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T. E. HARRIS AND BRANCHING PROCESSES

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T. E. Harris was a pioneer par excellence in many fields of probability theory. In this paper, we give a brief survey of the many fundamental contributions of Harris to the theory of branching processes, starting with his doctoral work at Princeton in the late forties and culminating in his fundamental book "The Theory of Branching Processes," published in 1963.

1. Introduction. T. E. Harris wrote the first definitive book [5] on branching processes, published in 1963. It covered much of the work on the subject up to that time, a sizeable part due to Harris himself. It identified the subject of branching processes and resulted in a great deal of interest in the subject, among both mathematicians and statisticians. Between 1963 and 1970, a vast number of papers on branching processes appeared in many good journals specializing in probability theory and mathematical statistics, and by 1971 more books on the subject appeared both in the U.S. and elsewhere [15, 18]. Harris himself moved on to work on other beautiful topics such as percolation and interacting particle systems. As with branching processes, his work in these other areas was profound. T. E. Harris was pioneer par excellence, creating many areas of research in which he laid the foundations that others built on. In what follows, we present a brief account of Harris's contribution to branching processes.

Harris's 1947 PhD dissertation at the mathematics department of Princeton University was on branching processes, titled "Some theorems on Bernoulli multiplicative processes." This was followed in 1948 by his basic paper [6] in the *Annals of Mathematical Statistics*. In [6], he used the term *branching processes*, a term which had also been used by Russian mathematicians; he treated the single type discrete time branching process. He also coined the term *Galton–Watson branching process* for this process. His main focus in [6] was on the supercritical case; we now give a description of this work.

2. Single type, discrete time case. Let $\{p_j\}_{j\geq 0}$ be a probability distribution. Let $\{\xi_{n,k}; n \geq 0, k \geq 1\}$ be an array of nonnegative integer valued random variables that are i.i.d. (independent and identically distributed) with distribution $\{p_j\}_{j\geq 0}$.

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Let Z_0 be a positive integer. Now set

$$Z_1 = \sum_{k=1}^{Z_0} \xi_{0,k} \tag{1}$$

and for $n \geq 1$, $Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n,k}$ if $Z_n > 0$ and 0 if $Z_n = 0$. Then the sequence $\{Z_n\}_{n\geq 0}$ is called a Galton–Watson branching process with initial population Z_0 and offspring distribution $\{p_j\}_{j\geq 0}$. Clearly, $\{Z_n\}_{n\geq 0}$ is a Markov chain with time homogeneous transition probabilities and the nonnegative integers as the state space. The transition probabilities are given by

$$p_{ij} = P\left(\sum_{r=1}^{i} \xi_r = j\right) \qquad \text{for } i \geq 1 \quad \text{and} \quad p_{00} = 1,$$

where $\{\xi_r\}_{r\geq 1}$ are i.i.d. with distribution $\{p_k\}_{k\geq 0}$.

One can interpret the sequence $\{Z_n\}_{n\geq 0}$ as follows. If Z_n is thought of as the number of individuals in the nth generation, then each one of them produces a random number of children with distribution $\{p_j\}_{j\geq 0}$ independently of others in the nth generation as well as any past ancestors. The total number Z_{n+1} of all these individuals is the size of the $(n+1)$st generation.

An important parameter in determining how the sequence $\{Z_n\}_{n\geq 0}$ behaves for n large is the offspring mean $m \equiv \sum_j jp_j$. Here are some basic results.

THEOREM 2.1. *Let* $0 < m \equiv \sum_{j=1}^{\infty} jp_j < \infty$ *and let* $P(0 < Z_0 < \infty) = 1$. *Then*:

(i) $m < 1 \Rightarrow P(Z_n \to 0 \textit{ as } n \to \infty) = 1$,
(ii) $m = 1, p_1 < 1 \Rightarrow P(Z_n \to 0 \textit{ as } n \to \infty) = 1$,
(iii) $m > 1 \Rightarrow P(Z_n \to 0 \textit{ as } n \to \infty \mid Z_0 = 1) \equiv q < 1$,

where q *is the unique root of the equation*

$$s = f(s) \equiv \sum_{j=0}^{\infty} p_j s^j, \qquad 0 \leq s < 1.$$

Further, $P(Z_n \to \infty \textit{ as } n \to \infty \mid Z_0 = 1) = 1 - q$, *and for any* $k \geq 1$, $P(Z_n \to 0 \textit{ as } n \to \infty \mid Z_0 = k) = q^k$.

Harris in his book [5] notes in that in 1874 in [19], Galton and Watson did notice that the *extinction probability* q satisfied $q = f(q)$, but failed to notice that if $m > 1$ the relevant root is less than one. Galton and Watson's work was motivated by the problem of the survival of British peerage names, posed by Galton in the *London Times* in the 1870s.

