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FAST TRANSPORT ASYMPTOTICS FOR STOCHASTIC RDES WITH BOUNDARY NOISE

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We consider a class of stochastic reaction-diffusion equations also having a stochastic perturbation on the boundary and we show that when the diffusion rate is much larger than the rate of reaction, it is possible to replace the SPDE by a suitable one-dimensional stochastic differential equation. This replacement is possible under the assumption of spectral gap for the diffusion and is a result of *averaging* in the fast spatial transport. We also study the fluctuations around the averaged motion.

1. Introduction. In classical chemical kinetics, the evolution of concentrations of various components in a reaction is described by ordinary differential equations. Such a description turns out to be unsatisfactory in a number of applications, especially in biology (see [12]).

There are several ways to construct a more adequate mathematical model. If the reaction is fast enough, one should take into account that the concentration is not constant in space in the volume where the reaction takes place. Then, the change of concentration due to spatial transport should be taken into account and the system of ODEs should be replaced by a system of PDEs of reaction-diffusion type. In some cases, one should also take into account random changes in time of the rates of reaction. Then, the ODE is replaced by a stochastic differential equation. If the rates change randomly not just in time but also in space, then evolution of concentrations can be described by a system of SPDEs.

On the other hand, the rates of chemical reactions in the system and the diffusion coefficients may, and as a rule do, have different orders. Some of them are much smaller than others and this allows one to apply various versions of the averaging principle and other asymptotic methods, thereby eventually obtaining a relatively simple description of the system.

In this paper, we study the case where the diffusion rate is much larger than the rate of reaction and we show that in this case, it is possible to replace SPDEs of reaction-diffusion type by suitable SDEs. Such an approximation is valid, in

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particular, if the reaction occurs only on the boundary of the domain (this means that the nonlinearity is included in the boundary conditions). This replacement is a result of *averaging* in the fast spatial transport. We would like to stress that our approach allows us also to calculate the main terms of deviations of the solution of the original problem from the simplified model. Notice, moreover, that the case where the diffusion coefficients and some of the reaction rates are large compared with other rates can be considered in a similar way.

More precisely, we are dealing with the following class of equations:

$$
\begin{cases}
\dfrac{\partial u_\varepsilon}{\partial t}(t,x) = \dfrac{1}{\varepsilon}\mathcal{A}u_\varepsilon(t,x) + f(t,x,u_\varepsilon(t,x)) & \\
\qquad\qquad + g(t,x,u_\varepsilon(t,x))\dfrac{\partial w^Q}{\partial t}(t,x), & t \geq 0, x \in D, \\
\dfrac{1}{\varepsilon}\dfrac{\partial u_\varepsilon}{\partial \nu}(t,x) = \sigma(t,x)\dfrac{\partial w^B}{\partial t}(t,x), & t \geq 0, x \in \partial D, \\
u_\varepsilon(0,x) = u_0(x), & x \in D,
\end{cases}
\tag{1.1}
$$

for some $0 < \varepsilon \ll 1$. These are reaction-diffusion equations perturbed by a noise of multiplicative type, where the diffusion term $\mathcal{A}$ is multiplied by a large parameter ε^{-1} and a noisy perturbation is also acting on the boundary of the domain D.

Here, D is a bounded open subset of $\mathbb{R}^d$, with $d \geq 1$, having a regular boundary (for more details, see Section 2) and, in the case $d = 1$, we take $D = [a, b]$. $\mathcal{A}$ is a uniformly elliptic second order operator and $\partial/\partial\nu$ is the corresponding conormal derivative. This is why the same constant ε^{-1}, which is in front of the operator $\mathcal{A}$, is also present in front of the conormal derivative $\partial/\partial\nu$. In what follows, we shall denote by A the realization in $L^2(D)$ of the differential operator $\mathcal{A}$, endowed with the conormal boundary condition.

The coefficients $f, g : [0,\infty) \times D \times \mathbb{R} \to \mathbb{R}$ are assumed to be measurable and satisfy a Lipschitz condition with respect to the third variable, uniformly with respect to the first two variables, and the mapping $\sigma : [0,\infty) \times \partial D \to \mathbb{R}$ is bounded with respect to the space variable.

The noisy perturbations are given by two independent cylindrical Wiener processes, w^Q and w^B, defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, which take values on $L^2(D)$ and $L^2(\partial D)$, respectively, and have covariance operators $Q \in \mathcal{L}^+(L^2(D))$ and $B \in \mathcal{L}^+(L^2(\partial D))$, respectively.[2] In space dimension $d = 1$, we can take Q equal to the identity operator so that we can deal with space–time white noise. Moreover, as $L^2(\{a, b\}) = \mathbb{R}^2$, in space dimension $d = 1$, we do not assume any condition on B.

Stochastic partial differential equations with a noisy term also acting on the boundary have been studied by several authors; see, for example, da Prato and Zabczyk [3], Freidlin and Wentzell [6] and Sowers [10]. The last two mentioned

[2]Here, and in what follows, given any Banach space E, we denote by $\mathcal{L}(E)$ the Banach space of bounded linear operators on E and by $\mathcal{L}^+(E)$ the subspace of nonnegative and symmetric operators.

papers also deal with some limiting results with respect to small parameters appearing in front of the noise. However, the limiting results which we are studying in the present paper seem to be completely new and we are not aware of any previous results dealing with the same sort of multiscaling problem, even in the simpler case of homogeneous boundary conditions (i.e., $\sigma = 0$).

As mentioned above, our interest is in studying the limiting behavior of the solution u_ε of problem (1.1) as the parameter ε goes to zero, under the assumption that the diffusion X_t associated with the operator $\mathcal{A}$, endowed with the conormal boundary condition [this corresponds to a diffusion X_t on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$ which reflect on the boundary of D], admits a unique invariant measure μ and a spectral gap occurs. That is, for any $h \in L^2(D, \mu)$,

$$\int_D \left| \hat{\mathbb{E}}^x h(X_t) - \int_D h(y)\mu(dy) \right|^2 \mu(dx) \le c e^{-2\gamma t} \int_D |h(y)|^2 \mu(dy)$$

for some constant $\gamma > 0$. This can be expressed in terms of the semigroup e^{tA} associated with the diffusion X_t, by saying that

$$\left| e^{tA} h - \int_D h(x)\mu(dx) \right|_{L^2(D,\mu)} \le c e^{-\gamma t} |h|_{L^2(D,\mu)}. \tag{1.2}$$

Moreover, as shown in Remark 2.1, the space $L^2(D)$ is continuously embedded into $L^2(D, \mu)$.

Our aim is to prove that equation (1.1) can be replaced by a suitable one-dimensional stochastic differential equation, whose coefficients are obtained by averaging the coefficients and the noises in (1.1) with respect to the invariant measure μ. More precisely, for any $h \in L^2(D, \mu)$, we define

$$\hat{F}(t, h) = \int_D f(t, x, h(x))\mu(dx), \qquad t \ge 0,$$

and for any $h \in L^2(D, \mu)$, $z \in L^2(D)$ and $k \in L^2(\partial D)$, we define

$$\hat{G}(t, h)z = \int_D g(t, x, h(x))z(x)\mu(dx), \qquad t \ge 0,$$

and

$$\hat{\Sigma}(t)k = \delta_0 \int_D N_{\delta_0}[\sigma(t, \cdot)k](x)\mu(dx), \qquad t \ge 0,$$

where N_{δ_0} is the Neumann map associated with $\mathcal{A}$ and δ_0 is a suitable constant (see Section 2, [8] and [9] for definitions). We prove that for any $t \ge 0$, the mappings $\hat{F}(t, \cdot) : L^2(D, \mu) \to \mathbb{R}$ and $\hat{G}(t, \cdot) : L^2(D, \mu) \to L^2(D)$ are both well defined and Lipschitz continuous, and $\hat{\Sigma}(t) \in L^2(\partial D)$, so that the stochastic ordinary differential equation

$$\begin{cases} dv(t) = \hat{F}(t, v(t))\, dt + \hat{G}(t, v(t))\, dw^Q(t) + \hat{\Sigma}(t)\, dw^B(t), \\ v(0) = \displaystyle\int_D u_0(x)\mu(dx), \end{cases} \tag{1.3}$$

admits, for any $T > 0$ and $p \geq 1$, a unique strong solution $u \in L^p(\Omega; C([0,T]))$ which is adapted to the filtration of the noises w^Q and w^B. Notice that (1.3) is a one-dimensional stochastic equation, in the sense that the space variables have disappeared. In Section 4, we show that it can be rewritten as

$$dv(t) = \hat{F}(t, v(t))\,dt + \Phi(t, v(t))\,d\beta_t,$$

where β_t is a standard Brownian motion and the diffusion coefficient Φ is explicitly given in terms of Q, G, B and Σ.

When we say that equation (1.1) can be replaced by (1.3), we mean that the solution u_ε of (1.1) can be approximated by the solution v of (1.3) in the following sense:

$$\lim_{\varepsilon\to 0} \mathbb{E} \sup_{t\in[\delta,T]} \left| \int_D |u_\varepsilon(t,x) - v(t)|^2 \mu(dx) \right|^p = 0 \tag{1.4}$$

for any fixed $0 < \delta < T$ and $p \geq 1/2$.

In order to prove (1.4), we first have to prove that for any fixed $\varepsilon > 0$, equation (1.1) admits a unique adapted mild solution in $L^p(\Omega, C([0,T]; L^2(D)))$, that is, there exists a unique adapted process u_ε such that

$$\begin{aligned} u_\varepsilon(t) = e^{tA/\varepsilon} u_0 &+ \int_0^t e^{(t-s)A/\varepsilon} F(s, u_\varepsilon(s))\,ds + \int_0^t e^{(t-s)A/\varepsilon} G(s, u_\varepsilon(s))\,dw^Q(s) \\ &+ w^\varepsilon_{A,B}(t), \end{aligned}$$

where $w^\varepsilon_{A,B}(t)$ is the boundary term (the *stochastic boundary convolution*)

$$w^\varepsilon_{A,B}(t) = (\delta_0 - A) \int_0^t e^{(t-s)A/\varepsilon} N_{\delta_0}[\Sigma(s)\,dw^B(s)], \qquad t \geq 0$$

(here, and in what follows, F and G denote the composition/multiplication operators associated with f and g, resp.). In particular, we have to show that the above term is well defined in $L^p(\Omega, C([0,T]; L^2(D)))$. Concerning the notion of mild solutions and existence and uniqueness results for SPDEs like (1.1), with fixed $\varepsilon > 0$, we refer to Da Prato and Zabczyk [3]. However, we would like to stress that in the present paper, we are not imposing the Hilbert–Schmidt condition on the covariance operators Q and B, and this makes the treatment of the stochastic convolution and of the stochastic boundary convolution more complicated, in view also of the a priori estimates with respect to $\varepsilon > 0$.

