The Annals of Probability
2011, Vol. 39, No. 2, 471–506
DOI: 10.1214/10-AOP556

HARMONIC FUNCTIONS, h-TRANSFORM AND LARGE DEVIATIONS FOR RANDOM WALKS IN RANDOM ENVIRONMENTS IN DIMENSIONS FOUR AND HIGHER

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We consider large deviations for nearest-neighbor random walk in a uniformly elliptic i.i.d. environment on $\mathbb{Z}^d$. There exist variational formulae for the quenched and averaged rate functions I_q and I_a, obtained by Rosenbluth and Varadhan, respectively. I_q and I_a are not identically equal. However, when $d \geq 4$ and the walk satisfies the so-called (**T**) condition of Sznitman, they have been previously shown to be equal on an open set $\mathcal{A}_{eq}$.

For every $\xi \in \mathcal{A}_{eq}$, we prove the existence of a positive solution to a Laplace-like equation involving ξ and the original transition kernel of the walk. We then use this solution to define a new transition kernel via the h-transform technique of Doob. This new kernel corresponds to the unique minimizer of Varadhan's variational formula at ξ. It also corresponds to the unique minimizer of Rosenbluth's variational formula, provided that the latter is slightly modified.

1. Introduction.

1.1. *The model.* Let $(e_i)_{i=1}^d$ be the canonical basis for the d-dimensional integer lattice $\mathbb{Z}^d$ with $d \geq 1$. Consider a discrete-time Markov chain on $\mathbb{Z}^d$ with nearest-neighbor steps, that is, with steps in $U := \{\pm e_i\}_{i=1}^d$. For every $x \in \mathbb{Z}^d$ and $z \in U$, denote the transition probability from x to $x + z$ by $\pi(x, x + z)$ and refer to the transition vector $\omega_x := (\pi(x, x + z))_{z \in U}$ as the *environment* at x. If the environment $\omega := (\omega_x)_{x \in \mathbb{Z}^d}$ is sampled from a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, then this process is called *random walk in a random environment* (RWRE). Here, $\mathcal{B}$ is the Borel σ-algebra corresponding to the product topology.

The environment is said to be *uniformly elliptic* if

$$\text{there exists a } \delta > 0 \text{ such that } \pi(0, z) \geq \delta \text{ for every } \omega \in \Omega \text{ and } z \in U. \tag{1.1}$$

For every $y \in \mathbb{Z}^d$, define the shift T_y on Ω by $(T_y\omega)_x := \omega_{x+y}$. Throughout this paper, we will assume that $\mathbb{P}$ is stationary and ergodic under $(T_z)_{z \in U}$. This condition is clearly satisfied when

$$\omega = (\omega_x)_{x \in \mathbb{Z}^d} \qquad \text{is an i.i.d. collection.} \tag{1.2}$$

Received January 2010; revised April 2010.
AMS 2000 subject classifications. 60K37, 60F10, 82C41.
Key words and phrases. Random walk, random environment, large deviations, harmonic functions, Doob h-transform.

For every $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, the Markov chain with environment ω induces a probability measure P_x^ω on the space of paths starting at x. Statements about P_x^ω that hold for $\mathbb{P}$-a.e. ω are referred to as *quenched*. Statements about the semidirect product $P_x := \mathbb{P} \otimes P_x^\omega$ are referred to as *averaged* (or *annealed*). Expectations under $\mathbb{P}$, P_x^ω and P_x are denoted by $\mathbb{E}$, E_x^ω and E_x, respectively.

See [28] for a survey of results on RWRE.

Because of the extra layer of randomness in the model, the standard questions of recurrence versus transience, the law of large numbers (LLN), the central limit theorem (CLT) and the large deviation principle (LDP)—which have well-known answers for classical random walk—become hard. However, it is possible, by taking the *point of view of the particle*, to treat the two layers of randomness as one: if we denote the random path of the particle by $X := (X_n)_{n\geq 0}$, then $(T_{X_n}\omega)_{n\geq 0}$ is a Markov chain (referred to as the *environment Markov chain*) on Ω with transition kernel $\overline{\pi}$ given by

$$\overline{\pi}(\omega, \omega') := \sum_{z\,:\,T_z\omega=\omega'} \pi(0, z).$$

This is a standard approach in the study of random media; see, for example, [9, 11] or [12].

Instead of viewing the environment Markov chain as an auxiliary construction, one can introduce it first and then deduce the particle dynamics from it.

DEFINITION 1.1. A function $\hat{\pi} : \Omega \times U \to \mathbb{R}^+$ is said to be an "environment kernel" if $\hat{\pi}(\cdot, z)$ is $\mathcal{B}$-measurable for each $z \in U$ and $\sum_{z\in U} \hat{\pi}(\cdot, z) = 1$. It can be viewed as a transition kernel on Ω via the following identification:

$$\overline{\pi}(\omega, \omega') := \sum_{z\,:\,T_z\omega=\omega'} \hat{\pi}(\omega, z).$$

Given $x \in \mathbb{Z}^d$, $\omega \in \Omega$ and any environment kernel $\hat{\pi}$, the quenched probability measure $P_x^{\hat{\pi},\omega}$ on the space of particle paths $(X_n)_{n\geq 0}$ starting at x in environment ω is defined by setting $P_x^{\hat{\pi},\omega}(X_o = x) = 1$ and

$$P_x^{\hat{\pi},\omega}(X_{n+1} = y + z | X_n = y) = \hat{\pi}(T_y\omega, z)$$

for all $n \geq 0$, $y \in \mathbb{Z}^d$ and $z \in U$. The semidirect product $P_x^{\hat{\pi}} := \mathbb{P} \otimes P_x^{\hat{\pi},\omega}$ is referred to as the averaged measure and expectations under $P_x^{\hat{\pi},\omega}$ and $P_x^{\hat{\pi}}$ are denoted by $E_x^{\hat{\pi},\omega}$ and $E_x^{\hat{\pi}}$, respectively.

1.2. *Summary of results*. In this paper, we will focus on the large deviation properties of multidimensional RWRE. Section 2 is a detailed survey of the previous results on this topic that are relevant to our purposes. The precise statements of our results are postponed to Section 3 because they rely heavily on the notation and theorems given in Section 2.

In this subsection, we will provide a short and less technical description of the key theorems in Section 2. References will be omitted for the sake of brevity. We will then highlight our main results.

1.2.1. *Summary of previous results.* In the case of quenched RWRE, the LDP holds for the mean velocity X_n/n of the particle. Rosenbluth gives a variational formula for the corresponding rate function I_q. For any $\xi \in \mathbb{R}^d$, $I_q(\xi)$ is equal to the infimum of $H(\hat{\pi}, \mathbb{Q})$, where $H(\cdot)$ is a relative entropy and $(\hat{\pi}, \mathbb{Q})$ varies over all pairs such that: (i) $\hat{\pi}$ is an environment kernel; (ii) $\mathbb{Q}$ is a $\hat{\pi}$-invariant probability measure on Ω; (iii) $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{B}$; (iv) the asymptotic mean velocity of the walk induced by $(\hat{\pi}, \mathbb{Q})$ is equal to ξ.

For averaged walks in i.i.d. environments, Varadhan proves the LDP for X_n/n and gives yet another variational formula for the corresponding rate function I_a. For any $\xi \neq 0$, $I_a(\xi)$ is the infimum of $\mathfrak{I}_a(\alpha)$, where $\mathfrak{I}_a(\cdot)$ is a relative entropy [not equal to $H(\cdot)$] and α varies over all $\mathbb{Z}^d$-valued transient processes with stationary and ergodic increments in U such that the mean drift of α is equal to ξ.

It is easily shown that (i) $I_a \leq I_q$ and (ii) I_q, I_a are not identically equal. When $d \geq 4$ and the walk satisfies the so-called (**T**) condition of Sznitman, I_q and I_a are known to be strictly convex, analytic and equal on an open set $\mathcal{A}_{eq}$. At every $\xi \in \mathcal{A}_{eq}$, Varadhan's variational formula for $I_a(\xi)$ has a unique minimizer.

1.2.2. *Summary of our results.* We will assume that the environment is i.i.d., $d \geq 4$ and the (**T**) condition of Sznitman holds. For every $\xi \in \mathcal{A}_{eq}$, we will prove the existence of an $h(\theta, \cdot) \in L^2(\mathbb{P})$ that solves a certain equation involving $\theta := \nabla I_a(\xi)$ and the original kernel π of the walk; see (3.2). Since (3.2) resembles the Laplace equation, we will refer to $h(\theta, \cdot)$ as *harmonic*. We will then use $h(\theta, \cdot)$ to define a new environment kernel $\hat{\pi}^\theta$ via the h-transform technique of Doob; see (3.3).

For every $\xi \in \mathcal{A}_{eq}$, we will prove the existence of a probability measure $\mathbb{Q}_\xi$ on Ω that is $\hat{\pi}^\theta$-invariant. The pair $(\hat{\pi}^\theta, \mathbb{Q}_\xi)$ corresponds to a stationary Markov chain with values in Ω. This Markov chain induces a $\mathbb{Z}^d$-valued transient process μ_ξ^∞ with stationary and ergodic increments in U. We will show that μ_ξ^∞ is the unique minimizer of Varadhan's variational formula for $I_a(\xi)$.

The pair $(\hat{\pi}^\theta, \mathbb{Q}_\xi)$ is a natural minimizer candidate for Rosenbluth's variational formula for $I_q(\xi)$. However, it is not known whether $\mathbb{Q}_\xi \ll \mathbb{P}$ on $\mathcal{B}$. We will resolve this issue by slightly modifying Rosenbluth's formula so that the infimum of $H(\cdot)$ will be taken over a larger class of pairs. Finally, we will show that $(\hat{\pi}^\theta, \mathbb{Q}_\xi)$ is the unique minimizer of this new formula.

2. Previous results on large deviations for RWRE.

2.1. *The quenched LDP.* Recall that a sequence $(Q_n)_{n \geq 1}$ of probability measures on a topological space $\mathbb{X}$ satisfies the *large deviation principle* (LDP) with rate function $I : \mathbb{X} \to [0, \infty]$ if I is lower semicontinuous and, for any measurable set G,

$$-\inf_{x \in G^o} I(x) \le \liminf_{n\to\infty} \frac{1}{n} \log Q_n(G) \le \limsup_{n\to\infty} \frac{1}{n} \log Q_n(G) \le -\inf_{x\in\overline{G}} I(x).$$

Here, G^o is the interior of G and $\overline{G}$ its closure. See [6] for general background regarding large deviations.

In this paper, the following theorem will be referred to as the quenched (level-1) LDP.

THEOREM 2.1 (Quenched LDP). *Assume* (1.1). *For* $\mathbb{P}$-*a.e.* ω, $(P_o^\omega(\frac{X_n}{n} \in \cdot))_{n\ge 1}$ *satisfies the LDP with a deterministic and convex rate function* I_q. (*The subscript stands for "quenched."*)

Greven and den Hollander [7] prove Theorem 2.1 for walks on $\mathbb{Z}$ in i.i.d. environments. They provide a formula for I_q and show that its graph typically has flat pieces. Comets, Gantert and Zeitouni [5] generalize the results in [7] to stationary and ergodic environments.

For $d \ge 1$, the first result on quenched large deviations is given by Zerner [29]. He uses a subadditivity argument for certain passage times to prove Theorem 2.1 in the case of *nestling* walks in i.i.d. environments.

DEFINITION 2.2. RWRE is said to be *nonnestling* relative to a unit vector $\hat{u} \in \mathcal{S}^{d-1}$ if

$$\operatorname*{ess\,inf}_{\mathbb{P}} \sum_{z\in U} \pi(0, z)\langle z, \hat{u}\rangle > 0. \tag{2.1}$$

It is said to be *nestling* if it is not nonnestling relative to any unit vector. In the latter case, the convex hull of the support of the law of $\sum_z \pi(0, z)z$ contains the origin.

By a more direct use of the subadditive ergodic theorem, Varadhan [22] drops the nestling assumption and generalizes Zerner's result to stationary and ergodic environments. The drawback of these approaches is that they do not lead to any formula for the rate function.

Kosygina, Rezakhanlou and Varadhan [10] consider diffusions on $\mathbb{R}^d$ (with $d \ge 1$) in stationary and ergodic environments. They prove the analog of Theorem 2.1 via a minimax argument and provide a variational formula for the quenched rate function. Rosenbluth [18] adapts their work to the context of RWRE. [See (2.7) below for Rosenbluth's variational formula for I_q.]

2.2. *The quenched level-2 LDP and Rosenbluth's variational formula.* The minimax argument of Kosygina et al. [10] can be generalized to establish a quenched LDP for the so-called *pair empirical measure* of the environment Markov chain. Below, we introduce some notation in order to give the precise statement of this theorem.

For any measurable space $(Y, \mathcal{F})$, write $M_1(Y, \mathcal{F})$ [or simply $M_1(Y)$ whenever no confusion occurs] for the space of probability measures on $(Y, \mathcal{F})$. Consider the random walk $X = (X_n)_{n\geq 0}$ on $\mathbb{Z}^d$ in a stationary and ergodic environment, let $Z_n = X_n - X_{n-1}$ and focus on

$$\nu_{n,X} := \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{T_{X_k}\omega, Z_{k+1}},$$

which is a random element of $M_1(\Omega \times U)$. The map $(\omega, z) \mapsto (\omega, T_z\omega)$ embeds $M_1(\Omega \times U)$ into $M_1(\Omega \times \Omega)$ and we therefore refer to $\nu_{n,X}$ as the pair empirical measure of the environment Markov chain. For any $\mu \in M_1(\Omega \times U)$, define the probability measures $(\mu)^1$ and $(\mu)^2$ on Ω by

$$d(\mu)^1(\omega) := \sum_{z\in U} d\mu(\omega, z) \quad \text{and} \quad d(\mu)^2(\omega) := \sum_{z\in U} d\mu(T_{-z}\omega, z), \tag{2.2}$$

respectively, which are the marginals of μ when μ is seen as an element of $M_1(\Omega \times \Omega)$. With this notation, let

$$M_1'(\Omega \times U) := \Big\{\mu \in M_1(\Omega \times U) : (\mu)^1 = (\mu)^2 \ll \mathbb{P}, \\ \frac{d\mu(\cdot, z)}{d(\mu)^1(\cdot)} > 0 \text{ for every } z \in U\Big\}.$$

THEOREM 2.3 (Quenched level-2 LDP, Yilmaz [25]). *Assume* (1.1). *For* $\mathbb{P}$-*a.e.* ω, $(P_o^\omega(\nu_{n,X} \in \cdot))_{n\geq 1}$ *satisfies the LDP with the rate function* $\mathfrak{I}_q^{**}$, *the double convex conjugate of* $\mathfrak{I}_q : M_1(\Omega \times U) \to \mathbb{R}$ *given by*

$$\mathfrak{I}_q(\mu) = \begin{cases} \displaystyle\int_\Omega \sum_{z\in U} d\mu(\omega, z) \log \frac{d\mu(\omega, z)}{d(\mu)^1(\omega)\pi(0, z)}, & \textit{if } \mu \in M_1'(\Omega \times U), \\ \infty, & \textit{otherwise}. \end{cases} \tag{2.3}$$

Rosenbluth's quenched LDP result is a corollary of Theorem 2.3. Indeed, for any $\mu \in M_1(\Omega \times U)$, set

$$\xi_\mu := \int \sum_{z\in U} d\mu(\omega, z) z. \tag{2.4}$$

For any $\xi \in \mathbb{R}^d$, define

$$A_\xi := \{\mu \in M_1(\Omega \times U) : \xi_\mu = \xi\}. \tag{2.5}$$

With this notation,

$$I_q(\xi) = \inf_{\mu\in A_\xi} \mathfrak{I}_q^{**}(\mu) \tag{2.6}$$

$$= \inf_{\mu\in A_\xi} \mathfrak{I}_q(\mu). \tag{2.7}$$

Here, (2.6) follows from Theorem 2.3 via the so-called contraction principle (see [6]). Note that, even though $\mathfrak{I}_q$ is convex, it may not be lower semicontinuous (see Appendix A of [25] for an example). Therefore, $\mathfrak{I}_q^{**}$ is not equal to $\mathfrak{I}_q$ in general. Nevertheless, (2.7) is valid (see [25]) and it is precisely equal to the variational formula obtained by Rosenbluth in [18].