In his paper [6], which is based on his doctoral thesis, Harris focused mainly on the *supercritical case*, that is, $m > 1$. The case $m = 1$ is called *critical* and $m < 1$ is the *subcritical case*. Let $\{p_j\}, m, \{Z_n\}$ be as in Theorem 1.

THEOREM 2.2 (Supercritical case, [6, 7]). *Assume* $p_0 = 0, p_1 < 1, m > 1$, $\sum_{j=1}^{\infty} j^2 p_j < \infty$ *and* $0 < Z_0 < \infty$. *Let* $W_n \equiv Z_n/m^n, n \geq 0$. *Then there exists a nonnegative random variable* W *such that*:

(i) $E((W_n - W)^2 \mid Z_0) \to 0$ *as* $n \to \infty$,
(ii) $P(W = 0) = 0$,
(iii) W *has an absolutely continuous distribution an* $(0, \infty)$ *with a continuous density*,
(iv) $E(W \mid Z_0 = 1) = 1$.

Harris [5] observes that J. L. Doob seems to have been the first to note that $\{W_n\}_{n \geq 0}$ is a martingale and, being nonnegative, converges a.s. as $n \to \infty$. Kesten and Stigum [13] improved on this, as follows.

THEOREM 2.3 [13]. *Let* $p_0 = 0, p_1 < 1, 0 < Z_0 < \infty, 1 < m$, *and* $W_n = \frac{Z_n}{m^n}$. *Then*:

(i) $\sum_1^{\infty} j(\log j) p_j < \infty \Rightarrow W_n \to W$ *a.s. and in mean*, *where* $P(W = 0) = 0$, $E(W \mid Z_0 = 1) = 1$ *and* W *has an absolutely continuous distribution on* $(0, \infty)$.
(ii) $\sum_1^{\infty} j(\log j) p_j = \infty \Rightarrow W_n \to 0$, *a.s.*

The work of A. N. Kolmogorov [14] in 1938 and A. M. Yaglom [20] in 1947 (see [5]) led to the following.

THEOREM 2.4 (Critical case). *Suppose* $m = 1, p_1 < 1$ *and* $\sum_1^{\infty} j^2 p_j < \infty$. *Then*, *as* $n \to \infty$,

(i) $nP(Z_n > 0 \mid Z_0 = 1) \to \frac{\sigma^2}{2}$, *where* $\sigma^2 \equiv \sum_1^{\infty} j^2 p_j - 1$,
(ii) $P(\frac{Z_n}{n} > x \mid Z_0 = 1, Z_n > 0) \to e^{-2/\sigma^2 x}$, *for all* $0 < x < \infty$.

THEOREM 2.5 (Subcritical case). *Let* $m < 1$. *Then for all* $j \geq 1$, $\lim_n P(Z_n = j \mid Z_n > 0) \equiv b_j$ *exists*, $0 < b_j < \infty$ *and* $\sum_{j=1}^{\infty} b_j = 1$.

In his book [5], Harris presents extensions of Theorems 2.1, 2.2, 2.4 and 2.5 to the multitype (finite type) case. In [13], Kesten and Stigum established the analog of Theorem 2.3 above for the multitype Galton–Watson process. See Athreya and Ney [1] for details; see also Sevastyanov [18] and Mode [15].

3. Single type, age dependent case. In 1948 Harris, with Richard Bellman [3, 10], formulated the theory of age dependent branching processes, where each individual lives a random length of time and on death creates a random number of individuals, and all individuals live and reproduce independently of each other. Assuming all moments on the offspring distribution and an absolutely continuous life time distribution, they established an integral equation for the probability generating function of $Z(t)$, the population size at time t. They showed that in the supercritical case, $Z(t)e^{-\alpha t}$ converges in probability to a limit random variable W, where α is the *Malthusian parameter* defined by $m \int_0^\infty e^{-\alpha u}\, dG(u) = 1$, with $G(\cdot)$ being the distribution function of the lifetime of an individual. They further showed that W is nontrivial and has an absolutely continuous distribution on $(0, \infty)$. There are analogs of Theorems 2.3 and 2.4 for this case, as well.

Conditions for the supercritical case were relaxed by later authors; see Athreya and Ney [1].

4. General type case. Harris also considered branching processes with arbitrary type space by using the point process approach. Here in any generation, one has a finite point process an some type space X. The basic branching property of independent production is retained. An individual located at $x \in X$ produces children according to a point process over X whose distribution depends on x. All individuals act independently of each other. For this, Harris used the method of moment generating functions. In [9], he established the analog of Theorem 2.2 in this context, and applied this to nuclear cascades and related processes, as well as a one-dimensional neutron model. For details on this, see Harris's book [5]. Harris mentions that J. E. Moyal worked on similar ideas. In the 1970s, Jagers and his colloborators in Sweden developed this topic further in great detail (see [11]). See also Ney [16, 17].