Actually, once we have a unique adapted mild solution u_ε for (1.1), we prove an a priori estimate of the following type:

$$\sup_{\varepsilon\in(0,1]} \mathbb{E}|u_\varepsilon(t)|^p_{C([0,T];L^2(D))} \leq c_{T,p}\big(1 + |u_0|^p_{L^2(D)}\big).$$

Due to (1.2), this allows us to proceed to the proof of (1.4).

After we have proven (1.4), in the final section, we study the fluctuations of u_ε from v. Namely, we introduce the random field

$$z_\varepsilon(t,x) := \frac{u_\varepsilon(t,x) - v(t)}{\sqrt{\varepsilon}}, \qquad (t,x) \in [0,+\infty) \times D,$$

and show that, under the assumption that the noisy perturbation in (1.1) is of additive type (i.e., the diffusion coefficient g is independent of u), for any $t > 0$,

$$z_\varepsilon(t) \rightharpoonup I_0(t) \qquad \text{in } L^2(D,\mu), \varepsilon \downarrow 0,$$

where $I_0(t,x)$ is the Gaussian random field taking values in $L^2(D,\mu)$ for any $t > 0$, defined by

$$\begin{aligned} I_0(t,x) := &\int_0^\infty \big(e^{sA} G(t)\, dw^Q(s,x) - \langle \hat{G}(t), dw^Q(s)\rangle_{L^2(D)}\big) \\ &+ \int_0^\infty \big((\delta_0 - A)e^{sA} N_{\delta_0}[\Sigma(t)\, dw^B(s)](x) - \langle \hat{\Sigma}(t), dw^B(s)\rangle_{L^2(\partial D)}\big). \end{aligned}$$

The random field $I_0(t,x)$ is well defined in $L^2(D,\mu)$ because of the spectral gap inequality (1.2) and, in the case where the coefficients g and σ do not depend on t, $I_0(t,x)$ also does not depend on t so that the weak limit of $z_\varepsilon(t,x)$ as $\varepsilon \downarrow 0$ depends only on the space variable x and is constant in time for any $t > 0$.

2. Notation and assumptions. Let D be a bounded domain in $\mathbb{R}^d$, with $d \geq 1$, satisfying the extension and exterior cone properties, and let ν be the outward normal at ∂D. We assume that ∂D is a C^∞ manifold and D is locally only on one side of ∂D. In the case $d = 1$, D is a bounded interval (a,b).

We define $H := L^2(D)$ and $Z := L^2(\partial D)$ and, for any $\alpha \geq 0$, we define $H^\alpha := H^\alpha(D)$ and $Z^\alpha := H^\alpha(\partial D)$ (in particular, $H^0 = H$ and $Z^0 = Z$).

We assume that $\mathcal{A}$ is a second order differential operator,

$$\mathcal{A}f = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial f}{\partial x_j}\right) + \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i},$$

satisfying the uniform ellipticity condition

$$\inf_{x\in D} \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq a_0 \sum_{i=1}^d \xi_i^2, \qquad \xi \in \mathbb{R}^d,$$

for some $a_0 > 0$. The coefficients a_{ij} and b_i are assumed to be smooth [for simplicity, we take them to be in $C^\infty(D)$]. In what follows, we shall denote by A the realization in H of the operator $\mathcal{A}$, endowed with the boundary condition

$$\frac{\partial h}{\partial \nu}(x) := \langle a(x)\nu(x), \nabla h(x)\rangle_{\mathbb{R}^d} = 0, \qquad x \in \partial D. \tag{2.1}$$

Namely,

$$\begin{cases} Ah = \mathcal{A}h, \qquad h \in D(A), \\ D(A) = \{h \in H^2(D); \langle a(x)\nu(x), \nabla h(x)\rangle_{\mathbb{R}^d} = 0, x \in \partial D\}. \end{cases}$$

As is well known, the operator A generates an analytic semigroup $\{e^{tA}\}_{t\geq 0}$ in H which is also strongly continuous. Moreover,

$$D(A^\alpha) = D((A^\star)^\alpha) \subset H^{2\alpha}, \qquad \alpha \geq 0,$$

and

$$D(A^\alpha) = H^{2\alpha}, \qquad 0 \leq \alpha < \tfrac{3}{4} \tag{2.2}$$

(for proofs, see [11] and [8], resp.).

If, for any $1 < p \leq \infty$, we denote by A_p the realization in $L^p(D)$ of the operator $\mathcal{A}$, endowed with the boundary condition (2.1), it can be proven that A_p generates a strongly continuous analytic semigroup e^{tA_p} in $L^p(D)$. Notice that all of these semigroups are consistent, so, in what follows, we shall denote them all by e^{tA}.

As proved in, for example, [5], Theorem 2.4.4, since $\mathcal{A}$ is uniformly elliptic and the domain D has the extension property, the semigroup e^{tA} admits an integral kernel $k_t(x, y)$. Due to the boundary condition, the kernel satisfies

$$0 \leq k_t(x, y) \leq c(t^{-d/2} + 1), \qquad t > 0, \tag{2.3}$$

for some constant $c > 0$, almost everywhere in $D \times D$.

As a consequence of our assumptions on $\mathcal{A}$ and D, it is possible to prove that there exists some $\delta_0 \in \mathbb{R}$ such that for any $\delta \geq \delta_0$ and $h \in Z$, the elliptic boundary value problem

$$\begin{cases} (\delta - \mathcal{A})v(x) = 0, & x \in D, \\ \langle a(x)\nu(x), \nabla v(x)\rangle_{\mathbb{R}^d} = h(x), & x \in \partial D, \end{cases} \tag{2.4}$$

admits a unique weak solution $v \in H$, which we will denote by $N_\delta h$. The application $N_\delta : Z \to H$ is known as the *Neumann map* associated with the operator $\mathcal{A}$. It is well known that N_δ maps Z into H as a bounded linear mapping. Moreover, according to elliptic theory for domains with smooth boundaries (for a proof, see [9], Theorem 7.4 of Volume I), we have

$$N_\delta \in \mathcal{L}(Z^\alpha, H^{\alpha+3/2}), \qquad \alpha \geq 0. \tag{2.5}$$

In what follows, we shall assume that e^{tA} has the following long-time behavior.

HYPOTHESIS 1. *The semigroup e^{tA}, $t \geq 0$, admits a unique invariant measure μ and there exists some $\gamma > 0$ such that, for any $h \in L^2(D, \mu)$,*

$$\left| e^{tA}h - \int_D h(y)\mu(dy) \right|_{L^2(D,\mu)} \leq ce^{-\gamma t}|h|_{L^2(D,\mu)}, \qquad t \geq 0. \tag{2.6}$$

In what follows, we shall set $H_\mu := L^2(D, \mu)$ and

$$\langle h, \mu \rangle := \int_D h(x)\mu(dx).$$

REMARK 2.1.

1. If $\mathcal{A}$ is a divergence-type operator, that is, $b_i \equiv 0$ for any $i = 1, \dots, d$, then the operator A is self-adjoint in H. This implies that it is possible to fix a complete orthonormal system $\{e_k\}_{k\geq 0}$ in H and an increasing sequence of nonnegative real numbers $\{\alpha_k\}_{k\geq 0}$ such that

$$Ae_k = -\alpha_k e_k, \qquad k \in \mathbb{N}.$$

Let e_0 be the constant eigenfunction corresponding to the $\alpha_0 = 0$ eigenvalue and let α_1 be the first positive eigenvalue. It is immediate to check that

$$\mu(dx) = e_0^2\, dx = |D|^{-1}\, dx \tag{2.7}$$

and, in particular, that $H = H_\mu$, with equivalence of norms. Moreover, as for any $x \in H$, we have

$$e^{tA}x - \langle x, \mu \rangle = \sum_{i=1}^{\infty} e^{-t\alpha_i} \langle x, e_i \rangle_H e_i$$

and $\alpha_1 \leq \alpha_i$ for any $i \geq 1$, it is immediate to check that

$$|e^{tA}x - \langle x, \mu \rangle|^2_{H_\mu} = |D|^{-1} \sum_{i=1}^{\infty} e^{-2t\alpha_i} \langle x, e_i \rangle^2_H \leq e^{-2t\alpha_1} |x|^2_{H_\mu},$$

so the constant γ in (2.6) coincides with α_1.

2. If A is self-adjoint, as above, for any $\delta > 0$ and $k \in \mathbb{N}$ it holds that

$$N^\star_\delta e_k = \frac{1}{\delta + \alpha_k} e_{k|_{\partial D}}. \tag{2.8}$$

Actually, for any $h \in Z$, we have

$$\begin{aligned}\langle N_\delta h, e_k \rangle_H &= \frac{1}{\delta + \alpha_k} \int_D N_\delta h(x)(\delta + \alpha_k) e_k(x)\, dx \\ &= \frac{1}{\delta + \alpha_k} \int_D N_\delta h(x)(\delta - \mathcal{A}) e_k(x)\, dx.\end{aligned}$$

Now, if we assume that $h \in Z^{1/2}$, according to (2.5), we have that $N_\delta h \in H^2$ and then, due to the Gauss–Green formula and to (2.4), we obtain

$$\int_D N_\delta h(x) \mathcal{A} e_k(x)\, dx = -\int_{\partial D} h(\sigma) e_k(\sigma)\, d\sigma + \int_D \mathcal{A} N_\delta h(x) e_k(x)\, dx.$$

This implies that

$$\begin{aligned}\langle N_\delta h, e_k\rangle_H &= \frac{1}{\delta+\alpha_k}\int_D (\delta-\mathcal{A})N_\delta h(x)e_k(x)\,dx + \frac{1}{\delta+\alpha_k}\int_{\partial D} h(\sigma)e_k(\sigma)\,d\sigma\\ &= \frac{1}{\delta+\alpha_k}\langle h, e_{k_{|\partial D}}\rangle_Z\end{aligned}$$

so that

$$\langle h, N_\delta^\star e_k\rangle_Z = \frac{1}{\delta+\alpha_k}\langle h, e_{k_{|\partial D}}\rangle_Z.$$

As $Z^{1/2}$ is dense in Z, we can conclude that (2.8) holds.