2.3. *The quenched level-3 LDP.* Theorem 2.3 can be generalized to establish a quenched LDP for the *empirical process*

$$\nu_{n,X}^{\infty} := \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{T_{X_k}\omega, Z_{k+1}^{\infty}},$$

which is a random element of $M_1(\Omega\times U^{\mathbb{N}})$. Here, Z_{k+1}^{∞} is shorthand notation for $(Z_{k+i})_{i\geq 1}$.

THEOREM 2.4 (Quenched level-3 LDP, Rassoul-Agha and Seppäläinen [17]). *Assume* (1.1). *For* $\mathbb{P}$*-a.e.* ω, $(P_o^{\omega}(\nu_{n,X}^{\infty}\in\cdot))_{n\geq 1}$ *satisfies the LDP with a deterministic and convex rate function* $I_{q,3}: M_1(\Omega\times U^{\mathbb{N}})\to\mathbb{R}$.

Rassoul-Agha and Seppäläinen actually obtain this result in greater generality, namely for bounded step size walks satisfying a weak ellipticity condition (see [17]). Also, they show that, just as in Theorem 2.3, the rate function $I_{q,3}$ is the lower semicontinuous regularization of a relative entropy. We choose not to state the precise formula of $I_{q,3}$ here, partly in order to keep the notation simple and partly because we will not need it in what follows.

2.4. *The averaged LDP and Varadhan's variational formula.* In this paper, the following theorem will be referred to as the averaged (level-1) LDP.

THEOREM 2.5 (Averaged LDP). *Assume* (1.1) *and* (1.2). $(P_o(\frac{X_n}{n}\in\cdot))_{n\geq 1}$ *satisfies the LDP with a convex rate function* I_a (*the subscript stands for "averaged"*).

Comets et al. [5] prove Theorem 2.5 for $d=1$ and obtain the following variational formula for I_a:

$$I_a(\xi) = \inf_{\mathbb{Q}}\{I_q^{\mathbb{Q}}(\xi) + |\xi| h_s(\mathbb{Q}|\mathbb{P})\}. \tag{2.8}$$

Here, the infimum is over all stationary and ergodic probability measures on Ω, $I_q^{\mathbb{Q}}(\cdot)$ denotes the rate function for the quenched LDP when the environment measure is $\mathbb{Q}$ and $h_s(\cdot|\cdot)$ is specific relative entropy. Similarly to the quenched picture, the graph of I_a is shown typically to have flat pieces.

Varadhan [22] proves Theorem 2.5 for any $d \geq 1$. He gives yet another variational formula for I_a. Below, we introduce some notation in order to write down this formula.

An infinite path $(x_i)_{i\leq 0}$ with nearest-neighbor steps $x_{i+1} - x_i$ is said to be in W_∞^{tr} if $x_o = 0$ and $\lim_{i\to-\infty} |x_i| = \infty$. For any $w \in W_\infty^{\mathrm{tr}}$, let n_o be the number of times w visits the origin, excluding the last visit. By the transience assumption, n_o is finite. For any $z \in U$, let $n_{o,z}$ be the number of times w jumps to z after a visit to the origin. Clearly, $\sum_{z\in U} n_{o,z} = n_o$. If the averaged walk starts from time $-\infty$ and its path $(X_i)_{i\leq 0}$ up to the present is conditioned to be equal to w, then the probability of the next step being equal to z is

$$q(w, z) := \frac{\mathbb{E}[\pi(0, z) \prod_{z'\in U} \pi(0, z')^{n_{o,z'}}]}{\mathbb{E}[\prod_{z'\in U} \pi(0, z')^{n_{o,z'}}]}, \tag{2.9}$$

by Bayes' rule.

Consider the map $T^* : W_\infty^{\mathrm{tr}} \to W_\infty^{\mathrm{tr}}$ that takes $(x_i)_{i\leq 0}$ to $(x_i - x_{-1})_{i\leq -1}$. Let $\mathcal{I}$ be the set of probability measures on W_∞^{tr} that are invariant under T^* and $\mathcal{E}$ be the set of extremal points of $\mathcal{I}$. Each $\alpha \in \mathcal{I}$ (resp., $\alpha \in \mathcal{E}$) corresponds to a transient process with stationary (resp., stationary and ergodic) increments and induces a probability measure Q_α on particle paths $(X_i)_{i\in\mathbb{Z}}$. The associated *mean drift* is $m(\alpha) := \int (x_o - x_{-1})\, d\alpha = Q_\alpha(X_1 - X_o)$. Define

$$Q_\alpha^w(\cdot) := Q_\alpha\big(\cdot|\sigma(X_i : i \leq 0)\big)(w) \quad \text{and} \quad q_\alpha(w, z) := Q_\alpha^w(X_1 = z) \tag{2.10}$$

for α-a.e. w and $z \in U$.

With this notation,

$$I_a(\xi) = \inf_{\substack{\alpha\in\mathcal{E}:\\ m(\alpha)=\xi}} \mathfrak{I}_a(\alpha) \tag{2.11}$$

for every $\xi \neq 0$, where

$$\mathfrak{I}_a(\alpha) := \int_{W_\infty^{\mathrm{tr}}} \left[\sum_{z\in U} q_\alpha(w, z) \log \frac{q_\alpha(w, z)}{q(w, z)}\right] d\alpha(w). \tag{2.12}$$

Rassoul-Agha [16] generalizes Varadhan's result to a class of mixing environments and also to some other models of random walk on $\mathbb{Z}^d$.

In Section 2.6, we will summarize the known qualitative properties of I_a. In particular, we will state some regularity results which are valid under a certain transience condition of Sznitman. The next subsection is devoted to introducing this condition, which involves what are called *regeneration times*.

2.5. *Regeneration times and Sznitman's condition.* Take a unit vector $\hat{u} \in \mathcal{S}^{d-1}$. Define a sequence $(\tau_m)_{m\geq 0} = (\tau_m(\hat{u}))_{m\geq 0}$ of random times, which are referred to as *regeneration times* (relative to $\hat{u}$), by $\tau_o := 0$ and

$$\tau_m := \inf\{j > \tau_{m-1} : \langle X_i, \hat{u}\rangle < \langle X_j, \hat{u}\rangle \leq \langle X_k, \hat{u}\rangle \quad \text{for all } i, k \text{ with } i < j < k\} \tag{2.13}$$

for every $m \geq 1$. (Regeneration times first appeared in the work of Kesten [8] on one-dimensional RWRE. They were adapted to the multidimensional setting by Sznitman and Zerner; see [21].) If the walk is directionally transient relative to $\hat{u}$, that is, if

$$P_o\Big(\lim_{n\to\infty} \langle X_n, \hat{u}\rangle = \infty\Big) = 1, \tag{2.14}$$

then $P_o(\tau_m < \infty) = 1$ for every $m \geq 1$. As shown in [21], the significance of $(\tau_m)_{m\geq 1}$ is due to the fact that

$$(X_{\tau_m+1} - X_{\tau_m}, X_{\tau_m+2} - X_{\tau_m}, \ldots, X_{\tau_{m+1}} - X_{\tau_m}, \tau_{m+1} - \tau_m)_{m\geq 1}$$

is an i.i.d. sequence under P_o when $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ is an i.i.d. collection.

The walk is said to satisfy Sznitman's transience condition $(\mathbf{T}, \hat{u})$ if (2.14) holds and

$$E_o\Big[\sup_{1\leq i\leq \tau_1(\hat{u})} \exp\{c|X_i|\}\Big] < \infty \qquad \text{for some } c > 0. \tag{2.15}$$

Define the *first backtracking time* of the walk to be

$$\beta = \beta(\hat{u}) := \inf\{i \geq 0 : \langle X_i, \hat{u}\rangle < \langle X_o, \hat{u}\rangle\}. \tag{2.16}$$

The following lemmas list some important facts regarding regenerations.

LEMMA 2.6 (Sznitman [20]). *Assume $d \geq 2$, (1.1), (1.2) and that $(\mathbf{T}, \hat{u})$ holds for some $\hat{u} \in \mathcal{S}^{d-1}$. Then*:

(a) *$P_o(\beta(\hat{u}) = \infty) > 0$ and $\tau_1(\hat{u})$ has finite P_o-moments of arbitrary order*;
(b) *the LLN holds with a limiting velocity ξ_o such that $\langle \xi_o, \hat{u}\rangle > 0$*;
(c) *$(\mathbf{T}, \hat{v})$ is satisfied for every $\hat{v} \in \mathcal{S}^{d-1}$ such that $\langle \xi_o, \hat{v}\rangle > 0$.*

LEMMA 2.7. *Assume (1.1) and (1.2). If the walk is nonnestling (see Definition 2.2) relative to some $\hat{u} \in \mathcal{S}^{d-1}$, then*

$$E_o[\exp\{c\tau_1(\hat{u})\}] < \infty \tag{2.17}$$

for some $c > 0$. In particular, $(\mathbf{T}, \hat{u})$ is satisfied. On the other hand, if the walk is nestling, then (2.17) fails to hold for every $\hat{u} \in \mathcal{S}^{d-1}$ and $c > 0$.

PROOF. The first statement is proved in [19]. The second statement follows immediately from the fact that $I_a(0) = 0$ when the walk is nestling (see [22]). □

LEMMA 2.8. *Assume* (1.1) *and* (1.2). *If the walk is nonnestling and some* $\hat{v} \in \mathcal{S}^{d-1}$ *satisfies* $\langle \xi_o, \hat{v} \rangle > 0$, *then*

$$E_o[\exp\{c\tau_1(\hat{v})\}] < \infty$$

for some $c > 0$.

PROOF. This is Lemma 8 of [26]. $\square$

COROLLARY 2.9. *Assume* $d \geq 2$, (1.1), (1.2) *and that* $(\mathbf{T}, \hat{u})$ *holds for some* $\hat{u} \in \mathcal{S}^{d-1}$. *Since* $\xi_o \neq 0$, *there exists a* $z \in U$ *such that* $\langle \xi_o, z \rangle > 0$. *Then*:

(a) $P_o(\beta(z) = \infty) > 0$ *and* $\tau_1(z)$ *has finite* P_o-*moments of arbitrary order*;

(b) *if the walk is nonnestling*, *then there exists a* $c_1 > 0$ *such that*

$$E_o[\exp\{2c_1\tau_1(z)\}] < \infty;$$

(c) *if the walk is nestling*, *then there exists a* $c_1 > 0$ *such that*

$$E_o\Big[\sup_{1 \leq i \leq \tau_1(z)} \exp\{c_1|X_i|\}\Big] < \infty.$$

2.6. *Qualitative properties of the quenched and the averaged rate functions*. Denote the zero-sets of I_q and I_a by $\mathcal{N}_q := \{\xi \in \mathbb{R}^d : I_q(\xi) = 0\}$ and $\mathcal{N}_a := \{\xi \in \mathbb{R}^d : I_a(\xi) = 0\}$, respectively. The following theorem summarizes some of the known qualitative properties of the quenched and the averaged rate functions when $d \geq 2$. The rest of the known properties are given in Section 2.7.

THEOREM 2.10. *Assume* $d \geq 2$, (1.1) *and* (1.2). *Then*:

(a) I_q *and* I_a *are convex*, $I_q(0) = I_a(0)$ *and* $\mathcal{N}_q = \mathcal{N}_a$ (*see* [22]);

(b) *if the walk is nonnestling*, *then*:

  (i) $\mathcal{N}_a$ *consists of the true velocity* ξ_o (*see* [22]);

  (ii) I_a *is strictly convex and analytic on an open set* $\mathcal{A}_a$ *containing* ξ_o (*see* [13, 23]);

(c) *if the walk is nestling*, *then* $\mathcal{N}_a$ *is a line segment containing the origin that can extend in one or both directions* (*see* [22]); *it cannot extend in both directions when* $d = 2$ (*see* [30]) *or when* $d \geq 5$ (*see* [1]);

(d) *if the walk is nestling*, *but* $(\mathbf{T}, \hat{u})$ *is satisfied for some* $\hat{u} \in \mathcal{S}^{d-1}$, *then*:

  (i) *the origin is an endpoint of* $\mathcal{N}_a$ (*see* [20]);

  (ii) I_a *is strictly convex and analytic on an open set* $\mathcal{A}_a$ (*see* [23]);

  (iii) *there exists a* $(d-1)$-*dimensional smooth surface patch* $\mathcal{A}_a^b$ *such that* $\xi_o \in \mathcal{A}_a^b \subset \partial\mathcal{A}_a$ (*see* [23]);

  (iv) *the unit vector* η_o *normal to* $\mathcal{A}_a^b$ (*and pointing in* $\mathcal{A}_a$) *at* ξ_o *satisfies* $\langle \eta_o, \xi_o \rangle > 0$ (*see* [23]);

  (v) $I_a(t\xi) = tI_a(\xi)$ *for every* $\xi \in \mathcal{A}_a^b$ *and* $t \in [0, 1]$ (*see* [13]).

2.7. *Comparing the quenched and the averaged rate functions.* Assume (1.1) and (1.2). It is clear that

$$\mathcal{D} := \{(\xi_1, \ldots, \xi_d) \in \mathbb{R}^d : |\xi_1| + \cdots + |\xi_d| \leq 1\} = \{\xi \in \mathbb{R}^d : I_a(\xi) < \infty\}$$
$$= \{\xi \in \mathbb{R}^d : I_q(\xi) \leq -\log \delta\}.$$

For any $\xi \in \mathbb{R}^d$, $I_a(\xi) \leq I_q(\xi)$ by Jensen's inequality and Fatou's lemma. Moreover, when the support of $\mathbb{P}$ is not a singleton, $I_a < I_q$ at some interior points of $\mathcal{D}$ (see Proposition 4 of [26]).

