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# LARGE DEVIATIONS FOR LOCAL TIMES AND INTERSECTION LOCAL TIMES OF FRACTIONAL BROWNIAN MOTIONS AND RIEMANN–LIOUVILLE PROCESSES

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In this paper, we prove exact forms of large deviations for local times and intersection local times of fractional Brownian motions and Riemann–Liouville processes. We also show that a fractional Brownian motion and the related Riemann–Liouville process behave like constant multiples of each other with regard to large deviations for their local and intersection local times. As a consequence of our large deviation estimates, we derive laws of iterated logarithm for the corresponding local times. The key points of our methods: (1) logarithmic *superadditivity* of a normalized sequence of moments of exponentially randomized local time of a fractional Brownian motion; (2) logarithmic *subadditivity* of a normalized sequence of moments of exponentially randomized intersection local time of Riemann–Liouville processes; (3) comparison of local and intersection local times based on embedding of a part of a fractional Brownian motion into the reproducing kernel Hilbert space of the Riemann–Liouville process.

**1. Introduction.** Let $B^H(t)$, $t \geq 0$, be a standard $d$-dimensional fractional Brownian motion with index $H \in (0,1)$. That is, $B^H(t)$ is a zero-mean Gaussian process with stationary increments and covariance function

$$\mathbb{E}[B^H(t)B^H(s)^\top] = \tfrac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\}I_d,$$

where $I_d$ is the identity matrix of size $d$. $B^H(t)$ is also a self-similar process with index $H$. The local time $L_t^x(B^H)$ of $B^H(t)$ at $x \in \mathbb{R}^d$ is defined heuristically as

$$L_t^x(B^H) = \int_0^t \delta_x(B^H(s))\,ds, \qquad t \geq 0.$$

It is known that $L_t^x(B^H)$ exists and is jointly continuous in $(t,x)$ as long as $Hd < 1$. By the self-similarity of a fractional Brownian motion, $L_t^x(B^H) \overset{d}{=}$

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$t^{1-Hd}L_1^{x/t^H}(B^H)$. In particular,

$$L_t^0(B^H) \stackrel{d}{=} t^{1-Hd}L_1^0(B^H). \tag{1.1}$$

Our first goal is to investigate large deviations associated with tail probabilities of $L_t^0(B^H)$. By the scaling given above, we may consider only $t = 1$. In the classical case, when $H = 1/2$ and $d = 1$, it is well known (see the book of Revuz and Yor [37], page 240) that $L_1^0(B^{1/2}) \stackrel{d}{=} |U|$ with $U \sim \mathcal{N}(0, 1)$. Consequently,

$$\lim_{a\to\infty} a^{-2} \log \mathbb{P}\{L_1^0(B^{1/2}) \geq a\} = -\tfrac{1}{2}.$$

In Theorem 2.1 we prove that for a fractional Brownian motion a nontrivial limit

$$\lim_{a\to\infty} a^{-1/Hd} \log \mathbb{P}\{L_1^0(B^H) \geq a\}$$

exists and we give bounds for this limit.

Closely related to the fractional Brownian motion is the Riemann–Liouville process $W^H(t)$ with index $H > 0$ which is defined as a stochastic convolution

$$W^H(t) = \int_0^t (t-s)^{H-1/2}\, dB(s), \qquad t \geq 0, \tag{1.2}$$

where $B(t)$ is a $d$-dimensional standard Brownian motion. $\{W^H(t)\}_{t\geq 0}$ is a self-similar zero-mean Gaussian process with index $H$, as is $B^H(t)$, but $W^H(t)$ does not have stationary increments and there is no upper bound restriction on index $H > 0$. If $L_t^0(W^H)$ denotes the local time of $W^H(t)$ at 0, then by the self-similarity we also have

$$L_t^0(W^H) \stackrel{d}{=} t^{1-Hd}L_1^0(W^H). \tag{1.3}$$

The relation between $W^H(t)$ and $B^H(t)$ becomes transparent when we write a moving average representation of $B^H(t)$, $t \in \mathbb{R}$, in the form

$$B^H(t) = c_H \int_{-\infty}^t [(t-s)^{H-1/2} - (-s)_+^{H-1/2}]\, dB(s), \tag{1.4}$$

where

$$c_H = \sqrt{2H}2^H B(1-H, H+1/2)^{-1/2}, \tag{1.5}$$

and $B(\cdot,\cdot)$ denotes the beta function. The analytic derivation of $c_H$ is given for completeness in the Appendix (a different but equivalent form of $c_H$ is also derived in Mishura [34], Lemma A.0.1, by a Fourier analytic method). From (1.4), we have a decomposition

$$c_H^{-1}B^H(t) = W^H(t) + Z^H(t), \tag{1.6}$$

where

$$Z^H(t) = \int_{-\infty}^0 [(t-s)^{H-1/2} - (-s)^{H-1/2}]\, dB(s) \tag{1.7}$$

is a process independent of $W^H(t)$.

This moving average representation for fractional Brownian motion was introduced in the pioneering work of Mandelbrot and Van Ness [32] and used extensively by many authors, sometimes with different normalizing constant $c_H$ in (1.5) (e.g., Li and Linde [28] uses $\Gamma(H+1/2)^{-1}$ for $c_H$).

We will show that paths of $Z^H(t)$, away from $t=0$, can be matched with functions in the reproducing kernel Hilbert space of $W^H(t)$ (Proposition 3.5, Section 3.2). This and the independence of $Z^H(t)$ from $W^H(t)$ will allow us to show that large deviation constants of tail probabilities of $L_1^0(W^H)$ and of $L_1^0(c_H^{-1}B^H) = c_H^d L_1^0(B^H)$ are the same (Theorem 2.2). In this context, we also want to mention Theorem 3.22 of Xiao [42], who established bounds for tail probabilities of the local time $L_1^0$ of the general Gaussian processes in the form

$$-C_1 \le \liminf_{a\to\infty} \frac{1}{\phi(a)} \log\{L_1^0 \ge a\} \le \limsup_{a\to\infty} \frac{1}{\phi(a)} \log\{L_1^0 \ge a\} \le -C_2$$

and raised a question on the existence of the limit (Question 3.25, [42]). Further, we cite the paper by Baraka, Mountford and Xiao [4] for some similar tail estimate of the local time of multi-parameter fractional Brownian motions.

Next, we will consider $p$ independent copies $B_1^H(t), \dots, B_p^H(t)$ of a standard $d$-dimensional fractional Brownian motion $B^H(t)$. Throughout this paper,

$$p^* := p/(p-1)$$

will stand for the conjugate to $p>1$. Our next and main goal is to investigate large deviations for intersection local time $\alpha^H(\cdot)$ of $B_1^H(t), \dots, B_p^H(t)$, which is a random measure on $(\mathbb{R}^+)^p$ given heuristically by

$$\alpha^H(A) = \int_A \prod_{j=1}^{p-1} \delta_0\big(B_j^H(s_j) - B_{j+1}^H(s_{j+1})\big)\, ds_1 \cdots ds_p, \qquad A \subset (\mathbb{R}^+)^p.$$

Quantities measuring the amount of self-intersection of a random walk, or of mutual intersection of several independent random walks, have been studied intensively for more than twenty years; see, for example, [9, 10, 13, 19, 25, 26, 33]. This research is motivated by the role these quantities play in quantum field theory (see, e.g., [14]) in our understanding of self-avoiding walks and polymer models (see, e.g., [21, 31]) or in the analysis of stochastic processes in random environments (see, e.g., [2, 15, 16, 20]). In the latter models, dependence between a moving particle and a random environment frequently comes from the particle's ability to revisit sites with an attractive (in some sense) environment. Consequently, measures of self-intersection quantify the degree of dependence between movement and environment. Typically, in high dimensions, this dependence gets weaker, as the movements become more transient and self-intersections less likely. Investigation of large deviations for intersection local times is closely related to asymptotics of the partition functions in above models.

There are two equivalent ways to construct $\alpha^H(A)$ rigorously. In the first way, $\alpha^H(A)$ is defined as the local time at zero of the multi-parameter process

$$X(t_1, \dots, t_p) = \big(B_1^H(t_1) - B_2^H(t_2), \dots, \\ B_{p-1}^H(t_{p-1}) - B_p^H(t_p)\big), \qquad (t_1, \dots, t_p) \in (\mathbb{R}^+)^p. \tag{1.8}$$

More precisely, consider the occupation measure

$$\mu_A(B) = \int_A \mathbf{1}_B\big(B_1^H(s_1) - B_2^H(s_2), \dots, \\ B_{p-1}^H(s_{p-1}) - B_p^H(s_p)\big)\, ds_1 \cdots ds_p, \qquad B \subset \mathbb{R}^{d(p-1)}.$$

By Theorem 7.1, as $Hd < p^*$, there is a density function $\alpha^H(A, \cdot)$ of $\mu_A(\cdot)$ such that if $A = [0, t_1] \times \cdots \times [0, t_p]$, then $\alpha^H([0, t_1] \times \cdots \times [0, t_p], x)$ is jointly continuous in $(t_1, \dots, t_p, x)$. We define $\alpha^H(A) := \alpha^H(A, 0)$.

For the second way of constructing $\alpha^H(A)$, write for any $\varepsilon > 0$

$$\alpha_\varepsilon^H(A) = \int_{\mathbb{R}^d} \int_A \prod_{j=1}^p p_\varepsilon\big(B_j^H(s_j) - x\big)\, ds_1 \cdots ds_p\, dx, \tag{1.9}$$

where $p_\varepsilon$ are probability densities approximating $\delta_0$ as $\varepsilon \to 0$. Notice that

$$\begin{aligned}\alpha_\varepsilon^H(A) &= \int_A h_\varepsilon\big(B_1^H(s_1) - B_2^H(s_2), \dots, B_{p-1}^H(s_{p-1}) - B_p^H(s_p)\big)\, ds_1 \cdots ds_p \\ &= \int_{\mathbb{R}^{d(p-1)}} h_\varepsilon(x) \alpha^H(A, x)\, dx,\end{aligned}$$

where

$$h_\varepsilon(x_1, \dots, x_{p-1}) = \int_{\mathbb{R}^d} p_\varepsilon(-x) \prod_{j=1}^{p-1} p_\varepsilon\left(\sum_{k=j}^{p-1} x_k - x\right)$$

is an probability density on $\mathbb{R}^{d(p-1)}$ approaching $\delta_0(x_1, \dots, x_{p-1})$ as $\varepsilon \to 0^+$.

By the continuity of $\alpha^H(A, x)$, $\lim_{\varepsilon \to 0^+} \alpha_\varepsilon^H(A) = \alpha^H(A)$ almost surely. Applying Proposition 3.1 to the Gaussian field given in (1.8), the convergence is also in $\mathcal{L}^m$ for all positive $m$. This way of constructing $\alpha^H(A)$ justifies the symbolic notation

$$\alpha^H(A) = \int_{\mathbb{R}^d} \int_A \prod_{j=1}^p \delta_0\big(B_j^H(s_j) - x\big)\, ds_1 \cdots ds_p\, dx.$$

In the special case $p = 2$ and $Hd < 2$, Nualart and Ortiz-Latorre [35] proved that $\alpha_\varepsilon^H([0, t_1] \times [0, t_2])$ converges in $\mathcal{L}^2$ as $\varepsilon \to 0^+$, with

$$p_\varepsilon(x) = (2\varepsilon\pi)^{-d/2} \exp\{-|x|^2/2\varepsilon\}. \tag{1.10}$$

For the Riemann–Liouville process $W^H(t)$ an analogous construction of the intersection local time

$$\tilde{\alpha}^H(A) = \int_A \prod_{j=1}^{p-1} \delta_0\big(W_j^H(s_j) - W_{j+1}^H(s_{j+1})\big)\, ds_1 \cdots ds_p$$

$$= \int_{\mathbb{R}^d} \int_A \prod_{j=1}^{p} \delta_0\big(W_j^H(s_j) - x\big)\, ds_1 \cdots ds_p\, dx, \qquad A \subset (\mathbb{R}^+)^p,$$

can be done under the same condition $Hd < p^*$.

By the self-similarity of $B^H(t)$ and $W^H(t)$, for any $t > 0$

$$\alpha^H([0,t]^p) \stackrel{d}{=} t^{p-Hd(p-1)} \alpha^H([0,1]^p) \tag{1.11}$$

and

$$\tilde{\alpha}^H([0,t]^p) \stackrel{d}{=} t^{p-Hd(p-1)} \tilde{\alpha}^H([0,1]^p). \tag{1.12}$$

Finally, we would like to discuss this research in a more general context of Markovian versus non-Markovian structures. Naturally, most of the existing results on large deviation for (intersection) local time have been obtained for Markov processes such as Brownian motions, Lévy stable processes, general Lévy processes and random walks. The underlying Markovian structure has been essential for the methods in these studies; see Chen [10] for references and a systematical account of such works. Departures from Markovian models are often driven by the underlying physics to match the required level of dependence (memory) and smoothness/roughness of sample paths. Fractional Brownian motion and Riemann–Liouville processes are the most natural candidates as extensions of Brownian motion into the non-Markovian world. They offer the existence of the intersection local time for any number $p$ of processes in any dimension $d$ as long as $H$ is sufficiently small. Therefore, they may help scientists to build more realistic and robust models while posing serious challenge to mathematicians due to the non-Markovian nature.

In this paper, we mainly use Gaussian techniques motivated from the study of continuity properties of local time, and more generally, from theory of Gaussian processes. It is also helpful to see connections between small ball probability estimates and tail behavior of the local time. Indeed, large value of the local time at zero means that the process stayed for a long time in a small neighborhood of zero. By this analogy, Propositions 3.1 and 3.3 can be motivated by the corresponding results for small balls (see comments preceding these propositions in Section 3.1).

## 2. Main results.

THEOREM 2.1. *Let $B^H(t)$ be a standard $d$-dimensional fractional Brownian motion with index $H$ such that $Hd < 1$. Then the limit*

$$\lim_{a\to\infty} a^{-1/(Hd)} \log \mathbb{P}\{L_1^0(B^H) \geq a\} = -\theta(H,d) \tag{2.1}$$

*exists and* $\theta(H,d)$ *satisfies the following bounds*:

$$(\pi c_H^2/H)^{1/(2H)}\theta_0(Hd) \le \theta(H,d) \le (2\pi)^{1/(2H)}\theta_0(Hd), \tag{2.2}$$

*where* $c_H$ *is given by* (1.5) *and*

$$\theta_0(\kappa) = \kappa\left(\frac{(1-\kappa)^{1-\kappa}}{\Gamma(1-\kappa)}\right)^{1/\kappa}. \tag{2.3}$$

Notice that in the classical case of one-dimensional Brownian motion, (2.2) becomes the equality. The fact that the lower bound is less than or equal to the upper bound in (2.2) is equivalent to $c_H^2 \le 2H$, which can also be seen directly. Indeed, from (3.16)

$$\frac{c_H^2}{2H} = \operatorname{Var}(B^H(1)|B^H(s), s \le 0) \le \operatorname{Var}(B^H(1)) = 1. \tag{2.4}$$

The equality only holds for a Brownian motion, that is, $H = 1/2$.

THEOREM 2.2. *Let* $W^H(t)$ *be a* $d$*-dimensional Riemann–Liouville process as in* (1.2) *such that* $Hd < 1$. *Then the limit*

$$\lim_{a\to\infty} a^{-1/(Hd)} \log \mathbb{P}\{L_1^0(W^H) \ge a\} = -\tilde{\theta}(H,d) \tag{2.5}$$

*exists with*

$$\tilde{\theta}(H,d) = (c_H)^{-1/H}\theta(H,d), \tag{2.6}$$

*where* $\theta(H,d)$ *is as in Theorem* 2.1 *and* $c_H$ *is given by* (1.5).

THEOREM 2.3. *Let* $\tilde{\alpha}^H(\cdot)$ *be the intersection local time of* $p$*-independent* $d$*-dimensional Riemann–Liouville process* $W_1^H(t), \dots, W_p^H(t)$, *where* $Hd < p^*$. *Then the limit*

$$\lim_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{P}\{\tilde{\alpha}^H([0,1]^p) \ge a\} = -\tilde{K}(H,d,p) \tag{2.7}$$

*exists and* $\tilde{K}(H,d,p)$ *satisfies the following bounds*:

$$\begin{aligned} & p\frac{Hd}{p^*}\left(1-\frac{Hd}{p^*}\right)^{1-p^*/(Hd)}\left(\frac{\pi}{H}\right)^{1/(2H)} p^{p^*/(2Hp)}\Gamma\left(1-\frac{Hd}{p^*}\right)^{-p^*/(Hd)} \\ &\quad \le \tilde{K}(H,d,p) \\ &\quad \le p\frac{Hd}{p^*}\left(1-\frac{Hd}{p^*}\right)^{1-p^*/(Hd)}\left(\frac{2\pi}{c_H^2 p^*}\right)^{1/(2H)} \\ &\qquad \times \left(\int_0^\infty (1+t^{2H})^{-d/2} e^{-t}\,dt\right)^{-p^*/(Hd)}, \end{aligned} \tag{2.8}$$

*where* $c_H$ *is given by* (1.5).

There is a direct way to show that the lower bound is less than or equal to the upper bound in (2.8). Observe that by Hölder's inequality, $1 + t^{2H} \geq p^{1/p}(p^*)^{1/p^*} t^{2H/p^*}$ which leads to

$$\int_0^\infty (1+t^{2H})^{-d/2} e^{-t}\, dt \leq p^{-d/(2p)}(p^*)^{-d/(2p^*)}\Gamma(1 - Hd/p^*).$$

After cancellation on both sides of (2.8), the problem is then reduced to examining the relation $c_H^2 \leq 2H$, which is given in (2.4).