3. As

$$e^{tA}h(x) = \int_D k_t(x,y)h(y)\,dy, \qquad x\in D,$$

and $e^{tA}1 = 1$, we have

$$|e^{tA}h(x)|^2 \le e^{tA}|h|^2(x), \qquad x\in D.$$

Due to the invariance of μ, this implies that for any $h\in H_\mu$,

$$\int_D |e^{tA}h(x)|^2\mu(dx) \le \int_D e^{tA}|h|^2(x)\mu(dx) = \int_D |h(x)|^2\mu(dx),$$

so e^{tA} acts on H_μ as a contraction, that is,

$$\|e^{tA}\|_{\mathcal{L}(H_\mu)} \le 1, \qquad t\ge 0. \tag{2.9}$$

4. We have that H is continuously embedded into H_μ. Actually, due to the invariance of μ and to the kernel representation of e^{tA}, for any $h\in H$, we have

$$\int_D |h(x)|^2\mu(dx) = \int_D e^{1A}|h|^2(x)\mu(dx) = \int_D\int_D k_1(x,y)|h(y)|^2\,dy\mu(dx).$$

Then, thanks to (2.3), we have

$$|h|^2_{H_\mu} = \int_D |h(x)|^2\mu(dx) \le c\int_D |h(y)|^2 dy = |h|^2_H.$$

5. As a matter of fact, there exists a nonnegative function $m\in L^\infty(D)$ such that

$$\mu(dx) = m(x)\,dx, \qquad x\in D.$$

Actually, let $\varphi,\psi\in C^2(\bar D)$, with φ fulfilling the boundary condition (2.1). Integrating by parts, we obtain

$$\langle\psi,\mathcal{A}\varphi\rangle_H = \langle\mathcal{A}^\star\psi,\varphi\rangle_H - \int_{\partial D}\langle a\nu,\nabla\psi\rangle_{\mathbb{R}^d}\varphi\,d\sigma + \int_{\partial D}\langle b,\nu\rangle_{\mathbb{R}^d}\varphi\psi\,d\sigma,$$

where

$$\mathcal{A}^\star\psi = \frac{\partial}{\partial x_j}\left(a_{ij}\,\frac{\partial\psi}{\partial x_i}\right) - \langle b, \nabla\psi\rangle_{\mathbb{R}^d} - \operatorname{div} b\psi.$$

Hence, the operator $\mathcal{A}^\star$, endowed with the boundary condition

$$\langle a(x)\nu(x), \nabla\psi(x)\rangle_{\mathbb{R}^d} - \langle b(x), \nu(x)\rangle_{\mathbb{R}^d}\psi(x) = 0, \qquad x \in \partial D, \tag{2.10}$$

is the formal adjoint of the operator $\mathcal{A}$, endowed with the boundary condition (2.1).

Now, the function $u = 1$ is a nonzero solution of the problem

$$\begin{cases} \mathcal{A}u(x) = 0, & x \in D, \\ \langle a(x)\nu(x), \nabla u(x)\rangle_{\mathbb{R}^d} = 0, & x \in \partial D. \end{cases}$$

Then, by the Fredholm alternative, there exists a nonzero weak solution $\varphi \in H^1$ to the adjoint problem

$$\begin{cases} \mathcal{A}^\star\varphi(x) = 0, & x \in D, \\ \langle a(x)\nu(x), \nabla\varphi(x)\rangle_{\mathbb{R}^d} - \langle b(x), \nu(x)\rangle_{\mathbb{R}^d}\varphi(x) = 0, & x \in \partial D. \end{cases}$$

By elliptic regularity results (cf. [7], Chapter 3), as the boundary of D and the coefficients of $\mathcal{A}$ (and hence of $\mathcal{A}^\star$) are of class C^∞, we have that φ is a classical solution to the adjoint problem. Hence, if $A^\star$ is the adjoint of A in H, for any λ sufficiently large, we have

$$(\lambda I - A^\star)^{-1}\varphi = \frac{1}{\lambda}\varphi$$

and by taking the inverse Laplace transform, we obtain $e^{tA^\star}\varphi = \varphi$ for any $t \geq 0$.

Now, due to the positivity of the semigroup e^{tA} (and hence of the semigroup $e^{tA^\star}$) and to the fact that e^{tA} is conservative, we have that the set

$$\Lambda := \{\varphi \in H : e^{tA^\star}\varphi = \varphi, t \geq 0\}$$

is a lattice, that is, $|\varphi| \in \Lambda$ for any $\varphi \in \Lambda$. Therefore, if we set

$$m(x) := \frac{|\varphi(x)|}{\int_D |\varphi(y)|\,dy}, \qquad x \in D,$$

we have that $e^{tA^\star}m = m$ for any $t \geq 0$ and hence $m(x)\,dx$ is a probability measure and is invariant for e^{tA}. As μ is the unique invariant measure for e^{tA}, we are done.

Concerning the coefficients f, g and σ we assume the following conditions.

HYPOTHESIS 2.

1. *The mappings* $f, g : [0, \infty) \times D \times \mathbb{R} \to \mathbb{R}$ *are measurable and the mappings* $f(t, x, \cdot), g(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ *are Lipschitz continuous, uniformly with respect to*

$(t,x) \in [0,T] \times D$, *for any* $T > 0$. *Namely, for any* $\xi, \eta \in \mathbb{R}$

$$\sup_{(t,x)\in[0,T]\times D} |f(t,x,\xi) - f(t,x,\eta)| \le L_{T,f}|\xi - \eta|,$$

$$\sup_{(t,x)\in[0,T]\times D} |g(t,x,\xi) - g(t,x,\eta)| \le L_{T,g}|\xi - \eta|.$$

2. *The mapping* $\sigma : [0,\infty) \times \partial D \to \mathbb{R}$ *is measurable and for any* $T > 0$,

$$\sup_{t\in[0,T]} |\sigma(t,\cdot)|_{L^\infty(\partial D)} =: c_{T,\sigma} < \infty.$$

In what follows, for any $t \ge 0$ and $h_1, h_2 \in H$, we shall define

$$F(t,h_1)(x) := f(t,x,h_1(x)), \qquad x \in D,$$

and

$$[G(t,h_1)h_2](x) := g(t,x,h_1(x))h_2(x), \qquad x \in D.$$

Due to Hypothesis 2, we have that $F(t,\cdot) : H \to H$, $G(t,\cdot) : H \to \mathcal{L}(H, L^1(D))$ and $G(t,\cdot) : H \to \mathcal{L}(L^\infty(D), H)$ are all Lipschitz continuous, uniformly with respect to $t \in [0,T]$, for any $T > 0$.

Notice that the same is true for the mappings $F(t,\cdot) : H_\mu \to H_\mu$, $G(t,\cdot) : H_\mu \to \mathcal{L}(H_\mu, L^1(D,\mu))$ and $G(t,\cdot) : H_\mu \to \mathcal{L}(L^\infty(D;\mu), H_\mu)$.

Analogously, if, for any $t \ge 0$ and $z \in Z$, we set

$$[\Sigma(t)z](x) := \sigma(t,x)z(x), \qquad x \in \partial D,$$

then we have that $\Sigma(t)$ is a bounded linear operator on Z and for any $T > 0$,

$$\|\Sigma(t)\|_{\mathcal{L}(Z)} \le c_{T,\sigma}, \qquad t \in [0,T]. \tag{2.11}$$

Finally, concerning the noisy perturbations $w^Q(t)$ and $w^B(t)$, we assume that they are two independent cylindrical Wiener processes defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, taking values in H and Z, respectively, with respective covariance operators $Q \in \mathcal{L}^+(H)$ and $B \in \mathcal{L}^+(Z)$. Namely,

$$w^Q(t) = \sum_{k\in\mathbb{N}} \lambda_k e_k \beta_k(t), \qquad w^B(t) = \sum_{k\in\mathbb{N}} \theta_k f_k \hat{\beta}_k(t),$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the orthonormal basis of H which diagonalizes Q, with eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$, $\{f_k\}_{k\in\mathbb{N}}$ is the orthonormal basis of Z which diagonalizes B, with eigenvalues $\{\theta_k\}_{k\in\mathbb{N}}$, and $\{\beta_k\}_{k\in\mathbb{N}}$ and $\{\hat{\beta}_k\}_{k\in\mathbb{N}}$ are two sequences of independent standard Brownian motions, both defined on the stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Notice that the two sequences above are not convergent in H and Z, but in any Hilbert spaces U and V which contain H and Z, respectively, with Hilbert–Schmidt embedding. Moreover, in the case $d = 1$, we have $Z = \mathbb{R}^2$ and hence

$$w^B(t) = \Theta\hat{\beta}(t),$$

where $\Theta = \operatorname{diag}(\theta_1, \theta_2)$ and $\hat{\beta}(t) = (\hat{\beta}_1(t), \hat{\beta}_2(t))$ is a two-dimensional standard Brownian motion.

In what follows, we shall assume the following summability conditions on the eigenvalues λ_k and θ_k and the sup-norm of the corresponding eigenfunctions.

HYPOTHESIS 3.

1. *If $d \geq 2$, then there exists $\rho < 2d/(d-2)$ such that*

$$\sum_{k \in \mathbb{N}} \lambda_k^{\rho} |e_k|_{\infty}^2 =: \kappa_Q < \infty. \tag{2.12}$$

2. *If $d \geq 2$, then there exists $\beta < 2d/(d-1)$ such that*

$$\sum_{k \in \mathbb{N}} \theta_k^{\beta} =: \kappa_B < \infty. \tag{2.13}$$

REMARK 2.2.

1. From the proofs of Lemmas 3.3, 4.3 and 5.4, it is possible to see that if the mapping $g : [0, T] \times D \times \mathbb{R} \to \mathbb{R}$ is uniformly bounded for any $T > 0$, then we do not need to require that the sequence $\{e_k\}_{k \in \mathbb{N}}$ is contained in $L^{\infty}(D)$ and condition (2.12) can be replaced by
$$\sum_{k \in \mathbb{N}} \lambda_k^{\rho} < \infty.$$
2. As both $d/(d-2)$ and $d/(d-1)$ are strictly greater than 1, neither Q nor B are required to be Hilbert–Schmidt operators in general. Moreover, in space dimension $d = 1$, we have no conditions on the eigenvalues $\{\lambda_k\}$ and we can take $Q = I$. This means that we can deal with space–time white noise.

3. A priori bounds for the solution of (1.1). In this section, we are concerned with uniform bounds for the pth moments of the $C([0, T]; H)$-norm of the mild solution u_ε of (1.1).

We first recall some general facts about the linear parabolic equation with non-homogeneous boundary conditions

$$\begin{cases} \dfrac{\partial y}{\partial t}(t, x) = \mathcal{A} y(t, x), & t \geq 0, x \in D, \\ \langle a(x) \nu(x), \nabla y(t, x) \rangle_{\mathbb{R}^d} = v(t, x), & t \geq 0, x \in \partial D, \\ y(0, x) = y_0(x), & x \in D, \end{cases} \tag{3.1}$$

where v is a Z-valued function. If $v(\cdot)$ is twice continuously differentiable and there exists $\delta_0 > 0$ such that $y_0 - N_\delta v(0) \in D(A)$ for $\delta > \delta_0$, then the solution of problem (3.1) is given by

$$y(t) = e^{tA} y_0 + (\delta - A) \int_0^t e^{(t-s)A} N_\delta v(s)\, ds \tag{3.2}$$

(for a proof, see, e.g., [4], Proposition 13.2.1).