The following theorem considers ballistic walks in dimensions four and higher, and says that the quenched and the averaged rate functions are identically equal on a set whose interior contains $\mathcal{N}_a \setminus \{0\}$.

THEOREM 2.11 (Yilmaz [26]). *Assume $d \geq 4$, (1.1), (1.2) and that $(\mathbf{T}, \hat{u})$ holds for some $\hat{u} \in \mathcal{S}^{d-1}$. Then*:

(a) *if the walk is nonnestling, $I_q = I_a$ on an open set $\mathcal{A}_{eq}$ containing ξ_o*;
(b) *if the walk is nestling*:
  (i) *$I_q = I_a$ on an open set $\mathcal{A}_{eq}$*;
  (ii) *there exists a $(d-1)$-dimensional smooth surface patch $\mathcal{A}_{eq}^b$ such that $\xi_o \in \mathcal{A}_{eq}^b \subset \partial\mathcal{A}_{eq}$*;
  (iii) *the unit vector η_o normal to $\mathcal{A}_{eq}^b$ (and pointing in $\mathcal{A}_{eq}$) at ξ_o satisfies $\langle \eta_o, \xi_o \rangle > 0$*;
  (iv) *$I_q(t\xi) = tI_q(\xi) = tI_a(\xi) = I_a(t\xi)$ for every $\xi \in \mathcal{A}_{eq}^b$ and $t \in [0, 1]$.*

Assuming $d = 1$, (1.1) and (1.2), Comets et al. [5] use (2.8) to show that $I_q(\xi) = I_a(\xi)$ if and only if $\xi = 0$ or $I_a(\xi) = 0$. In particular, Theorem 2.11 cannot be generalized to $d \geq 1$. It turns out that it cannot be generalized to $d \geq 2$ or 3, either. Indeed, for $d = 2, 3$, Yilmaz and Zeitouni [27] provide examples of nonnestling walks in uniformly elliptic i.i.d. environments for which the quenched and the averaged rate functions are not identically equal on any open set containing the true velocity ξ_o.

2.8. *Dual results for the logarithmic moment generating functions.* For every $\theta \in \mathbb{R}^d$, consider the logarithmic moment generating functions

$$\Lambda_q(\theta) := \lim_{n\to\infty} \frac{1}{n} \log E_o^\omega[\exp\{\langle \theta, X_n\rangle\}] \quad \text{and}$$
$$\Lambda_a(\theta) := \lim_{n\to\infty} \frac{1}{n} \log E_o[\exp\{\langle \theta, X_n\rangle\}]. \tag{2.18}$$

By Varadhan's lemma (see [6]), $\Lambda_q(\theta) = \sup_{\xi\in\mathbb{R}^d}\{\langle\theta, \xi\rangle - I_q(\xi)\} = I_q^*(\theta)$, the convex conjugate of I_q at θ. Similarly, $\Lambda_a(\theta) = I_a^*(\theta)$.

For every $c > 0$, define

$$\mathcal{C}(c) := \begin{cases} \{\theta \in \mathbb{R}^d : |\theta| < c\}, & \text{if the walk is nonnestling,} \\ \{\theta \in \mathbb{R}^d : |\theta| < c, \Lambda_a(\theta) > 0\}, & \text{if the walk is nestling.} \end{cases} \tag{2.19}$$

In the latter case, $I_a(0) = 0$; see Theorem 2.10. It follows from convex duality that

$$0 = I_a(0) = \sup_{\theta \in \mathbb{R}^d} \{\langle \theta, 0 \rangle - \Lambda_a(\theta)\} = - \inf_{\theta \in \mathbb{R}^d} \Lambda_a(\theta).$$

In other words, $\Lambda_a(\theta) \geq 0$ for every $\theta \in \mathbb{R}^d$. The zero-level set $\{\theta \in \mathbb{R}^d : \Lambda_a(\theta) = 0\}$ of the convex function Λ_a is convex and $\mathcal{C}(c)$ is an open ball minus this convex set.

The following theorems state some of the known qualitative properties of Λ_q and Λ_a.

THEOREM 2.12 (Peterson and Zeitouni [13], Yilmaz [23]). *Assume $d \geq 2$, (1.1) and (1.2). Recall (2.19). If $(\mathbf{T}, \hat{u})$ holds for some $\hat{u} \in \mathcal{S}^{d-1}$, then Λ_a is analytic on $\mathcal{C}_a := \mathcal{C}(c_1)$, where c_1 is as in Corollary 2.9. Moreover, the Hessian $\mathcal{H}_a$ of Λ_a is positive definite on $\mathcal{C}_a$.*

THEOREM 2.13 (Yilmaz [26]). *Assume $d \geq 4$, (1.1) and (1.2). Recall (2.19). If $(\mathbf{T}, \hat{u})$ holds for some $\hat{u} \in \mathcal{S}^{d-1}$, then there exists a $c_2 \in (0, c_1)$ such that $\Lambda_q = \Lambda_a$ on $\mathcal{C}_{eq} := \mathcal{C}(c_2)$.*

In fact, the regularity properties of I_a that are stated in Theorem 2.10 are obtained from Theorem 2.12 via convex duality (see [13, 23]) and $\mathcal{A}_a = \{\nabla \Lambda_a(\theta) : \theta \in \mathcal{C}_a\}$. Similarly, note that Theorem 2.11 is a corollary of Theorem 2.13 and $\mathcal{A}_{eq} = \{\nabla \Lambda_a(\theta) : \theta \in \mathcal{C}_{eq}\}$.

3. Our results.

In this paper, we will obtain new results concerning the large deviation properties of RWRE on $\mathbb{Z}^d$ under the conditions of Theorems 2.11 and 2.13. In other words, we will assume that

(3.1) $d \geq 4$, the environment is uniformly elliptic and i.i.d. [see (1.1) and (1.2)] and $(\mathbf{T}, e_1)$ holds.

Here, we have chosen e_1 for convenience. However, there is no loss of generality, that is, we could have chosen any $\hat{u} \in \mathcal{S}^{d-1}$; see Lemma 2.6.

3.1. *Existence of harmonic functions: h-transform.*

Given any $\theta \in \mathbb{R}^d$, define $\pi^\theta : \Omega \times U \to \mathbb{R}$ by setting

$$\pi^\theta(\omega, z) := \pi(0, z) \exp\{\langle \theta, z \rangle - \Lambda_a(\theta)\}$$

for every $\omega \in \Omega$ and $z \in U$. Our first result concerns the existence of positive harmonic functions for π^θ. (Here, we use the term *harmonic* in analogy with the continuum case where π^θ is replaced by a second order elliptic operator.)

THEOREM 3.1. *Assume* (3.1). *Recall Theorem* 2.13. *For every* $\theta \in \mathcal{C}_{eq}$, *there exists an* $h(\theta, \cdot) \in L^2(\mathbb{P})$ *such that* $\mathbb{P}(h(\theta, \cdot) > 0) = 1$ *and*

$$h(\theta, \omega) = \sum_{z \in U} \pi(0, z) \exp\{\langle \theta, z \rangle - \Lambda_a(\theta)\} h(\theta, T_z \omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{3.2}$$

Note that π^θ would correspond to a Markov chain on Ω if $\sum_{z \in U} \pi^\theta(\omega, z) = 1$ were true for $\mathbb{P}$-a.e. ω. However, as we will see, the latter condition is not satisfied unless $\theta = 0$. Nevertheless, (3.2) enables us to define an environment kernel (as in Definition 1.1) related to π^θ via the so-called *h-transform* technique of Doob; see [14].

DEFINITION 3.2. Assume (3.1). For every $\theta \in \mathcal{C}_{eq}$, define a new environment kernel $\hat{\pi}^\theta : \Omega \times U \to \mathbb{R}^+$ by setting

$$\hat{\pi}^\theta(\omega, z) := \pi(0, z) \exp\{\langle \theta, z \rangle - \Lambda_a(\theta)\} \frac{h(\theta, T_z \omega)}{h(\theta, \omega)} \tag{3.3}$$

for every $\omega \in \Omega$ and $z \in U$. This technique is called *h-transform*.

3.2. *The unique minimizer of Varadhan's variational formula*. Recall the sets $\mathcal{A}_a$ and $\mathcal{A}_{eq}$ which were introduced in Theorems 2.10 and 2.11, respectively. Whenever $\xi \in \mathcal{A}_a$, it is shown in [23] that there is a unique minimizer of Varadhan's variational formula (2.11) for $I_a(\xi)$. Our second result reveals the hidden Markovian structure of this minimizer when (3.1) holds and $\xi \in \mathcal{A}_{eq}$.

Before stating this theorem, we need to introduce a family of sub-σ-algebras of $\mathcal{B}$: for any $\hat{v} \in \mathcal{S}^{d-1}$ and $n \geq 0$, let

$$\mathcal{B}_n^+(\hat{v}) := \sigma(\omega_x : \langle x, \hat{v} \rangle \geq -n). \tag{3.4}$$

THEOREM 3.3. *Assume* (3.1). *Recall Theorem* 2.11 *and Definition* 3.2. *For every* $\xi \in \mathcal{A}_{eq}$, *there exists a unique* $\theta \in \mathcal{C}_{eq}$ *such that* $\xi = \nabla \Lambda_a(\theta)$. [*By convex duality*, $\theta = \nabla I_a(\xi)$.]

(a) *There exists a unique* $\mathbb{Q}_\xi \in M_1(\Omega, \mathcal{B})$ *that satisfies the following*:
  (i) $\mathbb{Q}_\xi$ *is* $\hat{\pi}^\theta$-*invariant, that is*, $\sum_{z \in U} d\mathbb{Q}_\xi(T_{-z}\omega) \hat{\pi}^\theta(T_{-z}\omega, z) = d\mathbb{Q}_\xi(\omega)$;
  (ii) $\mathbb{Q}_\xi \ll \mathbb{P}$ *on* $\mathcal{B}_n^+(e_1)$ *for every* $n \geq 0$; *see* (3.4).
  The pair $(\hat{\pi}^\theta, \mathbb{Q}_\xi)$ *corresponds to a stationary Markov chain* (*with values in* Ω) *which can be identified with a* $\hat{\mu}_\xi^\infty \in M_1(\Omega \times U^{\mathbb{N}})$. *The marginal on* Ω *of* $\hat{\mu}_\xi^\infty$ *is* $\mathbb{Q}_\xi$ *and* $\hat{\pi}^\theta$ *is the conditional of* z_1 *given* ω.

(b) $\hat{\mu}_\xi^\infty$ *induces a* $\mathbb{Z}^d$-*valued transient process with stationary increments in* U *via the map*

$$(\omega, z_1, z_2, z_3, \ldots) \mapsto (z_1, z_1 + z_2, z_1 + z_2 + z_3, \ldots).$$

Extend this process to a probability measure on doubly infinite paths $(x_i)_{i \in \mathbb{Z}}$ *and refer to its restriction to* W_∞^{tr} *as* μ_ξ^∞. *With this notation*, μ_ξ^∞ *is the unique minimizer of Varadhan's variational formula* (2.11).

In words, when a particle under P_o is conditioned to have asymptotic mean velocity equal to any given $\xi \in \mathcal{A}_{eq}$, the environment Markov chain chooses to switch from its original kernel $\overline{\pi}$ to the tilted kernel $\hat{\pi}^\theta$ given in (3.3), where $\theta = \nabla I_a(\xi) \in \mathcal{C}_{eq}$. The most economical tilt in terms of averaged large deviations is realized by an h-transform.

REMARK 3.4. There is an alternative characterization of $\hat{\mu}_\xi^\infty$ [see (5.2)] which involves regeneration times. That formula (or, rather, its analog for the marginal on $U^{\mathbb{N}}$ of $\hat{\mu}_\xi^\infty$) has already appeared in Definition 9 of [23]. If one takes (5.2) as the definition of $\hat{\mu}_\xi^\infty$, then part (b) of Theorem 3.3 becomes essentially a restatement of Theorem 10 of [23] (see Theorem 5.2 of the current paper for details). In other words, the novelty of Theorem 3.3 lies in part (a).

3.3. *Equality of the quenched and the averaged minimizers.* The quenched level-3 LDP stated in Theorem 2.4 implies the quenched (level-1) LDP (i.e., Theorem 2.1) via the contraction principle. Indeed, for any $\xi \in \mathbb{R}^d$, define

$$A_\xi^\infty := \left\{\hat{\alpha} \in M_1(\Omega \times U^{\mathbb{N}}) : \int \sum_{(z_i)_{i\geq 1} \in U^{\mathbb{N}}} d\hat{\alpha}(\omega, (z_i)_{i\geq 1}) z_1 = \xi\right\}. \tag{3.5}$$

With this notation,

$$I_q(\xi) = \inf_{\hat{\alpha} \in A_\xi^\infty} I_{q,3}(\hat{\alpha}). \tag{3.6}$$

Our third result is as follows.

THEOREM 3.5. *Assume* (3.1). *For every $\xi \in \mathcal{A}_{eq}$, the measure $\hat{\mu}_\xi^\infty$ (which is obtained in Theorem* 3.3*) is the unique minimizer of* (3.6).

We already know from Theorem 2.11 that the quenched and the averaged rate functions I_q and I_a are equal on $\mathcal{A}_{eq}$. The natural interpretation of Theorem 3.5 is that, for $\mathbb{P}$-a.e. ω, when a particle under P_o^ω is conditioned to have asymptotic mean velocity equal to any given $\xi \in \mathcal{A}_{eq}$, the environment Markov chain chooses to switch from its original kernel $\overline{\pi}$ to the tilted kernel $\hat{\pi}^\theta$. Compare this with the last paragraph of the previous subsection.

Since the contraction from level-3 to level-1 may be done in two steps (instead of one), the following is an immediate consequence of Theorem 3.5.

COROLLARY 3.6. *Assume* (3.1). *For every $\xi \in \mathcal{A}_{eq}$, let $\hat{\mu}_\xi \in M_1(\Omega \times U)$ be the marginal of $\hat{\mu}_\xi^\infty \in M_1(\Omega \times U^{\mathbb{N}})$. With this notation, $\hat{\mu}_\xi$ is the unique minimizer of the variational formula*

$$I_q(\xi) = \inf_{\mu \in A_\xi} \mathfrak{I}_q^{**}(\mu)$$

given in (2.6).

3.4. *Modifying Rosenbluth's variational formula*. Recall Rosenbluth's variational formula

$$I_q(\xi) = \inf_{\mu \in A_\xi} \mathfrak{I}_q(\mu)$$

given in (2.7). Its advantage over (2.6) is that $\mathfrak{I}_q$ has a simple formula, whereas $\mathfrak{I}_q^{**}$ does not. Corollary 3.6 identifies the unique minimizer of (2.6) when (3.1) holds and $\xi \in \mathcal{A}_{eq}$. We would like to obtain an analogous result for (2.7). However, as we illustrate below, there is a problem.