THEOREM 2.4. *Let $\alpha^H(\cdot)$ be the intersection local time of $p$-independent standard $d$-dimensional fractional Brownian motions $B_1^H(t), \dots, B_p^H(t)$, where $Hd < p^*$. Then the limit*

$$\lim_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{P}\{\alpha^H([0,1]^p) \geq a\} = -K(H,d,p) \tag{2.9}$$

*exists with*

$$K(H,d,p) = c_H^{1/H}\tilde{K}(H,d,p). \tag{2.10}$$

Our results seem to be closely related to the large deviations of the self-intersection local times heuristically written as

$$\beta^H([0,t]^p_<) = \int_{[0,t]^p_<} \prod_{j=1}^{p-1} \delta_0\big(B^H(s_j) - B^H(s_{j+1})\big)\, ds_1 \cdots ds_p,$$

where

$$[0,t]^p_< = \{(s_1,\dots,s_p) \in [0,t]^p;\ s_1 < \cdots < s_p\}.$$

In the case when $Hd < 1$, we can rewrite

$$\beta^H([0,t]^p_<) = \frac{1}{p!}\int_{\mathbb{R}^d} [L_t^x(B^H)]^p\, dx.$$

To see the connection between $\alpha^H$ and $\beta^H$, notice that by Hölder's inequality and arithmetic and geometric mean inequality,

$$(\alpha^H([0,1]^p))^{1/p} = \left(\int_{\mathbb{R}^d} \prod_{j=1}^p L_1^x(B_j^H)\, dx\right)^{1/p} \leq \frac{1}{p}\sum_{j=1}^p \left(\int_{\mathbb{R}^d} [L_1^x(B_j^H)]^p\, dx\right)^{1/p}.$$

Thus, for any $\theta > 0$,

$$\begin{aligned}&\mathbb{E}\exp\{\theta a^{(p^*-Hd)/(Hdp)}(\alpha^H([0,1]^p))^{1/p}\}\\ &\qquad \leq \left[\mathbb{E}\exp\left\{\theta p^{-1} a^{(p^*-Hd)/(Hdp)}\left(\int_{\mathbb{R}^d}[L_1^x(B^H)]^p\, dx\right)^{1/p}\right\}\right]^p.\end{aligned}$$

On the other hand, by Theorem 2.4 and Varadhan's integral lemma,

$$\begin{aligned}
&\lim_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{E}\exp\{\theta p^{-1} a^{(p^*-Hd)/(Hdp)} (\alpha^H([0,1]^p))^{1/p}\} \\
&\quad = \sup_{\lambda>0}\{\theta p^{-1}\lambda^{1/p} - K(H,d,p)\lambda^{p^*/(Hdp)}\} \\
&\quad = \bigl(Hd/(p^* K(H,d,p))\bigr)^{Hd/(p^*-Hd)}(1-Hd/p^*)(\theta/p)^{p^*/(p^*-Hd)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\liminf_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{E}\exp\biggl\{\theta a^{(p^*-Hd)/(Hdp)} \biggl(\int_{\mathbb{R}^d} [L_1^x(B^H)]^p\,dx\biggr)^{1/p}\biggr\} \\
&\qquad \ge p^{-1}\bigl(Hd/(p^* K(H,d,p))\bigr)^{Hd/(p^*-Hd)} \qquad (2.11)\\
&\qquad\quad \times (1-Hd/p^*)(\theta/p)^{p^*/(p^*-Hd)}.
\end{aligned}$$

If this can be strengthened into equality with limits, then by Gärtner–Ellis theorem, for any $\lambda > 0$,

$$\begin{aligned}
&\lim_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{P}\biggl\{\biggl(\int_{\mathbb{R}^d} [L_1^x(B^H)]^p\,dx\biggr)^{1/p} \ge \lambda a^{1/p}\biggr\} \\
&\quad = -\sup_{\theta>0}\{\lambda\theta - p^{-1}\bigl(Hd/(p^* K(H,d,p))\bigr)^{(Hd)/(p^*-Hd)} \\
&\qquad\qquad \times (1-Hd/p^*)(\theta/p)^{p^*/(p^*-Hd)}\} \\
&\quad = -p^{-1}K(H,d,p)\lambda^{p^*/(Hd)}.
\end{aligned}$$

In particular,

$$\lim_{a\to\infty} a^{-p^*/(Hdp)} \log \mathbb{P}\biggl\{\int_{\mathbb{R}^d} [L_1^x(B^H)]^p\,dx \ge a\biggr\} = -p^{-1}K(H,d,p). \tag{2.12}$$

The conjecture (2.12) is partially supported by a recent result of Hu, Nualart and Song (Theorem 1, [23]) which states that when $Hd < 1$ and $p = 2$,

$$\mathbb{E}\biggl\{\int_{\mathbb{R}^d} [L_1^x(B^H)]^2\,dx\biggr\}^n \le C^n (n!)^{Hd}, \qquad n = 1, 2, \dots,$$

for some $C > 0$. Indeed, a standard application of Chebyshev inequality and Stirling formula leads to the upper bound of the form

$$\limsup_{a\to\infty} a^{-1/(Hd)} \log \mathbb{P}\biggl\{\int_{\mathbb{R}^d} [L_1^x(B^H)]^2\,dx \ge \lambda a\biggr\} \le -l,$$

where $l$ is a positive constant. This rate of decay of tail probabilities is sharp by comparing it with (2.11) for $p = 2$.

In the case $Hd \geq 1$, $\beta^H([0,t]^p_<)$ cannot be properly defined. On the other hand, this problem can be fixed in some cases by renormalization. For simplicity we consider the case $p = 2$. Hu and Nualart prove (Theorem 1, [22]) that for $1 \leq Hd < 3/2$, the renormalized self-intersection local time formally given as

$$\gamma^H([0,t]^2_<) = \iint_{\{0\leq r<s\leq t\}} \delta_0\big(B^H(r) - B^H(s)\big)\,dr\,ds \\ - \mathbb{E}\iint_{\{0\leq r<s\leq t\}} \delta_0\big(B^H(r) - B^H(s)\big)\,dr\,ds$$

exists with the scaling property

$$\gamma^H([0,t]^2_<) \stackrel{d}{=} t^{2-Hd}\gamma^H([0,1]^2_<). \tag{2.13}$$

We also point that an earlier work by Rosen [38] in the special case $d = 2$.

Based on a similar but more heuristic reasoning, it seems plausible to expect that

$$\lim_{a\to\infty} a^{-1/(Hd)} \log \mathbb{P}\{\gamma^H([0,1]^2_<) \geq a\} = -2^{(Hd)^{-1}-1}K(H,d,2). \tag{2.14}$$

We refer the interested reader to Theorem 4, [23] for some exponential integrabilities established by Hu, Nualart and Song based on Clark–Ocone's formula. We leave these problems for future investigation.

Our large deviations estimates can be applied to obtain the law of the iterated logarithm.

THEOREM 2.5. *When $Hd < 1$,*

$$\limsup_{t\to\infty} t^{-(1-Hd)}(\log\log t)^{-Hd} L_t^0(B^H) = \theta(H,d)^{-Hd} \qquad a.s., \tag{2.15}$$

$$\limsup_{t\to\infty} t^{-(1-Hd)}(\log\log t)^{-Hd} L_t^0(W^H) = \tilde{\theta}(H,d)^{-Hd} \qquad a.s. \tag{2.16}$$

*When $Hd < p^*$,*

$$\limsup_{t\to\infty} t^{-p(1-Hd/p^*)}(\log\log t)^{-Hd(p-1)}\alpha^H([0,t]^p) \\ = K(H,d,p)^{-Hd(p-1)} \qquad a.s., \tag{2.17}$$

$$\limsup_{t\to\infty} t^{-p(1-Hd/p^*)}(\log\log t)^{-Hd(p-1)}\tilde{\alpha}^H([0,t]^p) \\ = \tilde{K}(H,d,p)^{-Hd(p-1)} \qquad a.s., \tag{2.18}$$

Even with the large deviations stated in Theorem 2.1–2.4, the proof of Theorem 2.5 appears to be highly nontrivial due to long-range dependency of the model. Here, we mention some previous results given in Baraka and Mountford [3], Baraka, Mountford and Xiao [4]. Using the large deviation estimate similar to (2.1), Baraka, Mountford and Xiao were able to establish some laws of the iterated logarithm which describe the short term behaviors (as $t \to 0^+$) of the local times of fractional Brownian motions. As pointed out by Baraka and Mountford (page 163, [3]), their method does not lead to the laws of the iterated logarithm of large time given in Theorem 2.5.

Theorem 2.5 will be proved in Section 6. The proof of the lower bound appears to be highly nontrivial due to long-range dependency of the model. The approach relies on a quantified use of Cameron–Martin formula.

Since all main theorems stated in this section have been known in the classic case $H = 1/2$ (see, e.g., [9] and [12]), we assume $H \neq 1/2$ in the remainder of the paper.

## 3. Basic tools.

In this section, we provide some basic results that will be used in our proofs. We state them separately for a convenient reference.

3.1. *Comparison of local times*. We will give general comparison results for local times for Gaussian processes. They are based on the standard Fourier analytic approach but go far beyond, motivated mainly by similar small deviation estimates. We start with an outline of the analytic method typically used in the study of local times for Gaussian processes, in particular on its the moments; see Berman [7] and Xiao [42].

For a fixed sample function and fixed time $t > 0$, the Fourier transform on space variable $x \in \mathbb{R}^d$ is the function of $\lambda \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} e^{i\lambda\cdot x} L(t,x)\,dx = \int_0^t e^{i\lambda\cdot X(s)}\,ds.$$

Thus, the local time $L(t,x)$ can be expressed as the inverse Fourier transform:

$$L(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\lambda\cdot x}\int_0^t e^{i\lambda\cdot X(s)}\,ds\,d\lambda.$$

The $m$th power of $L(t,x)$ is

$$\begin{aligned} L(t,x)^m &= \frac{1}{(2\pi)^{md}}\int_{\mathbb{R}^{md}} e^{-ix\cdot\sum_{k=1}^m \lambda_k} \\ &\qquad \times \int_{[0,t]^m} \exp\Biggl(i\sum_{k=1}^m \lambda_k\cdot X(s_k)\Biggr)\,ds_1\cdots ds_m\,d\lambda_1\cdots d\lambda_m. \end{aligned}$$

Take the expected value under the sign of integration: the second exponential in the above integral is replaced by the joint characteristic function of $X(s_1),\dots,X(s_m)$.

In the Gaussian case, we obtain

$$
\begin{aligned}
&\mathbb{E}L(t,x)^m \\
&\quad = \frac{1}{(2\pi)^{md}} \\
&\qquad \times \int_{\mathbb{R}^{md}} e^{-ix\cdot\sum_{k=1}^m \lambda_k} \\
&\qquad\qquad \times \int_{[0,t]^m} \exp\biggl(-\frac{1}{2}\operatorname{Var}\biggl(\sum_{k=1}^m \lambda_k\cdot X(s_k)\biggr)\biggr)\, ds_1\cdots ds_m\, d\lambda_1\cdots d\lambda_m.
\end{aligned}
$$

Interchanging integration and applying the characteristic function inversion formula, we can get a more explicit (but somewhat less useful) expression in terms of integration associated with $\det(\mathbb{E}X(s_i)X(s_j))^{-1/2}$. Estimates of the moments of local time $L(t,x)$ thus depend on the rate of decrease to 0 of $\det(\mathbb{E}X(s_i)X(s_j))$ as $s_j - s_{j-1} \to 0$ for some $j$. Here in our approach, we have to make proper adjustment by approximating $L(t,x)$.

Consider now random fields $X(\mathbf{t})$ taking values in $\mathbb{R}^d$, where $\mathbf{t} = (t_1,\dots,t_p) \in (\mathbb{R}^+)^p$. For a fixed Borel set $A \subset (\mathbb{R}^+)^p$, recall that the local time formally given as

$$
L_X(A,x) = \int_A \delta_x(X(\mathbf{s}))\, d\mathbf{s} \tag{3.1}
$$

is defined as the density of the occupation measure

$$
\mu_A(B) = \int_A 1_B(X(\mathbf{s}))\, d\mathbf{s}, \qquad B \subset \mathbb{R}^d,
$$

if $\mu_A(\cdot)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Given a nondegenerate Gaussian probability density $h(x)$ on $\mathbb{R}^d$ and $\varepsilon > 0$, the function $h_\varepsilon(x) = \varepsilon^{-d/2}h(\varepsilon^{-1/2}x)$ is also a probability density. Define the smoothed local time

$$
L_X(A,x,\varepsilon) = \int_A h_\varepsilon\bigl(X(\mathbf{s}) - x\bigr)\, d\mathbf{s}. \tag{3.2}
$$

Our first proposition provides moment comparison (3.6) which can be viewed as analogy of Anderson's inequality in the small ball analog: For independent Gaussian vectors $X$, $Y$, $X$ symmetric,

$$
\mathbb{P}(\|X+Y\| \le \varepsilon) \le \mathbb{P}(\|X\| \le \varepsilon).
$$

See Li and Shao [30] for various application of this useful inequality.

PROPOSITION 3.1. *Let $A \subset (\mathbb{R}^+)^p$ be a fixed bounded Borel set. Let $X(\mathbf{t})$ $[\mathbf{t} = (t_1,\dots,t_p) \in (\mathbb{R}^+)^p]$ be a zero-mean $\mathbb{R}^d$-valued Gaussian random field with* *the local time $L_X(A,x)$ continuous in $x \in \mathbb{R}^d$. Assume that for every $m = 1, 2, \dots,$*

$$\int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\right)\right\} < \infty. \tag{3.3}$$

*Then $L_X(A,0) \in \mathcal{L}^m$ (i.e., finite mth moment), with*

$$\begin{aligned}\mathbb{E}L_X(A,0)^m &= \frac{1}{(2\pi)^{md}} \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \\ &\quad\times \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\right)\right\}\end{aligned} \tag{3.4}$$

*and*

$$\lim_{\varepsilon \to 0^+} \mathbb{E}|L_X(A,0,\varepsilon) - L_X(A,0)|^m = 0. \tag{3.5}$$

*If $Y(\mathbf{t})$ $[\mathbf{t} = (t_1, \dots, t_p) \in (\mathbb{R}^+)^p]$ is another zero-mean $\mathbb{R}^d$-valued Gaussian random field independent of $X(\mathbf{t})$ such that the local time $L_{X+Y}(A,x)$ of $X(\mathbf{t}) + Y(\mathbf{t})$ is continuous in $x$, then*

$$\mathbb{E}[L_{X+Y}(A,0)^m] \le \mathbb{E}[L_X(A,0)^m]. \tag{3.6}$$

PROOF. By Fourier inversion, we have from (3.2)

$$L_X(A,0,\varepsilon) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\lambda \exp\left\{-\frac{\varepsilon}{2}(\lambda \cdot \Gamma\lambda)\right\} \int_A e^{-i\lambda \cdot X(\mathbf{s})}\, d\mathbf{s},$$

where $\Gamma$ is the covariance matrix of Gaussian density $h(x)$. Using Fubini's theorem,

$$\begin{aligned}\mathbb{E}L_X(A,0,\varepsilon)^m &= \frac{1}{(2\pi)^{md}} \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \\ &\quad\times \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \\ &\quad\times \exp\left\{-\frac{\varepsilon}{2}\sum_{k=1}^m \lambda_k \cdot \Gamma\lambda_k\right\} \mathbb{E}\exp\left\{-i\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\right\} \\ &= \frac{1}{(2\pi)^{md}} \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \\ &\quad\times \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \\ &\quad\times \exp\left\{-\frac{\varepsilon}{2}\sum_{k=1}^m \lambda_k \cdot \Gamma\lambda_k\right\} \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k \cdot X(\mathbf{s}_k)\right)\right\}.\end{aligned} \tag{3.7}$$

By monotonic convergence theorem, the right-hand side converges to the right-hand side of (3.4) as $\varepsilon \to 0^+$. In particular, the family

$$\mathbb{E}L_X(A, 0, \varepsilon)^m \qquad (\varepsilon > 0)$$

is bounded for $m = 1, 2, \dots$. Consequently, this family is uniformly integrable for $m = 1, 2, \dots$. Therefore, (3.4) and (3.5) follow from the fact that $L_X(A, 0, \varepsilon)$ converges to $L_X(A, 0)$, which is led by the continuity of $L_X(A, x)$.

Finally, (3.6) follows from the comparison

$$\int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda \cdot \big(X(\mathbf{s}_k) + Y(\mathbf{s}_k)\big)\right)\right\}$$
$$\leq \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda \cdot X(\mathbf{s}_k)\right)\right\}. \qquad \square$$

In certain situations we can also reverse bound in (3.6) as a result of the Cameron–Martin formula. In small ball setting, this is motivated by the Chen–Li's inequality [11] which can be used to estimate small ball probabilities under any norm via a relatively easier $L_2$-norm estimate. See also the survey of Li and Shao [30]. Let $X$ and $Y$ be any two centered independent Gaussian random vectors in a separable Banach space $B$ with norm $\|\cdot\|$. We use $|\cdot|_{\mu(X)}$ to denote the inner product norm induced on $H_\mu$ by $\mu = \mathcal{L}(X)$. Then for any $\lambda > 0$ and $\varepsilon > 0$,

$$\mathbb{P}(\|X + Y\| \leq \varepsilon) \geq \mathbb{P}(\|X\| \leq \varepsilon) \cdot \mathbb{E}\exp\{-2^{-1}|Y|^2_{\mu(X)}\}$$

and

$$\mathbb{P}(\|Y\| \leq \varepsilon) \geq \mathbb{P}(\|X\| \leq \lambda\varepsilon) \cdot \mathbb{E}\exp\{-2^{-1}\lambda^2|Y|^2_{\mu(X)}\}.$$

Next, we provide the local time counterpart of this inequality, which is crucial in our estimates. Suppose that the process $X(\mathbf{t})$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$, where $\mathbf{T} = (T_1, \dots, T_p) \in (\mathbb{R}_+)^p$, can be viewed as a Gaussian random vector in a separable Banach space $B$ such that the evaluations $x \mapsto x(\mathbf{t})$ are measurable (say $B = C([\mathbf{0}, \mathbf{T}]; \mathbb{R}^d)$, for concreteness). Let $\mathcal{H}(X)$ denote the reproducing kernel Hilbert space (RKHS) of $X(\mathbf{t})$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$, equipped with the norm $\|\cdot\|$. Now we will make a crucial assumption that the independent process $Y(\mathbf{t})$, $\mathbf{t} \in [\mathbf{0}, \mathbf{T}]$, has almost all paths in $\mathcal{H}(X)$.

