad+\operatorname{var}[a]\sum_{x}\sum_{x''}4\eta_L(x)\eta_L(x'')\underbrace{\sum_{x'}\nabla_j\nabla_i\bar{G}(x-x')\nabla_j\nabla_i\bar{G}(x''-x')}_{=-\nabla_i\nabla_i\sum_{x'}\bar{G}(x-x')\nabla_j\nabla_j\bar{G}(x''-x')}.\end{aligned}$$

Summing in j and using that $-\triangle G(x)=\delta(x)$, this turns into

$$\begin{aligned}\sum_{j=1}^{d}\operatorname{var}[\mathcal{A}_{ij}]&=\operatorname{var}[a]\sum_{x}\sum_{x'}\big(\delta(x-x')-4\nabla_i\nabla_i\bar{G}(x-x')\big)\eta_L(x)\eta_L(x')\\&\quad+\operatorname{var}[a]\sum_{x}\sum_{x''}4\eta_L(x)\eta_L(x'')\nabla_i\nabla_i\bar{G}(x-x'')\\&=\operatorname{var}[a]\sum_{x}\sum_{x'}\delta(x-x')\eta_L(x)\eta_L(x')\\&=\operatorname{var}[a]\sum_{x}\eta_L(x)^2,\end{aligned}$$

from which we deduce (A.1).

Acknowledgments. A. Gloria acknowledges full support and F. Otto acknowledges partial support of the Hausdorff Center for Mathematics, Bonn, Germany.

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PROJECT-TEAM SIMPAF
INRIA LILLE-NORD EUROPE
FRANCE
E-MAIL: antoine.gloria@inria.fr

MAX PLANCK INSTITUTE
FOR MATHEMATICS IN THE SCIENCES
LEIPZIG
GERMANY
E-MAIL: otto@mis.mpg.de