Such a formula can be extended by continuity to less regular functions v. In particular, for each $\varepsilon > 0$, we can consider the problem

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) = \dfrac{1}{\varepsilon}\mathcal{A}y(t,x), & t \geq 0, x \in D, \\ \langle a(x)\nu(x), \nabla y(t,x)\rangle_{\mathbb{R}^d} = \varepsilon\sigma(t,x)\dfrac{\partial w^B}{\partial t}(t,x), & t \geq 0, x \in \partial D, \\ y(0,x) = 0, & x \in D, \end{cases} \tag{3.3}$$

where w^B is the cylindrical Wiener process defined in Z, introduced in Section 2. In analogy to formula (3.2), by taking $\delta = \delta_0/\varepsilon$ and $v(t) = \varepsilon\Sigma(t)\,\partial w^B/\partial t$, we say that for any $\varepsilon \in (0, 1]$, the process

$$w^{\varepsilon}_{A,B}(t) = (\delta_0 - A)\int_0^t e^{(t-s)A/\varepsilon} N_{\delta_0}[\Sigma(s)\,dw^B(s)], \qquad t \geq 0,$$

is a *mild solution* to problem (3.3). The process $w^{\varepsilon}_{A,B}(t)$ can be interpreted as a *boundary Ornstein–Uhlenbeck* process and can be written as the infinite series

$$w^{\varepsilon}_{A,B}(t) = \sum_{k\in\mathbb{N}}(\delta_0 - A)\int_0^t e^{(t-s)A/\varepsilon} N_{\delta_0}[\Sigma(s)Bf_k]\,d\hat{\beta}_k(s), \qquad t \geq 0.$$

As proved in the next lemma, such a series is well defined in $L^p(\Omega; C([0,T]; H))$ for any $T > 0$ and $p \geq 1$. Moreover, a uniform estimate with respect to $\varepsilon \in (0, 1]$ holds.

LEMMA 3.1. *Under part* 2 *of Hypothesis* 3, *the process* $w^{\varepsilon}_{A,B}$ *belongs to* $L^p(\Omega; C([0,T]; H))$ *for any* $T > 0$, $p \geq 1$ *and* $\varepsilon \in (0, 1]$, *and*

$$\sup_{\varepsilon\in(0,1]} \mathbb{E}|w^{\varepsilon}_{A,B}|^p_{C([0,T];H)} =: c_{T,p} < \infty. \tag{3.4}$$

PROOF. As a consequence of the stochastic Fubini theorem and of the elementary identity

$$\int_\sigma^t (t-s)^{\alpha-1}(s-\sigma)^{-\alpha}\,ds = \frac{\pi}{\sin\pi\alpha}, \qquad 0 \leq \sigma \leq t, \alpha \in (0,1),$$

we have the factorization formula

$$w^{\varepsilon}_{A,B}(t) = \frac{\sin\pi\alpha}{\pi}\int_0^t (t-s)^{\alpha-1} e^{(t-s)A/\varepsilon} Y_{\varepsilon,\alpha}(s)\,ds,$$

where

$$Y_{\varepsilon,\alpha}(s) = \int_0^s (s-r)^{-\alpha}(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r)\,dw^B(r)]$$

(for a proof, see [2]). By the Hölder inequality, this implies that for any $\alpha > 1/p$,

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} |w^{\varepsilon}_{A,B}(t)|^p_H \\
&\quad \le c_{T,p,\alpha} \int_0^T \mathbb{E}|Y_{\varepsilon,\alpha}(s)|^p_H \, ds \\
&\quad \le c_{T,p,\alpha} \int_0^T \Bigg(\int_0^s (s-r)^{-2\alpha} \\
&\qquad\qquad \times \sum_{k\in\mathbb{N}} \theta_k^2 \big|(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r) f_k]\big|^2_H \, dr \Bigg)^{p/2} ds,
\end{aligned} \tag{3.5}$$

the last inequality following from the Burkholder–Davis–Gundy inequality.

Now, assume that $d > 1$ (the case $d = 1$ is simpler). According to (2.13), we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \theta_k^2 \big|(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r) f_k]\big|^2_H \\
&\quad \le \kappa_B^{2/\beta} \Bigg(\sum_{k\in\mathbb{N}} \big|(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r) f_k]\big|^2_H \Bigg)^{1/\zeta} \\
&\qquad \times \sup_{k\in\mathbb{N}} \big|(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r) f_k]\big|^{2(\zeta-1)/\zeta}_H,
\end{aligned} \tag{3.6}$$

where $\zeta := \beta/(\beta - 2)$. Thanks to (2.2) and (2.5), for any $\rho > 0$, we have

$$S_\rho := (\delta_0 - A)^{(3-\rho)/4} N_{\delta_0} \in \mathcal{L}(Z, H). \tag{3.7}$$

Hence, for any $\varepsilon > 0$ and $0 \le r \le s \le T$, due to (2.11), we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \big|(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r) f_k]\big|^2_H \\
&\quad = \sum_{k\in\mathbb{N}} \big|e^{(s-r)/2A/\varepsilon} (\delta_0 - A)^{(1+\rho)/4} e^{(s-r)/2A/\varepsilon} S_\rho \Sigma(r) f_k\big|^2_H \\
&\quad = \sum_{k\in\mathbb{N}} \sum_{h\in\mathbb{N}} \big|\langle f_k, \Sigma(r) S_\rho^\star [(\delta_0 - A)^{(1+\rho)/4} e^{(s-r)/2A/\varepsilon}]^\star e^{(s-r)/2A^\star/\varepsilon} e_h \rangle_Z\big|^2 \\
&\quad = \sum_{h\in\mathbb{N}} \big|\Sigma(r) S_\rho^\star [(\delta_0 - A)^{(1+\rho)/4} e^{(s-r)/2A/\varepsilon}]^\star e^{(s-r)/2A^\star/\varepsilon} e_h\big|^2_Z \\
&\quad \le c_{T,\rho} \Bigg[\bigg(\frac{\varepsilon}{s-r} \bigg)^{(1+\rho)/2} + 1 \Bigg] \sum_{h\in\mathbb{N}} \big|e^{(s-r)/2A^\star/\varepsilon} e_h\big|^2_H.
\end{aligned} \tag{3.8}$$

As the semigroup e^{tA} admits an integral kernel $k_t(x, y)$, that is,

$$e^{tA} f(x) = \int_D k_t(x, y) f(y) \, dy, \qquad x \in D,$$

we have

$$e^{tA^\star}h(y) = \int_D k_t(x,y)h(x)\,dx, \qquad y \in D.$$

This implies

$$\begin{aligned}\sum_{h\in\mathbb{N}} |e^{(s-r)/2A^\star/\varepsilon} e_h|_H^2 &= \sum_{h\in\mathbb{N}} \int_D |e^{(s-r)/2A^\star/\varepsilon} e_h(y)|^2\,dy \\ &= \sum_{h\in\mathbb{N}} \int_D \left|\int_D k_{(s-r)/(2\varepsilon)}(x,y)e_h(x)\,dx\right|^2 dy \\ &= \sum_{h\in\mathbb{N}} \int_D |\langle k_{(s-r)/(2\varepsilon)}(\cdot,y), e_h\rangle_H|^2\,dy \\ &= \int_D |k_{(s-r)/(2\varepsilon)}(\cdot,y)|_H^2\,dy.\end{aligned} \tag{3.9}$$

Now, due to (2.3), for any $t > 0$ and $y \in D$, we have

$$|k_t(\cdot,y)|_H^2 = \int_D |k_t(x,y)|^2\,dx \le c(t^{-d/2}+1)\int_D k_t(x,y)\,dx$$

and hence

$$\int_D |k_t(\cdot,y)|_H^2\,dy \le c(t^{-d/2}+1)\int_{D\times D} k_t(x,y)\,dx\,dy = c|D|(t^{-d/2}+1).$$

This implies that for any $\varepsilon > 0$,

$$\sum_{h\in\mathbb{N}} |e^{(s-r)/2A^\star/\varepsilon} e_h|_H^2 \le c|D|\left[\left(\frac{\varepsilon}{s-r}\right)^{d/2} + 1\right],$$

so, thanks to (3.8), we have

$$\begin{aligned}&\left(\sum_{k\in\mathbb{N}} |(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r)f_k]|_H^2\right)^{1/\zeta} \\ &\qquad \le c_{T,\rho}\left[\left(\frac{\varepsilon}{s-r}\right)^{(d+1+\rho)/(2\zeta)} + 1\right].\end{aligned} \tag{3.10}$$

Next, by proceeding as in (3.8), we have

$$\begin{aligned}&\sup_{k\in\mathbb{N}} |(\delta_0 - A)e^{(s-r)A/\varepsilon} N_{\delta_0}[\Sigma(r)f_k]|_H^{2(\zeta-1)/\zeta} \\ &\qquad \le c_{T,\rho}\left[\left(\frac{\varepsilon}{s-r}\right)^{(1+\rho)(\zeta-1)/(2\zeta)} + 1\right].\end{aligned} \tag{3.11}$$

Therefore, thanks to (3.5), (3.6), (3.10) and (3.11), we can conclude that for any $\varepsilon \in (0,1]$,

$$\mathbb{E}\sup_{t\in[0,T]} |w^\varepsilon_{A,B}(t)|_H^p \le c_{T,p,\alpha,\rho}\left(\int_0^T \left[s^{-(2\alpha+(d+\zeta)/(2\zeta)+\rho/2)} + 1\right] ds\right)^{p/2}.$$

Now, as in Hypothesis 3, we are assuming that $\beta < 2d/(d-1)$, so we have $(d+\zeta)/2\zeta < 1$. This means that we can fix $\bar{\alpha} > 0$ and $\bar{\rho} > 0$ such that

$$2\bar{\alpha} + \frac{d+\zeta}{2\zeta} + \frac{\bar{\rho}}{2} < 1$$

and then, for any $p > \bar{p} := 1/\bar{\alpha}$ we obtain

$$\sup_{\varepsilon\in(0,1]} \mathbb{E} \sup_{t\in[0,T]} |w^{\varepsilon}_{A,B}(t)|^p_H \le c_{T,p}.$$

The estimate for general $p \ge 1$ follows from the Hölder inequality. □

Next, we pass to (1.1).

DEFINITION 3.2. Let $T > 0$ and $p \ge 1$. An adapted process $u_\varepsilon \in L^p(\Omega; C([0, T]; H))$ is a *mild* solution of (1.1) if, for any $t \in [0, T]$,

$$u_\varepsilon(t) = e^{tA/\varepsilon}u_0 + \int_0^t e^{(t-s)A/\varepsilon} F(s, u_\varepsilon(s))\, ds + w^{\varepsilon}_{A,Q}(u_\varepsilon)(t) + w^{\varepsilon}_{A,B}(t),$$

where, for any $u \in L^p(\Omega; C([0, T]; H))$, we define

$$w^{\varepsilon}_{A,Q}(u)(t) := \int_0^t e^{(t-s)A/\varepsilon} G(s, u(s))\, dw^Q(s), \qquad t \ge 0.$$

As is well known, $w^{\varepsilon}_{A,Q}(u)$ is the unique mild solution of the problem

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) = \dfrac{1}{\varepsilon}\mathcal{A}y(t,x) + g(t,x,u(t,x))\dfrac{\partial w^Q}{\partial t}(t,x), & t \ge 0, x \in D, \\ \langle a(x)\nu(x), \nabla y(t,x)\rangle_{\mathbb{R}^d} = 0, & t \ge 0, x \in \partial D, \\ y(0,x) = 0, & x \in D, \end{cases} \tag{3.12}$$

where w^Q is the cylindrical Wiener process with values in H, introduced in Section 2.