PROPOSITION 3.2. *In the above setting, under the assumptions of Proposition* 3.1, *we have*

$$\mathbb{E}[L_{X+Y}(A, 0)^m] \geq \mathbb{E}e^{-1/2\|Y\|^2}\mathbb{E}[L_X(A, 0)^m] \tag{3.8}$$

*for every* $A \subset [\mathbf{0}, \mathbf{T}]$ *and* $m \in \mathbb{N}$.

PROOF. Applying Lemma 3.7(ii), for $g(x) = \prod_{k=1}^m h_\varepsilon(x(\mathbf{s}_k))$, $x \in B$, we get

$$\begin{aligned}\mathbb{E}[L_{X+Y}(A,0,\varepsilon)^m] &= \int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m\, \mathbb{E}\prod_{k=1}^m h_\varepsilon\big(X(\mathbf{s}_k)+Y(\mathbf{s}_k)\big)\\ &\ge \mathbb{E}e^{-1/2\|Y\|^2}\int_{A^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m\, \mathbb{E}\prod_{k=1}^m h_\varepsilon(X(\mathbf{s}_k))\\ &= \mathbb{E}e^{-1/2\|Y\|^2}\mathbb{E}[L_X(A,0,\varepsilon)^m].\end{aligned}$$

Applying (3.5) for both processes, $X$ and $X+Y$, we get (3.8). □

3.2. *RKHS of $W^H(t)$ and the remainder $Z^H(t)$.* Let $H \in (0,1/2)\cup(1/2,1)$ and recall decomposition (1.6):

$$c_H^{-1}B^H(t) = W^H(t) + Z^H(t), \qquad t \ge 0,$$

where the remainder process $Z^H(t)$ can be written as

$$Z^H(t) = \int_0^\infty \{(t+s)^{H-1/2} - s^{H-1/2}\}\, d\bar{B}(s), \tag{3.9}$$

with $\bar{B}(s) := B(-s)$, $s \ge 0$. Clearly, $Z^H(t)$ is a self-similar process with index $H$ and the processes $W^H(t)$ and $Z^H(t)$ are independent. In this section, we develop a technique allowing us to treat sample paths of $Z^H(t)$ as, essentially, elements of the reproducing kernel Hilbert space (RKHS) of $W^H(t)$.

The RKHS $\mathbb{H}[0,T]$ of the the Riemann–Liouville process $\{W^H(t)\}_{t\in[0,T]}$ with index $H > 0$, viewed as a random element in $C[0,T]$, follows standard theory of RKHS; see [5] and [29]. Van der Vaart and van Zanten [40], Lemma 10.2, proved that

$$\mathbb{H}[0,T] = I_{0+}^{H+1/2}(L_2[0,T]), \tag{3.10}$$

where

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\, ds, \qquad t \in [0,T], \tag{3.11}$$

is the Riemann–Liouville fractional integral of order $\alpha > 0$; for $\alpha = 0$, $I_{0+}^0 f := f$.

PROPOSITION 3.3. *$\{Z^H(t)\}_{t\ge a}$ has $C^\infty$-sample paths a.s. for any $a > 0$. However, for every $T > 0$,*

$$\mathbb{P}\big(\{Z^H(t)\}_{t\in[0,T]} \in \mathbb{H}[0,T]\big) = 0.$$

PROOF. Formal $n$-tuple differentiation of $Z^H(t)$ gives

$$\frac{\partial^n}{\partial t^n} Z^H(t) = \prod_{k=1}^n \left(H - \frac{2k-1}{2}\right)\int_0^\infty (t+s)^{H-(2n+1)/2}\, d\bar{B}(s), \qquad t > 0.$$

The right-hand side is a well-defined Gaussian process with locally square integrable sample paths. By consecutive integration of this process over $[a,t]$, we prove that $\{Z^H(t)\}_{t\geq a}$ has $C^{(n-1)}$-sample paths, $n\geq 1$, which proves the first part of the proposition.

To prove the second part, observe that

$$Z^H(t) = (I_{0+}^{H+1/2}V^H)(t), \qquad t\geq 0, \tag{3.12}$$

where $V^H(t)$ is a Gaussian process given by

$$V^H(t) = \frac{H-1/2}{\Gamma(3/2-H)}\int_0^\infty \frac{t^{-H-1/2}u^{H-1/2}}{t+u}\,d\bar{B}(u), \qquad t\geq 0.$$

Direct computation gives $\mathbb{E}[(V^H(t))^2] = Ct^{-1}$, where $C$ depends only on $H$. Hence, $\mathbb{E}\|V^H\|^2_{L_2[0,T]} = \infty$ but $\mathbb{E}\|V^H\|_{L_1[0,T]} < \infty$. Combining the fact that $I_{0+}^{H+1/2}$ is one-to-one on $L_1[0,T]$ (see [39], Theorem 2.4) with (3.10) and (3.12) we get

$$\mathbb{P}(\{Z^H(t)\}_{t\in[0,T]}\in\mathbb{H}[0,T]) = \mathbb{P}(\{V^H(t)\}_{t\in[0,T]}\in L_2[0,T]) = 0,$$

where the last equality follows from a zero-one law and integrability of Gaussian norms. $\square$

Direct verification whether a given function belongs to $\mathbb{H}[0,T]$ can be difficult. Therefore, we give below a simple to check sufficient condition. Let $AC_2^m[0,T]$ denote the space of functions $f$ which have continuous derivatives up to order $m-1$ on $[0,T]$, with $f^{(m-1)}$ absolutely continuous on $[0,T]$, and $f^{(m)}\in L_2[0,T]$, $m\in\mathbb{N}$.

PROPOSITION 3.4. *Let $m=\lceil H+1/2\rceil$. If $f\in AC_2^m[0,T]$ is such that $f^{(k)}(0)=0$ for $0\leq k<m$, then $f\in\mathbb{H}[0,T]$ and*

$$\|f\|_{\mathbb{H}[0,T]} = k_H\|I_{0+}^{m-(H+1/2)}f^{(m)}\|_{L_2[0,T]}, \tag{3.13}$$

*where $k_H=\Gamma(H+1/2)^{-1}$.*

PROOF. By our assumption $f = I_{0+}^m f^{(m)}$, where $f^{(m)}\in L_2[0,T]$. Put $g = I_{0+}^{m-(H+1/2)}f^{(m)}$. Since the operators of fractional integration $\{I_{0+}^\alpha:\alpha\geq 0\}$ form a strongly continuous semigroup on $L_2[0,1]$ (see [39], Theorem 2.6), we get that $g\in L_2[0,T]$ and

$$I_{0+}^{H+1/2}g = I_{0+}^{H+1/2}(I_{0+}^{m-(H+1/2)}f^{(m)}) = I_{0+}^m f^{(m)} = f.$$

In view of (3.10), $f\in\mathbb{H}_T$ and from [40], Lemma 10.2,

$$\|f\|_{\mathbb{H}[0,T]} = k_H\|g\|_{L_2[0,T]} = k_H\|I_{0+}^{m-(H+1/2)}f^{(m)}\|_{L_2[0,T]}. \qquad \square$$

The remainder $Z^H$ is not in $\mathbb{H}[0,T]$ by Proposition 3.3. The next result shows the way to circumvent this problem, which is crucial to our technique.

PROPOSITION 3.5. *For any $a > 0$, there is a Gaussian process $\{Z_a^H(t)\}_{t\geq 0}$ such that*:

(i) $Z_a^H(t) = Z^H(t)$ *for all* $t \geq a$;
(ii) *for any* $T > 0$

$$\mathbb{P}\big(\{Z_a^H(t)\}_{t\in[0,T]} \in \mathbb{H}[0,T]\big) = 1.$$

PROOF. First, consider $H \in (0, \frac{1}{2})$, so that $m = \lceil H + 1/2 \rceil = 1$. Define

$$Z_a^H(t) = \begin{cases} At, & 0 \leq t \leq a, \\ Z^H(t), & t > a, \end{cases}$$

where $A = a^{-1} Z^H(a)$. Since $Z_a^H(t)$ has paths in $AC_2^1[0,T]$ (see the first part of Proposition 3.3) and $Z_a^H(0) = 0$, (ii) holds by Proposition 3.4.

Now we consider $H \in (\frac{1}{2}, 1)$, so that $m = \lceil H + 1/2 \rceil = 2$. Define

$$Z_a^H(t) = \begin{cases} B_1 t^2 + B_2 t^3, & 0 \leq t \leq a, \\ Z^H(t), & t > a, \end{cases}$$

where $B_1 = 3a^{-2} Z^H(a) - a^{-1} \dot{Z}^H(a)$, $B_2 = -2a^{-3} Z^H(a) + a^{-2} \dot{Z}^H(a)$ and $\dot{Z}^H(t) := \frac{\partial}{\partial} Z^H(t)$. As in the previous case, part (ii) follows by Proposition 3.4. Indeed, $Z_a^H(t)$ has paths in $AC_2^2[0,T]$, $Z_a^H(0) = 0$ and $\dot{Z}_a^H(0) = 0$. $\square$

The above method of modifying of $Z^H$ in a neighborhood of 0 will also be used in Section 6 for other processes and the $\mathbb{H}[0,T]$-norm of such modifications will to be estimated. For this purpose, the next lemma will be useful.

LEMMA 3.6. *Let $m = \lceil H + 1/2 \rceil$. If $f \in AC_2^m[0,T]$ and $f^{(k)}(0) = 0$ for $0 \leq k < m$, then for every $a \in (0,T)$*

$$\|f\|_{\mathbb{H}[0,T]}^2 \leq C\bigg\{(T^{2m-2H} - a^{2m-2H})\|f^{(m)}\|_{L_\infty[0,a]}^2 + \int_a^T \bigg|\int_a^T (t-s)^{m-H-3/2} f^{(m)}(s)\,ds\bigg|^2 dt\bigg\},$$

*where $C$ depends only on $H$.*

PROOF. Put $\kappa = m - (H + 1/2)$. In view of (3.13), we get

$$\begin{aligned}
\|f\|_{\mathbb{H}[0,T]}^2 &= k_H^2 \big\|I_{0+}^\kappa\big(f^{(m)}\mathbf{1}_{[0,a]} + f^{(m)}\mathbf{1}_{[a,T]}\big)\big\|_{L_2[0,T]}^2 \\
&\leq 2k_H^2 \|I_{0+}^\kappa \mathbf{1}_{[0,a]}\|_{L_2[0,T]}^2 \|f^{(m)}\|_{L_\infty[0,a]}^2 + 2k_H^2 \big\|I_{0+}^\kappa\big(f^{(m)}\mathbf{1}_{[a,T]}\big)\big\|_{L_2[0,T]}^2 \\
&\leq C(T^{2m-2H} - a^{2m-2H})\|f^{(m)}\|_{L_\infty[0,a]}^2 \\
&\quad + 2k_H^2 \int_a^T \bigg|\int_a^T (t-s)^{\kappa-1} f^{(m)}(s)\,ds\bigg|^2 dt.
\end{aligned}$$

$\square$

3.3. *Technical lemmas*. The following auxiliary results and formulas are used in the proofs of the main theorems. They are given here for a convenient reference.

LEMMA 3.7. *Let $\mu$ be a centered Gaussian measure in a separable Banach space $B$. Let $g: B \mapsto \mathbb{R}_+$ be a measurable function. Then*:

(i) *if $\{x \in B : g(x) \geq t\}$ is symmetric and convex for every $t > 0$, then for every $y \in B$*

$$\int_B g(x+y)\mu(dx) \leq \int_B g(x)\mu(dx);$$

(ii) *if $g$ is symmetric $[g(-x) = g(x), x \in B]$, then for every $y$ in the RKHS $\mathcal{H}_\mu$ of $\mu$*

$$\int_B g(x+y)\mu(dx) \geq \exp\left\{-\frac{1}{2}\|y\|_\mu^2\right\} \int_B g(x)\mu(dx),$$

*where $\|y\|_\mu$ denotes the norm in $\mathcal{H}_\mu$.*

PROOF. Part (i) follows from Anderson's inequality

$$\begin{aligned}\int_B g(x+y)\mu(dx) &= \int_0^\infty \mu\{x \in B : g(x+y) \geq t\}\, dt \\ &\leq \int_0^\infty \mu\{x \in B : g(x) \geq t\}\, dt = \int_B g(x)\mu(dx).\end{aligned}$$

Part (ii) uses Cameron–Martin formula and the convexity of exponential function

$$\begin{aligned}\int_B g(x+y)\mu(dx) &= \int_B g(x) \exp\left\{\langle x, y\rangle_\mu - \frac{1}{2}\|y\|_\mu^2\right\}\mu(dx) \\ &= \frac{1}{2}\int_B g(x) \exp\left\{\langle x, y\rangle_\mu - \frac{1}{2}\|y\|_\mu^2\right\}\mu(dx) \\ &\quad + \frac{1}{2}\int_B g(x) \exp\left\{-\langle x, y\rangle_\mu - \frac{1}{2}\|y\|_\mu^2\right\}\mu(dx) \\ &\geq \exp\left\{-\frac{1}{2}\|y\|_\mu^2\right\} \int_B g(x)\mu(dx).\end{aligned}$$

□

The next lemma is well known and goes back at least to 1950s in equivalent forms; see Anderson [1], page 42, Berman [6], page 293, and [7], page 71. The basic fact is that conditional distribution of $X_k$ given all the $X_i, 1 \leq i < k$, is a univariate Gaussian distribution with (conditional) mean $\mathbb{E}(X_k|X_1, \ldots, X_{k-1})$ and (conditional) variance

$$\det(\mathrm{Cov}(X_1, \ldots, X_k)) / \det(\mathrm{Cov}(X_1, \ldots, X_{k-1}))$$

for $1 \leq k \leq m$.

LEMMA 3.8. *Let* $(X_1,\dots,X_m)$ *be a mean-zero Gaussian random vector. Then*

$$\det(\mathrm{Cov}(X_1,\dots,X_m)) = \mathrm{Var}(X_1)\,\mathrm{Var}(X_2|X_1)\cdots\mathrm{Var}(X_m|X_{m-1},\dots,X_1).$$

Let $B^H(t)$ be given by its moving average representation (1.4). By the deconvolution formula of Pipiras and Taqqu [36], we also have

$$B(t) = c_H^* \int_{-\infty}^t \bigl((t-s)_+^{1/2-H} - (-s)_+^{1/2-H}\bigr)\, dB^H(s), \tag{3.14}$$

where $c_H^* = \{c_H\Gamma(H+1/2)\Gamma(3/2-H)\}^{-1}$ and the integral with respect to $B^H(t)$ is well-defined in the $L^2$-sense. It follows from (1.4) and (3.14) that for every $t \in \mathbb{R}$

$$\mathcal{F}_t := \sigma\{B^H(s);\ -\infty < s \le t\} = \sigma\{B(s);\ -\infty < s \le t\}, \tag{3.15}$$

where the second equality holds modulo sets of probability zero. Then for every $s < t$

$$\mathbb{E}(B^H(t)|\mathcal{F}_s) = c_H \int_{-\infty}^s \bigl((t-u)^{H-1/2} - (-u)_+^{H-1/2}\bigr)\, dB(u). \tag{3.16}$$

If $d = 1$, then for every $s < t$

$$\begin{aligned}\mathrm{Var}(B^H(t)|\mathcal{F}_s) &= \mathbb{E}\{[B^H(t) - \mathbb{E}(B^H(t)|\mathcal{F}_s)]^2|\mathcal{F}_s\}\\ &= \mathbb{E}\biggl\{\int_s^t (t-u)^{H-1/2}\, dB(u)|\mathcal{F}_s\biggr\}\\ &= c_H^2 \int_s^t (t-u)^{2H-1}\, du = \frac{c_H^2}{2H}(t-s)^{2H}.\end{aligned} \tag{3.17}$$

For the reader's convenience, we also quote the following lemma due to König and Mörters [24], Lemma 2.3.

LEMMA 3.9. *Let* $Y \ge 0$ *be a random variable and let* $\gamma > 0$. *If*

$$\lim_{m\to\infty} \frac{1}{m} \log\biggl(\frac{1}{(m!)^\gamma}\mathbb{E}Y^m\biggr) = \kappa \tag{3.18}$$

*for some* $\kappa \in \mathbb{R}$, *then*

$$\lim_{y\to\infty} \frac{1}{y^{1/\gamma}} \log \mathbb{P}\{Y \ge y\} = -\gamma e^{-\kappa/\gamma}. \tag{3.19}$$

## 4. Large deviations for local times.

4.1. *Proof of Theorem 2.1—Superadditivity argument.* In light of Lemma 3.9, it is enough to show that the limit in (3.18) exists for $Y = L_1^0(B^H)$ and for $\gamma = Hd$. We will prove it by a superadditivity argument. Let $\tau$ be an exponential time independent of $B^H(t)$. We will first show that for any integer $m, n \ge 1$,

$$\mathbb{E}[L_\tau^0(B^H)^{m+n}] \ge \binom{m+n}{m} \mathbb{E}[L_\tau^0(B^H)^m]\mathbb{E}[L_\tau^0(B^H)^n]. \tag{4.1}$$

Let $t > 0$ be fixed. Notice that by Theorem 7.1, the Gaussian process $B^H(t)$ satisfies the condition (3.3) posted in Proposition 3.1. By (3.4), therefore,

$$\begin{aligned}\mathbb{E}[L_t^0(B^H)^m] &= \frac{1}{(2\pi)^{md}} \int_{[0,t]^m} ds_1 \cdots ds_m \\ &\quad \times \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \exp\left\{-\frac{1}{2} \operatorname{Var}\left(\sum_{k=1}^m \lambda_k \cdot B^H(s_k)\right)\right\} \\ &= \frac{1}{(2\pi)^{md}} \int_{[0,t]^m} ds_1 \cdots ds_m \\ &\quad \times \left[\int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \exp\left\{-\frac{1}{2} \operatorname{Var}\left(\sum_{k=1}^m \lambda_k B_0^H(s_k)\right)\right\}\right]^d,\end{aligned}$$

where $B_0^H(t)$ is 1-dimensional fractional Brownian motion.