We express Rosenbluth's formula in the following way:

$$I_q(\xi) = \inf\{H(\mu) : \mu \in A_\xi \cap M_1'(\Omega \times U)\}, \tag{3.7}$$

where

$$H(\mu) := \int_\Omega \sum_{z \in U} d\mu(\omega, z) \log \frac{d\mu(\omega, z)}{d(\mu)^1(\omega)\pi(0, z)} \tag{3.8}$$

denotes relative entropy and

$$M_1'(\Omega \times U) := \left\{\mu \in M_1(\Omega \times U) : (\mu)^1 = (\mu)^2 \ll \mathbb{P}, \frac{d\mu(\cdot, z)}{d(\mu)^1(\cdot)} > 0 \text{ for every } z \in U\right\}.$$

In light of Corollary 3.6, a natural minimizer candidate for (3.7) is $\hat{\mu}_\xi$. Note that $\hat{\mu}_\xi$ is an element of $M_1'(\Omega \times U)$ if and only if its marginal $\mathbb{Q}_\xi$ is absolutely continuous relative to $\mathbb{P}$ on $\mathcal{B}$. However, all we know is that $\mathbb{Q}_\xi \ll \mathbb{P}$ on $\mathcal{B}_n^+(e_1)$ for every $n \geq 0$; see Theorem 3.3.

Instead of trying to show that $\hat{\mu}_\xi$ is an element of $M_1'(\Omega \times U)$, we will replace $M_1'(\Omega \times U)$ by a larger set that contains $\hat{\mu}_\xi$.

DEFINITION 3.7. A measure $\mu \in M_1(\Omega \times U)$ is said to be in $M_1''(\Omega \times U)$ if it satisfies the following conditions:

(a) $(\mu)^1 = (\mu)^2$; see (2.2);
(b) $\hat{\pi}(\cdot, z) := \frac{d\mu(\cdot, z)}{d(\mu)^1(\cdot)} > 0$ for every $z \in U$;
(c) there exists a $\hat{v} \in \mathcal{S}^{d-1}$ such that $P_o^{\hat{\pi}}(\lim_{n\to\infty}\langle X_n, \hat{v}\rangle = \infty) = 1$; see Definition 1.1;
(d) $(\mu)^1 \ll \mathbb{P}$ on $\mathcal{B}_n^+(\hat{v})$ for every $n \geq 0$; see (3.4).

THEOREM 3.8. *Assume* (1.1). *Recall* (3.8) *and Definition* 3.7. *For every* $\xi \neq 0$,

$$I_q(\xi) = \inf\{H(\mu) : \mu \in A_\xi \cap M_1''(\Omega \times U)\}. \tag{3.9}$$

Our last result is the following theorem.

THEOREM 3.9. *Assume* (3.1). *For every* $\xi \in \mathcal{A}_{eq}$, $\hat{\mu}_\xi$ *is the unique minimizer of* (3.9).

Note that (3.9) does not involve any complex conjugation and, therefore, is simpler (i.e., more explicit) than (2.6). Because of this, we believe that Theorem 3.9 is more useful than Corollary 3.6.

3.5. *Some questions and comments*.

1. When (3.1) holds and $\xi \in \mathcal{A}_{eq}$, Theorem 3.3 states that $\mathbb{Q}_\xi \ll \mathbb{P}$ on $\mathcal{B}_n^+(e_1)$ for every $n \geq 0$. On the other hand, it is not known if $\mathbb{Q}_\xi \ll \mathbb{P}$ on $\mathcal{B}$. Is the latter statement true? Note that, when $\xi = \xi_o$, this question is of great interest (in its own right) because $\mathbb{Q}_{\xi_o}$ is the invariant measure from the point of view of the particle.

   Bolthausen and Sznitman [3] prove that $\mathbb{Q}_\xi \ll \mathbb{P}$ on $\mathcal{B}$ when $\xi = \xi_o$ and the disorder in the environment is low. One expects their argument to work when $|\xi - \xi_o|$ is small. However, their technique does not generalize to the case where the disorder is not low.
2. The limitation of our results is that they are valid when (3.1) holds and $\xi \in \mathcal{A}_{eq}$, and their proofs break down if any of these assumptions are weakened. Therefore, it is natural to ask the following question: in the context of multidimensional RWRE, does the connection between h-transform and large deviations exist under more general conditions? Note that such a connection has been established (i) for walks with bounded jumps on $\mathbb{Z}$ in stationary and ergodic environments (see [25]), and (ii) for space–time walks in dimensions $3+1$ and higher (see [24]).

The rest of this paper is devoted to the proofs of our results. Most of our efforts are focused on Theorems 3.1 and 3.3, which are established in Sections 4 and 5, respectively. The remaining results (i.e., Theorems 3.5, 3.8 and 3.9) are obtained in Section 6.

4. Proof of the existence of harmonic functions.

4.1. *An L^2 estimate*. Assume (3.1). Recall (2.16) and (2.18). For every $n \geq 1$, $\theta \in \mathbb{R}^d$ and $\omega \in \Omega$, define

$$g_n(\theta,\omega) := E_o^\omega[\exp\{\langle\theta, X_{H_n}\rangle - \Lambda_a(\theta)H_n\}, H_n = \tau_k \text{ for some } k \geq 1, \beta = \infty] \tag{4.1}$$

and

$$h_n(\theta,\omega) := E_o^\omega[\exp\{\langle\theta, X_{H_n}\rangle - \Lambda_a(\theta)H_n\}, H_n = \tau_k \text{ for some } k \geq 1], \tag{4.2}$$

where

$$H_n := \inf\{i \geq 0 : \langle X_i, e_1\rangle \geq n\}.$$

LEMMA 4.1 (Yilmaz [26]). *Assume* (3.1). *There exists a* $c_2 \in (0, c_1)$ *such that*

$$\liminf_{n\to\infty} \mathbb{E}\{g_n(\theta, \cdot)\} > 0 \tag{4.3}$$

and

$$\sup_{n\geq 1} \mathbb{E}\{g_n(\theta, \cdot)^2\} < \infty \tag{4.4}$$

for every $\theta \in \mathcal{C}_{eq} := \mathcal{C}(c_2)$; *see* (2.19).

PROOF. This constitutes the core of the proof of Theorem 2.13. For the convenience of the reader, we will give a sketch of the argument. See Lemmas 11 and 12 of [26] for the complete proof.

It is shown in Lemma 12 of [23] that for every $\theta \in \mathcal{C}_a := \mathcal{C}(c_1)$,

$$E_o[\exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}|\beta = \infty] = 1. \tag{4.5}$$

[Note that Theorem 2.12 follows from (4.5) by the implicit function theorem.] For every $y \in \mathbb{Z}^d$, let

$$q^\theta(y) := E_o[\exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}, X_{\tau_1} = y|\beta = \infty].$$

Since $\sum_{y\in\mathbb{Z}^d} q^\theta(y) = 1$ by (4.5), $(q^\theta(y))_{y\in\mathbb{Z}^d}$ defines a random walk $(Y_k)_{k\geq 0}$ on $\mathbb{Z}^d$. For every $n \geq 1$, $\mathbb{E}\{g_n(\theta, \cdot)\}/P_o(\beta = \infty)$ is equal to the probability of the event $\{\langle Y_k, e_1\rangle = n$ for some $k \geq 1\}$. By renewal theory, this probability is easily shown to converge to a nonzero limit. In particular, (4.3) follows.

For every $x, \tilde{x} \in \mathbb{Z}^d$, consider two independent walks $X = X(x) := (X_i)_{i\geq 0}$ and $\tilde{X} = \tilde{X}(\tilde{x}) := (\tilde{X}_j)_{j\geq 0}$, starting at x and $\tilde{x}$, respectively, in the same environment. Denote their joint quenched law and joint averaged law by $P^\omega_{x,\tilde{x}} := P^\omega_x \otimes P^\omega_{\tilde{x}}$ and $P_{x,\tilde{x}}(\cdot) := \mathbb{E}\{P^\omega_{x,\tilde{x}}(\cdot)\}$, respectively. As usual, $E^\omega_{x,\tilde{x}}$ and $E_{x,\tilde{x}}$ refer to expectations under $P^\omega_{x,\tilde{x}}$ and $P_{x,\tilde{x}}$, respectively.

Clearly, $P_{x,\tilde{x}} \neq P_x \otimes P_{\tilde{x}}$. On the other hand, the two walks do not know that they are in the same environment unless their paths intersect. In particular, for any event A involving X and $\tilde{X}$,

$$P_{x,\tilde{x}}(A \cap \{\gamma_1 = \infty\}) = P_x \otimes P_{\tilde{x}}(A \cap \{\gamma_1 = \infty\}), \tag{4.6}$$

if $x \neq \tilde{x}$, where

$$\gamma_1 := \inf\{m \in \mathbb{Z} : X_i = \tilde{X}_j \text{ for some } i \geq 0,\ j \geq 0, \text{ and } \langle X_i, e_1\rangle = m\}. \tag{4.7}$$

Similar to the random times $(H_n)_{n\geq 0}$ and β for X, define $(\tilde{H}_n)_{n\geq 0}$ and $\tilde{\beta}$ for $\tilde{X}$. The proof of (4.4) makes use of the *joint regeneration levels* of X and $\tilde{X}$, which are elements of

$$\mathcal{L} := \{n \geq 0 : \langle X_i, e_1\rangle \geq n \text{ and } \langle \tilde{X}_j, e_1\rangle \geq n \text{ for every } i \geq H_n \text{ and } j \geq \tilde{H}_n\}.$$

Note that if the starting points x and $\tilde{x}$ are both in $\mathbb{V}_d := \{z \in \mathbb{Z}^d : \langle z, e_1\rangle = 0\}$, then

$$0 \in \mathcal{L} \iff \beta = \tilde{\beta} = \infty \iff l_1 := \inf \mathcal{L} = 0.$$

For every $n \geq 1$ and $\theta \in \mathcal{C}_a$, define

$$f(\theta, n, X, \tilde{X}) := \exp\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta) H_n\} \exp\{\langle \theta, \tilde{X}_{\tilde{H}_n}\rangle - \Lambda_a(\theta)\tilde{H}_n\}.$$

With this notation,

$$\mathbb{E}\{g_n(\theta, \cdot)^2\} = E_{o,o}[f(\theta, n, X, \tilde{X}), n \in \mathcal{L}, l_1 = 0]. \tag{4.8}$$

By Lemma 4.2 (stated below), the random paths X and $\tilde{X}$ intersect finitely many times and the probability that they intersect far away from the origin is exponentially small. Conditioned on the first joint regeneration level after the last intersection, the right-hand side of (4.8) can be written as a product of two terms. The first term is shown to be finite, by renewal theory, when $\theta \in \mathcal{C}(c_2)$ with a small enough $c_2 \in (0, c_1)$, and the second term is bounded from above by $\mathbb{E}\{g_n(\theta, \cdot)\}^2 \leq 1$ since the walks can be thought of as taking place in independent environments. □

As mentioned in the sketch above, the following lemma is central to the proof of (4.4).

LEMMA 4.2. *Assume* (3.1). *Recall* (4.7) *and let* $\mathbb{V}_d' := \mathbb{V}_d \setminus \{0\}$. *Then*

$$\inf_{z \in \mathbb{V}_d'} P_{o,z}(l_1 = 0) \geq \inf_{z \in \mathbb{V}_d'} P_{o,z}(\gamma_1 = \infty, l_1 = 0) > 0. \tag{4.9}$$

PROOF. Assume (3.1). We saw in part (a) of Corollary 2.9 that τ_1 has finite moments of arbitrary order. Therefore, the second inequality follows from Proposition 3.1 (for $d \geq 5$) and Proposition 3.4 (for $d = 4$) of the recent work of Berger and Zeitouni [2]. (The proofs of these propositions are based on certain Green's function estimates which fail to hold unless $d \geq 4$.) Since the first inequality is clear, we are done. □

REMARK 4.3. It is easy to see that the first infimum in (4.9) is positive when $d = 2, 3$ as well. However, we will not need this fact in what follows.

LEMMA 4.4. *Assume* (3.1). *Recall* (4.2). *For every* $\theta \in \mathcal{C}_{eq}$,

$$\liminf_{n \to \infty} \mathbb{E}\{h_n(\theta, \cdot)\} > 0 \tag{4.10}$$

and

$$\sup_{n \geq 1} \mathbb{E}\{h_n(\theta, \cdot)^2\} < \infty. \tag{4.11}$$

PROOF. Recall the notation in the sketch of the proof of Lemma 4.1. By definition, $h_n(\theta,\omega) \geq g_n(\theta,\omega)$ for every $n \geq 1$, $\theta \in \mathcal{C}_{eq}$ and $\omega \in \Omega$. Hence, (4.10) follows immediately from (4.3).

For every $n \geq 1$ and $\theta \in \mathcal{C}_{eq}$,

$$\begin{aligned}
\mathbb{E}\{h_n(\theta,\cdot)^2\} &= E_{o,o}[f(\theta,n,X,\tilde{X}), n \in \mathcal{L}] \\
&= \sum_{k=0}^{n} \sum_{z \in \mathbb{V}_d} E_{o,o}[f(\theta,n,X,\tilde{X}), l_1 = k, \tilde{X}_{\tilde{H}_k} - X_{H_k} = z, n \in \mathcal{L}] \\
&= \sum_{k=0}^{n} \sum_{z \in \mathbb{V}_d} E_{o,o}[f(\theta,k,X,\tilde{X}), l_1 = k, \tilde{X}_{\tilde{H}_k} - X_{H_k} = z] \qquad (4.12)\\
&\qquad \times e^{-\langle\theta,z\rangle} E_{o,z}[f(\theta,n-k,X,\tilde{X}), n-k \in \mathcal{L} | l_1 = 0] \\
&\leq E_{o,o}[f(\theta,l_1,X,\tilde{X})]\Big(\inf_{z \in \mathbb{V}_d} P_{o,z}(l_1 = 0)\Big)^{-1} \qquad (4.13)\\
&\qquad \times \sup_{\substack{0 \leq k \leq n \\ z \in \mathbb{V}_d}} \mathbb{E}\{g_{n-k}(\theta,\cdot) g_{n-k}(\theta,T_z\cdot)\} \\
&\leq E_{o,o}[f(\theta,l_1,X,\tilde{X})]\Big(\inf_{z \in \mathbb{V}_d} P_{o,z}(l_1 = 0)\Big)^{-1} \qquad (4.14)\\
&\qquad \times \sup_{m \geq 1} \mathbb{E}\{g_m(\theta,\cdot)^2\}.
\end{aligned}$$

Indeed, we have (4.12) by the independence structure which is still valid for common regeneration blocks. (4.13) follows by noting that

$$\begin{aligned}
&e^{-\langle\theta,z\rangle} E_{o,z}[f(\theta,n-k,X,\tilde{X}), n-k \in \mathcal{L} | l_1 = 0] \\
&\quad = \big(P_{o,z}(l_1 = 0)\big)^{-1} \mathbb{E}\{g_{n-k}(\theta,\cdot) g_{n-k}(\theta,T_z\cdot)\} \\
&\quad \leq \Big(\inf_{z \in \mathbb{V}_d} P_{o,z}(l_1 = 0)\Big)^{-1} \sup_{\substack{0 \leq k \leq n \\ z \in \mathbb{V}_d}} \mathbb{E}\{g_{n-k}(\theta,\cdot) g_{n-k}(\theta,T_z\cdot)\}.
\end{aligned}$$

The third term in (4.14) is obtained using the Schwarz inequality and it is finite by Lemma 4.1. Similarly, the second term in (4.14) is finite by Lemma 4.2. Therefore, to prove (4.11), it suffices to show that the first term in (4.14) is also finite.