5. Cosmic-ray cascades. Harris studied the theory of cosmic-rays cascades and supplemented the work of nuclear physicists; Chapter 7 of his 1963 book [5] deals with this topic. We present a brief summary of Harris's work on cosmic-ray cascades as discussed in his paper [8]. Here are the model assumptions:

(1) A photon of positive energy ε, moving through homogeneous material, has probability $\lambda\, dt + o(dt)$ of being transformed in the thickness interval $(t, t + dt)$ into two electrons, positive or negative, which receive energies εU and $\varepsilon(1 - U)$, respectively, where U is a random variable with an absolutely continuous distribution in $(0, 1)$. Note that the role of time parameter is played by the thickness of the material.

(2) An electron loses (by "collision" or "ionization") a deterministic amount of energy βt in an interval of length t.

(3) An electron radiates photons in such a way that the probability that an electron of energy ε emits a photon of energy between εu and $\varepsilon(u + du)$ in a small thickness interval of length dt is $k(u)\, du\, dt$. Further, the energy that goes to the radiated photon is subtracted from that of the parent photon. A special case of interest for $k(\cdot)$ is $k(u) = \frac{\mu}{u} + k_0(u)$ with $|\frac{dk_o(u)}{du}| \le c(1 - u)^{-b}, 0 < u < 1$, where c and μ are constants and $b < 2$.

Here λ and β are constants independent of t and ε.

Under the above assumptions, Harris shows that if $\beta = 0$ and $\varepsilon_0(t)$ is the energy at time t of an electron with $\varepsilon_0(0) = 1$ then, for $t > 0$, $X_0(t) \equiv -\log \varepsilon_0(t)$ has an infinitely divisible distribution with probability density $h_t(x), x > 0$, whose characteristic function (Fourier transform) is given by

$$\int_0^\infty e^{i\theta x} h_t(x)\,dx = \exp\left(t\int_0^\infty (e^{i\theta u}-1)k(1-e^{-u})e^{-u}\,du\right).$$

Harris observes that the special case when $k(u) = -\frac{\mu}{\log(1-u)}$ and $h_t(x)$ is the Gamma density $\frac{x^{\mu t-1}e^{-x}}{\Gamma(\mu t)}$, this was given by Bhabha and Heitler [4].

Next, Harris considers the random process $N(\varepsilon, t)$, the total number of electrons at t whose energies are greater than ε, for $\varepsilon > 0$. Let $f_i(s, \varepsilon, t) \equiv E_i(s^{N(\varepsilon,t)}), 0 \le s \le 1$, where E_i stands for expectation when the starting particle is of energy 1, and is a photon for $i = 1$ and an electron for $i = 2$. Harris shows that the following integro-differential equation holds:

$$\frac{\partial f_2}{\partial t}(s,\varepsilon,t) = \int_0^1 \left[f_1\left(s,\frac{\varepsilon}{u},t\right)f_2\left(s,\frac{\varepsilon}{1-u},t\right) - f_2(s,\varepsilon,t)\right]k(u)\,du,$$

with $f_1(s,\varepsilon,0) = f_1(s,1,t) = f_2(s,l,t) = 1$ for $t > 0$ and $f_2(s,\varepsilon,0) = s$ for $\varepsilon < 1$. Harris shows that the earlier results of Bartlett and Kendall [2] and of Janossy [12] could be deduced from the above.

Harris introduces a vector valued Markov process $(I(t), \zeta(t)), t \ge 0$, where $I(t)$ is the condition of a single particle at time t which can be a photon ($I = 1$) or an electron ($I = 2$) and has energy $\zeta(t)$. He then derives the limiting distribution of the process $(I(t), \zeta(t))$ as $t \to \infty$ (assuming $\beta = 0$) and is able to deduce the earlier results of other authors as special cases.

Harris also obtained results for cascades with $\beta > 0$. In particular, he shows that when $\beta > 0$, the energy $\varepsilon_1(t)$ of an electron at time t can be represented by

$$\varepsilon_1(t) = \max\left\{0, \varepsilon_0(t)\left(1 - \beta\int_0^t \frac{ds}{\varepsilon_0(s)}\right)\right\}.$$

6. Concluding remarks. T. E. Harris was deeply involved in the development of all aspects of contemporary branching process theory. He laid a rigorous foundation to areas where it had been lacking. His 1963 book [5] is a beautiful and major work of scholarship. One can substantially credit to its publication the explosion of work on branching processes in the 1960s and 1970s and up to the present. It set the impetus and direction of research on the subject for many years. The present authors owe T. E. Harris a deep debt of gratitude for this.

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