By integration with respect to Gaussian measures,

$$\begin{aligned}&\int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \exp\left\{-\frac{1}{2} \operatorname{Var}\left(\sum_{k=1}^m \lambda_k B_0^H(s_k)\right)\right\} \\ &\qquad = (2\pi)^{m/2} \det\{\operatorname{Cov}(B_0^H(s_1), \dots, B_0^H(s_m))\}^{-1/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}[L_t^0(B^H)^m] &= \frac{m!}{(2\pi)^{md/2}} \int_{[0,t]_<^m} ds_1 \cdots ds_m \\ &\quad \times \det\{\operatorname{Cov}(B_0^H(s_1), \dots, B_0^H(s_m))\}^{-d/2}.\end{aligned} \tag{4.2}$$

In (4.2) and elsewhere, for any $A \subset \mathbb{R}^+$ and an integer $m \ge 1$, we define

$$A_<^m = \{(s_1, \dots, s_m) \in A^m;\ s_1 < \cdots < s_m\}.$$

Put

$$\mathcal{A}(s_1, \dots, s_k) = \sigma\{B_0^H(s_1), \dots, B_0^H(s_k)\}, \qquad k = 1, \dots, m,$$

and $\mathcal{A}(s_1, \dots, s_k) = \{\varnothing, \Omega\}$ when $k = 0$. By Lemma 3.8,

$$\mathbb{E}[L_t^0(B^H)^m] = \frac{m!}{(2\pi)^{md/2}} \int_{[0,t]_<^m} ds_1 \cdots ds_m\, \varphi_m(s_1, \dots, s_m),$$

where

$$\varphi_m(s_1, \dots, s_m) = \prod_{k=1}^m \operatorname{Var}(B_0^H(s_k) | B_0^H(s_1), \dots, B_0^H(s_{k-1}))^{-d/2}$$

with the convention that the first term is $\mathrm{Var}(B_0^H(s_1))$ for $k=1$. We are ready to establish (4.1). Let $m, n \ge 1$ be integers. Then, for any $s_1 < \cdots < s_{n+m}$ and $n+1 \le k \le n+m$,

$$\begin{aligned}
&\mathrm{Var}(B_0^H(s_k)|B_0^H(s_1),\ldots,B_0^H(s_{k-1}))\\
&\quad = \mathrm{Var}\bigl(B_0^H(s_k)-B_0^H(s_n)|B_0^H(s_1),\ldots,B_0^H(s_{k-1})\bigr)\\
&\quad = \mathrm{Var}\bigl(B_0^H(s_k)-B_0^H(s_n)|B_0^H(s_1),\ldots,B_0^H(s_n),\\
&\qquad\qquad B_0^H(s_{n+1})-B_0^H(s_n),\ldots,B_0^H(s_{k-1})-B_0^H(s_n)\bigr)\\
&\quad \le \mathrm{Var}\bigl(B_0^H(s_k)-B_0^H(s_n)|B_0^H(s_{n+1})-B_0^H(s_n),\ldots,B_0^H(s_{k-1})-B_0^H(s_n)\bigr)\\
&\quad = \mathrm{Var}\bigl(B_0^H(s_k-s_n)|B_0^H(s_{n+1}-s_n),\ldots,B_0^H(s_{k-1}-s_n)\bigr),
\end{aligned}$$

where the last step follows from the stationarity of increments. Thus,

$$\varphi_{n+m}(s_1,\ldots,s_{n+m}) \ge \varphi_n(s_1,\ldots,s_n)\varphi_m(s_{n+1}-s_n,\ldots,s_{n+m}-s_n).$$

Notice that from (4.2)

$$\begin{aligned}
\mathbb{E}[L_\tau^0(B^H)^m] &= \frac{m!}{(2\pi)^{md/2}}\mathbb{E}\int_{[0,\tau]^m_<} ds_1\cdots ds_m\,\varphi_m(s_1,\ldots,s_m)\\
&= \frac{m!}{(2\pi)^{md/2}}\mathbb{E}\int_{s_1<\cdots<s_m} 1_{s_m<\tau}\,ds_1\cdots ds_m\,\varphi_m(s_1,\ldots,s_m) \qquad (4.3)\\
&= \frac{m!}{(2\pi)^{md/2}}\int_{(\mathbb{R}^+)^m_<} ds_1\cdots ds_m\,\varphi_m(s_1,\ldots,s_m)e^{-s_m}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}[L_\tau^0(B^H)^{n+m}]\\
&\quad = \frac{(n+m)!}{(2\pi)^{(n+m)d/2}}\int_{(\mathbb{R}^+)^{n+m}_<} ds_1\cdots ds_{n+m}\,\varphi_{n+m}(s_1,\ldots,s_{n+m})e^{-s_{n+m}}\\
&\quad \ge \frac{(n+m)!}{(2\pi)^{(n+m)d/2}}\int_{(\mathbb{R}^+)^{n+m}_<} ds_1\cdots ds_{n+m}\\
&\qquad\quad \times\varphi_n(s_1,\ldots,s_n)e^{-s_n}\varphi_m(s_{n+1}-s_n,\ldots,s_{n+m}-s_n)e^{-(s_{n+m}-s_n)}\\
&\quad = \frac{(n+m)!}{(2\pi)^{(n+m)d/2}}\int_{(\mathbb{R}^+)^{n}_<} ds_1\cdots ds_n\,\varphi_n(s_1,\ldots,s_n)e^{-s_n}\\
&\qquad\quad \times\int_{(\mathbb{R}^+)^{m}_<} dt_1\cdots dt_m\,\varphi_m(t_1,\ldots,t_m)e^{-t_m}\\
&\quad = \binom{n+m}{m}\mathbb{E}[L_\tau^0(B^H)^n]\mathbb{E}[L_\tau^0(B^H)^m].
\end{aligned}$$

We proved relation (4.1) which says that the sequence $m \mapsto \log(\frac{1}{m!}\mathbb{E}[L_\tau^0(B^H)^m])$ is super-additive. By Fekete's lemma, the limit

$$\lim_{m\to\infty}\frac{1}{m}\log\left(\frac{1}{m!}\mathbb{E}[L_\tau^0(B^H)^m]\right) = \sup_{m\geq 1}\frac{1}{m}\log\left(\frac{1}{m!}\mathbb{E}[L_\tau^0(B^H)^m]\right) \tag{4.4}$$
$$= \log L,$$

exists, possibly as an extended number. By the scaling property (1.1),

$$\mathbb{E}[L_\tau^0(B^H)^m] = \mathbb{E}[\tau^{(1-Hd)m}]\mathbb{E}[L_1^0(B^H)^m]$$
$$= \Gamma(1+(1-Hd)m)\mathbb{E}[L_1^0(B^H)^m].$$

From (4.4) and Stirling's formula, we get

$$\lim_{m\to\infty}\frac{1}{m}\log\left(\frac{1}{(m!)^{Hd}}\mathbb{E}[L_1^0(B^H)^m]\right) = \log\{(1-Hd)^{-(1-Hd)}L\}. \tag{4.5}$$

Applying Lemma 3.9, we establish (2.1) with

$$\theta(H,d) = Hd(1-Hd)^{-1+1/Hd}L^{-1/Hd}. \tag{4.6}$$

To obtain (2.2) and complete the proof it is enough to show that

$$(2\pi)^{-d/2}\Gamma(1-Hd) \leq L \leq (H^{-1}\pi c_H^2)^{-d/2}\Gamma(1-Hd). \tag{4.7}$$

By (4.1)

$$\frac{1}{m!}\mathbb{E}[L_\tau^0(B^H)^m] \geq \{\mathbb{E}L_\tau^0(B^H)\}^m = \{(2\pi)^{-d/2}\Gamma(1-Hd)\}^m,$$

where the equality comes from (4.3) (for $m=1$). This proves the lower bound in (4.7).

To prove the upper bound, we first notice that

$$\mathrm{Var}(B_0^H(s_k)|B_0^H(s_1),\dots,B_0^H(s_{k-1}))$$
$$\geq \mathrm{Var}(B_0^H(s_k)|B_0(s), s\leq s_{k-1}) \tag{4.8}$$
$$= \frac{c_H^2}{2H}(s_k - s_{k-1})^{2H},$$

where we used (3.17). Hence, the function $\varphi$ defined above satisfies, with $s_0 = 0$,

$$\varphi_m(s_1,\dots,s_m) \leq (2H/c_H^2)^{md/2}\prod_{k=1}^m (s_k - s_{k-1})^{-Hd},$$

and by (4.3),

$$(\pi c_H^2/H)^{md/2}\mathbb{E}[L_\tau^0(B^H)^m] \leq m!\int_{(\mathbb{R}^+)^m_<} ds_1\cdots ds_m \prod_{k=1}^m (s_k - s_{k-1})^{-Hd}e^{-s_m}$$
$$= m!\left\{\int_0^\infty t^{-Hd}e^{-t}\,dt\right\}^m \tag{4.9}$$
$$= m!\Gamma(1-Hd)^m.$$

This establishes (4.7) and completes the proof.

4.2. *Proof of Theorem 2.2—Comparison argument*. First, we note that

$$L_t^0(c_H^{-1}B^H) = c_H^d L_t^0(B^H). \tag{4.10}$$

Thus, from the decomposition (1.6) and (3.6) for every $m \in \mathbb{N}$,

$$c_H^{md}\mathbb{E}[L_1^0(B^H)^m] \le \mathbb{E}[L_1^0(W^H)^m]. \tag{4.11}$$

To prove a reverse inequality (up to a multiplicative constant) we use notation (3.1). Fix $a \in (0, 1)$ and let let $Z_a^H(t)$, $t \ge 0$, be the process specified in Proposition 3.5 that is also independent of $W^H(t)$, $t \ge 0$. We have

$$c_H^d L_1^0(B^H) = L_{c_H^{-1}B^H}([0,1],0) \ge L_{c_H^{-1}B^H}([a,1],0) = L_{W^H+Z_a^H}([a,1],0).$$

Thus, by (3.8) we get

$$\begin{aligned}
c_H^{md}\mathbb{E}[L_1^0(B^H)^m] &\ge \mathbb{E}[L_{W^H+Z_a^H}([a,1],0)^m] \\
&\ge K_a\mathbb{E}[L_{W^H}([a,1],0)^m] \\
&= K_a\mathbb{E}[(L_1^0(W^H) - L_a^0(W^H))^m] \\
&\ge K_a\{\mathbb{E}[L_1^0(W^H)^m]^{1/m} - \mathbb{E}[L_a^0(W^H)^m]^{1/m}\}^m \\
&= K_a(1-a^{1-Hd})^m\mathbb{E}[L_1^0(W^H)^m],
\end{aligned}$$

where the last equality uses self-similarity (1.3) and $K_a = \mathbb{E}\exp\{-\frac{1}{2}\|Z_a^H\|^2\}$. Here $\|Z_a^H\| < \infty$ a.s. is the RKHS norm associated with $\{W^H(t)\}_{t\in[0,1]}$ and computed for paths of $\{Z_a^H(t)\}_{t\in[0,1]}$. This together with (4.11) yields

$$c_H^{md}\mathbb{E}[L_1^0(B^H)^m] \le \mathbb{E}[L_1^0(W^H)^m] \le K_a^{-1}(1-a^{1-Hd})^{-m}c_H^{md}\mathbb{E}[L_1^0(B^H)^m].$$

Applying the limit as in (4.5) to both sides and then passing $a \to 0$ gives

$$\lim_{m\to\infty}\frac{1}{m}\log\left(\frac{1}{(m!)^{Hd}}\mathbb{E}[L_1^0(W^H)^m]\right) = \log\{c_H^d(1-Hd)^{-(1-Hd)}L\}.$$

Therefore, by Lemma 3.9, the limit in (2.5) exists and $\tilde{\theta}(H,d) = c_H^{-1/H}\theta(H,d)$ by (4.6).

## 5. Large deviations for intersection local times.

5.1. *Proof of Theorem 2.3—Subadditivity argument*. Let $\tilde{\alpha}_\varepsilon^H(A)$ be defined analogously to (1.9) by

$$\tilde{\alpha}_\varepsilon^H(A) = \int_{\mathbb{R}^d}\int_A \prod_{j=1}^p p_\varepsilon(W_j^H(s_j) - x)\,ds_1\cdots ds_p\,dx,$$

where $p_\varepsilon$ is as in (1.10). We will first prove the subadditivity property: for every $m, n \in \mathbb{N}$,

$$
\begin{aligned}
&\mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])^{m+n}\big] \\
&\quad\le \binom{m+n}{m}^p \mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])^{m}\big] \\
&\qquad\times \mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])^{n}\big],
\end{aligned}
\tag{5.1}
$$

where $\tau_1, \dots, \tau_p$ are i.i.d. exponential random variables with mean 1 and independent of $W_1^H(t), \dots, W_p^H(t)$. Indeed, since

$$
\begin{aligned}
&\tilde{\alpha}_\varepsilon^H([0,t_1]\times\cdots\times[0,t_p])^m \\
&\quad= \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \prod_{j=1}^p \prod_{k=1}^m \int_0^{t_j} p_\varepsilon\big(W_j^H(s_{j,k}) - x_k\big)\, ds_{j,k},
\end{aligned}
$$

we can write

$$
\begin{aligned}
&\mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])^{m+n}\big] \\
&\quad= \int_{(\mathbb{R}^d)^{m+n}} dx_1 \cdots dx_{m+n}\, \xi(x_1, \dots, x_{m+n})^p,
\end{aligned}
\tag{5.2}
$$

where

$$
\xi(x_1,\dots,x_{m+n}) = \int_0^\infty dt\, e^{-t} \int_{[0,t]^{m+n}} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\big(W^H(s_k) - x_k\big).
$$

Let

$$
D_t = \big\{(s_1,\dots,s_{m+n}) \in [0,t]^{m+n} : \max\{s_1,\dots,s_m\} \le \min\{s_{m+1},\dots,s_{m+n}\}\big\}.
$$

There are exactly $\binom{m+n}{m}$ permutations $\sigma_i$ of $\{1,\dots,m+n\}$ such that $\bigcup_i \sigma_i^{-1} D_t = [0,t]^{m+n}$ and $\sigma_i^{-1} D_t$ are disjoint modulo sets of measure zero [here, $\sigma(s_1, \dots, s_{m+n}) := (s_{\sigma(1)}, \dots, s_{\sigma(m+n)})$]. Therefore,

$$
\begin{aligned}
&\int_{[0,t]^{m+n}} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\big(W^H(s_k) - x_k\big) \\
&\quad= \sum_i \int_{\sigma_i^{-1} D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\big(W^H(s_k) - x_k\big) \\
&\quad= \sum_i \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\big(W^H(s_k) - x_{\sigma_i(k)}\big),
\end{aligned}
$$

which gives by Hölder's inequality

$$
\begin{aligned}
&\xi(x_1, \dots, x_{m+n})^p \\
&\quad = \left\{ \sum_i \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\bigl(W^H(s_k) - x_{\sigma_i(k)}\bigr) \right\}^p \\
&\quad \le \binom{m+n}{m}^{p-1} \\
&\qquad \times \sum_i \left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\bigl(W^H(s_k) - x_{\sigma_i(k)}\bigr) \right\}^p.
\end{aligned}
$$

Substituting into (5.2) yields

$$
\begin{aligned}
&\mathbb{E}\bigl[\tilde{\alpha}_\varepsilon^H([0, \tau_1] \times \cdots \times [0, \tau_p])^{m+n}\bigr] \\
&\quad \le \binom{m+n}{m}^{p-1} \\
&\qquad \times \sum_i \int_{(\mathbb{R}^d)^{m+n}} dx_1 \cdots dx_{m+n} \\
&\qquad\qquad \times \left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\bigl(W^H(s_k) - x_{\sigma_i(k)}\bigr) \right\}^p \\
&\quad = \binom{m+n}{m}^{p} \int_{(\mathbb{R}^d)^{m+n}} dx_1 \cdots dx_{m+n} \\
&\qquad \times \left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\bigl(W^H(s_k) - x_k\bigr) \right\}^p.
\end{aligned}
$$