By Hölder's inequality,

$$\begin{aligned}
&E_{o,o}[f(\theta,l_1,X,\tilde{X})] \\
&\quad = \sum_{k=0}^{\infty} E_{o,o}[\exp\{\langle\theta, X_{H_k}\rangle - \Lambda_a(\theta) H_k\} \\
&\qquad\qquad \times \exp\{\langle\theta, \tilde{X}_{\tilde{H}_k}\rangle - \Lambda_a(\theta) \tilde{H}_k\}, l_1 = k]
\end{aligned}$$

$$\le \sum_{k=0}^{\infty} E_{o,o}[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}]^{1/4}$$

$$\times E_{o,o}[\exp\{4\langle\theta, \tilde{X}_{\tilde{H}_k}\rangle - 4\Lambda_a(\theta)\tilde{H}_k\}]^{1/4} P_{o,o}(l_1 = k)^{1/2}$$

$$= \sum_{k=0}^{\infty} E_o[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}]^{1/2} P_{o,o}(l_1 = k)^{1/2}. \tag{4.15}$$

For any $k \ge 1$,

$$\begin{aligned}
&E_o[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}] \\
&\quad = E_o[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}, H_k \le \tau_1] \\
&\quad\quad + E_o[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}, \tau_1 < H_k] \\
&\quad = E_o[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}, H_k \le \tau_1] \\
&\quad\quad + \sum_{j=1}^{k-1} E_o[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}, \tau_1 = H_j] \\
&\quad = E_o[\exp\{4\langle\theta, X_{H_k}\rangle - 4\Lambda_a(\theta)H_k\}, H_k \le \tau_1] \\
&\quad\quad + \sum_{j=1}^{k-1} E_o[\exp\{4\langle\theta, X_{\tau_1}\rangle - 4\Lambda_a(\theta)\tau_1\}, \tau_1 = H_j] \\
&\quad\quad\quad \times E_o[\exp\{4\langle\theta, X_{H_{k-j}}\rangle - 4\Lambda_a(\theta)H_{k-j}\}|\beta = \infty] \\
&\quad \le E_o\Big[\sup_{1\le n\le\tau_1} \exp\{4|\theta||X_n| - 4(0 \wedge \Lambda_a(\theta))\tau_1\}\Big] \\
&\quad\quad \times \Big(1 + \sup_{1\le i<k} E_o[\exp\{4\langle\theta, X_{H_i}\rangle - 4\Lambda_a(\theta)H_i\}|\beta = \infty]\Big)
\end{aligned}$$

$$\le E_o\Big[\sup_{1\le n\le\tau_1} \exp\{4|\theta||X_n| - 4(0 \wedge \Lambda_a(\theta))\tau_1\}\Big]\Big(1 + \sup_{1\le i<k} i e^{a_1|\theta|i}\Big) \tag{4.16}$$

$$\le K_1(1 + k e^{a_1|\theta|k}). \tag{4.17}$$

Indeed, if the walk is nonnestling, we have $4|\theta||X_n| - 4(0 \wedge \Lambda_a(\theta))\tau_1 \le 8|\theta|\tau_1$ for every $n \le \tau_1$. On the other hand, if the walk is nestling, then $4|\theta||X_n| - 4(0 \wedge \Lambda_a(\theta))\tau_1 = 4|\theta||X_n|$ since $\Lambda_a(\theta) \ge 0$. Therefore, in both cases, the first term in (4.16) is finite (provided that $4|\theta| < c_1$) and it is denoted by K_1 in (4.17). The second term in (4.16) is obtained using Lemma 28 of [26], where $a_1 > 0$ is a constant.

It is shown in (the proof of) Lemma 30 of [26] that $E_{o,o}[e^{a_3 l_1}] < \infty$ for some $a_3 > 0$. For any $k \ge 1$,

$$P_{o,o}(l_1 = k) \le E_{o,o}[e^{a_3 l_1}]e^{-a_3 k} =: K_2 e^{-a_3 k}. \tag{4.18}$$

Putting (4.15), (4.17) and (4.18) together, we conclude that

$$
\begin{aligned}
E_{o,o}[f(\theta, l_1, X, \tilde{X})] &\le \sum_{k=0}^{\infty} K_1^{1/2}\big(1 + ke^{a_1|\theta|k}\big)^{1/2} K_2^{1/2} e^{-a_3 k/2} \\
&\le 2(K_1K_2)^{1/2} \sum_{k=0}^{\infty} k^{1/2} e^{(a_1|\theta| - a_3)k/2} \\
&< \infty,
\end{aligned}
$$

provided that $|\theta| < a_3/a_1$.

The constant c_2 is chosen in [26] such that it satisfies $c_2 < \min(c_1/4, a_3/a_1)$, along with a few other conditions. Thus, (4.11) holds for every $\theta \in \mathcal{C}_{eq} = \mathcal{C}(c_2)$. □

4.2. *Proof of Theorem* 3.1. For every $n \ge 2$, $\theta \in \mathcal{C}_{eq}$ and $\omega \in \Omega$,

$$
\begin{aligned}
h_n(\theta, \omega) &= E_o^{\omega}[\exp\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta) H_n\}, H_n = \tau_k \text{ for some } k \ge 1] \\
&= \sum_{z \in U} E_o^{\omega}[\exp\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta) H_n\}, \\
&\qquad\qquad X_1 = z, H_n = \tau_k \text{ for some } k \ge 1] \\
&= \sum_{z \in U} \pi(0, z) \exp\{-\Lambda_a(\theta)\} \\
&\qquad \times E_z^{\omega}[\exp\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta) H_n\}, H_n = \tau_k \text{ for some } k \ge 1] \\
&= \sum_{z \in U} \pi(0, z) \exp\{\langle \theta, z\rangle - \Lambda_a(\theta)\} h_{n - \langle z, e_1\rangle}(\theta, T_z \omega). \qquad (4.19)
\end{aligned}
$$

Here, (4.19) is obtained by shifting the environment by z.

Define a new function $\bar{h}_n(\theta, \cdot) : \Omega \to \mathbb{R}$ by

$$
\bar{h}_n(\theta, \omega) := \frac{1}{n-1} \sum_{i=2}^{n} h_i(\theta, \omega). \qquad (4.20)
$$

Since $(\bar{h}_n(\theta, \cdot))_{n \ge 1}$ is bounded in $L^2(\mathbb{P})$ by (4.11), it has a subsequence $(\bar{h}_{n_k}(\theta, \cdot))_{k \ge 1}$ that converges weakly to some $h(\theta, \cdot) \in L^2(\mathbb{P})$.

It follows immediately from (4.19) that

$$
\begin{aligned}
\bar{h}_n(\theta, \omega) &= \sum_{z \in U} \pi(0, z) \exp\{\langle \theta, z\rangle - \Lambda_a(\theta)\} \bar{h}_n(\theta, T_z \omega) \\
&\quad + \frac{1}{n-1}\big(h_1(\theta, T_{e_1}\omega) - h_2(\theta, T_{-e_1}\omega) \qquad (4.21) \\
&\qquad - h_n(\theta, T_{e_1}\omega) + h_{n+1}(\theta, T_{-e_1}\omega)\big).
\end{aligned}
$$

Set $n = n_k$ and take the weak limit of both sides of (4.21) as $k \to \infty$. Since the term on the second line converges (strongly and, hence, weakly) to zero in $L^2(\mathbb{P})$, we conclude that

$$h(\theta, \omega) = \sum_{z \in U} \pi(0, z) \exp\{\langle \theta, z\rangle - \Lambda_a(\theta)\} h(\theta, T_z\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{4.22}$$

Note that $h(\theta, \omega) \geq 0$ for $\mathbb{P}$-a.e. ω. Equation (4.22) [in combination with (1.1)] implies that the set $\{\omega \in \Omega : h(\theta, \omega) = 0\}$ is invariant under $(T_z)_{z \in U}$. Since (1.2) ensures that the environment is ergodic under these shifts, $\mathbb{P}(h(\theta, \cdot) = 0) \in \{0, 1\}$. However, $\mathbb{E}\{h(\theta, \cdot)\} > 0$ by (4.10). Therefore, we conclude that $\mathbb{P}(h(\theta, \cdot) > 0) = 1$. We have thus proven Theorem 3.1.

4.3. *A useful representation.* Define a function $\varphi : \mathcal{C}_{eq} \times \Omega \times \mathbb{Z}^d \to \mathbb{R}^+$ by setting

$$\varphi(\theta, \omega, x) := \frac{E_o^\omega[\exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}, X_{\tau_1} = x]}{P_o^{T_x\omega}(\beta = \infty)} \tag{4.23}$$

for every $\theta \in \mathcal{C}_{eq}$, $\omega \in \Omega$ and $x \in \mathbb{Z}^d$. Note that $\varphi(\theta, \omega, x) = 0$ unless $\langle x, e_1\rangle \geq 1$.

The following lemma will be useful in the next section.

LEMMA 4.5. *For every $\theta \in \mathcal{C}_{eq}$, there exists a $\mathcal{B}_o^+(e_1)$-measurable $g(\theta, \cdot) \in L^2(\mathbb{P})$ such that*

$$h(\theta, \omega) = \sum_{x \in \mathbb{Z}^d} \varphi(\theta, \omega, x) g(\theta, T_x\omega) \qquad \textit{for } \mathbb{P}\textit{-a.e. } \omega. \tag{4.24}$$

PROOF. Recall (4.1) and (4.2). For every $n \geq 2$ and $\theta \in \mathcal{C}_{eq}$, define $\bar{g}_n(\theta, \cdot) \in L^2(\mathbb{P})$ analogously to (4.20). Since $(\bar{g}_n(\theta, \cdot))_{n\geq 1}$ is bounded in $L^2(\mathbb{P})$ by (4.4), it has a subsequence $(\bar{g}_{n_k}(\theta, \cdot))_{k \geq 1}$ that converges weakly to some $g(\theta, \cdot) \in L^2(\mathbb{P})$. [Choose $(n_k)_{k\geq 1}$ to be a further subsequence of the subsequence in the proof of Theorem 3.1 so that $(\bar{h}_{n_k}(\theta, \cdot))_{k\geq 1}$ converges weakly to $h(\theta, \cdot) \in L^2(\mathbb{P})$.] Note that $g_n(\theta, \cdot)$ is $\mathcal{B}_o^+(e_1)$-measurable for every $n \geq 1$ since the event $\{\beta = \infty\}$ is part of the definition of $g_n(\theta, \cdot)$. Hence, $g(\theta, \cdot)$ is $\mathcal{B}_o^+(e_1)$-measurable.

For every $N \geq 1$, $n \geq N$, $\theta \in \mathcal{C}_{eq}$ and $\omega \in \Omega$,

$$\begin{aligned} h_n(\theta, \omega) &= E_o^\omega[\exp\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta)H_n\}, H_n = \tau_k \text{ for some } k \geq 1] \\ &= E_o^\omega[\exp\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta)H_n\}, \\ &\qquad |X_{\tau_1}| \geq N, H_n = \tau_k \text{ for some } k \geq 1] \\ &\quad + \sum_{|x|<N} E_o^\omega[\exp\{\langle \theta, X_{H_n}\rangle - \Lambda_a(\theta)H_n\}, \\ &\qquad\qquad X_{\tau_1} = x, H_n = \tau_k \text{ for some } k \geq 1]. \end{aligned} \tag{4.25}$$

Denote the first term in (4.25) by $R_{N,n}(\theta,\omega)$. It follows immediately from (4.5), the renewal structure and the monotone convergence theorem that

$$\begin{aligned}\lim_{N\to\infty}\sup_{n\geq N}\mathbb{E}\{R_{N,n}(\theta,\cdot)\}\\ \leq \lim_{N\to\infty} E_o[\exp\{\langle\theta, X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\}, |X_{\tau_1}|\geq N]=0.\end{aligned}\tag{4.26}$$

Recall (4.23) and observe that, for every $n\geq N$,

$$\begin{aligned}h_n(\theta,\omega) &= R_{N,n}(\theta,\omega)\\ &\quad+\sum_{|x|<N} E_o^{\omega}[\exp\{\langle\theta, X_{H_n}\rangle-\Lambda_a(\theta)H_n\},\\ &\qquad\qquad X_{\tau_1}=x, H_n=\tau_k \text{ for some } k\geq 1]\\ &= R_{N,n}(\theta,\omega)\\ &\quad+\sum_{|x|<N} E_o^{\omega}[\exp\{\langle\theta, X_{\tau_1}\rangle-\Lambda_a(\theta)\tau_1\}, X_{\tau_1}=x]\\ &\qquad\times e^{-\langle\theta,x\rangle}E_x^{\omega}[\exp\{\langle\theta, X_{H_n}\rangle-\Lambda_a(\theta)H_n\},\\ &\qquad\qquad H_n=\tau_k \text{ for some } k\geq 1|\beta=\infty]\\ &= R_{N,n}(\theta,\omega)+\sum_{|x|<N}\varphi(\theta,\omega,x)g_{n-\langle x,e_1\rangle}(\theta,T_x\omega).\end{aligned}$$

Therefore, whenever $n_k\geq N$,

$$\begin{aligned}\frac{1}{n_k}\sum_{i=N}^{n_k} h_i(\theta,\omega) &= \frac{1}{n_k}\sum_{i=N}^{n_k} R_{N,i}(\theta,\omega)\\ &\quad+\sum_{|x|<N}\varphi(\theta,\omega,x)\frac{1}{n_k}\sum_{i=N}^{n_k} g_{i-\langle x,e_1\rangle}(\theta,T_x\omega).\end{aligned}\tag{4.27}$$

Multiplying both sides of (4.27) by any indicator function $\chi\in L^{\infty}(\mathbb{P})$, integrating against $\mathbb{P}$ and letting k tend to infinity, we arrive at the following inequality:

$$\left|\int h(\theta,\omega)\chi(\omega)\,d\mathbb{P}-\int\sum_{|x|<N}\varphi(\theta,\omega,x)g(\theta,T_x\omega)\chi(\omega)\,d\mathbb{P}\right|\leq\sup_{n\geq N}\mathbb{E}\{R_{N,n}(\theta,\cdot)\}.$$

Finally, let N tend to infinity. The monotone convergence theorem and (4.26) imply the desired result. □

5. Proof of our results on averaged large deviations. We will start this section by stating two results concerning the unique minimizer of Varadhan's variational formula (2.11). We will then give a series of lemmas. Finally, we will combine everything and prove Theorem 3.3.