As for $w^{\varepsilon}_{A,B}$, we show that $w^{\varepsilon}_{A,Q}$ satisfies a bound in $L^p(\Omega; C([0, T]; H))$ which is uniform with respect to $\varepsilon \in (0, 1]$.

LEMMA 3.3. *Assume Hypothesis 2 and part 1 of Hypothesis 3. Then, $w^{\varepsilon}_{A,Q}$ is Lipschitz continuous from $L^p(\Omega; C([0, T]; H))$ into itself for any $T > 0$ and $p \ge 1$, and*

$$\sup_{\varepsilon\in(0,1]} \mathbb{E}|w^{\varepsilon}_{A,Q}(u)|^p_{C([0,T];H)} \le c_{T,p}\left(1 + \mathbb{E}\int_0^T |u(s)|^p_H\, ds\right). \tag{3.13}$$

PROOF. The proof of the Lipschitz continuity of $w^{\varepsilon}_{A,Q}$ in $L^p(\Omega; C([0, T]; H))$ is classical and can be found in, for example, [1]. Concerning estimate (3.13),

as in the proof of Lemma 3.1, we use a factorization argument and, for any $\alpha > 1/p$, we get

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} |w^{\varepsilon}_{A,Q}(t)|^p_H \\
&\quad \le c_{T,p,\alpha}\mathbb{E}\int_0^T \left(\int_0^s (s-r)^{-2\alpha} \sum_{k\in\mathbb{N}} \lambda_k^2 |e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|^2_H \, dr\right)^{p/2} ds.
\end{aligned}$$

According to (2.12), if we set $\zeta := \rho/(\rho-2)$, then we have

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} \lambda_k^2 |e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|^2_H \\
&\quad \le \kappa_Q^{2/\rho}\left(\sum_{k\in\mathbb{N}} |e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|^2_H\right)^{1/\zeta} \\
&\qquad \times \sup_{k\in\mathbb{N}} |e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|^{2(\zeta-1)/\zeta}_H |e_k|^{-4/\rho}_\infty.
\end{aligned} \tag{3.14}$$

As in the proof of (3.9), we have

$$\sum_{k\in\mathbb{N}} |e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|^2_H = \int_D |k_{(s-r)/\varepsilon}(x,\cdot)g(r,\cdot,u(r))|^2_H \, dx.$$

Now, thanks to (2.3), for any $t > 0$, $x \in D$ and $h \in H$, we have

$$\begin{aligned}
|k_t(x,\cdot)h|^2_H &= \int_D |k_t(x,y)h(y)|^2 \, dy \\
&\le c(t^{-d/2}+1)\int_D k_t(x,y)h^2(y)\, dy \\
&= c(t^{-d/2}+1)e^{tA}h^2(x)
\end{aligned} \tag{3.15}$$

and this is meaningful since e^{tA} is well defined in $L^1(D)$. In particular, for any $\varepsilon > 0$,

$$\begin{aligned}
&\sum_{k\in\mathbb{N}} |e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|^2_H \\
&\quad \le c\left[\left(\frac{\varepsilon}{s-r}\right)^{d/2}+1\right]\int_D e^{(s-r)A/\varepsilon} g^2(r,\cdot,u(r))(x)\, dx \\
&\quad = c\left[\left(\frac{\varepsilon}{s-r}\right)^{d/2}+1\right]|e^{(s-r)A/\varepsilon} g^2(r,\cdot,u(r))|_{L^1(D)} \\
&\quad \le c\left[\left(\frac{\varepsilon}{s-r}\right)^{d/2}+1\right]|g(r,\cdot,u(r))|^2_H
\end{aligned}$$

and, due to the linear growth of g,

$$\begin{aligned}\Big(\sum_{k\in\mathbb{N}}|e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|_H^2\Big)^{1/\zeta}\\ \le c_T\Big[\Big(\frac{\varepsilon}{s-r}\Big)^{d/(2\zeta)}+1\Big](1+|u(r)|_H^{2/\zeta}).\end{aligned}\tag{3.16}$$

By analogous arguments, we have

$$\sup_{k\in\mathbb{N}}|e^{(s-r)A/\varepsilon}[G(r,u(r))e_k]|_H^{2(\zeta-1)/\zeta}|e_k|_\infty^{-4/\rho}\le c_T\big(1+|u(r)|_H^{2(\zeta-1)/\zeta}\big)\tag{3.17}$$

and then, thanks to (3.14), (3.16) and (3.17), we get, for any $\varepsilon\in(0,1]$,

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}|w_{A,Q}^\varepsilon(t)|_H^p\\ &\quad\le c_{T,p,\alpha}\mathbb{E}\int_0^T\Big(\int_0^s\Big[\Big(\frac{1}{s-r}\Big)^{2\alpha+d/(2\zeta)}+1\Big](1+|u(r)|_H^2)\,dr\Big)^{p/2}ds.\end{aligned}$$

As we are assuming $\rho<2d/(d-2)$, we can find $\bar\alpha>0$ such that $2\bar\alpha+d/(2\zeta)<1$. Due to the Young inequality, this implies (3.13) for all $p>\bar p=1/\bar\alpha$ and hence for all $p\ge1$. □

According to Lemmas 3.1 and 3.3, we have the following result.

THEOREM 3.4. *Under Hypotheses 1, 2 and 3, for any $T>0$ and $p\ge1$, and for any $u_0\in H$ and $\varepsilon>0$, equation (1.1) admits a unique adapted mild solution $u_\varepsilon\in L^p(\Omega;C([0,T];H))$. Moreover,*

$$\sup_{\varepsilon\in(0,1]}\mathbb{E}|u_\varepsilon|_{C([0,T];H)}^p\le c_{T,p}(1+|u_0|_H^p).\tag{3.18}$$

PROOF. As both $F(t,\cdot):H\to H$ and $w_{A,Q}^\varepsilon:L^p(\Omega;C([0,T];H))\to L^p(\Omega;C([0,T];H))$ are Lipschitz continuous and $w_{A,B}^\varepsilon\in L^p(\Omega;C([0,T];H))$, we have that the mapping Φ_ε defined by

$$\Phi_\varepsilon(u)(t)=e^{tA/\varepsilon}u_0+\int_0^t e^{(t-s)A/\varepsilon}F(s,u(s))\,ds+w_{A,Q}^\varepsilon(u)(t)+w_{A,B}^\varepsilon(t)$$

is Lipschitz continuous from the space of adapted processes in $L^p(\Omega;C([0,T];H))$ into itself. Therefore, by a classical fixed point argument, equation (1.1) admits a unique adapted mild solution $u_\varepsilon\in L^p(\Omega,C([0,T];H))$.

Next, for any $\varepsilon>0$, we have

$$\begin{aligned}|u_\varepsilon(t)|_H^p\le c_p\Big(&|u_0|_H^p+ct^{p-1}\int_0^t(1+|u_\varepsilon(s)|_H^p)\,ds\\ &+|w_{A,Q}^\varepsilon(u_\varepsilon)(t)|_H^p+|w_{A,B}^\varepsilon(t)|_H^p\Big)\end{aligned}$$

and then, according to (3.4) and (3.13), we conclude that

$$\mathbb{E} \sup_{t\in[0,T]} |u_\varepsilon(t)|_H^p \le c_{T,p}\Big(1 + |u_0|_H^p + \int_0^T \mathbb{E} \sup_{r\in[0,s]} |u_\varepsilon(r)|_H^p \, ds\Big).$$

The Gronwall lemma allows us to obtain (3.18). □

4. The averaging result. In this section, we show that for any $0 < \delta < T$ and $p \ge 1$, the sequence $\{u_\varepsilon\}_{\varepsilon\in(0,1]}$ converges in $L^p(\Omega; C([\delta, T]; H_\mu))$ to the solution of a suitable one-dimensional stochastic differential equation. In what follows, we first introduce the limiting equation by constructing the coefficients and by describing a situation in which they are given by a simple expression. In the second part of this section, we prove the convergence result.

We start with the drift term. For each $t \ge 0$ and $h \in H$, we define

$$\hat{F}(t,h) := \langle F(t,h), \mu\rangle = \int_D f(t,x,h(x))\mu(dx), \tag{4.1}$$

where $\mu(dx)$ is the unique invariant measure associated with the semigroup e^{tA} (see Section 2 and Hypothesis 1). According to Hypothesis 2, for any $T > 0$ and $h_1, h_2 \in H$, we have

$$|f(t,x,h_1(x)) - f(t,x,h_2(x))| \le L_{T,f}|h_1(x) - h_2(x)|, \qquad (t,x) \in [0,T] \times D,$$

so that

$$\hat{F}(t,\cdot) : H_\mu \to \mathbb{R}$$

is Lipschitz continuous, uniformly with respect to $t \in [0,T]$, for any $T > 0$. Notice that, as $H \subset H_\mu$, this implies that $\hat{F}(t,\cdot) : H \to \mathbb{R}$ is also Lipschitz continuous.

Next, we construct the term arising from the stochastic convolution $w^\varepsilon_{A,Q}(u)(t)$. For each $t \ge 0$ and $h \in H$, we introduce the linear mapping

$$z \in H \mapsto \sum_{k\in\mathbb{N}} \langle G(t,h)e_k, \mu\rangle\langle z, e_k\rangle_H = \langle G(t,h)z, \mu\rangle \in \mathbb{R}.$$

As H is continuously embedded into H_μ, for any $T > 0$, we have

$$|\langle G(t,h)z, \mu\rangle| \le |g(t,\cdot,h)|_{H_\mu}|z|_{H_\mu} \le c_T(1 + |h|_{H_\mu})|z|_H, \qquad t \le T.$$

This means that there exists $\hat{G}(t,h) \in H$ such that

$$\langle \hat{G}(t,h), z\rangle_H = \langle G(t,h)z, \mu\rangle, \qquad z \in H.$$

Moreover, since for any $h_1, h_2 \in H_\mu$ and $T > 0$,

$$\begin{aligned} &|\langle G(t,h_1)z, \mu\rangle - \langle G(t,h_2)z, \mu\rangle| \\ &\qquad \le |g(t,\cdot,h_1) - g(t,\cdot,h_2)|_{H_\mu}|z|_H \\ &\qquad \le c_T|h_1 - h_2|_{H_\mu}|z|_H, \qquad t \le T, \end{aligned}$$

we have that the mapping $\hat{G}(t,\cdot): H_\mu \to H$ is Lipschitz continuous, uniformly with respect to $t \in [0,T]$, for any $T > 0$.