Since the last integrand can be written as

$$
\begin{aligned}
&\left\{ \int_0^\infty dt\, e^{-t} \int_{D_t} ds_1 \cdots ds_{m+n}\, \mathbb{E} \prod_{k=1}^{m+n} p_\varepsilon\bigl(W^H(s_k) - x_k\bigr) \right\}^p \\
&\quad = \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p\, e^{-(t_1 + \cdots + t_p)} \\
&\qquad \times \int_{D_{t_1} \times \cdots \times D_{t_p}} \left( \prod_{j=1}^p ds_{j,1} \cdots ds_{j,m+n} \right) \mathbb{E} \prod_{k=1}^{m+n} \prod_{j=1}^p p_\varepsilon\bigl(W_j^H(s_{j,k}) - x_k\bigr),
\end{aligned}
$$

after integrating with respect to $x_1, \dots, x_{m+n}$ we get

$$\begin{aligned}\mathbb{E}[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])^{m+n}] \le \binom{m+n}{m}^p \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p\, e^{-(t_1+\cdots+t_p)} \\ \times \int_{D_{t_1}\times\cdots\times D_{t_p}} \left(\prod_{j=1}^p ds_{j,1}\cdots ds_{j,m+n}\right) \\ \times \mathbb{E}\prod_{k=1}^{m+n} g_\varepsilon(W_1^H(s_{1,k}),\dots,W_p^H(s_{p,k})),\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}g_\varepsilon(y_1,\dots,y_p) &:= \int_{\mathbb{R}^d}\prod_{j=1}^p p_\varepsilon(y_j-x)\,dx \\ &= (2\pi\varepsilon)^{-dp/2}\int_{\mathbb{R}^d} e^{-(|x|^2-2x\cdot\overline{y}+p^{-1}\sum_{i=1}^p|y_i|^2)p/(2\varepsilon)} \\ &= (2\pi\varepsilon)^{-d(p-1)/2}p^{-d/2}\exp\left\{-\frac{1}{2\varepsilon}\sum_{j=1}^p|y_j-\overline{y}|^2\right\},\end{aligned} \tag{5.4}$$

and $\overline{y} := p^{-1}\sum_{i=1}^p y_i$ for $y_1,\dots,y_p\in\mathbb{R}^d$. Moreover,

$$\begin{aligned}&\int_{D_{t_1}\times\cdots\times D_{t_p}}\left(\prod_{j=1}^p ds_{j,1}\cdots ds_{j,m+n}\right)\mathbb{E}\prod_{k=1}^{m+n} g_\varepsilon(W_1^H(s_{1,k}),\dots,W_p^H(s_{p,k})) \\ &\quad= \int_{[\mathbf{0},\mathbf{t}]^m}\left(\prod_{j=1}^p ds_{j,1}\cdots ds_{j,m}\right)\int_{[\mathbf{0},\mathbf{t}-\mathbf{s}^*]^n}\left(\prod_{j=1}^p ds_{j,m+1}\cdots ds_{j,m+n}\right) \\ &\qquad\times \mathbb{E}\prod_{k=1}^m g_\varepsilon(W_1^H(s_{1,k}),\dots,W_p^H(s_{p,k})) \\ &\qquad\times \prod_{k=m+1}^{m+n} g_\varepsilon\big(W_1^H(s_1^*+s_{1,k}),\dots,W_p^H(s_p^*+s_{p,k})\big),\end{aligned} \tag{5.5}$$

where

$$\mathbf{t}=(t_1,\dots,t_p), \qquad \mathbf{s}^*=(s_1^*,\dots,s_p^*)$$

and

$$s_j^* = \max\{s_{j,k}: 1\le k\le m\}.$$

Assuming that $W_j^H(t)$ are given by (1.2) with independent Brownian motions $B_j(t)$, define $\mathcal{F}_{\mathbf{s}^*} = \sigma\{B_j(u_j): u_j\le s_j^*, j=1,\dots,p\}$. Put also

$$Y_j(s_j^*,s) = \int_{s_j^*}^{s_j^*+s}(s_j^*+s-u)^{H-1/2}\,dB_j(u)$$

and

$$Z(s_j^*, s) = \int_0^{s_j^*} (s_j^* + s - u)^{H-1/2} \, dB_j(u),$$

so that $W_j(s_j^* + s) = Y_j(s_j^*, s) + Z_j(s_j^*, s)$. The last expectation can be written as

$$\begin{aligned}
&\mathbb{E}\Biggl\{\prod_{k=1}^{m} g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})) \\
&\qquad \times \mathbb{E}\Biggl[\prod_{k=m+1}^{m+n} g_\varepsilon\bigl(Y_1^H(s_1^*, s_{1,k}) + Z_1^H(s_1^*, s_{1,k}), \dots, \\
&\qquad\qquad\qquad Y_p^H(s_p^*, s_{p,k}) + Z_p^H(s_p^*, s_{p,k})\bigr) \Big| \mathcal{F}_{s^*}\Biggr]\Biggr\} \\
&\quad \le \mathbb{E}\Biggl[\prod_{k=1}^{m} g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k}))\Biggr] \\
&\qquad \times \mathbb{E}\Biggl[\prod_{k=m+1}^{m+n} g_\varepsilon(Y_1^H(s_1^*, s_{1,k}), \dots, Y_p^H(s_p^*, s_{p,k}))\Biggr] \\
&\quad = \mathbb{E}\Biggl[\prod_{k=1}^{m} g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k}))\Biggr] \\
&\qquad \times \mathbb{E}\Biggl[\prod_{k=m+1}^{m+n} g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k}))\Biggr],
\end{aligned}$$

where the inequality follows from Lemma 3.7(i) [see the evaluation of $g_\varepsilon$ in (5.4) and the positive quadratic form associated with it] and the last equality follows from

$$(Y_1(s_1^*, s_{1,k}), \dots, Y_p(s_p^*, s_{p,k})) \stackrel{d}{=} (W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})).$$

Combining the above bound with (5.5) and then with (5.3) we obtain

$$\begin{aligned}
&\mathbb{E}[\tilde{\alpha}_\varepsilon^H([0, \tau_1] \times \cdots \times [0, \tau_p])^{m+n}] \\
&\quad \le \binom{m+n}{m}^p \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \, e^{-(t_1 + \cdots + t_p)} \\
&\qquad \times \int_{[\mathbf{0}, \mathbf{t}]^m} \Biggl(\prod_{j=1}^{p} ds_{j,1} \cdots ds_{j,m}\Biggr) \mathbb{E} \prod_{k=1}^{m} g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})) \\
&\qquad \times \int_{[\mathbf{0}, \mathbf{t} - \mathbf{s}^*]^n} \Biggl(\prod_{j=1}^{p} ds_{j,m+1} \cdots ds_{j,m+n}\Biggr)
\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E} \prod_{k=m+1}^{m+n} g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})) \\
&= \binom{m+n}{m}^p \int_{(\mathbb{R}_+)^m} \left(\prod_{j=1}^p ds_{j,1} \cdots ds_{j,m}\right) \\
&\quad \times \mathbb{E} \prod_{k=1}^m g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})) e^{-(s_1^*+\cdots+s_p^*)} \\
&\quad \times \int_{[\mathbf{s}^*,\infty]^p} dt_1 \cdots dt_p \, e^{-[(t_1-s_1^*)+\cdots+(t_p-s_p^*)]} \\
&\quad \times \int_{[\mathbf{0},\mathbf{t}-\mathbf{s}^*]^n} \left(\prod_{j=1}^p ds_{j,m+1} \cdots ds_{j,m+n}\right) \\
&\quad \times \mathbb{E} \prod_{k=m+1}^{m+n} g_\varepsilon(W_1^H(s_{1,k}), \dots, W_p^H(s_{p,k})) \\
&= \binom{m+n}{m}^p \mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1] \times \cdots \times [0,\tau_p])^m\big] \\
&\quad \times \mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1] \times \cdots \times [0,\tau_p])^n\big],
\end{aligned}
$$

where in the last equality we use

$$
e^{-(s_1^*+\cdots+s_p^*)} = \int_{(\mathbb{R}_+)^p} e^{-(t_1+\cdots+t_p)} \prod_{k=1}^m \mathbf{1}_{[\mathbf{s}^*,\mathbf{t}]}(s_{1,k}, \dots, s_{p,k}) \, dt_1 \cdots dt_p
$$

and the definition of $g_\varepsilon$ in (5.4). The subadditivity (5.1) is thus proved for any $\varepsilon > 0$.

Now we would like to take $\varepsilon \to 0^+$ on the both sides of (5.1) in an attempt to establish

$$
\begin{aligned}
\mathbb{E}\tilde{\alpha}^H([0,\tau_1] \times \cdots \times [0,\tau_p])^{m+n} \le \binom{m+n}{n}^p & \mathbb{E}\tilde{\alpha}^H([0,\tau_1] \times \cdots \times [0,\tau_p])^m \\
& \times \mathbb{E}\tilde{\alpha}^H([0,\tau_1] \times \cdots \times [0,\tau_p])^n.
\end{aligned} \tag{5.6}
$$

To this end, we need to show that for any $m \ge 1$, $\tilde{\alpha}^H([0,\tau_1] \times \cdots \times [0,\tau_p])$ is indeed in $\mathcal{L}^m(\Omega, \mathcal{A}, \mathbb{P})$ and

$$
\lim_{\varepsilon \to 0^+} \mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1] \times \cdots \times [0,\tau_p])^m\big] = \mathbb{E}\big[\tilde{\alpha}^H([0,\tau_1] \times \cdots \times [0,\tau_p])^m\big]. \tag{5.7}
$$

Indeed, using (5.1) repeatedly we have that

$$
\mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1] \times \cdots \times [0,\tau_p])^m\big] \le (m!)^p \mathbb{E}\big[\tilde{\alpha}_\varepsilon^H([0,\tau_1] \times \cdots \times [0,\tau_p])\big].
$$

Notice that

$$\begin{aligned}\mathbb{E}[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])] &= \int_{\mathbb{R}^d}\left[\int_0^\infty e^{-t}\mathbb{E}p_\varepsilon\bigl(W^H(t)-x\bigr)\,dt\right]^p dx\\ &= \int_{\mathbb{R}^d}\left[\int_0^\infty e^{-t}\int_{\mathbb{R}^d}p_\varepsilon(y-x)p_{t^*}(y)\,dy\right]^p dx,\end{aligned}$$

where $t^* = (2H)^{-1}t^{2H}$ and the last step follows from the easy-to-check fact that $W^H(t)\sim N(0,(2H)^{-1}t^{2H}I_d)$. By Jensen's inequality, the right-hand side is less than or equal to

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}p_\varepsilon(y-x)\left[\int_0^\infty e^{-t}\int_{\mathbb{R}^d}p_{t^*}(y)\,dy\right]^p dy\,dx\\ &\quad= \int_0^\infty\cdots\int_0^\infty dt_1\cdots dt_p\, e^{-(t_1+\cdots+t_p)}\int_{\mathbb{R}^d}\prod_{j=1}^p p_{t_j^*}(x)\,dx\\ &\quad= (H/\pi)^{d(p-1)/2}\\ &\qquad\times\int_0^\infty\cdots\int_0^\infty e^{-(t_1+\cdots+t_p)}\left(\sum_{j=1}^p\prod_{1\le k\ne j\le p}t_k^{2H}\right)^{-d/2}dt_1\cdots dt_p,\end{aligned}$$

where the last step follows from a routine Gaussian integration.

By arithmetic-geometric mean inequality,

$$\frac{1}{p}\sum_{j=1}^p\prod_{1\le k\ne j\le p}t_k^{2H}\ge\prod_{j=1}^p\prod_{1\le k\ne j\le p}t_k^{2H/p}=\prod_{j=1}^p t_k^{2H(p-1)/p}.$$

So we have

$$\begin{aligned}\mathbb{E}\bigl[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])\bigr] &\le (H/\pi)^{d(p-1)/2}p^{-d/2}\left(\int_0^\infty t^{-Hd(p-1)/p}e^{-t}\,dt\right)^p\\ &= (H/\pi)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p.\end{aligned}$$

Summarizing our computation, we obtain

$$\begin{aligned}&(m!)^{-p}\mathbb{E}\bigl[\tilde{\alpha}_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m\bigr]\\ &\qquad\le\bigl((H/\pi)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p\bigr)^m.\end{aligned}\tag{5.8}$$

By Theorem 7.1, the process

$$X^H(t_1,\dots,t_p)=\bigl(W_1^H(t_1)-W_2^H(t_2),\dots,W_{p-1}^H(t_{p-1})-W_p^H(t_p)\bigr)$$

satisfies the condition (3.3) with $A=[\mathbf{0},\mathbf{t}]=[0,t_1]\times\cdots\times[0,t_p]$ for any $t_1,\dots,t_p\ge 0$ and

$$\begin{aligned}&\tilde{\alpha}_\varepsilon^H([0,t_1]\times\cdots\times[0,t_p])\\ &\qquad=\int_{[\mathbf{0},\mathbf{t}]}h_\varepsilon\bigl(W_1^H(s_1)-W_2^H(s_2),\dots,W_{p-1}^H(s_{p-1})-W_p^H(s_p)\bigr)\,ds_1\cdots ds_p,\end{aligned}$$

where

$$h_\varepsilon(x_1,\dots,x_{p-1})=\int_{\mathbb{R}^d}p_\varepsilon(-x)\prod_{j=1}^{p-1}p_1\left(\sum_{k=j}^{p-1}x_k-x\right)dx$$

is a nondegenerate normal density on $\mathbb{R}^{d(p-1)}$. By Proposition 3.1, $\tilde\alpha_\varepsilon^H([0,t_1]\times\cdots\times[0,t_p])\in\mathcal{L}^m(\Omega,\mathcal{A},\mathbb{P})$ and

$$\lim_{\varepsilon\to0^+}\mathbb{E}\big[\tilde\alpha_\varepsilon^H([0,t_1]\times\cdots\times[0,t_p])^m\big]=\mathbb{E}\big[\tilde\alpha^H([0,t_1]\times\cdots\times[0,t_p])^m\big]. \tag{5.9}$$

In addition, by the representation (3.7) one can see that for any $\varepsilon'<\varepsilon$,

$$\mathbb{E}\big[\tilde\alpha_\varepsilon^H([0,t_1]\times\cdots\times[0,t_p])^m\big]\le\mathbb{E}\big[\tilde\alpha_{\varepsilon'}^H([0,t_1]\times\cdots\times[0,t_p])^m\big].$$

Thus, (5.7) follows from monotonic convergence theorem and the identities

$$\begin{aligned}&\mathbb{E}\big[\tilde\alpha_\varepsilon^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m\big]\\&\quad=\int_{(\mathbb{R}^+)^p}e^{-(t_1+\cdots+t_p)}\mathbb{E}\big[\tilde\alpha_\varepsilon^H([0,t_1]\times\cdots\times[0,t_p])^m\big]\,dt_1\cdots dt_p\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}&\mathbb{E}\big[\tilde\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m\big]\\&\quad=\int_{(\mathbb{R}^+)^p}e^{-(t_1+\cdots+t_p)}\mathbb{E}\big[\tilde\alpha^H([0,t_1]\times\cdots\times[0,t_p])^m\big]\,dt_1\cdots dt_p.\end{aligned}$$

Further, by (5.8) we obtain the bound

$$\begin{aligned}&(m!)^{-p}\mathbb{E}\big[\tilde\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m\big]\\&\qquad\le\big((H/\pi)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p\big)^m.\end{aligned} \tag{5.11}$$

Inequality (5.6) implies that the sequence $m\mapsto\log((m!)^{-p}\mathbb{E}\tilde\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m)$ is sub-additive. Hence, the limit

$$\lim_{m\to\infty}\frac{1}{m}\log\big((m!)^{-p}\mathbb{E}\tilde\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m\big)=c(H,d,p) \tag{5.12}$$

exists, possibly as an extended number. Further, by (5.11),

$$c(H,d,p)\le\log\big\{(H/\pi)^{d(p-1)/2}p^{-d/2}\Gamma(1-Hd/p^*)^p\big\}. \tag{5.13}$$

Now we will deduce the moments behavior of $\tilde\alpha^H([0,1]^p)$.

Notice that $\tau_*=\min\{\tau_1,\dots,\tau_p\}$ is an exponential time with parameter $p$:

$$\begin{aligned}\mathbb{E}\tilde\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m&\ge\mathbb{E}\tilde\alpha^H([0,\tau_*]^p)^m\\&=\mathbb{E}\tau_*^{(p-Hd(p-1))m}\mathbb{E}\tilde\alpha^H([0,1]^p)^m\\&=p^{-(p-Hd(p-1))m}\Gamma\big(1+\big(p-Hd(p-1)\big)m\big)\\&\qquad\times\mathbb{E}\tilde\alpha^H([0,1]^p)^m.\end{aligned}$$

By Stirling's formula,

$$\limsup_{m\to\infty}\frac{1}{m}\log\bigl\{(m!)^{-Hd(p-1)}\mathbb{E}\tilde{\alpha}^H([0,1]^p)^m\bigr\}$$
$$\le c(H,d,p)-p(1-Hd/p^*)\log(1-Hd/p^*).$$

On the other hand, for every $t_1,\dots,t_p>0$,

$$\begin{aligned}
&\mathbb{E}\tilde{\alpha}_\varepsilon^H([0,t_1]\times\cdots\times[0,t_p])^m\\
&\quad=\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\prod_{j=1}^p\int_{[0,t_j]^m}ds_1\cdots ds_m\,\mathbb{E}\prod_{k=1}^m p_\varepsilon\bigl(W^H(s_k)-x_k\bigr)\\
&\quad\le\prod_{j=1}^p\left\{\int_{(\mathbb{R}^d)^m}dx_1\cdots dx_m\left(\int_{[0,t_j]^m}ds_1\cdots ds_m\,\mathbb{E}\prod_{k=1}^m p_\varepsilon\bigl(W^H(s_k)-x_k\bigr)\right)^p\right\}^{1/p}\\
&\quad=\prod_{j=1}^p\{\mathbb{E}\tilde{\alpha}_\varepsilon^H([0,t_j]^p)^m\}^{1/p}.
\end{aligned}$$

Letting $\varepsilon\to0^+$, from (5.9) we get

$$\begin{aligned}
\mathbb{E}\tilde{\alpha}^H([0,t_1]\times\cdots\times[0,t_p])^m&\le\prod_{j=1}^p\{\mathbb{E}\tilde{\alpha}^H([0,t_j]^p)^m\}^{1/p}\\
&=\mathbb{E}\tilde{\alpha}^H([0,1]^p)^m\cdot\prod_{j=1}^p t_j^{m(1-Hd/p^*)},
\end{aligned}$$

where the last equality uses self-similarity (1.12). Hence,

$$\begin{aligned}
&\mathbb{E}\tilde{\alpha}^H([0,\tau_1]\times\cdots\times[0,\tau_p])^m\\
&\quad=\int_{(\mathbb{R}_+)^p}dt_1\cdots dt_p\,e^{-(t_1+\cdots+t_p)}\mathbb{E}\tilde{\alpha}^H([0,t_1]\times\cdots\times[0,t_p])^m\\
&\quad\le\mathbb{E}\tilde{\alpha}^m([0,1]^p)^m\int_{(\mathbb{R}_+)^p}dt_1\cdots dt_p\,e^{-(t_1+\cdots+t_p)}(t_1\cdots t_p)^{m(1-Hd/p^*)}\\
&\quad=\mathbb{E}\tilde{\alpha}^H([0,1]^p)^m\,\Gamma\bigl(1+m(1-Hd/p^*)\bigr)^p.
\end{aligned}\tag{5.14}$$

By Stirling's formula again,

$$\liminf_{m\to\infty}\frac{1}{m}\log\bigl\{(m!)^{-Hd(p-1)}\mathbb{E}\tilde{\alpha}^H([0,1]^p)^m\bigr\}$$
$$\ge c(H,d,p)-p(1-Hd/p^*)\log(1-Hd/p^*).$$

We have shown that

$$\lim_{m\to\infty}\frac{1}{m}\log\bigl\{(m!)^{-Hd(p-1)}\mathbb{E}\tilde{\alpha}^H([0,1]^p)^m\bigr\}=C(H,d,p),\tag{5.15}$$

where by (5.13),

$$C(H,d,p) = c(H,d,p) - p(1 - Hd/p^*)\log(1 - Hd/p^*)$$
$$\leq \log\{(H/\pi)^{d(p-1)/2} p^{-d/2}\Gamma(1 - Hd/p^*)^p \times (1 - Hd/p^*)^{-p(1-Hd/p^*)}\}. \tag{5.16}$$

On the other hand, let $\bar{\alpha}^H(A)$ be the intersection local time generated by $c_H^{-1}B_1^H(t), \dots, c_H^{-1}B_p^H(t)$. We have that

$$\bar{\alpha}^H(A) = c_H^{d(p-1)}\alpha^H(A), \qquad A \subset (\mathbb{R}^+)^p. \tag{5.17}$$

In view of the decomposition (1.6), by Proposition 3.1 we have that

$$\mathbb{E}[\tilde{\alpha}^H([0,1]^p)^m] \geq \mathbb{E}[\bar{\alpha}^H([0,1]^p)^m] = c_H^{d(p-1)m}\mathbb{E}[\alpha^H([0,1]^p)^m]. \tag{5.18}$$

It follows from (5.24) below that

$$C(H,d,p) \geq p\log\left\{c_H^{d/p^*}(1 - Hd/p^*)^{-(1-Hd/p^*)}\left(\frac{p^*}{2\pi}\right)^{d/(2p^*)} \times \int_0^\infty (1 + t^{2H})^{-d/2}e^{-t}\,dt\right\}. \tag{5.19}$$

Applying Lemma 3.9 leads the first conclusion (2.7) of our theorem with

$$\tilde{K}(H,d,p) = Hd(p-1)\exp\left\{-\frac{C(H,d,p)}{Hd(p-1)}\right\}$$

and therefore the bounds given in (2.8) follows from (5.16) and (5.19).