5.1. *The unique minimizer of Varadhan's variational formula*. Assume (3.1). Take any $\xi \in \mathcal{A}_a$. Since the Hessian $\mathcal{H}_a$ of Λ_a is positive definite on $\mathcal{C}_a$ by Theorem 2.12, there exists a unique $\theta \in \mathcal{C}_a$ satisfying $\xi = \nabla\Lambda_a(\theta)$. In the next paragraph, we define a probability measure $\hat{\mu}_\xi^\infty \in M_1(\Omega \times U^{\mathbb{N}})$ by specifying the integrals of certain test functions against this measure.

For every $N, M, K \geq 0$, take any bounded function $f : \Omega \times U^{\mathbb{N}} \to \mathbb{R}$ such that $f(\cdot, (z_i)_{i\geq 1})$ is independent of $(z_i)_{i>K}$ and is measurable with respect to

$$\mathcal{B}_N^M(e_1) = \mathcal{B}_N^M := \sigma(\omega_x : -N \leq \langle x, e_1\rangle \leq M). \tag{5.1}$$

Define $\hat{\mu}_\xi^\infty \in M_1(\Omega \times U^{\mathbb{N}})$ by setting

$$\begin{aligned}\int f\, d\hat{\mu}_\xi^\infty := \sum_{j=0}^{\infty} E_o[\tau_N \leq j < \tau_{N+1},\\ f(T_{X_j}\omega, Z_{j+1}^\infty)\exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\} | \beta = \infty]\\ \times \big(E_o[\tau_1 \exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} | \beta = \infty]\big)^{-1},\end{aligned} \tag{5.2}$$

where $J := N + M + K + 1$ and $Z_{j+1}^\infty = (Z_{j+i})_{i\geq 1} := (X_i - X_{i-1})_{i\geq 1}$. (The measure $\hat{\mu}_\xi^\infty$ is well defined. See the proof of Theorem 5.1.)

The following theorem states that the empirical process

$$\nu_{n,X}^\infty := \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{1}_{T_{X_k}\omega, Z_{k+1}^\infty}$$

of the walk under P_o converges to $\hat{\mu}_\xi^\infty$ when the particle is conditioned to have mean velocity ξ.

THEOREM 5.1. *Assume* (3.1). *For every* $\xi \in \mathcal{A}_a$, $\varepsilon > 0$, $N, M, K \geq 0$ *and* $f : \Omega \times U^{\mathbb{N}} \to \mathbb{R}$ *bounded such that* $f(\cdot, (z_i)_{i\geq 1})$ *is independent of* $(z_i)_{i>K}$ *and is* $\mathcal{B}_N^M$*-measurable, the following holds*:

$$\begin{aligned}\limsup_{\delta'\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_o\bigg(\bigg|\int f\, d\nu_{n,X}^\infty - \int f\, d\hat{\mu}_\xi^\infty\bigg| > \varepsilon \bigg|\\ \bigg|\frac{X_n}{n} - \xi\bigg| \leq \delta'\bigg) < 0.\end{aligned} \tag{5.3}$$

PROOF. Definition 9 of [23] introduces a probability measure $\bar{\mu}_\xi^\infty \in M_1(U^{\mathbb{N}})$ by the formula in (5.2), except that the test functions do not depend on ω in that case. Proposition 16 of [23] shows that $\bar{\mu}_\xi^\infty$ is well defined, and Theorem 17 of [23] establishes the analog of (5.3) for $\bar{\mu}_\xi^\infty$. The proofs of these results generalize to our setting without any nontrivial change. Therefore, we omit the proof of Theorem 5.1. Also, note that Theorem 5.1 is proved in [24] for the related model of space–time RWRE. □

In the first paragraph of Section 3.2, we mentioned an existence and uniqueness result for the minimizer of Varadhan's variational formula (2.11) for the averaged rate function I_a. The following is the precise statement.

THEOREM 5.2. *Assume* (3.1). *For every* $\xi \in \mathcal{A}_a$, $\hat{\mu}_\xi^\infty$ *induces a* $\mathbb{Z}^d$*-valued transient process with stationary and ergodic increments in* U *via the map*

$$(\omega, z_1, z_2, z_3, \dots) \mapsto (z_1, z_1 + z_2, z_1 + z_2 + z_3, \dots).$$

Extend this process to a probability measure on doubly infinite paths $(x_i)_{i\in\mathbb{Z}}$ *and refer to its restriction to* W_∞^{tr} *as* μ_ξ^∞. *With this notation,* μ_ξ^∞ *is the unique minimizer of Varadhan's variational formula* (2.11).

PROOF. This is Theorem 10 of [23], with the following difference: that result is concerned with $\bar{\mu}_\xi^\infty$ (which was mentioned in the proof of Theorem 5.1) and it uses the map

$$(z_1, z_2, z_3, \dots) \mapsto (z_1, z_1 + z_2, z_1 + z_2 + z_3, \dots)$$

to induce a $\mathbb{Z}^d$-valued transient process with stationary and ergodic increments in U. However, since $\bar{\mu}_\xi^\infty$ is the marginal of $\hat{\mu}_\xi^\infty$ on $U^{\mathbb{N}}$, the $\mathbb{Z}^d$-valued process induced by $\bar{\mu}_\xi^\infty$ is nothing but μ_ξ^∞. $\square$

5.2. *The Markovian structure of the minimizer.* Assume (3.1). Take any $\xi \in \mathcal{A}_{eq}$. Let $\theta \in \mathcal{C}_{eq}$ denote the unique solution of $\xi = \nabla\Lambda_a(\theta)$. Recall the environment kernel $\hat{\pi}^\theta$ defined in (3.3) via h-transform. For any $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, abbreviate the notation introduced in Definition 1.1 by writing $P_x^{\theta,\omega}$ and $E_x^{\theta,\omega}$ instead of $P_x^{\hat{\pi}^\theta,\omega}$ and $E_x^{\hat{\pi}^\theta,\omega}$, respectively.

For every $n \geq 1$, define $\hat{\mu}_{n,\xi}^\infty \in M_1(\Omega \times U^{\mathbb{N}})$ as follows:

$$\hat{\mu}_{n,\xi}^\infty(\cdot) := \frac{\mathbb{E}\{h(\theta,\omega) P_o^{\theta,\omega}((T_{X_n}\omega, Z_{n+1}^\infty) \in \cdot)\}}{\mathbb{E}\{h(\theta,\omega)\}}. \tag{5.4}$$

LEMMA 5.3. *For every* $N, M, K \geq 0$ *and* $f : \Omega \times U^{\mathbb{N}} \to \mathbb{R}$ *bounded such that* $f(\cdot, (z_i)_{i\geq 1})$ *is independent of* $(z_i)_{i>K}$ *and is* $\mathcal{B}_N^M$*-measurable,*

$$\int f \, d\hat{\mu}_{n,\xi}^\infty = \frac{E_o[f(T_{X_n}\omega, Z_{n+1}^\infty) \exp\{\langle\theta, X_{\tau_L}\rangle - \Lambda_a(\theta)\tau_L\}]}{E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}]} \tag{5.5}$$

for every $L \geq n + M + K + 1$.

PROOF. For every $N, M, K \geq 0$, take a bounded function $f : \Omega \times U^{\mathbb{N}} \to \mathbb{R}$ such that $f(\cdot, (z_i)_{i\geq 1})$ is independent of $(z_i)_{i>K}$ and is $\mathcal{B}_N^M$-measurable; see (5.1).

For every $L \geq n + M + K + 1$,

$$\begin{aligned}
&\mathbb{E}\{h(\theta,\omega)\}\int f\,d\hat{\mu}_{n,\xi}^{\infty} \\
&\quad = \mathbb{E}\{h(\theta,\omega)E_o^{\theta,\omega}[f(T_{X_n}\omega, Z_{n+1}^{\infty})]\} \\
&\quad = \mathbb{E}\bigg\{h(\theta,\omega)E_o^{\omega}\bigg[f(T_{X_n}\omega, Z_{n+1}^{\infty}) \\
&\qquad\qquad \times \exp\{\langle\theta, X_{n+K}\rangle - \Lambda_a(\theta)(n+K)\}\frac{h(\theta, T_{X_{n+K}}\omega)}{h(\theta,\omega)}\bigg]\bigg\} \qquad (5.6)\\
&\quad = E_o[f(T_{X_n}\omega, Z_{n+1}^{\infty})\exp\{\langle\theta, X_{n+K}\rangle - \Lambda_a(\theta)(n+K)\}h(\theta, T_{X_{n+K}}\omega)] \\
&\quad = E_o[f(T_{X_n}\omega, Z_{n+1}^{\infty})\exp\{\langle\theta, X_{H_L}\rangle - \Lambda_a(\theta)H_L\}h(\theta, T_{X_{H_L}}\omega)] \qquad (5.7)\\
&\quad = \sum_{\langle x,e_1\rangle\geq 1} E_o[f(T_{X_n}\omega, Z_{n+1}^{\infty})\exp\{\langle\theta, X_{H_L}\rangle - \Lambda_a(\theta)H_L\} \\
&\qquad\qquad \times \varphi(\theta, T_{X_{H_L}}\omega, x)g(\theta, T_{X_{H_L}+x}\omega)]. \qquad (5.8)
\end{aligned}$$

Explanation: (5.6) follows from the definition of $\hat{\pi}^{\theta}$ by noting that $f(T_{X_n}\omega, Z_{n+1}^{\infty})$ depends only on the first $n+K$ steps of the walk. (5.7) holds because H_L is a stopping time and $H_L \geq n+K$. The representation of $h(\theta,\cdot)$ in (4.24) gives (5.8).

For any $x \in \mathbb{Z}^d$ such that $\langle x, e_1\rangle \geq 1$,

$$\begin{aligned}
&E_o[f(T_{X_n}\omega, Z_{n+1}^{\infty}) \\
&\quad \times \exp\{\langle\theta, X_{H_L}\rangle - \Lambda_a(\theta)H_L\}\varphi(\theta, T_{X_{H_L}}\omega, x)g(\theta, T_{X_{H_L}+x}\omega)] \\
&\quad = \sum_{\langle y,e_1\rangle = L} \mathbb{E}\big\{E_o^{\omega}[f(T_{X_n}\omega, Z_{n+1}^{\infty})\exp\{\langle\theta, y\rangle - \Lambda_a(\theta)H_L\} \\
&\qquad\qquad \times \varphi(\theta, T_y\omega, x), X_{H_L} = y] \\
&\qquad\qquad \times g(\theta, T_{y+x}\omega)\big\} \qquad (5.9)\\
&\quad = \sum_{\langle y,e_1\rangle = L} E_o[f(T_{X_n}\omega, Z_{n+1}^{\infty})\exp\{\langle\theta, y\rangle - \Lambda_a(\theta)H_L\} \\
&\qquad\qquad \times \varphi(\theta, T_y\omega, x), X_{H_L} = y]\mathbb{E}\{g(\theta,\omega)\} \qquad (5.10)\\
&\quad = E_o[f(T_{X_n}\omega, Z_{n+1}^{\infty})\exp\{\langle\theta, X_{H_L}\rangle - \Lambda_a(\theta)H_L\}\varphi(\theta, T_{X_{H_L}}\omega, x)] \\
&\qquad \times \mathbb{E}\{g(\theta,\omega)\}. \qquad (5.11)
\end{aligned}$$

Explanation: for any $y \in \mathbb{Z}^d$ such that $\langle y, e_1\rangle = L$, the random quantities $E_o^{\omega}[\ldots, X_{H_L} = y]$ and $g(\theta, T_{y+x}\omega)$ appearing in (5.9) are independent because the former is measurable with respect to $\sigma(\omega_{x'} : \langle x' - x, e_1\rangle < L)$, whereas the latter is $\mathcal{B}^+_{L+\langle x,e_1\rangle}(e_1)$-measurable; see Lemma 4.5. This independence (in combination with the stationarity of $\mathbb{P}$) gives (5.10).

By plugging (5.11) into (5.8), we see that

$$\begin{aligned}
&\frac{\mathbb{E}\{h(\theta,\cdot)\}}{\mathbb{E}\{g(\theta,\cdot)\}}\int f\,d\hat{\mu}^\infty_{n,\xi}\\
&\qquad=\sum_{\langle x,e_1\rangle\ge 1}\sum_{k=1}^{L+1}\mathbb{E}\big\{E^\omega_o[f(T_{X_n}\omega,Z^\infty_{n+1})\exp\{\langle\theta,X_{\tau_k}\rangle-\Lambda_a(\theta)\tau_k\},\\
&\qquad\qquad\tau_{k-1}\le H_L<\tau_k, X_{\tau_k}=x]\\
&\qquad\qquad\times\big(P^{T_x\omega}_o(\beta=\infty)\big)^{-1}\big\}
\end{aligned}\tag{5.12}$$

$$\begin{aligned}
&\qquad=\frac{1}{P_o(\beta=\infty)}\\
&\qquad\quad\times\sum_{\langle x,e_1\rangle\ge 1}\sum_{k=1}^{L+1}E_o[f(T_{X_n}\omega,Z^\infty_{n+1})\exp\{\langle\theta,X_{\tau_k}\rangle-\Lambda_a(\theta)\tau_k\},\\
&\qquad\qquad\tau_{k-1}\le H_L<\tau_k, X_{\tau_k}=x]
\end{aligned}\tag{5.13}$$

$$\qquad=\frac{1}{P_o(\beta=\infty)}E_o[f(T_{X_n}\omega,Z^\infty_{n+1})\exp\{\langle\theta,X_{\tau_L}\rangle-\Lambda_a(\theta)\tau_L\}].\tag{5.14}$$

Explanation: (5.12) follows from the definition of $\varphi(\theta,\cdot,\cdot)$ given in (4.23). For every $x\in\mathbb{Z}^d$ such that $\langle x,e_1\rangle\ge 1$, the random quantity $P^{T_x\omega}_o(\beta=\infty)$ is independent of the ratio $\frac{E^\omega_o[\ldots,X_{\tau_k}=x]}{P^{T_x\omega}_o(\beta=\infty)}$ appearing in (5.12) since the latter is easily seen to be equal to an expectation involving the stopping time $H_{\langle x,e_1\rangle}$ (and nothing beyond that). This independence implies (5.13). Using (4.5), the τ_k in the exponential can be replaced first by τ_{L+1} and then by τ_L. This gives (5.14).

Finally, observe that (5.14) agrees with (5.5), except that the normalization constant has to be simplified. However, it is clear that the constant in (5.5) is correct [take $f\equiv 1$ and apply (4.5)]. □

LEMMA 5.4. *For every $f:\Omega\times U^{\mathbb{N}}\to\mathbb{R}$ bounded such that $f(\cdot,(z_i)_{i\ge1})$ is $\mathcal{B}$-measurable, the following convergence takes place*:

$$\lim_{n\to\infty}\int f\,d\hat{\mu}^\infty_{n,\xi}=\int f\,d\hat{\mu}^\infty_\xi.$$

In particular, $(\hat{\mu}^\infty_{n,\xi})_{n\ge1}$ converges weakly to $\hat{\mu}^\infty_\xi$.