This, in particular, implies that the mapping $\hat{G}(t,\cdot)$ is also Lipschitz continuous, both in H and in H_μ, uniformly for $t \in [0,T]$.

Finally, we construct the term arising from the boundary convolution $w^\varepsilon_{A,B}(t)$. For each fixed $t \geq 0$, we introduce the mapping

$$h \in Z \mapsto \delta_0 \langle N_{\delta_0}[\Sigma(t)h], \mu\rangle = \delta_0 \int_D N_{\delta_0}[\sigma(t,\cdot)h](x)\mu(dx) \in \mathbb{R}.$$

As N_{δ_0} is a bounded linear operator from Z into H, $\Sigma(t)$ is bounded and linear in Z and H is continuously embedded in H_μ, such a mapping is bounded and linear from Z into $\mathbb{R}$ and then, for any $t \geq 0$, there exists $\hat{\Sigma}(t) \in Z$ such that for any $h \in Z$, we have

$$\langle \hat{\Sigma}(t), h\rangle_Z = \delta_0 \langle N_{\delta_0}[\Sigma(t)h], \mu\rangle = \delta_0 \int_D N_{\delta_0}[\sigma(t,\cdot)h](x)\mu(dx). \tag{4.2}$$

We can now introduce the limiting equation. It is the one-dimensional stochastic differential equation

$$\begin{cases} dv(t) = \hat{F}(t, v(t))\,dt + \langle \hat{G}(t, v(t)), dw^Q(t)\rangle_H + \langle \hat{\Sigma}(t), dw^B(t)\rangle_Z, \\ v(0) = \langle u_0, \mu\rangle. \end{cases} \tag{4.3}$$

As the mappings $\hat{F}(t,\cdot): \mathbb{R} \to \mathbb{R}$ and $\hat{G}(t,\cdot): \mathbb{R} \to H$ are both Lipschitz continuous, uniformly with respect to $t \in [0,T]$, for any $T > 0$, equation (4.3) admits a unique strong solution $v \in L^p(\Omega; C([0,T];\mathbb{R}))$ for any $p \geq 1$ and $T > 0$, that is, there exists a unique adapted process in $L^p(\Omega; C([0,T];\mathbb{R}))$ which is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that

$$v(t) = \langle u_0, \mu\rangle + \int_0^t \hat{F}(s, v(s))\,ds + \hat{w}_{A,Q}(v)(t) + \hat{w}_{A,B}(t),$$

where

$$\hat{w}_{A,Q}(v)(t) := \int_0^t \langle \hat{G}(s, v(s)), dw^Q(s)\rangle_H, \qquad \hat{w}_{A,B}(t) := \int_0^t \langle \hat{\Sigma}(s), dw^B(s)\rangle_Z.$$

Notice that both $\hat{w}_{A,Q}(v)(t)$ and $\hat{w}_{A,B}(t)$ are $\mathcal{F}_t$-martingales having zero mean. Moreover, we have

$$\mathbb{E}|\hat{w}_{A,Q}(v)(t)|^2 = \int_0^t \mathbb{E}|Q\hat{G}(s, v(s))|_H^2\,ds \tag{4.4}$$

and

$$\mathbb{E}|\hat{w}_{A,B}(t)|^2 = \int_0^t \mathbb{E}|B\hat{\Sigma}(s)|_Z^2\,ds. \tag{4.5}$$

In particular, as w^Q and w^B are independent, we have that $\hat{w}_{A,Q}(v)(t) + \hat{w}_{A,B}(t)$ is an $\mathcal{F}_t$-martingale having zero mean and covariance

$$\int_0^t \big(\mathbb{E}|Q\hat{G}(s, v(s))|_H^2 + |B\hat{\Sigma}(s)|_Z^2\big)\, ds \tag{4.6}$$

so that there exists some Brownian motion β_t defined on some stochastic basis $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}})$ such that the solution of problem (4.3) coincides in law with the solution of the problem

$$\begin{cases} dv(t) = \hat{F}(t, v(t))\, dt + \Phi(t, v(t))\, d\beta_t, \\ v(0) = \langle u_0, \mu\rangle, \end{cases}$$

where

$$\Phi(t, v) = \big(|Q\hat{G}(t, v)|_H^2 + |B\hat{\Sigma}(t)|_Z\big)^{1/2}. \tag{4.7}$$

As shown in Remark 2.1, in the case where the operator A is self-adjoint, we have

$$\mu(dx) = \frac{1}{|D|}\, dx$$

so that, due to the definition of $\hat{G}(t, v)$, we get

$$|Q\hat{G}(t, v)|_H^2 = \frac{1}{|D|^2}|Qg(t, \cdot, v)|_H^2 = \frac{1}{|D|^2}\int_D |[Qg(t, \cdot, v)](x)|^2\, dx.$$

Concerning the boundary term, due to (2.8), we have

$$\begin{aligned} |B\hat{\Sigma}(t)|_Z^2 &= \frac{\delta_0^2}{|D|^2}\sum_{k\in\mathbb{N}} |\langle N_{\delta_0}[\Sigma(t)Bf_k], 1\rangle_H|^2 \\ &= \sum_{k\in\mathbb{N}} \frac{\delta_0^2}{|D|^2}|\langle[\Sigma(t)Bf_k], N_{\delta_0}^\star 1\rangle_Z|^2 \\ &= \sum_{k\in\mathbb{N}} \frac{1}{|D|^2}|\langle f_k, B\sigma(t, \cdot)\rangle_Z|^2 = \frac{1}{|D|^2}|B\sigma(t, \cdot)|_Z^2 \\ &= \frac{1}{|D|^2}\int_{\partial D} |[B\sigma(t, \cdot)](\eta)|^2\, d\eta. \end{aligned}$$

Therefore, in the self-adjoint case, we have

$$\Phi(t, v) = \frac{1}{|D|}\bigg(\int_D |[Qg(t, \cdot, v)](x)|^2\, dx + \int_{\partial D} |[B\sigma(t, \cdot)](\eta)|^2\, d\eta\bigg)^{1/2}.$$

Now that we have described the candidate limit equation, we prove that u_ε in fact converges to its solution.

THEOREM 4.1. *Assume Hypotheses 1, 2 and 3. Then, for any $u_0 \in H$, $p \geq 1$, $T > 0$ and $\theta < 1$, and for any $\delta > 0$, we have*

$$\begin{aligned}\mathbb{E} \sup_{t\in[\delta,T]} |u_\varepsilon(t) - v(t)|_{H_\mu}^p &\leq c_{T,p,\theta}(\varepsilon + \varepsilon^{p\theta/2})(1 + |u_0|_{H_\mu}^p) \\ &\quad + e^{-\gamma p\delta/\varepsilon}|u_0|_{H_\mu}^p,\end{aligned} \tag{4.8}$$

where v is the solution of the one-dimensional problem (4.3). In particular,

$$\lim_{\varepsilon\to 0} \mathbb{E} \sup_{t\in[\delta,T]} |u_\varepsilon(t) - v(t)|_{H_\mu}^p = 0.$$

PROOF. We have

$$\begin{aligned}u_\varepsilon(t) - v(t) &= (e^{tA/\varepsilon}u_0 - \langle u_0, \mu\rangle) + \int_0^t \big(\hat{F}(s, u_\varepsilon(s)) - \hat{F}(s, v(s))\big)\, ds \\ &\quad + \int_0^t \big\langle \big(\hat{G}(s, u_\varepsilon(s)) - \hat{G}(s, v(s))\big), dw^Q(s)\big\rangle_H + R_\varepsilon(t),\end{aligned}$$

where

$$\begin{aligned}R_\varepsilon(t) &:= \int_0^t e^{(t-s)A/\varepsilon} F(s, u_\varepsilon(s))\, ds - \int_0^t \hat{F}(s, u_\varepsilon(s))\, ds \\ &\quad + w_{A,Q}^\varepsilon(u_\varepsilon)(t) - \hat{w}_{A,Q}(u_\varepsilon)(t) + w_{A,B}^\varepsilon(t) - \hat{w}_{A,B}(t).\end{aligned} \tag{4.9}$$

This yields

$$\begin{aligned}&|u_\varepsilon(t) - v(t)|_{H_\mu}^p \\ &\quad \leq c_{T,p}\bigg(|e^{tA/\varepsilon}u_0 - \langle u_0, \mu\rangle|_{H_\mu}^p \\ &\qquad + \int_0^t |\hat{F}(s, u_\varepsilon(s)) - \hat{F}(s, v(s))|^p\, ds \\ &\qquad + \bigg|\int_0^t \big\langle \big(\hat{G}(s, u_\varepsilon(s)) - \hat{G}(s, v(s))\big), dw^Q(s)\big\rangle_H\bigg|^p \\ &\qquad\qquad + |R_\varepsilon(t)|_{H_\mu}^p\bigg).\end{aligned} \tag{4.10}$$

Due to the Lipschitz continuity of $\hat{F}(t, \cdot) : H_\mu \to \mathbb{R}$, for any $0 \leq t \leq T$, we have

$$\begin{aligned}&\mathbb{E} \sup_{s\in[0,t]} \int_0^s |\hat{F}(r, u_\varepsilon(r)) - \hat{F}(r, v(r))|^p\, dr \\ &\quad \leq c_{T,p} \int_0^t \mathbb{E}|u_\varepsilon(r) - v(r)|_{H_\mu}^p\, dr.\end{aligned} \tag{4.11}$$

Analogously, due to the Lipschitz continuity of $\hat{G}(t,\cdot): H_\mu \to H$ and the Burkholder–Davis–Gundy inequality, for any $0 \le t \le T$, we easily obtain

$$\begin{aligned}\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^s \langle \hat{G}(r, u_\varepsilon(r)) - \hat{G}(r, v(r)), dw^Q(r)\rangle_H \right|^p \\ \le c_{T,p} \int_0^t \mathbb{E}|u_\varepsilon(r) - v(r)|^p_{H_\mu}\, dr.\end{aligned} \tag{4.12}$$

Then, thanks to condition (2.6), for any $0 \le t \le T$,

$$\begin{aligned}&\mathbb{E}|u_\varepsilon(t) - v(t)|^p_{H_\mu} \\ &\quad \le c_{T,p}\Big(e^{-\gamma p t/\varepsilon}|u_0|^p_{H_\mu} + \mathbb{E}\sup_{t\in[0,T]} |R_\varepsilon(t)|^p_{H_\mu} + \int_0^t \mathbb{E}|u_\varepsilon(s) - v(s)|^p_{H_\mu}\, ds\Big)\end{aligned}$$

and, by comparison, this yields

$$\int_0^t \mathbb{E}|u_\varepsilon(s) - v(s)|^p_{H_\mu}\, ds \le c_{T,p}\Big(\varepsilon|u_0|^p_{H_\mu} + \mathbb{E}\sup_{t\in[0,T]} |R_\varepsilon(t)|^p_{H_\mu}\Big). \tag{4.13}$$

In view of (4.10), thanks to (4.11) and (4.12), for any $0 < \delta < T$, we obtain

$$\begin{aligned}&\mathbb{E}\sup_{t\in[\delta,T]} |u_\varepsilon(t) - v(t)|^p_{H_\mu} \\ &\quad \le c e^{-\gamma p \delta/\varepsilon}|u_0|^p_{H_\mu} + c_{T,p}\int_0^T \mathbb{E}|u_\varepsilon(s) - v(s)|^p_{H_\mu}\, dt \\ &\qquad + c_{T,p}\mathbb{E}\sup_{t\in[0,T]} |R_\varepsilon(t)|^p_{H_\mu}.\end{aligned}$$

Therefore, if we show that, for any $T > 0$, $p \ge 1$ and $\theta \in (0,1)$,

$$\mathbb{E}\sup_{t\in[0,T]} |R_\varepsilon(t)|^p_{H_\mu} \le c_{T,p,\theta}\varepsilon^{p\theta/2}(1 + |u_0|^p_H), \tag{4.14}$$

then we can conclude that (4.8) holds. □

Due to (4.9), in order to prove (4.14) and hence complete the proof of Theorem 4.1, we need the following three lemmas.