5.2. *Proof of Theorem 2.4—Comparison argument.* In connection to (5.15), we first show that

$$\lim_{m\to\infty}\frac{1}{m}\log\{(m!)^{-Hd(p-1)}\mathbb{E}\alpha^H([0,1]^p)^m\} = C(H,d,p) - d(p-1)\log c_H. \tag{5.20}$$

The upper bound follows immediately from (5.15) and the comparison (5.18). To establish the lower bound, we once again consider the intersection local time $\bar{\alpha}^H(A)$ generated by the normalized fractional Brownian motions

$$\bar{B}_1^H(t) = c_H^{-1}B_1^H(t), \qquad \dots, \qquad \bar{B}_p^H(t) = c_H^{-1}B_p^H(t).$$

For any $\varepsilon > 0$, define

$$\bar{\alpha}_\varepsilon^H(A) = \int_{\mathbb{R}^d}\int_A \prod_{j=1}^p p_\varepsilon(\bar{B}_j^H(s_j) - x)\,ds_1\cdots ds_p\,dx, \tag{5.21}$$

Let $0 < \delta < 1$ be a small but fixed number. Notice

$$\begin{aligned}\mathbb{E}\bar{\alpha}_\varepsilon^H([0,1]^p)^m &\geq \mathbb{E}\alpha_\varepsilon^H([\delta,1]^p)^m\\ &= \int_{([\delta,1]^p)^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m\, \mathbb{E}\prod_{k=1}^m g_\varepsilon(\bar{B}_1^H(s_{1,k}),\dots,\bar{B}_p^H(s_{p,k}))\end{aligned}$$

where $g_\varepsilon(x_1,\dots,x_p)$ is defined by (5.4) and we adopt the notation $\mathbf{s}_k = (s_{1,k},\dots, s_{p,k})$.

Consider $(W_1^H(t_1),\dots,W_p^H(t_p))$ ($\mathbf{t} = (t_1,\dots,t_p) \in [0,1]^p$) as a Gaussian random variable taking values in the Banach space $\bigotimes_{j=1}^p C\{[0,1]^p,\mathbb{R}^d\}$. Then the reproducing kernel Hilbert space of $(W_1^H(t_1),\dots,W_p^H(t_p))$ is $\tilde{H}_W = \bigotimes_{j=1}^p H_W$. For each $(f_1(t_1),\dots,f_p(t)) \in \tilde{H}_W$

$$\|(f_1(t_1),\dots,f_p(t))\|_{\tilde{H}_W}^2 = \sum_{j=1}^p \|f_j\|_{H_W}^2,$$

where $\|\cdot\|_{H_W}$ is the reproducing kernel Hilbert norm of $H_W$.

Let $Z_{\delta,1}^H(t),\dots,Z_{\delta,p}^H(t)$ be the processes constructed in Lemma 3.5 (with $a = \delta$) by $Z_1^H(t),\dots,Z_p^H(t)$, respectively. For each $(\mathbf{s}_1,\dots,\mathbf{s}_m) \in ([\delta,1]^p)^m$ by the decomposition (1.6) we have

$$\begin{aligned}&\mathbb{E}\prod_{k=1}^m g_\varepsilon(\bar{B}_1^H(s_{1,k}),\dots,\bar{B}_p^H(s_{p,k}))\\ &\qquad = \mathbb{E}\prod_{k=1}^m g_\varepsilon\big(W_1^H(s_{1,k}) + Z_{\delta,1}^H(s_{1,k}),\dots,W_p^H(s_{p,k}) + Z_{\delta,p}^H(s_{p,k})\big).\end{aligned}$$

Fixed $(\mathbf{s}_1,\dots,\mathbf{s}_m) \in ([\delta,1]^p)^m$. Applying Lemma 3.7(ii) to the functional $g(f_1,\dots,f_p)$ on $\bigotimes_{j=1}^p C\{[0,1]^p,\mathbb{R}^d\}$ defined by

$$g(f_1,\dots,f_p) \equiv \prod_{k=1}^m g_\varepsilon(f_1(s_{1,k}),\dots,f_p(s_{p,k})),$$

$$(f_1,\dots,f_p) \in \bigotimes_{j=1}^p c\{[0,1]^p,\mathbb{R}^d\},$$

then the right-hand side is greater than

$$\begin{aligned}&\left(\mathbb{E}\exp\left\{-\frac{1}{2}\|Z_\delta^H\|_{H_W}^2\right\}\right)^p \mathbb{E}g(W_1^H,\dots,W_p^H)\\ &\qquad = \left(\mathbb{E}\exp\left\{-\frac{1}{2}\|Z_\delta^H\|_{H_W}^2\right\}\right)^p \mathbb{E}\prod_{k=1}^m g_\varepsilon(W_1^H(s_{1,k}),\dots,W_p^H(s_{p,k})).\end{aligned}$$

Summarizing our estimate, we have

$$\begin{aligned}
&\mathbb{E}\bar{\alpha}_\varepsilon^H([0,1]^p)^m \\
&\quad \geq \left(\mathbb{E}\exp\left\{-\frac{1}{2}\|Z_\delta^H\|_{H_W}^2\right\}\right)^p \\
&\qquad \times \int_{([\delta,1]^p)^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \, \mathbb{E}\prod_{k=1}^m g_\varepsilon(W_1^H(s_{1,k}),\dots,W_p^H(s_{p,k})) \\
&\quad = \left(\mathbb{E}\exp\left\{-\frac{1}{2}\|Z_\delta^H\|_{H_W}^2\right\}\right)^p \mathbb{E}\tilde{\alpha}_\varepsilon^H([\delta,1]^p)^m.
\end{aligned}$$

By Proposition 3.1, letting $\varepsilon \to 0^+$ on both sides yields

$$\mathbb{E}\bar{\alpha}^H([0,1]^p)^m \geq \left(\mathbb{E}\exp\{-\tfrac{1}{2}\|Z_\delta^H\|_{H_W}^2\}\right)^p \mathbb{E}\tilde{\alpha}_\varepsilon^H([\delta,1]^p)^m.$$

In view of (5.17),

$$\begin{aligned}
&\liminf_{m\to\infty}\frac{1}{m}\log\bigl((m!)^{-Hd(p-1)}\mathbb{E}[\alpha^H([0,1]^p)^m]\bigr) \\
&\quad \geq -d(p-1)\log c_H + \liminf_{m\to\infty}\frac{1}{m}\log\bigl((m!)^{Hd(p-1)}\mathbb{E}[\tilde{\alpha}^H([\delta,1]^p)^m]\bigr).
\end{aligned} \tag{5.22}$$

To establish the lower bound for (5.20), therefore, it remains to show that

$$\liminf_{\delta\to 0^+}\liminf_{m\to\infty}\frac{1}{m}\log\frac{1}{(m!)^{Hd(p-1)}}\mathbb{E}[\tilde{\alpha}^H([\delta,1]^p)^m] \geq C(H,d,p). \tag{5.23}$$

Write

$$\tilde{\alpha}^H([0,1]^p) = \tilde{\alpha}^H([\delta,1]\times[0,1]^{p-1}) + \tilde{\alpha}^H([0,\delta]\times[0,1]^{p-1}).$$

By the triangular inequality,

$$\begin{aligned}
&\{\mathbb{E}[\tilde{\alpha}^H([0,1]^p)^m]\}^{1/m} \\
&\quad \leq \bigl\{\mathbb{E}\bigl[\tilde{\alpha}^H([\delta,1]\times[0,1]^{p-1})^m\bigr]\bigr\}^{1/m} + \bigl\{\mathbb{E}\bigl[\tilde{\alpha}^H([0,\delta]\times[0,1]^{p-1})^m\bigr]\bigr\}^{1/m}.
\end{aligned}$$

Given $\varepsilon > 0$,

$$\begin{aligned}
&\mathbb{E}\bigl[\tilde{\alpha}_\varepsilon^H([\delta,1]\times[0,1]^{p-1})^m\bigr] \\
&\quad = \int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m \left[\int_{[\delta,1]^m}\mathbb{E}\prod_{k=1}^m p_\varepsilon\bigl(W^H(s_k)-x_k\bigr)\,ds_1\cdots ds_m\right] \\
&\qquad \times \left[\int_{[0,1]^m}\mathbb{E}\prod_{k=1}^m p_\varepsilon\bigl(W^H(s_k)-x_k\bigr)\,ds_1\cdots ds_m\right]^{p-1} \\
&\quad \leq \left\{\int_{(\mathbb{R}^d)^m} dx_1\cdots dx_m\left[\int_{[\delta,1]^m}\mathbb{E}\prod_{k=1}^m p_\varepsilon\bigl(W^H(s_k)-x_k\bigr)\,ds_1\cdots ds_m\right]^p\right\}^{1/p}
\end{aligned}$$

$$\times \left\{ \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m \right.$$

$$\left. \times \left[ \int_{[0,1]^m} \mathbb{E} \prod_{k=1}^m p_\varepsilon\big(W^H(s_k) - x_k\big)\, ds_1 \cdots ds_m \right]^p \right\}^{(p-1)/p}$$

$$= \{\mathbb{E}[\tilde{\alpha}_\varepsilon^H([\delta, 1]^p)^m]\}^{1/p} \{\mathbb{E}[\tilde{\alpha}_\varepsilon^H([0, 1]^p)^m]\}^{(p-1)/p}.$$

Letting $\varepsilon \to 0^+$ yields

$$\mathbb{E}\big[\tilde{\alpha}^H([\delta, 1] \times [0, 1]^{p-1})^m\big] \le \{\mathbb{E}[\tilde{\alpha}^H([\delta, 1]^p)^m]\}^{1/p} \{\mathbb{E}[\tilde{\alpha}^H([0, 1]^p)^m]\}^{(p-1)/p}.$$

Similarly,

$$\mathbb{E}\big[\tilde{\alpha}^H([0, \delta] \times [0, 1]^{p-1})^m\big] \le \{\mathbb{E}[\tilde{\alpha}^H([0, \delta]^p)^m]\}^{1/p} \{\mathbb{E}[\tilde{\alpha}^H([0, 1]^p)^m]\}^{(p-1)/p}.$$

So we have

$$\{\mathbb{E}[\tilde{\alpha}^H([0, 1]^p)^m]\}^{1/mp} \le \{\mathbb{E}[\tilde{\alpha}^H([\delta, 1]^p)^m]\}^{1/mp} + \{\mathbb{E}[\tilde{\alpha}^H([0, \delta]^p)^m]\}^{1/mp}.$$

By scaling,

$$\mathbb{E}[\tilde{\alpha}^H([0, \delta]^p)^m] = \delta^{(p - Hd(p-1))m} \mathbb{E}[\tilde{\alpha}^H([0, 1]^p)^m].$$

Thus,

$$\mathbb{E}[\tilde{\alpha}^H([\delta, 1]^p)^m] \ge \big[1 - \delta^{1 - Hd(p-1)/p}\big]^{mp} \mathbb{E}[\tilde{\alpha}^H([0, 1]^p)^m].$$

Therefore, (5.23) follows from (5.15).

To bound the limit in (5.20) from below, we claim that

$$\lim_{m\to\infty} \frac{1}{m} \log\{(m!)^{-Hd(p-1)} \mathbb{E}\alpha^H([0, 1]^p)^m\}$$

$$\ge p \log\Big\{(1 - Hd/p^*)^{-(1-Hd/p^*)} (p^*)^{d/(2p^*)} (2\pi)^{-d/(2p^*)} \times \int_0^\infty (1 + t^{2H})^{-d/2} e^{-t}\, dt\Big\}. \tag{5.24}$$

Let $\tau_1, \ldots, \tau_p$ be i.i.d. exponential times independent of $B_1^H(t), \ldots, B_p^H(t)$. Given $\varepsilon > 0$,

$$\mathbb{E}\big[\alpha_\varepsilon^H([0, \tau_1] \times \cdots \times [0, \tau_p])^m\big] = \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m\, Q_\varepsilon^p(x_1, \ldots, x_m),$$

where

$$Q_\varepsilon(x_1, \ldots, x_m) = \int_0^\infty e^{-t} \left[ \int_{[0,t]^m} ds_1 \cdots ds_m\, \mathbb{E} \prod_{k=1}^m p_\varepsilon\big(B^H(s_k) - x_k\big) \right] dt.$$

Let $f(x_1, \dots, x_m)$ be a rapidly decreasing function on $(\mathbb{R}^d)^m$ such that

$$\int_{(\mathbb{R}^d)^m} |f(x_1, \dots, x_m)|^{p^*} dx_1 \cdots dx_m = 1.$$

By Hölder's inequality,

$$\begin{aligned}
&\{\mathbb{E}[\alpha_\varepsilon^H([0, \tau_1] \times \cdots \times [0, \tau_p])]^m\}^{1/p} \\
&\qquad \ge \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m\, f(x_1, \dots, x_m) Q_\varepsilon(x_1, \dots, x_m) \\
&\qquad = \int_0^\infty e^{-t} \int_{[0,t]^m} \biggl[\int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m\, f(x_1, \dots, x_m) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times H_{\mathbf{s},\varepsilon}(x_1, \dots, x_m)\biggr] ds_1 \cdots ds_m\, dt,
\end{aligned}$$

where

$$H_{\mathbf{s},\varepsilon}(x_1, \dots, x_m) = \mathbb{E} \prod_{k=1}^m p_\varepsilon\bigl(B^H(s_k) - x_k\bigr), \qquad \mathbf{s} = (s_1, \dots, s_m).$$

Consider the Fourier transform

$$\widehat{f}(\lambda_1, \dots, \lambda_m) = \int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m\, f(x_1, \dots, x_m) \exp\biggl\{i \sum_{k=1}^m \lambda_k \cdot x_k\biggr\}.$$

It is easy to see that

$$\widehat{H}_{\mathbf{s},\varepsilon}(\lambda_1, \dots, \lambda_m) = \exp\biggl\{-\frac{\varepsilon}{2} \sum_{k=1}^m |\lambda_k|^2 - \frac{1}{2} \operatorname{Var}\biggl(\sum_{k=1}^m \lambda_k \cdot B^H(s_k)\biggr)\biggr\}.$$

By Parseval's identity,

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^m} dx_1 \cdots dx_m\, f(x_1, \dots, x_m) H_{\mathbf{s},\varepsilon}(x_1, \dots, x_m) \\
&\qquad = \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m\, \widehat{f}(\lambda_1, \dots, \lambda_m) \\
&\qquad\qquad \times \exp\biggl\{-\frac{\varepsilon}{2} \sum_{k=1}^m |\lambda_k|^2 - \frac{1}{2} \operatorname{Var}\biggl(\sum_{k=1}^m \lambda_k \cdot B^H(s_k)\biggr)\biggr\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\{\mathbb{E}[\alpha_\varepsilon^H([0, \tau_1] \times \cdots \times [0, \tau_p])]^m\}^{1/p} \\
&\qquad \ge \frac{1}{(2\pi)^{md}} \int_0^\infty e^{-t}\, dt \int_{[0,t]^m} ds_1 \cdots ds_m
\end{aligned}$$

$$\times \left[ \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \widehat{f}(\lambda_1, \dots, \lambda_m) \times \exp\left\{ -\frac{\varepsilon}{2} \sum_{k=1}^m |\lambda_k|^2 - \frac{1}{2} \operatorname{Var}\left( \sum_{k=1}^m \lambda_k \cdot B^H(s_k) \right) \right\} \right].$$

We now let $\varepsilon \to 0^+$ on the both hand sides. Noticing that the left-hand side falls into an obvious similarity to (5.7),

$$\begin{aligned}
&\{\mathbb{E}[\alpha^H([0, \tau_1] \times \cdots \times [0, \tau_p])]^m\}^{1/p} \\
&\quad \geq \frac{1}{(2\pi)^{md}} \int_0^\infty e^{-t}\, dt \int_{[0,t]^m} ds_1 \cdots ds_m \\
&\qquad \times \left[ \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \widehat{f}(\lambda_1, \dots, \lambda_m) \right. \\
&\qquad\qquad \left. \times \exp\left\{ -\frac{1}{2} \operatorname{Var}\left( \sum_{k=1}^m \lambda_k \cdot B^H(s_k) \right) \right\} \right].
\end{aligned} \tag{5.25}$$

We now specify the function $f(x_1, \dots, x_m)$ as

$$f(x_1, \dots, x_m) = C^m \prod_{k=1}^m p_1(x_k),$$

where

$$C = (p^*)^{d/(2p^*)} (2\pi)^{d(p^*-1)/(2p^*)}.$$

We have

$$\begin{aligned}
&\int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m\, \widehat{f}(\lambda_1, \dots, \lambda_m) \exp\left\{ -\frac{1}{2} \operatorname{Var}\left( \sum_{k=1}^m \lambda_k \cdot B^H(s_k) \right) \right\} \\
&\quad = C^m \left[ \int_{\mathbb{R}^m} d\gamma_1 \cdots d\gamma_m\, \exp\left\{ -\frac{1}{2} \sum_{k=1}^m \gamma_k^2 - \frac{1}{2} \operatorname{Var}\left( \sum_{k=1}^m \gamma_k B_0^H(s_k) \right) \right\} \right]^d,
\end{aligned}$$

where $B_0^H(t)$ is an 1-dimensional fractional Brownian motion.