PROOF. For any $N,M,K\ge 0$, take a bounded function $f:\Omega\times U^{\mathbb{N}}\to\mathbb{R}$ such that $f(\cdot,(z_i)_{i\ge1})$ is independent of $(z_i)_{i>K}$ and $\mathcal{B}^M_N$-measurable. Let $J:=N+$

$M + K + 1$.

$$E_o[\exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}] \lim_{n\to\infty} \int f\, d\hat{\mu}_{n,\xi}^{\infty} \tag{5.15}$$

$$= \lim_{n\to\infty} E_o[n < \tau_N, f(T_{X_n}\omega, Z_{n+1}^{\infty}) \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\}] \tag{5.16}$$

$$+ \lim_{n\to\infty} \sum_{i=0}^{\infty} E_o[\tau_{N+i} \le n < \tau_{N+i+1}, \\ f(T_{X_n}\omega, Z_{n+1}^{\infty}) \exp\{\langle\theta, X_{\tau_{J+i}}\rangle - \Lambda_a(\theta)\tau_{J+i}\}]$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} \left(\sum_{i=0}^{\infty} E_o[\tau_i = n - j, \exp\{\langle\theta, X_{\tau_i}\rangle - \Lambda_a(\theta)\tau_i\}] \right) \\ \times E_o[\tau_N \le j < \tau_{N+1}, \\ f(T_{X_j}\omega, Z_{j+1}^{\infty}) \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\} | \beta = \infty] \tag{5.17}$$

$$= S(\theta) \sum_{j=0}^{\infty} E_o[\tau_N \le j < \tau_{N+1}, \\ f(T_{X_j}\omega, Z_{j+1}^{\infty}) \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\} | \beta = \infty]$$

$$= S(\theta) E_o[\tau_1 \exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} | \beta = \infty] \int f\, d\hat{\mu}_{\xi}^{\infty}. \tag{5.18}$$

Explanation: (5.16) follows from (5.5). The first term in (5.16) goes to zero as $n \to \infty$ by the dominated convergence theorem. The renewal theorem for aperiodic sequences (see [4], Theorem 10.8) implies that the sum $(\sum_{i=0}^{\infty} E_o[\cdots])$ in (5.17) converges to some constant $S(\theta)$ as $n \to \infty$. Observe that the constants in (5.15) and (5.18) have to agree because $\hat{\mu}_{n,\xi}^{\infty}$ and $\hat{\mu}_{\xi}^{\infty}$ are known to be probability measures.

Thus far, we have shown that $\lim_{n\to\infty} \int f\, d\hat{\mu}_{n,\xi}^{\infty} = \int f\, d\hat{\mu}_{\xi}^{\infty}$ for a separating class of test functions. However, this is sufficient to conclude that $(\hat{\mu}_{n,\xi}^{\infty})_{n\ge 1}$ converges weakly to $\hat{\mu}_{\xi}^{\infty}$ since $M_1(\Omega \times U^{\mathbb{N}})$ is compact. □

Let $\mathbb{Q}_\xi \in M_1(\Omega)$ be the marginal of $\hat{\mu}_{\xi}^{\infty} \in M_1(\Omega \times U^{\mathbb{N}})$.

LEMMA 5.5. *$\mathbb{Q}_\xi$ is $\hat{\pi}^\theta$-invariant, that is,*

$$\sum_{z\in U} d\mathbb{Q}_\xi(T_{-z}\omega)\hat{\pi}^\theta(T_{-z}\omega, z) = d\mathbb{Q}_\xi(\omega).$$

PROOF. For any $f \in L^\infty(\mathbb{P})$, define $\hat{\pi}^\theta f : \Omega \to \mathbb{R}$ in the usual way:

$$(\hat{\pi}^\theta f)(\omega) := \sum_{z\in U} \hat{\pi}^\theta(\omega, z) f(T_z\omega).$$

Recall (5.4). For every $n \geq 1$,

$$\begin{aligned}\int (\hat{\pi}^{\theta} f)\, d\hat{\mu}^{\infty}_{n,\xi} &= \frac{\mathbb{E}\{h(\theta,\omega) E_o^{\theta,\omega}[(\hat{\pi}^{\theta} f)(T_{X_n}\omega)]\}}{\mathbb{E}\{h(\theta,\omega)\}} \\ &= \frac{\mathbb{E}\{h(\theta,\omega) E_o^{\theta,\omega}[f(T_{X_{n+1}}\omega)]\}}{\mathbb{E}\{h(\theta,\omega)\}} \\ &= \int f\, d\hat{\mu}^{\infty}_{n+1,\xi},\end{aligned}$$

by the Markov property. Let n tend to infinity, use Lemma 5.4 and conclude that

$$\int (\hat{\pi}^{\theta} f)\, d\mathbb{Q}_{\xi} = \int (\hat{\pi}^{\theta} f)\, d\hat{\mu}^{\infty}_{\xi} = \int f\, d\hat{\mu}^{\infty}_{\xi} = \int f\, d\mathbb{Q}_{\xi}.$$

This is equivalent to the desired result. □

LEMMA 5.6. *$\hat{\mu}^{\infty}_{\xi}$ induces, via the map*

$$(\omega, z_1, z_2, z_3, \ldots) \mapsto (\omega, T_{z_1}\omega, T_{z_1+z_2}\omega, T_{z_1+z_2+z_3}\omega, \ldots),$$

an Ω-valued stationary Markov process with marginal $\mathbb{Q}_{\xi}$ and transition kernel

$$\overline{\pi}^{\theta}(\omega, \omega') := \sum_{z\,:\, T_z\omega = \omega'} \hat{\pi}^{\theta}(\omega, z).$$

PROOF. For any $n \geq 1$, $K \geq 0$ and any two bounded measurable functions $f : \Omega^{K+1} \to \mathbb{R}$ and $g : \Omega^{\mathbb{N}} \to \mathbb{R}$, it follows from (5.4) and the Markov property that

$$\begin{aligned}&\mathbb{E}\{h(\theta,\omega)\} \int f(\omega, T_{z_1}\omega, \ldots, T_{z_1+\cdots+z_K}\omega) \\ &\qquad\qquad \times g(T_{z_1+\cdots+z_K}\omega, T_{z_1+\cdots+z_{K+1}}\omega, \ldots)\, d\hat{\mu}^{\infty}_{n,\xi} \\ &\quad = \mathbb{E}\{h(\theta,\omega) E_o^{\theta,\omega}[f(T_{X_n}\omega, \ldots, T_{X_{n+K}}\omega) g(T_{X_{n+K}}\omega, T_{X_{n+K+1}}\omega, \ldots)]\} \\ &\quad = \mathbb{E}\big\{h(\theta,\omega) E_o^{\theta,\omega}\big[f(T_{X_n}\omega, \ldots, T_{X_{n+K}}\omega) \\ &\qquad\qquad\qquad \times E_o^{\theta,\omega}[g(T_{X_{n+K}}\omega, T_{X_{n+K+1}}\omega, \ldots) | X_{n+K}]\big]\big\} \\ &\quad = \mathbb{E}\{h(\theta,\omega)\} \\ &\qquad \times \int f(\omega, T_{z_1}\omega, \ldots, T_{z_1+\cdots+z_K}\omega) \\ &\qquad\qquad \times E_o^{\theta, T_{z_1+\cdots+z_K}\omega}[g(T_{z_1+\cdots+z_K}\omega, T_{z_1+\cdots+z_K+X_1}\omega, \ldots)]\, d\hat{\mu}^{\infty}_{n,\xi}.\end{aligned}$$

Let n tend to infinity, use Lemma 5.4 and conclude that $\hat{\mu}^{\infty}_{\xi}$ indeed induces an Ω-valued Markov process with marginal $\mathbb{Q}_{\xi}$ and transition kernel $\overline{\pi}^{\theta}$. Finally, note that the stationarity of this process follows from a straightforward generalization of Lemma 5.5. □

LEMMA 5.7. *$\mathbb{Q}_\xi \ll \mathbb{P}$ on $\mathcal{B}_N^+(e_1)$ for every $N \geq 0$; see* (3.4).

PROOF. For any $N \geq 0$, take an $f \in L^\infty(\mathbb{P})$ such that f is nonnegative and $\mathcal{B}_N^M$-measurable for some $M \geq 0$. Let $J := N + M + 1$. It follows from (5.2) and the Schwarz inequality that

$$
\begin{aligned}
&E_o[\tau_1 \exp\{\langle\theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\}, \beta = \infty] \int f \, d\mathbb{Q}_\xi \\
&\quad = E_o\Biggl[\Biggl(\sum_{j=\tau_N}^{\tau_{N+1}-1} f(T_{X_j}\omega)\Biggr) \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\}, \beta = \infty\Biggr] \\
&\quad \leq \sum_{k=1}^{\infty} E_o\Biggl[\tau_{N+1} = k, \Biggl(\sum_{|x| \leq k} f(T_x\omega)\Biggr) \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\}, \beta = \infty\Biggr] \\
&\quad = \sum_{k=1}^{\infty} \sum_{l=1}^{k} \mathbb{E}\Biggl\{\Biggl(\sum_{|x| \leq k} f(T_x\omega)\Biggr) \\
&\qquad\qquad \times E_o^\omega[\tau_{N+1} = k = H_l, \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\}, \beta = \infty]\Biggr\} \\
&\quad \leq \sum_{k=1}^{\infty} \sum_{l=1}^{k} (2k+1)^d \|f\|_{L^2(\mathbb{P})} \\
&\qquad\qquad \times \mathbb{E}\{E_o^\omega[\tau_{N+1} = k = H_l, \\
&\qquad\qquad\qquad \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\}, \beta = \infty]^2\}^{1/2}. \qquad (5.19)
\end{aligned}
$$

There exist constants $C'_N < \infty$ and $a_4 > 0$ such that, for every $k \geq 1$ and $l \in \{1, \ldots, k\}$,

$$
\begin{aligned}
&\mathbb{E}\{E_o^\omega[\tau_{N+1} = k = H_l, \exp\{\langle\theta, X_{\tau_J}\rangle - \Lambda_a(\theta)\tau_J\}, \beta = \infty]^2\} \\
&\quad \leq \mathbb{E}\{E_o^\omega[\tau_{N+1} = k = H_l, \exp\{\langle\theta, X_{\tau_{N+1}}\rangle - \Lambda_a(\theta)\tau_{N+1}\}, \beta = \infty]^2\} \qquad (5.20) \\
&\qquad \times \Bigl(\inf_{z \in \mathbb{V}_d} P_{o,z}(l_1 = 0)\Bigr)^{-1} \\
&\qquad \times \mathbb{E}\{E_o^\omega[\exp\{\langle\theta, X_{\tau_M}\rangle - \Lambda_a(\theta)\tau_M\}, \beta = \infty]^2\} \\
&\quad \leq C'_N e^{-a_4 k}. \qquad (5.21)
\end{aligned}
$$

Explanation: the first term in (5.20) is bounded from above by $C_N e^{-a_4 k}$ for some $C_N < \infty$ and $a_4 > 0$. The second term in (5.20) is finite by Lemma 4.2. Using the technique in the proof of Lemma 4.1, the third term in (5.20) can be shown to be bounded from above by a constant that is independent of M. We leave the details to the reader.

Plugging (5.21) into (5.19), we see that

$$\int f\,d\mathbb{Q}_\xi \le C''_N \|f\|_{L^2(\mathbb{P})}$$

for some finite constant C''_N that is independent of M. Since the functions we have considered are dense in $L^2(\Omega, \mathcal{B}_N^+(e_1), \mathbb{P})$, it follows from the Riesz representation theorem that

$$\left.\frac{d\mathbb{Q}_\xi}{d\mathbb{P}}\right|_{\mathcal{B}_N^+(e_1)} \in L^2(\mathbb{P}). \tag{5.22}$$

□

Combining all of the results in this section, we get the following proof.

PROOF OF THEOREM 3.3. Recall that $\mathbb{Q}_\xi \in M_1(\Omega)$ denotes the marginal of $\hat{\mu}_\xi^\infty \in M_1(\Omega \times U^{\mathbb{N}})$ which, in turn, is defined in (5.2). We have seen that:

(i) $\mathbb{Q}_\xi$ is $\hat{\pi}^\theta$-invariant (see Lemma 5.5);
(ii) $\mathbb{Q}_\xi \ll \mathbb{P}$ on $\mathcal{B}_n^+(e_1)$ for every $n \ge 0$; see Lemma 5.7.

It follows from Lemma 5.8 (stated below) that $\mathbb{Q}_\xi$ is the unique element of $M_1(\Omega)$ that satisfies these two properties. We have also proven that $\hat{\mu}_\xi^\infty$ induces an Ω-valued stationary Markov process with marginal $\mathbb{Q}_\xi$ and transition kernel $\overline{\pi}^\theta$; see Lemma 5.6. These results imply part (a) of Theorem 3.3.

Note that part (b) of Theorem 3.3 is a special case of Theorem 5.2 since $\mathcal{A}_{eq} \subset \mathcal{A}_a$. □

In the proof above, we used the following generalization of a classical homogenization result which is originally due to Kozlov [11].

LEMMA 5.8 (Rassoul-Agha [15]). *Given any $\mathbb{Q} \in M_1(\Omega)$ and any environment kernel $\hat{\pi}$, define a measure $\mu \in M_1(\Omega \times U)$ by setting*

$$d\mu(\cdot, z) := d\mathbb{Q}(\cdot)\hat{\pi}(\cdot, z)$$

for each $z \in U$. Recall Definition 3.7. If $\mu \in M_1''(\Omega \times U)$, then the following hold:

(a) *the measures $\mathbb{P}$ and $\mathbb{Q}$ are, in fact, mutually absolutely continuous on $\mathcal{B}_n^+(\hat{v})$ for every $n \ge 0$;*
(b) *the environment Markov chain with kernel $\hat{\pi}$ and initial distribution $\mathbb{Q}$ is stationary and ergodic;*
(c) *$\mathbb{Q}$ is the unique $\hat{\pi}$-invariant probability measure on Ω that satisfies $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{B}_n^+(\hat{v})$ for every $n \ge 0$;*
(d) *the following LLN is satisfied: $P_o^{\hat{\pi}}(\lim_{n\to\infty} \frac{X_n}{n} = \int \sum_{z\in U} \hat{\pi}(\omega, z) z\, d\mathbb{Q}) = 1$.*

6. Proof of our results on quenched large deviations.

6.1. *Equality of the quenched and the averaged minimizers.* We start this section by stating the quenched version of Theorem 5.1.