LEMMA 4.2. *Assume Hypotheses 1, 2 and 3. Then, for any $T > 0$ and $p \ge 1$, and for any $\varepsilon \in (0,1]$, we have*

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]} \left| \int_0^t e^{(t-s)A/\varepsilon} F(s, u_\varepsilon(s))\, ds - \int_0^t \hat{F}(s, u_\varepsilon(s))\, ds \right|^p_{H_\mu} \\ \le c_{T,p}(1 + |u_0|^p_H)\varepsilon^p.\end{aligned} \tag{4.15}$$

PROOF. Due to Hypothesis 1, for any $t \in [0, T]$, we have

$$\begin{aligned}
&\big|e^{(t-s)A/\varepsilon} F(s, u_\varepsilon(s)) - \hat{F}(s, u_\varepsilon(s))\big|_{H_\mu} \\
&\quad \le c e^{-\gamma(t-s)/\varepsilon} |F(s, u_\varepsilon(s))|_{H_\mu} \\
&\quad \le c e^{-\gamma(t-s)/\varepsilon} |F(s, u_\varepsilon(s))|_{H} \\
&\quad \le c_T e^{-\gamma(t-s)/\varepsilon} \Big(1 + \sup_{s \le T} |u_\varepsilon(s)|_H\Big).
\end{aligned}$$

This implies that, for any $t \in [0, T]$,

$$\begin{aligned}
&\left|\int_0^t e^{(t-s)A/\varepsilon} F(s, u_\varepsilon(s))\, ds - \int_0^t \hat{F}(s, u_\varepsilon(s))\, ds\right|_{H_\mu}^p \\
&\quad \le c_{T,p}\Big(1 + \sup_{s \le T} |u_\varepsilon(s)|_H^p\Big)\left(\int_0^t e^{-\gamma s/\varepsilon}\, ds\right)^p
\end{aligned}$$

so that, thanks to (3.18), for any $\varepsilon \in (0, 1]$, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{t \in [0,T]} \left|\int_0^t e^{(t-s)A/\varepsilon} F(s, u_\varepsilon(s))\, ds - \int_0^t \hat{F}(s, u_\varepsilon(s))\, ds\right|_{H_\mu}^p \\
&\quad \le c_{T,p}(1 + |u_0|_H^p)\varepsilon^p.
\end{aligned}$$

□

LEMMA 4.3. *Assume Hypotheses 1, 2 and 3, and fix $T > 0$, $p \ge 1$ and $\theta < 1$. Then, there exists some constant $c_{T,p,\theta} > 0$ such that for any $\varepsilon \in (0, 1]$,*

$$\mathbb{E} \sup_{t \in [0,T]} |w_{A,Q}^\varepsilon(u_\varepsilon)(t) - \hat{w}_{A,Q}(u_\varepsilon)(t)|_{H_\mu}^p \le c_{T,p,\theta} \varepsilon^{p\theta/2}(1 + |u_0|_H^p). \tag{4.16}$$

PROOF. As in the proofs of Lemmas 3.1 and 3.3, we use a factorization argument. Since $e^{tA}1 = 1$, for any $t \ge 0$ and $\alpha > 0$, we have

$$\begin{aligned}
&w_{A,Q}^\varepsilon(u_\varepsilon)(t) - \int_0^t \langle \hat{G}(s, u_\varepsilon(s)), dw^Q(s)\rangle_H \\
&\quad = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} e^{(t-s)A/\varepsilon} Y_{\varepsilon,\alpha}(s)\, ds,
\end{aligned}$$

where

$$Y_{\varepsilon,\alpha}(s) := \int_0^s (s-r)^{-\alpha} e^{(s-r)A/\varepsilon} \Psi(r, u_\varepsilon(r))\, dw^Q(r)$$

and, for any $h_1, h_2 \in H$,

$$\Psi(r, h_1)h_2 := G(r, h_1)h_2 - \langle \hat{G}(r, h_1), h_2\rangle_H.$$

Hence, due to (2.9), e^{tA} is a contraction in H_μ for any $t \geq 0$, and by proceeding as in the proofs of Lemmas 3.1 and 3.3, for $\alpha < 1/p$, we obtain

$$\mathbb{E} \sup_{t\in[0,T]} \left| w^\varepsilon_{A,Q}(u_\varepsilon)(t) - \int_0^t \langle \hat{G}(s, u_\varepsilon(s)), dw^Q(s)\rangle_H \right|^p_{H_\mu}$$
$$\leq c_{T,p,\alpha} \mathbb{E} \int_0^T \left(\int_0^s (s-r)^{-2\alpha} \sum_{k\in\mathbb{N}} \lambda_k^2 |e^{(s-r)A/\varepsilon} \Psi(r, u_\varepsilon(r)) e_k|^2_{H_\mu} \, dr \right)^{p/2} ds.$$

Due to the invariance of μ and condition (2.6), we have

$$\begin{aligned} &|e^{(s-r)A/\varepsilon} \Psi(r, u_\varepsilon(r)) e_k|_{H_\mu} \\ &\quad = |e^{(s-r)A/\varepsilon} [G(r, u_\varepsilon(r)) e_k] - \langle \hat{G}(r, u_\varepsilon(r)), e_k\rangle_H|_{H_\mu} \\ &\quad = |e^{(s-r)/2A/\varepsilon} \big(e^{(s-r)/2A/\varepsilon} [G(r, u_\varepsilon(r)) e_k]\big) \\ &\qquad\qquad - \big\langle e^{(s-r)/2A/\varepsilon} [G(r, u_\varepsilon(r)) e_k], \mu\big\rangle|_{H_\mu} \\ &\quad \leq c e^{-\gamma(s-r)/(2\varepsilon)} |e^{(s-r)/2A/\varepsilon} [G(r, u_\varepsilon(r)) e_k]|_{H_\mu} \end{aligned}$$

so that

$$\mathbb{E} \sup_{t\in[0,T]} \left| w^\varepsilon_{A,Q}(u_\varepsilon)(t) - \int_0^t \langle \hat{G}(s, u_\varepsilon(s)), dw^Q(s)\rangle_H \right|^p_{H_\mu}$$
$$\leq c_{T,p,\alpha} \mathbb{E} \int_0^T \left(\int_0^s (s-r)^{-2\alpha} e^{-\gamma(s-r)/\varepsilon} \right.$$
$$\left. \times \sum_{k\in\mathbb{N}} \lambda_k^2 |e^{(s-r)/2A/\varepsilon} [G(r, u_\varepsilon(r)) e_k]|^2_{H_\mu} \, dr \right)^{p/2} ds. \tag{4.17}$$

Using the same arguments that were used in the proof of Lemma 3.3 [see (3.16) and (3.17)], for any $0 \leq r \leq s \leq T$, we get

$$\sum_{k\in\mathbb{N}} \lambda_k^2 |e^{(s-r)/2A/\varepsilon} [G(r, u_\varepsilon(r)) e_k]|^2_{H_\mu} \leq c_T \left[\left(\frac{\varepsilon}{s-r} \right)^{d/(2\zeta)} + 1 \right] (1 + |u_\varepsilon(r)|^2_H),$$

with $\zeta = \rho/(\rho - 2)$ if $d > 1$ and with $\zeta = 1$ if $d = 1$. Thanks to (3.18), this yields

$$\mathbb{E} \sup_{t\in[0,T]} \left| w^\varepsilon_{A,Q}(u_\varepsilon)(t) - \int_0^t \langle \hat{G}(s, u_\varepsilon(s)), dw^Q(s)\rangle_H \right|^p_{H_\mu}$$
$$\leq c_{T,p,\alpha} (1 + |u_0|^p_H) \varepsilon^{-\alpha p} \left(\int_0^T \left[\left(\frac{\varepsilon}{t} \right)^{(2\alpha + d/(2\zeta))} + 1 \right] e^{-\gamma t/\varepsilon} \, dt \right)^{p/2}.$$

Now, according to the first condition in Hypothesis 3, we have $d/2\zeta < 1$ so that, for any $\theta < 1$, we can fix $\bar{\alpha} > 0$ such that

$$1 - 2\bar{\alpha} > \theta, \qquad 2\bar{\alpha} + \frac{d}{2\zeta} < 1.$$

Then, with a change of variable, we easily obtain

$$E \sup_{t\in[0,T]} \left| w^{\varepsilon}_{A,Q}(u_\varepsilon)(t) - \int_0^t \langle \hat{G}(s, u_\varepsilon(s)), dw^Q(s)\rangle_H \right|^p_{H_\mu} \le c_{T,p,\theta}\varepsilon^{p\theta/2}(1 + |u_0|^p_H)$$

for any $p > \bar{p} := 1/\bar{\alpha}$. By the Hölder inequality, we obtain an analogous estimate for any $p \ge 1$ and (4.16) then follows. □

LEMMA 4.4. *Assume Hypotheses 1, 2 and 3, and fix any $T > 0$, $p \ge 1$ and $\theta < 1$. Then, there exists some constant $c_{T,p,\theta} > 0$ such that for any $\varepsilon \in (0, 1]$,*

$$\mathbb{E} \sup_{t\in[0,T]} |w^{\varepsilon}_{A,B}(t) - \hat{w}_{A,B}(t)|^p_{H_\mu} \le c_{T,p,\theta}\varepsilon^{p\theta/2}. \tag{4.18}$$