Let $\xi_1, \dots, \xi_m$ be i.i.d. standard normal random variable independent of $B_0^H(t)$. Write

$$\eta_k = \xi_k + B_0^H(s_k), \qquad k = 1, \dots, m.$$

We have

$$\frac{1}{2} \sum_{k=1}^m \gamma_k^2 + \frac{1}{2} \operatorname{Var}\left( \sum_{k=1}^m \gamma_k B_0^H(s_k) \right) = \frac{1}{2} \operatorname{Var}\left( \sum_{k=1}^m \gamma_k \eta_k \right).$$

And thus by Gaussian integration,

$$\int_{\mathbb{R}^m} d\gamma_1 \cdots d\gamma_m \exp\left\{-\frac{\sigma^2}{2}\sum_{k=1}^m \gamma_k^2 - \frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \gamma_k B_0^H(s_k)\right)\right\}$$
$$= (2\pi)^{m/2}\det\{\operatorname{Cov}(\eta_1,\dots,\eta_m)\}^{-1/2},$$

with convention that $s_0 = 0$.

Write $s_0 = 0$ and assume $s_1 < \cdots < s_m$. By Lemma 3.8,

$$\begin{aligned}\det\{\operatorname{Cov}(\eta_1,\dots,\eta_m)\} &= \operatorname{Var}(\eta_1)\prod_{k=2}^m \operatorname{Var}(\eta_k|\eta_1,\dots,\eta_{k-1}) \\ &= \{1+\operatorname{Var}(B_0(s_1))\} \\ &\quad\times \prod_{k=2}^m\{1+\operatorname{Var}(B_0^H(s_k)|B_0^H(s_1),\dots,B_0^H(s_{k-1}))\} \\ &\le \prod_{k=1}^m\{1+(s_k-s_{k-1})^{2H}\},\end{aligned}$$

where the last step follows from the computation

$$\begin{aligned}&\operatorname{Var}(B_0^H(s_k)|B_0^H(s_1),\dots,B_0^H(s_{k-1})) \\ &\qquad= \operatorname{Var}\bigl(B_0^H(s_k)-B_0^H(s_{k-1})|B_0^H(s_1),\dots,B_0^H(s_{k-1})\bigr) \\ &\qquad\le \operatorname{Var}\bigl(B_0^H(s_k)-B_0^H(s_{k-1})\bigr) = (s_k-s_{k-1})^{2H}.\end{aligned}$$

Summarizing our argument since (5.25), we obtain

$$\begin{aligned}&\bigl\{\mathbb{E}\bigl[\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])\bigr]^m\bigr\}^{1/p} \\ &\qquad\ge m!(C(2\pi)^{-d/2})^m\int_0^\infty e^{-t}\,dt\int_{[0,t]^m_<} ds_1\cdots ds_m \\ &\qquad\quad\times\prod_{k=1}^m\{1+(s_k-s_{k-1})^{2H}\}^{-d/2} \\ &\qquad= m!(C(2\pi)^{-d/2})^m\left[\int_0^\infty(1+t^{2H})^{-d/2}e^{-t}\,dt\right]^m.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\mathbb{E}\bigl[\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])\bigr]^m \\ &\qquad\ge (m!)^p(C(2\pi)^{-d/2})^{mp}\left[\int_0^\infty(1+t^{2H})^{-d/2}e^{-t}\,dt\right]^{pm}.\end{aligned}\tag{5.26}$$

On the other hand, with obvious similarity to (5.14)

$$\begin{aligned}&\mathbb{E}[\alpha^H([0,\tau_1]\times\cdots\times[0,\tau_p])]^m\\&\quad\le\mathbb{E}[\alpha^H([0,1]^p)]^m\{\Gamma(1+m(1-Hd/p^*))\}^p.\end{aligned}$$

Hence, (5.24) follows from (5.26) and Stirling's formula.

By (5.16) and (5.24), the limit given in (5.20) is finite. By Lemma 3.9, the large deviation given in (2.9) holds with

$$\begin{aligned}K(H,d,p)&=Hd(p-1)\exp\left\{-\frac{C(H,d,p)-d(p-1)\log c_H}{Hd(p-1)}\right\}\\&=c_H^{1/H}\tilde{K}(H,d,p).\end{aligned}$$

**6. The law of the iterated logarithm.** We will prove Theorem 2.5 in this section. Due to the similarity of arguments, we will only establish (2.18). By the self-similarity property (1.12), the large deviation limit of Theorem 2.3 can be rewritten as

$$\begin{aligned}&\lim_{t\to\infty}(\log\log t)^{-1}\log\mathbb{P}\{\tilde{\alpha}^H([0,t]^p)\ge\lambda t^{p-Hd(p-1)}(\log\log t)^{Hd(p-1)}\}\\&\qquad=-\tilde{K}(H,d,p)\lambda^{p^*/Hdp}\qquad(\lambda>0).\end{aligned}\tag{6.1}$$

Therefore, the upper bound

$$\begin{aligned}&\limsup_{t\to\infty}t^{Hd(p-1)-p}(\log\log t)^{-Hd(p-1)}\tilde{\alpha}^H([0,t]^p)\\&\qquad\le\tilde{K}(H,d,p)^{-Hd(p-1)}\qquad\text{a.s.}\end{aligned}$$

is a consequence of the standard argument using Borel–Cantelli lemma.

To show the lower bound, we proceed in several steps. First, let $N>1$ be a large but fixed number and write $t_n=N^n$ $(n=1,2,\dots)$. Define the $d$-dimensional process

$$Q_n^H(t)=\int_0^{t_n}(t+u)^{H-1/2}\,dB(u),\qquad t\ge 0,$$

where $B(u)$ is a standard $d$-dimensional Brownian motion. Recall that $\mathbb{H}[0,T]$ denotes the RKHS of $\{W^H(t)\}_{t\in[0,T]}$. Combining Propositions 3.3 and 3.5 we can deduce that $\{Q_n^H(t)\}_{t\in[0,T]}$ is not in $\mathbb{H}[0,T]$, $T>0$. For that reason, similarly as in Proposition 3.5, we define the following modifications of $Q_n^H(t)$. When $H\in(0,1/2)$, put

$$G_n^H(t)=\begin{cases}A_nt, & 0\le t\le t_n,\\ Q_n^H(t), & t>t_n,\end{cases}$$

where $A_n=t_n^{-1}Q_n^H(t_n)$. When $H\in(\frac{1}{2},1)$, put

$$G_n^H(t)=\begin{cases}B_{1,n}t^2+B_{2,n}t^3, & 0\le t\le t_n,\\ Q_n^H(t), & t>t_n,\end{cases}$$

where $B_{1,n} = 3t_n^{-2}Q_n^H(t_n) - t_n^{-1}\dot{Q}_n^H(t_n)$ and $B_{2,n} = -2t_n^{-3}Q_n^H(t_n) + t_n^{-2}\dot{Q}_n^H(t_n)$.

LEMMA 6.1. *For every $n \ge 1$, $\mathbb{P}(\{G_n^H(t)\}_{t\in[0,t_{n+1}]} \subset \mathbb{H}[0,t_{n+1}]) = 1$. Furthermore*,

$$\sup_n \mathbb{E}\|G_n^H\|^2_{\mathbb{H}[0,t_{n+1}]} < \infty. \tag{6.2}$$

PROOF. Obviously, it suffices to consider the case $d = 1$. The first part of the lemma follows by the same argument as in Proposition 3.5. For the second part, we use Lemma 3.6 with $a = t_n$ and $T = t_{n+1}$. A constant $C > 0$ below will depend only on $H$ but it will be allowed to be different at different places.

First, consider $H \in (0, 1/2)$, so that $m = \lceil H + 1/2\rceil = 1$. In this case, we get

$$\|\dot{G}_n^H\|_{L_\infty[0,t_n]} = |A_n|$$

and

$$\begin{aligned}
&\int_{t_n}^{t_{n+1}} \left|\int_{t_n}^{t} (t-s)^{-H-1/2}\dot{G}_n^H(s)\,ds\right|^2 dt \\
&\quad = C\int_{t_n}^{t_{n+1}} \left|\int_0^{t_n}\left(\int_{t_n}^{t}(t-s)^{-H-1/2}(s+u)^{H-3/2}\,ds\right)dB(u)\right|^2 dt.
\end{aligned}$$

Therefore,

$$\mathbb{E}\|\dot{G}_n^H\|^2_{L_\infty[0,t_n]} = t_n^{-2}\mathbb{E}Q_n^H(t_n)^2 = Ct_n^{2H-2}, \tag{6.3}$$

and

$$\begin{aligned}
&\mathbb{E}\int_{t_n}^{t_{n+1}} \left|\int_{t_n}^{t} (t-s)^{-H-1/2}\dot{G}_n^H(s)\,ds\right|^2 dt \\
&\quad = C\int_{t_n}^{t_{n+1}}\int_0^{t_n}\left(\int_{t_n}^{t}(t-s)^{-H-1/2}(s+u)^{H-3/2}\,ds\right)^2 du\,dt \\
&\quad = C\int_{t_n}^{t_{n+1}}\int_0^{t_n}\frac{(t-t_n)^{1-2H}(u+t_n)^{2H-1}}{(t+u)^2}\,du\,dt \\
&\quad \le C\int_{t_n}^{t_{n+1}}\left(\frac{t}{t_n}-1\right)^{1-2H}\frac{t_n}{t^2}\,dt \\
&\quad \le C\left(\frac{t_{n+1}}{t_n}-1\right)^{1-2H}\int_{t_n}^{t_{n+1}}\frac{1}{t}\,dt \\
&\quad \le C\left(\frac{t_{n+1}}{t_n}\right)^{1-2H}\log\frac{t_{n+1}}{t_n}.
\end{aligned} \tag{6.4}$$

Using bounds (6.3)–(6.4) with Lemma 3.6, we get

$$\begin{aligned}\mathbb{E}\|G_n^H\|^2_{\mathbb{H}[0,t_{n+1}]} &\le C(t_{n+1}^{2-2H} - t_n^{2-2H})t_n^{2H-2} + C(t_{n+1}/t_n)^{1-2H}\log(t_{n+1}/t_n)\\ &\le C(N^{2-2H} + N^{1-2H}\log N),\end{aligned}$$

which proves (6.2) in the case $H \in (0, 1/2)$. The proof in the case $H \in (1/2, 1)$ follows the same line of computations, and thus is omitted. □

For simplicity of notation, from now on write $\mathbb{H}_n$ for $\mathbb{H}[0, t_{n+1}]$. Define the sigma field

$$\mathcal{F}_t = \sigma\{(B_1(s), \dots, B_p(s));\ s \le t\}.$$

To complete the proof of Theorem 2.5, that is, to establish the lower bound in (2.18), it is enough to show that for any $\lambda < \tilde{K}(H, d, p)^{-Hd(p-1)}$ there is an $N$, sufficiently large, such that

$$\begin{aligned}&\sum_n \mathbb{P}\{\tilde{\alpha}^H([2t_n, t_{n+1}]^p)\\ &\qquad \ge \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)} | \mathcal{F}_{t_n}\} = \infty \qquad \text{a.s.}\end{aligned} \tag{6.5}$$

Indeed, by [8], Corollary 5.29, page 96, (6.5) implies that

$$\limsup_{n\to\infty} t_{n+1}^{Hd(p-1)-p}(\log\log t_{n+1})^{-Hd(p-1)}\tilde{\alpha}^H([2t_n, t_{n+1}]^p) \ge \lambda \qquad \text{a.s.}$$

which leads to

$$\limsup_{t\to\infty} t^{Hd(p-1)-p}(\log\log t)^{-Hd(p-1)}\tilde{\alpha}^H([0, t]^p) \ge \lambda \qquad \text{a.s.}$$

Letting $\lambda \to \tilde{K}(H, d, p)^{-Hd(p-1)}$ on the right-hand side yields the lower bound as claimed.

Now let $\varepsilon > 0$ be fixed and write

$$\begin{aligned}\tilde{\alpha}_\varepsilon^H([2t_n, t_{n+1}]^p) &= \int_{[2t_n, t_{n+1}]^p} ds_1 \cdots ds_p\, g_\varepsilon(W_1^H(s_1), \dots, W_p^H(s_p))\\ &= \int_{[t_n, t_{n+1}-t_n]^p} ds_1 \cdots ds_p\, g_\varepsilon\big(W_1^H(t_n + s_1), \dots, W_p^H(t_n + s_p)\big)\\ &= \int_{[t_n, t_{n+1}-t_n]^p} ds_1 \cdots ds_p\\ &\qquad \times g_\varepsilon\big(Y_1^H(s_1) + Z_1^H(s_1), \dots, Y_p^H(s_p) + Z_p^H(s_p)\big),\end{aligned}$$

where $g_\varepsilon(x_1, \dots, x_p)$ is given in (5.4) and for $j = 1, \dots, p$,

$$\begin{aligned}Y_j^H(t) &= \int_{t_n}^{t_n+t} (t_n + t - s)^{H-1/2}\, dB_j(s),\\ Z_j^H(t) &= \int_0^{t_n} (t_n + t - s)^{H-1/2}\, dB_j(s).\end{aligned}$$

Consider a symmetric set $A \subset \bigotimes_{j=1}^{p} C\{[0, t_{n+1}], \mathbb{R}^d\}$ defined by

$$A = \left\{ (f_1, \dots, f_p) \in \bigotimes_{j=1}^{p} C\{[0, t_{n+1}], \mathbb{R}^d\}; \right.$$
$$\int_{[t_n, t_{n+1}-t_n]^p} ds_1 \cdots ds_p \, g_\varepsilon(f_1(s_1), \dots, f_p(s_p))$$
$$\left. \geq \lambda t_{n+1}^{p-Hd(p-1)} (\log\log t_{n+1})^{Hd(p-1)} \right\}.$$

For any $(f_1, \dots, f_p) \in \bigotimes_{j=1}^{p} \mathbb{H}_n$, applying Lemma 3.7(ii) to the indicator of $A$ leads to

$$\mathbb{P}\{(W_1^H + f_1, \dots, W_p^H + f_p) \in A\}$$
$$\geq \exp\left\{ -\frac{1}{2} \sum_{j=1}^{p} \|f\|_{H_n}^2 \right\} \mathbb{P}\{(W_1^H, \dots, W_p^H) \in A\},$$

if $f_1, \dots, f_p \in \mathbb{H}_n$.

Notice that

$$\{Z^H(t);\ t_n \leq t \leq t_{n+1}\} \stackrel{d}{=} \{Q_n^H(t);\ t_n \leq t \leq t_{n+1}\}$$
$$= \{G_n^H(t);\ t_n \leq t \leq t_{n+1}\},$$
$$\{Y^H(t);\ t_n \leq t \leq t_{n+1}\} \stackrel{d}{=} \{W^H(t);\ t_n \leq t \leq t_{n+1}\}$$

and $Y^H(t)$ and $Z^H(t)$ are independent. By Lemma 6.1,

$$\mathbb{P}\{(Y_1^H + Z_1^H, \dots, Y_p^H + Z_p^H) \in A | \mathcal{F}_{t_n}\}$$
$$\geq \exp\left\{ -\frac{1}{2} \sum_{j=1}^{p} \|G_{n,j}^H\|_{\mathbb{H}_n}^2 \right\} \mathbb{P}\{(W_1^H, \dots, W_p^H) \in A\}$$

or

$$\mathbb{P}\{\tilde{\alpha}_\varepsilon^H([2t_n, t_{n+1}]^p) \geq \lambda t_{n+1}^{p-Hd(p-1)} (\log\log t_{n+1})^{Hd(p-1)} | \mathcal{F}_{t_n}\}$$
$$\geq \exp\left\{ -\frac{1}{2} \sum_{j=1}^{p} \|G_{n,j}^H\|_{\mathbb{H}_n}^2 \right\}$$
$$\times \mathbb{P}\{\tilde{\alpha}_\varepsilon^H([t_n, t_{n+1} - t_n]^p) \geq \lambda t_{n+1}^{p-Hd(p-1)} (\log\log t_{n+1})^{Hd(p-1)}\}.$$

Letting $\varepsilon \to 0^+$ on both sides yields

$$\begin{aligned}
\mathbb{P}\{\tilde{\alpha}^H([2t_n, t_{n+1}]^p) &\ge \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)} | \mathcal{F}_{t_n}\} \\
&\ge \exp\left\{-\frac{1}{2}\sum_{j=1}^{p} \|G_{n,j}^H\|_{\mathbb{H}_n}^2\right\} \\
&\quad \times \mathbb{P}\{\tilde{\alpha}^H([t_n, t_{n+1} - t_n]^p) \ge \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\}.
\end{aligned}$$

By (6.1) and by an argument similar to the one used for (5.23), for $\lambda < \tilde{K}(H, d, p)^{-Hd(p-1)}$ and any small $\delta > 0$, one can take $N$ sufficiently large so that, for large $n$,

$$\begin{aligned}
\mathbb{P}\{\tilde{\alpha}^H([t_n, t_{n+1} - t_n]^p) &\ge \lambda t_{n+1}^{p-Hd(p-1)}(\log\log t_{n+1})^{Hd(p-1)}\} \\
&\ge \exp\{-(1-\delta)\log\log t_{n+1}\} = (n \log N)^{-1+\delta}.
\end{aligned}$$

To establish (6.5), therefore, it suffices to show that for any $\varepsilon, \delta > 0$,

$$\sum_n \frac{1}{n^{1-\delta}} \mathbf{1}\left\{\sum_{j=1}^{p} \|G_{n,j}^H\|_{\mathbb{H}_n}^2 \le \varepsilon \log\log t_{n+1}\right\} = \infty \qquad \text{a.s.} \tag{6.6}$$

Indeed, by Lemma 6.1, $G_n^H$ can be viewed as a Gaussian sequence taking values in $H_n$. By the Gaussian tail estimate (see [27], page 59) there is $u = u(\varepsilon) > 0$ such that

$$\mathbb{P}\left\{\sum_{j=1}^{p} \|G_{n,j}^H\|_{\mathbb{H}_n}^2 \ge \varepsilon \log\log t_{n+1}\right\} \le \frac{1}{n^u}$$

for large $n$. Then for $0 < \delta < u$, we obtain (6.6), which yields (6.5). The proof is complete.