THEOREM 6.1. *Assume* (3.1). *For every* $\xi \in \mathcal{A}_{eq}$, $\varepsilon > 0$, $N, M, K \geq 0$ *and* $f : \Omega \times U^{\mathbb{N}} \to \mathbb{R}$ *bounded such that* $f(\cdot, (z_i)_{i\geq 1})$ *is independent of* $(z_i)_{i>K}$ *and is* $\mathcal{B}_N^M$*-measurable*, *the following holds*:

$$\limsup_{\delta'\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_o^\omega\left(\left|\int f\,d\nu_{n,X}^\infty-\int f\,d\hat\mu_\xi^\infty\right|>\varepsilon\,\middle|\,\left|\frac{X_n}{n}-\xi\right|\leq\delta'\right)<0 \tag{6.1}$$

for $\mathbb{P}$*-a.e.* ω.

PROOF. This is Theorem 3 of [24], except that [24] is concerned with space–time RWRE. However, the result is obtained directly from Theorem 5.1 by a standard application of the Borel–Cantelli lemma and Chebyshev's inequality. In other words, the proof in [24] makes no use of the space–time assumption. [The only notational difference is that, in the space–time case, $\Lambda_a(\theta)$ is equal to $\log\phi(\theta)$ for some explicit function $\phi(\cdot)$, but this does not play any role in the proof.] □

Now, we are ready to give the following proof.

PROOF OF THEOREM 3.5. Take any $\xi \in \mathcal{A}_{eq}$. Recall (3.5). If an $\hat\alpha \in A_\xi^\infty$ is not equal to $\hat\mu_\xi^\infty$, then

$$\left|\int f\,d\hat\alpha-\int f\,d\hat\mu_\xi^\infty\right|>\varepsilon$$

for some $\varepsilon > 0$, $N, M, K \geq 0$ and $f : \Omega \times U^{\mathbb{N}} \to \mathbb{R}$ bounded such that $f(\cdot, (z_i)_{i\geq 1})$ is independent of $(z_i)_{i>K}$ and $\mathcal{B}_N^M$-measurable.

For every $\delta' > 0$ and $\mathbb{P}$-a.e. ω, (the lower bound of) the quenched level-3 LDP (i.e., Theorem 2.4) implies that

$$-I_{q,3}(\hat\alpha)\leq\liminf_{n\to\infty}\frac{1}{n}\log P_o^\omega\left(\left|\int f\,d\nu_{n,X}^\infty-\int f\,d\hat\mu_\xi^\infty\right|>\varepsilon,\left|\frac{X_n}{n}-\xi\right|<\delta'\right).$$

On the other hand,

$$\lim_{\delta'\to 0}\lim_{n\to\infty}\frac{1}{n}\log P_o^\omega\left(\left|\frac{X_n}{n}-\xi\right|\leq\delta'\right)=-I_q(\xi),$$

by the quenched level-1 LDP (i.e., Theorem 2.1). Therefore,

$$\begin{aligned}
&-I_{q,3}(\hat\alpha)+I_q(\xi)\\
&\quad\leq\limsup_{\delta'\to 0}\limsup_{n\to\infty}\frac{1}{n}\log P_o^\omega\left(\left|\int f\,d\nu_{n,X}^\infty-\int f\,d\hat\mu_\xi^\infty\right|>\varepsilon\,\middle|\,\left|\frac{X_n}{n}-\xi\right|\leq\delta'\right)\\
&\quad<0,
\end{aligned}$$

by (6.1). In words, $\hat\alpha$ is not a minimizer of (3.6). However, since $I_{q,3}$ is lower semicontinuous and A_ξ^∞ is compact, there is a minimizer. We conclude that $\hat\mu_\xi^\infty$ is the unique minimizer of (3.6). □

6.2. *Modifying Rosenbluth's variational formula.*

LEMMA 6.2. *Assume* (1.1). *Recall* (3.8) *and Definition* 3.7. *For every* $\mu \in M_1''(\Omega \times U)$,

$$\mathfrak{I}_q^{**}(\mu) \le H(\mu). \tag{6.2}$$

PROOF. Fix a sequence of test functions, denoted by $(f_i)_{i\ge 1}$, that separate $M_1(\Omega \times U)$. For every $i \ge 1$ and $z \in U$, assume that $f_i(\cdot, z) : \Omega \to \mathbb{R}$ is measurable with respect to $\sigma(\omega_x : |x| \le i)$. Take any $\mu \in M_1''(\Omega \times U)$. For every $N \ge 1$,

$$G_{\mu,N} := \left\{ \nu \in M_1(\Omega \times U) : \left| \int f_i \, d\nu - \int f_i \, d\mu \right| < \frac{1}{N} \ \forall i \in \{1, \dots, N\} \right\}$$

is an open set. Recall $\hat{v} \in \mathcal{S}^{d-1}$ and the environment kernel $\hat{\pi}$ corresponding to μ; see Definition 3.7. Let $\mathbb{Q} := (\mu)^1$ so that $d\mu(\cdot, z) = d\mathbb{Q}(\cdot)\hat{\pi}(\cdot, z)$ for each $z \in U$. For every $n \ge 1$, introduce a new measure $R_{o,n}^{\hat{\pi},\omega}$ by setting

$$dR_{o,n}^{\hat{\pi},\omega} := \frac{\mathbb{1}_{\nu_{n,X} \in G_{\mu,N}, \beta > n}}{P_o^{\hat{\pi},\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n)} \, dP_o^{\hat{\pi},\omega},$$

where $\beta = \beta(\hat{v}) := \inf\{i \ge 0 : \langle X_i, \hat{v} \rangle < \langle X_o, \hat{v} \rangle\}$. With this notation, for $\mathbb{Q}$-a.e. ω,

$$\begin{aligned}
&\log P_o^{\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n) \\
&\quad = \log E_o^{\hat{\pi},\omega}\left[\nu_{n,X} \in G_{\mu,N}, \beta > n, \frac{dP_o^{\omega}}{dP_o^{\hat{\pi},\omega}} \right] \\
&\quad = \log P_o^{\hat{\pi},\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n) + \log \int \frac{dP_o^{\omega}}{dP_o^{\hat{\pi},\omega}} \, dR_{o,n}^{\hat{\pi},\omega} \\
&\quad \ge \log P_o^{\hat{\pi},\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n) - \int \log \frac{dP_o^{\hat{\pi},\omega}}{dP_o^{\omega}} \, dR_{o,n}^{\hat{\pi},\omega} \\
&\quad = \log P_o^{\hat{\pi},\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n) \\
&\qquad - \frac{1}{P_o^{\hat{\pi},\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n)} \\
&\qquad \times E_o^{\hat{\pi},\omega}\left[\nu_{n,X} \in G_{\mu,N}, \beta > n, \log \frac{dP_o^{\hat{\pi},\omega}}{dP_o^{\omega}} \right],
\end{aligned} \tag{6.3}$$

by a change of measure and Jensen's inequality.

It follows from Lemma 5.8 and the ergodic theorem that

$$\mathbb{Q} \otimes P_o^{\hat{\pi},\omega}(\nu_{n,X} \in G_{\mu,N} \text{ for sufficiently large } n) = 1$$

and

$$\mathbb{Q} \otimes P_o^{\hat{\pi},\omega}\left(\lim_{n\to\infty} \frac{1}{n} \log \frac{dP_o^{\hat{\pi},\omega}}{dP_o^{\omega}}(X_1,\ldots,X_n) = H(\mu)\right) = 1.$$

Hence, for $\mathbb{Q}$-a.e. ω,

$$\lim_{n\to\infty} P_o^{\hat{\pi},\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n) = P_o^{\hat{\pi},\omega}(\beta = \infty) \tag{6.4}$$

and

$$\begin{aligned} &\limsup_{n\to\infty} E_o^{\hat{\pi},\omega}\left[\nu_{n,X} \in G_{\mu,N}, \beta > n, \frac{1}{n} \log \frac{dP_o^{\hat{\pi},\omega}}{dP_o^{\omega}}\right] \\ &\qquad \le P_o^{\hat{\pi},\omega}(\beta = \infty) H(\mu). \end{aligned} \tag{6.5}$$

Here, (6.5) follows from Fatou's lemma since

$$\frac{1}{n} \log \frac{dP_o^{\hat{\pi},\omega}}{dP_o^{\omega}}(X_1,\ldots,X_n) = \frac{1}{n} \sum_{i=0}^{n-1} \log \frac{\hat{\pi}(T_{X_i}\omega, Z_{i+1})}{\pi(X_i, X_{i+1})} \le -\log \delta,$$

by uniform ellipticity; see (1.1).

It follows from parts (b) and (c) of Definition 3.7 that $P_o^{\hat{\pi},\omega}(\beta = \infty) > 0$ for $\mathbb{P}$-a.e. ω. Since $P_o^{\hat{\pi},\omega}(\beta = \infty)$ is $\mathcal{B}_o^+(\hat{v})$-measurable, part (d) of Definition 3.7 implies that

$$P_o^{\hat{\pi},\omega}(\beta = \infty) > 0 \qquad \text{for } \mathbb{Q}\text{-a.e. } \omega. \tag{6.6}$$

Combining (6.3), (6.4), (6.5) and (6.6), we see that

$$\liminf_{n\to\infty} \frac{1}{n} \log P_o^{\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n) \ge -H(\mu) \tag{6.7}$$

for $\mathbb{Q}$-a.e. ω. However, since $P_o^{\omega}(\nu_{n,X} \in G_{\mu,N}, \beta > n)$ is $\mathcal{B}_N^+(\hat{v})$-measurable for every $n \ge 1$, Lemma 5.8 implies that (6.7) holds for $\mathbb{P}$-a.e. ω as well. Therefore,

$$\liminf_{n\to\infty} \frac{1}{n} \log P_o^{\omega}(\nu_{n,X} \in G_{\mu,N}) \ge -H(\mu) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

For every $N \ge 1$ and $\mathbb{P}$-a.e. ω,

$$\limsup_{n\to\infty} \frac{1}{n} \log P_o^{\omega}(\nu_{n,X} \in G_{\mu,N}) \le -\inf_{\nu \in \overline{G_{\mu,N}}} \mathfrak{I}_q^{**}(\nu)$$

by the quenched level-2 LDP, that is, Theorem 2.3. Hence,

$$\inf_{\nu \in \overline{G_{\mu,N}}} \mathfrak{I}_q^{**}(\nu) \le H(\mu).$$

Sending N to infinity implies (6.2) since $\mathfrak{I}_q^{**}(\cdot)$ is lower semicontinuous and $(f_i)_{i\ge1}$ separates $M_1(\Omega \times U)$. □

PROOF OF THEOREM 3.8. Fix $\xi \neq 0$. For any $\mu \in A_\xi \cap M_1'(\Omega \times U)$, let $\hat{\pi}$ be the environment kernel given by $\hat{\pi}(\cdot, z) := \frac{d\mu(\cdot,z)}{d(\mu)^1(\cdot)}$ for each $z \in U$. It is shown in [11] that

$$P_o^{\hat{\pi}}\left(\lim_{n\to\infty} \frac{X_n}{n} = \xi\right) = 1.$$

Hence, $\mu \in M_1''(\Omega \times U)$. [For part (c) of Definition 3.7, take any $\hat{v} \in S^{d-1}$ such that $\langle \xi, \hat{v}\rangle > 0$.] In other words,

$$A_\xi \cap M_1'(\Omega \times U) \subset A_\xi \cap M_1''(\Omega \times U). \tag{6.8}$$

It follows from (2.6), (6.2), (6.8) and (3.7) that

$$\begin{aligned} I_q(\xi) &= \inf_{\mu \in A_\xi} \mathfrak{I}_q^{**}(\mu) \leq \inf\{\mathfrak{I}_q^{**}(\mu) : \mu \in A_\xi \cap M_1''(\Omega \times U)\} \\ &\leq \inf\{H(\mu) : \mu \in A_\xi \cap M_1''(\Omega \times U)\} \\ &\leq \inf\{H(\mu) : \mu \in A_\xi \cap M_1'(\Omega \times U)\} \\ &= I_q(\xi). \end{aligned}$$

In particular, $I_q(\xi) = \inf\{H(\mu) : \mu \in A_\xi \cap M_1''(\Omega \times U)\}$. □

PROOF OF THEOREM 3.9. Fix $\xi \in \mathcal{A}_{eq}$. Then, $\xi \neq 0$. Indeed, by differentiating both sides of (4.5) with respect to θ at $\theta = \nabla I_a(\xi)$, we see that

$$\langle \xi, e_1\rangle = \langle \nabla\Lambda_a(\theta), e_1\rangle = \frac{E_o[\langle X_{\tau_1}, e_1\rangle \exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} | \beta = \infty]}{E_o[\tau_1 \exp\{\langle \theta, X_{\tau_1}\rangle - \Lambda_a(\theta)\tau_1\} | \beta = \infty]} > 0.$$

Recall that $\hat{\mu}_\xi \in M_1(\Omega \times U)$ is the marginal of $\hat{\mu}_\xi^\infty \in M_1(\Omega \times U^{\mathbb{N}})$ and $d\hat{\mu}_\xi(\cdot, z) = d\mathbb{Q}_\xi(\cdot)\hat{\pi}^\theta(\cdot, z)$ for each $z \in U$. It is clear from Theorem 3.3 that $\hat{\mu}_\xi \in A_\xi \cap M_1''(\Omega \times U)$. Observe that

$$H(\hat{\mu}_\xi) = \int \sum_{z\in U} \hat{\pi}^\theta(\omega, z) \log \frac{\hat{\pi}^\theta(\omega, z)}{\pi(0, z)} \, d\mathbb{Q}_\xi(\omega) \tag{6.9}$$

$$= \int \sum_{z\in U} \hat{\pi}^\theta(\omega, z)\left(\langle \theta, z\rangle - \Lambda_a(\theta) + \log \frac{h(\theta, T_z\omega)}{h(\theta, \omega)}\right) d\mathbb{Q}_\xi(\omega) \tag{6.10}$$

$$= \langle \theta, \xi\rangle - \Lambda_a(\theta) + \int \sum_{z\in U} \hat{\pi}^\theta(\omega, z) \log \frac{h(\theta, T_z\omega)}{h(\theta, \omega)} \, d\mathbb{Q}_\xi(\omega) \tag{6.11}$$

$$= \langle \theta, \xi\rangle - \Lambda_a(\theta) \tag{6.12}$$

$$= I_q(\xi). \tag{6.13}$$

Explanation: (6.9), (6.10) and (6.11) follow from (3.8), (3.3) and (2.4), respectively. Since $\mathbb{Q}_\xi$ is $\hat{\pi}^\theta$-invariant by Lemma 5.5, it is easy to see that the integral in (6.11) is zero. Finally, (6.12) is equal to (6.13) because $\xi = \nabla\Lambda_a(\theta)$ and $I_q(\xi) = I_a(\xi)$.

Thus far, we have shown that $H(\hat{\mu}_\xi) = I_q(\xi)$. Now, take any $\nu \in A_\xi \cap M_1''(\Omega \times U)$. If $\nu \neq \hat{\mu}_\xi$, then

$$I_q(\xi) < \mathfrak{I}_q^{**}(\nu) \leq H(\nu),$$

by Corollary 3.6 and (6.2). We conclude that $\hat{\mu}_\xi$ is the unique minimizer of (3.9). □

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