**7. Local times of Gaussian fields.** We begin by mentioning the work of Geman, Horowitz and Rosen [17] on the condition for the existence and continuity of the local times of the Gaussian fields; see also recent work of Wu and Xiao [41]. Let $X(\mathbf{t})$ [$\mathbf{t} \in (\mathbb{R}^+)^p$] be a mean zero Gaussian field taking values in $\mathbb{R}^d$ such that there is a $\gamma > 0$ such that for any $t > 0$ and $m \in \mathbb{N}$,

$$\begin{aligned}
&\int_{([0,t]^p)^m} d\mathbf{s}_1 \cdots d\mathbf{s}_m \\
&\quad \times \int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^{m} |\lambda_k|^\gamma\right) \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^{m} \lambda_k \cdot X(\mathbf{s}_k)\right)\right\} < \infty.
\end{aligned} \tag{7.1}$$

Geman, Horowitz and Rosen (Theorem 2.8 in [17]) proved that the occupation time

$$\mu_{\mathbf{t}}(B) = \int_{[\mathbf{0},\mathbf{t}]} \mathbf{1}_{\{X(\mathbf{s}) \in B\}}\, d\mathbf{s}, \qquad B \subset \mathbb{R}^d,$$

is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. Further, the correspondent density function formally written as

$$\alpha([\mathbf{0},\mathbf{t}],x)=\int_{[\mathbf{0},\mathbf{t}]}\delta_x(X(\mathbf{s}))\,d\mathbf{s}$$

is jointly continuous in $(\mathbf{t},x)$. For fixed $x$, the distribution function $\alpha([\mathbf{0},\mathbf{t}],x)$ $[\mathbf{t}\in(\mathbb{R}^+)^p]$ generates a (random) measure $\alpha(A,x)$ $[A\subset(\mathbb{R}^+)^p]$ on $(\mathbb{R}^+)^p$ which is called the local time of $X(\mathbf{t})$.

In this paper, the result of Geman, Horowitz and Rosen is applied to the following four Gaussian fields:

1. The $d$-dimensional fractional Brownian motion $X_1(t)=B^H(t)$.
2. The $d$-dimensional Riemann–Liouville process $X_2(t)=W^H(t)$.
3. The $d(p-1)$-dimension Gaussian field

$$X_3(t_1,\dots,t_p)=\bigl(B_1^H(t_1)-B_2^H(t_2),\dots,B_{p-1}^H(t_{p-1})-B_p^H(t_p)\bigr).$$

4. The $d(p-1)$-dimension Gaussian field

$$X_4(t_1,\dots,t_p)=\bigl(W_1^H(t_1)-W_2^H(t_2),\dots,W_{p-1}^H(t_{p-1})-W_p^H(t_p)\bigr).$$

THEOREM 7.1. *Under $Hd<1$, $X_1(t)$ and $X_2(t)$ satisfy condition* (7.1); *under $Hd<p^*$, $X_3(\mathbf{t})$ and $X_4(\mathbf{t})$ satisfy condition* (7.1). *Consequently, $X_1$, $X_2$, $X_3$ and $X_4$ have continuous* (*jointly in time and space variables*) *local times.*

PROOF. Due to similarity we only verify (7.1) for $X_3$, which becomes

$$\begin{aligned}&\int_{([0,t]^p)^m}d\mathbf{s}_1\cdots d\mathbf{s}_m\\ &\quad\times\int_{(\mathbb{R}^{d(p-1)})^m}d\tilde\lambda_1\cdots d\tilde\lambda_m\exp\biggl\{-\frac{1}{2}\operatorname{Var}\biggl(\sum_{k=1}^m\tilde\lambda_k\cdot X(\mathbf{s}_k)\biggr)\biggr\}\prod_{k=1}^m|\tilde\lambda_k|^\gamma<\infty,\end{aligned}\tag{7.2}$$

where we use the notation

$$\mathbf{s}_k=(s_{k,1},\dots,s_{k,p})\quad\text{and}\quad\tilde\lambda_k=(\lambda_{k,1},\dots,\lambda_{k,p-1}).$$

Notice that

$$\operatorname{Var}\biggl(\sum_{k=1}^m\tilde\lambda_k\cdot X(\mathbf{s}_k)\biggr)=\sum_{j=1}^p\operatorname{Var}\biggl(\sum_{k=1}^m(\lambda_{k,j}-\lambda_{k,j-1})\cdot B^H(s_{k,j})\biggr)$$

with the convention $\lambda_{k,0}=\lambda_{k,p}=0$. By suitable substitution and using the bound

$$|\tilde\lambda_k|\le C\prod_{j=1}^p\max\{1,|\lambda_{k,j}-\lambda_{k,j-1}|\},$$

we have

$$\int_{(\mathbb{R}^{d(p-1)})^m} d\tilde{\lambda}_1 \cdots d\tilde{\lambda}_m \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \tilde{\lambda}_k \cdot X(\mathbf{s}_k)\right)\right\} \prod_{k=1}^m |\tilde{\lambda}_k|^\gamma$$
$$\le C \int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{j=1}^p H_j(\bar{\lambda}_j),$$

where

$$H_j(\bar{\lambda}_j) = \left(\prod_{k=1}^m \max\{1, |\lambda_{k,j}|^\gamma\}\right) \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_{k,j} \cdot B^H(s_{k,j})\right)\right\}$$

for $\bar{\lambda}_j = (\lambda_{1,j}, \dots, \lambda_{m,j})$ $(1 \le j \le p-1)$ and $\bar{\lambda}_p = -(\bar{\lambda}_1 + \cdots + \bar{\lambda}_{p-1})$.

Write

$$\prod_{j=1}^p H_j(\bar{\lambda}_j) = \prod_{j=1}^p \prod_{1 \le k \ne j \le p} H_k(\bar{\lambda}_k)^{1/(p-1)}.$$

By Hölder's inequality,

$$\int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{j=1}^p H_j(\bar{\lambda}_j)$$
$$\le \prod_{j=1}^p \left\{\int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{1 \le k \ne j \le p} H_k(\bar{\lambda}_k)^{p^*}\right\}^{1/p}.$$

When $j = p$,

$$\int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{1 \le k < p} H_k(\bar{\lambda}_k)^{p^*} = \prod_{k=1}^{p-1} \int_{\mathbb{R}^{md}} H_k(\bar{\lambda})^{p^*} d\bar{\lambda}.$$

As for $1 \le j \le p-1$, recall that $\bar{\lambda}_p = -(\bar{\lambda}_1 + \cdots + \bar{\lambda}_{p-1})$. By translation invariance,

$$\int_{\mathbb{R}^{md}} H_p(\bar{\lambda}_p)^{p^*} d\bar{\lambda}_j = \int_{\mathbb{R}^{md}} H_p(\bar{\lambda})^{p^*} d\bar{\lambda}.$$

By Fubini's theorem, for fixed $j$,

$$\int_{(\mathbb{R}^{md})^{p-1}} d\bar{\lambda}_1 \cdots d\bar{\lambda}_{p-1} \prod_{1 \le k \ne j \le p} H_k(\bar{\lambda}_k)^{p^*} = \prod_{1 \le k \ne j \le p} \int_{\mathbb{R}^{md}} H_k(\bar{\lambda})^{p^*} d\bar{\lambda}.$$

Summarizing our argument, the left-hand side of (7.2) is bounded by

$$C\left\{\int_{[0,t]^m} ds_1 \cdots ds_m \left[\int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m \max\{1, |\lambda_k|^{p^*\gamma}\}\right) \right.\right.$$
$$\left.\left. \times \exp\left\{-\frac{p^*}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k \cdot B^H(s_k)\right)\right\}\right]^{1/p^*}\right\}^p.$$

Hence, all we need is to find $\gamma > 0$ such that

$$\int_{[0,t]^m} ds_1 \cdots ds_m \left[\int_{(\mathbb{R}^d)^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m |\lambda_k|^\gamma\right) \times \exp\left\{-\frac{p^*}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k \cdot B^H(s_k)\right)\right\}\right]^{1/p^*} < \infty \tag{7.3}$$

for all $m \in \mathbb{N}$. Further separating variable and substituting variable, the above is reduced to

$$\int_{[0,t]^m} ds_1 \cdots ds_m \left[\int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m |\lambda_k|^\gamma\right) \times \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k B_0^H(s_k)\right)\right\}\right]^{d/p^*} < \infty. \tag{7.4}$$

By (4.8), for any $0 = s_0 < s_1 < \cdots < s_k$,

$$\begin{aligned}
&\operatorname{Var}\big(B_0^H(s_k) - B_0^H(s_{k-1}) | B_0^H(s_1), \ldots, B_0^H(s_{k-1})\big) \\
&\quad \ge \frac{1}{2H}(s_k - s_{k-1})^{2H} = \frac{1}{2H}\operatorname{Var}\big(B_0^H(s_k) - B_0^H(s_{k-1})\big).
\end{aligned}$$

This property is generalized into the notion known as local nondeterminism. By Lemma 2.3 in Berman [7], there is constant $c_m > 0$ such that for any $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and any $s_1 < \cdots < s_m$

$$\operatorname{Var}\left(\sum_{k=1}^m \lambda_k\big(B_0^H(s_k) - B_0^H(s_{k-1})\big)\right) \ge c_m \sum_{k=1}^m (s_k - s_{k-1})^{2H}\lambda_k^2.$$

Consequently, with notation $\lambda_0 = 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m |\lambda_k|^\gamma\right)\exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k B_0^H(s_k)\right)\right\} \\
&\quad = \int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m |\lambda_k - \lambda_{k-1}|^\gamma\right) \\
&\qquad \times \exp\left\{-\frac{1}{2}\operatorname{Var}\left(\sum_{k=1}^m \lambda_k\big(B_0^H(s_k) - B_0^H(s_{k-1})\big)\right)\right\} \\
&\quad \le \int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m |\lambda_k - \lambda_{k-1}|^\gamma\right)\exp\left\{-c_m \sum_{k=1}^m (s_k - s_{k-1})^{2H}\lambda_k^2\right\}.
\end{aligned}$$

Using the triangle inequality (for which we take $\gamma \le 1$),

$$\prod_{k=1}^m |\lambda_k - \lambda_{k-1}|^\gamma \le \prod_{k=1}^m (|\lambda_k|^\gamma + |\lambda_{k-1}|^\gamma) = \sum_{j_1,\dots,j_m} \prod_{k=1}^m |\lambda_k|^{\delta_{j_k}},$$

where $\delta_{j_k} = 0$, $\gamma$ or $2\gamma$. Notice that

$$\prod_{k=1}^m |\lambda_k|^{\delta_{j_k}} \le \prod_{k=1}^m (1 \vee |\lambda_k|)^{\delta_{j_k}} \le \prod_{k=1}^m (1 \vee |\lambda_k|)^{2\gamma}.$$

Notice the number of the terms in the previous summation is at most $2^m$. Thus,

$$\prod_{k=1}^m |\lambda_k - \lambda_{k-1}|^\gamma \le 2^m \prod_{k=1}^m (1 \vee |\lambda_k|)^{2\gamma}.$$

In this way, the problem is reduced to finding $\gamma > 0$ such that

$$\int_{[0,t]^m_<} ds_1 \cdots ds_m \left[\int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m |\lambda_k|^\gamma\right) \times \exp\left\{-c_m \sum_{k=1}^m (s_k - s_{k-1})^{2H} \lambda_k^2\right\}\right]^{d/p^*} < \infty. \tag{7.5}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}^m} d\lambda_1 \cdots d\lambda_m \left(\prod_{k=1}^m |\lambda_k|^\gamma\right) \exp\left\{-c_m \sum_{k=1}^m (s_k - s_{k-1})^{2H} \lambda_k^2\right\} \\
&= \prod_{k=1}^m \int_{-\infty}^\infty |\lambda|^\gamma e^{-c_m (s_k - s_{k-1})^{2H}\lambda^2} \, d\lambda \\
&= \left\{\int_{-\infty}^\infty |\lambda|^\gamma e^{-c_m \lambda^2} \, d\lambda\right\}^m \prod_{k=1}^m (s_k - s_{k-1})^{-(1+\gamma)H}.
\end{aligned}$$

Therefore, we need to choose $\gamma > 0$ such that

$$\int_{[0,t]^m_<} ds_1 \cdots ds_m \prod_{k=1}^m (s_k - s_{k-1})^{-(1+\gamma)Hd/p^*} < \infty.$$

This is always possible because $Hd < p^*$, so that $(1+\gamma)Hd < p^*$ for some $\gamma > 0$. The proof is complete. □

## APPENDIX

LEMMA A.1. *Let $\{B^H(t)\}_{t \in \mathbb{R}}$ be a standard fractional Brownian motion given by*

$$B^H(t) = c_H \int_{-\infty}^t \left((t-s)^{H-1/2} - (-s)_+^{H-1/2}\right) dB(s), \tag{A.1}$$

*where* $\{B(t)\}_{t\in\mathbb{R}}$ *is a standard Brownian motion. Then*

$$c_H = \sqrt{2H}2^H B(1-H, H+1/2)^{-1/2}, \tag{A.2}$$

*where* $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$ *is the usual beta function.*

PROOF. Since $\mathrm{Var}(B^H(1)) = 1$ we get

$$c_H = \left\{\int_0^\infty ((1+x)^{H-1/2} - x^{H-1/2})^2\,dx + \frac{1}{2H}\right\}^{-1/2}. \tag{A.3}$$

Put

$$I = \int_0^\infty ((1+x)^{H-1/2} - x^{H-1/2})^2\,dx.$$

Then

$$\begin{aligned}
I &= \lim_{\mu\to 0^+}\int_0^\infty ((1+x)^{H-1/2} - x^{H-1/2})^2 e^{-\mu x}\,dx\\
&= \lim_{\mu\to 0^+}\biggl\{(e^\mu+1)\mu^{-2H}\Gamma(2H) - e^\mu\mu^{-2H}\gamma(2H,\mu)\\
&\qquad\qquad - 2\int_0^\infty (1+x)^{H-1/2}x^{H-1/2}e^{-\mu x}\,dx\biggr\}\\
&= -\frac{1}{2H} + \lim_{\mu\to 0^+}\left\{2e^{\mu/2}\mu^{-2H}\Gamma(2H) - 2\int_0^\infty (1+x)^{H-1/2}x^{H-1/2}e^{-\mu x}\,dx\right\}\\
&= -\frac{1}{2H} + \lim_{\mu\to 0^+}\left\{2e^{\mu/2}\mu^{-2H}\Gamma(2H) - \frac{2}{\sqrt{\pi}}e^{\mu/2}\Gamma\left(H+\frac{1}{2}\right)\mu^{-H}K_{-H}\left(\frac{\mu}{2}\right)\right\},
\end{aligned}$$

where $\gamma(z,x)$ and $K_\nu(z)$ are the incomplete gamma function and modified Bessel function of the second kind, respectively. The third equality uses the facts that $e^\mu\mu^{-2H}\gamma(2H,\mu) = \frac{1}{2H} + o(1)$, and that $(e^\mu+1)\mu^{-2H} = 2e^{\mu/2}\mu^{-2H} + o(1)$ for $H < 1$, as $\mu \to 0$. The forth equality applies formula 3.3838 in [18].

Using the duplication formula $\Gamma(2H) = \frac{2^{2H-1}}{\sqrt{\pi}}\Gamma(H)\Gamma(H+\frac{1}{2})$ (see [18], formula 8.3351), we get

$$I = -\frac{1}{2H} + \frac{1}{\sqrt{\pi}}\Gamma\left(H+\frac{1}{2}\right)\lim_{\mu\to 0^+}\left\{\mu^{-2H}2^{2H}\Gamma(H) - 2\mu^{-H}K_H\left(\frac{\mu}{2}\right)\right\}.$$

Since

$$\mu^{-2H}2^{2H}\Gamma(H) = \int_0^\infty x^{H-1}e^{-(\mu^2)/(4)x}\,dx$$

and

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu\int_0^\infty t^{-\nu-1}e^{-t-z^2/(4t)}\,dt$$

(see [18], formula 3.4326), we obtain

$$
\begin{aligned}
I &= -\frac{1}{2H} + \frac{1}{\sqrt{\pi}}\Gamma\left(H+\frac{1}{2}\right)\lim_{\mu\to 0^+}\int_0^\infty x^{H-1}e^{-(\mu^2)/(4)x}\big(1-e^{-1/(4x)}\big)\,dx \\
&= -\frac{1}{2H} + \frac{1}{\sqrt{\pi}}\Gamma\left(H+\frac{1}{2}\right)\int_0^\infty x^{H-1}\big(1-e^{-1/(4x)}\big)\,dx \qquad \text{(A.4)}\\
&= -\frac{1}{2H} + \frac{\Gamma(1-H)\Gamma(H+1/2)}{\sqrt{\pi}4^H H}.
\end{aligned}
$$

Combining (A.4) with (A.3) and using the well-known formula $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ (see, e.g., [18], formula 8.3841), we get (A.2). □

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