PROOF. Notice that $(\delta_0 - A)e^{tA}1 = \delta_0$ for any $t \ge 0$. Then, as in Lemma 3.1, by factorization, we obtain

$$w^{\varepsilon}_{A,B}(t) - \int_0^t \langle \hat{\Sigma}(s), dw^B(s)\rangle_Z = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} e^{(t-s)A/\varepsilon} Y_{\varepsilon,\alpha}(s)\, ds,$$

where

$$Y_{\varepsilon,\alpha}(s) := \int_0^s (s-r)^{-\alpha}(\delta_0 - A)e^{(s-r)A/\varepsilon}\Psi(r)\, dw^B(r),$$

and for any $h \in Z$,

$$\Psi(r)h := N_{\delta_0}[\Sigma(r)h] - \frac{1}{\delta_0}\langle \hat{\Sigma}(r), h\rangle_Z.$$

Hence, according to (2.9), by arguing as in the proofs of Lemmas 3.1 and 3.3, for any $p > 1/\alpha$, we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \left| w^{\varepsilon}_{A,B}(t) - \int_0^t \langle \hat{\Sigma}(s), dw^B(s)\rangle_Z \right|^p_{H_\mu} \\
&\quad \le c_{T,p,\alpha} \int_0^T \Bigg(\int_0^s (s-r)^{-2\alpha} \\
&\qquad\qquad \times \sum_{k\in\mathbb{N}} \theta_k^2 |(\delta_0 - A)e^{(s-r)A/\varepsilon}[\Psi(r)f_k]|^2_{H_\mu}\, dr \Bigg)^{p/2} ds.
\end{aligned}$$

Due to the invariance of μ and to condition (2.6), we have

$$\begin{aligned}
&|(\delta_0 - A)e^{(s-r)A/\varepsilon}[\Psi(r)f_k]|_{H_\mu} \\
&\quad = |(\delta_0 - A)e^{(s-r)A/\varepsilon}N_{\delta_0}[\Sigma(r)f_k] - \delta_0\langle N_{\delta_0}[\Sigma(r)f_k], \mu\rangle|_{H_\mu} \\
&\quad = |e^{(s-r)/2A/\varepsilon}\big((\delta_0 - A)e^{(s-r)/2A/\varepsilon}N_{\delta_0}[\Sigma(r)f_k]\big) \\
&\qquad\qquad - \big\langle (\delta_0 - A)e^{(s-r)/2A/\varepsilon}N_{\delta_0}[\Sigma(r)f_k], \mu\big\rangle|_{H_\mu} \\
&\quad \le ce^{-\gamma(s-r)/(2\varepsilon)}|(\delta_0 - A)e^{(s-r)/2A/\varepsilon}N_{\delta_0}[\Sigma(r)f_k]|_{H_\mu}.
\end{aligned}$$

This implies that

$$\mathbb{E}\sup_{t\in[0,T]}\left|w^{\varepsilon}_{A,B}(t)-\int_0^t\langle\hat{\Sigma}(s),dw^B(s)\rangle_Z\right|^p_{H_\mu}$$
$$\le c_{T,p,\alpha}\int_0^T\Bigg(\int_0^s(s-r)^{-2\alpha}e^{-\gamma(s-r)/\varepsilon}$$
$$\times\sum_{k\in\mathbb{N}}\theta_k^2\big|(\delta_0-A)e^{(s-r)/2A/\varepsilon}N_{\delta_0}[\Sigma(r)f_k]\big|^2_{H_\mu}\,dr\Bigg)^{p/2}ds$$

and, hence, by proceeding as in the proof of Lemma 3.1, we conclude that

$$\mathbb{E}\sup_{t\in[0,T]}\left|w^{\varepsilon}_{A,B}(t)-\int_0^t\langle\hat{\Sigma}(s),dw^B(s)\rangle_Z\right|^p_{H_\mu}$$
$$\le c_{T,p,\alpha,\rho}\varepsilon^{-\alpha p}\left(\int_0^T\left[\left(\frac{\varepsilon}{s}\right)^{2\alpha+(d\,\mathrm{sign}(d-1)+\zeta)/(2\zeta)+\rho/2}+1\right]e^{-\gamma s/\varepsilon}\,ds\right)^{p/2},$$

where ρ is a positive constant to be chosen and where $\zeta=\beta/(\beta-2)$ if $d>1$ and $\zeta=1$ if $d=1$. Now, as we are assuming $\beta<2d/(d-1)$ when $d\ge 2$, for any $\theta<1$, we can fix $\bar{\alpha}$ and $\bar{\rho}$ both positive such that

$$1-2\bar{\alpha}>\theta,\qquad 2\bar{\alpha}+\frac{d\,\mathrm{sign}(d-1)+\zeta}{2\zeta}+\frac{\bar{\rho}}{2}<1.$$

Then, with a change of variable, for any $p>\bar{p}=1/\bar{\alpha}$,

$$\mathbb{E}\sup_{t\in[0,T]}\left|w^{\varepsilon}_{A,B}(t)-\int_0^t\langle\hat{\Sigma}(s),dw^B(s)\rangle_Z\right|^p_{H_\mu}\le c_{T,p}\varepsilon^{p\theta/2}$$

and this implies (4.18) for any $p\ge 1$. □

REMARK 4.5.

1. Notice that from (4.13), we have

$$\mathbb{E}|u_\varepsilon-v|^p_{L^p(0,T;H_\mu)}\le c_{T,p,\theta}(\varepsilon^{p\theta/2}+\varepsilon)(1+|u_0|^p_{H_\mu}) \tag{4.19}$$

so that

$$\lim_{\varepsilon\to 0}\mathbb{E}|u_\varepsilon-v|^p_{L^p(0,T;H_\mu)}=0.$$

2. If we take $u_0=\langle u_0,\mu\rangle$, then, for any $p\ge 1$, $T>0$ and $\theta<1$, we have the stronger estimate

$$\mathbb{E}\sup_{t\in[0,T]}|u_\varepsilon(t)-v(t)|^p_{H_\mu}\le c_{T,p,\theta}\varepsilon^{p\theta/2}(1+|u_0|^p). \tag{4.20}$$

3. From the proofs of Lemmas 4.3 and 4.4, we easily see that for any $T > 0$ and $p \geq 1$,

$$\sup_{t\in[0,T]} \mathbb{E}|w^{\varepsilon}_{A,Q}(u_\varepsilon)(t) - \hat{w}_{A,Q}(t)|^p_{H_\mu} \leq c_{T,p}\varepsilon^{p/2}(1 + |u_0|^p_H) \tag{4.21}$$

and

$$\sup_{t\in[0,T]} \mathbb{E}|w^{\varepsilon}_{A,B}(t) - \hat{w}_{A,B}(t)|^p_{H_\mu} \leq c_{T,p}\varepsilon^{p/2}. \tag{4.22}$$

Then, for any $T > 0$ and $p \geq 1$,

$$\sup_{t\in[0,T]} \mathbb{E}|R_\varepsilon(t)|^p_{H_\mu} \leq c_{T,p}\varepsilon^{p/2}(1 + |u_0|^p_{H_\mu}), \qquad \varepsilon \in (0, 1].$$

Then, by repeating the arguments used in the proof of Theorem 4.1, we have

$$\sup_{t\in[\delta,T]} \mathbb{E}|u_\varepsilon(t) - v(t)|^p_{H_\mu} \leq c_{T,p}(\varepsilon + \varepsilon^{p/2})(1 + |u_0|^p_{H_\mu}) + e^{-\gamma p\delta/\varepsilon}|u_0|^p_{H_\mu}.$$

Moreover, if $u_0 = \langle u_0, \mu\rangle$, as in (4.20), we have

$$\sup_{t\in[0,T]} \mathbb{E}|u_\varepsilon(t) - v(t)|^p_{H_\mu} \leq c_{T,p}\varepsilon^{p/2}(1 + |u_0|^p). \tag{4.23}$$

5. Fluctuations around the averaged motion. In this section, we analyze the fluctuations of the motion u_ε around the averaged motion v. More precisely, we will study the limiting behavior of the random field

$$z_\varepsilon(t, x) := \frac{u_\varepsilon(t, x) - v(t)}{\sqrt{\varepsilon}}, \qquad t \geq 0, x \in D, \tag{5.1}$$

as the parameter ε goes to zero.

In what follows, in addition to Hypothesis 2, we shall assume that the coefficients f and g satisfy the following conditions.

HYPOTHESIS 4.

1. *The mapping $f(t, x, \cdot) : \mathbb{R} \to \mathbb{R}$ is of class C^1, with Lipschitz continuous derivative, uniformly with respect to $x \in D$ and $t \in [0, T]$, for any $T > 0$.*
2. *The mapping g does not depend on the third variable, that is, $g(t, x, \eta) = g(t, x)$ for any $t \geq 0$, $x \in D$ and $\eta \in \mathbb{R}$.*
3. *For any $x \in D$, the mappings $g(\cdot, x) : [0, \infty) \to \mathbb{R}$ and $\sigma(\cdot, x) : \mathbb{R} \to \mathbb{R}$ are Hölder continuous of exponent $\alpha > 0$ and*

$$\begin{aligned} &\sup_{x\in D}[g(\cdot, x)]_{C^\alpha([0,+\infty))} = L_g < \infty, \\ &\sup_{\eta\in\partial D}[\sigma(\cdot, \eta)]_{C^\alpha([0,+\infty))} = L_\sigma < \infty. \end{aligned} \tag{5.2}$$

From Hypothesis 4, we easily obtain that the mapping $\hat{F}(t,\cdot): H_\mu \to \mathbb{R}$ is Fréchet differentiable and, for any $t \geq 0$ and $h, k \in H_\mu$, we have

$$D\hat{F}(t,h)k = \int_D \frac{\partial f}{\partial \xi}(t,x,h(x))k(x)\mu(dx) = \left\langle \frac{\partial f}{\partial \xi}(t,\cdot,h)k, \mu \right\rangle.$$

Moreover, $D\hat{F}(t,\cdot): H_\mu \to H$ is Lipschitz continuous, uniformly for $t \in [0,T]$.

THEOREM 5.1. *Assume Hypotheses 1–4. Then, for any $t > 0$,*

$$z_\varepsilon(t,x) \rightharpoonup I_0(t,x), \qquad \varepsilon \downarrow 0, \tag{5.3}$$

in H_μ, where $I_0(t,x)$ is the Gaussian random field defined for any $t > 0$ and $x \in D$ by

$$\begin{aligned} I_0(t,x) := &\int_0^\infty \Pi e^{sA} G(t)\, dw^Q(s,x) \\ &+ \int_0^\infty \Pi(\delta_0 - A) e^{sA} N_{\delta_0}[\Sigma(t)\, dw^B(s)](x). \end{aligned} \tag{5.4}$$

[*For any $x \in H_\mu$, we have set $\Pi x := x - \langle x, \mu \rangle$. Notice that, due to the invariance of μ,*

$$\Pi e^{tA} h = e^{tA} \Pi h, \qquad t \geq 0, h \in H_\mu, \qquad \Pi A h = A \Pi h, \qquad h \in D(A).]$$

We now define

$$I_G(t) := \int_0^\infty \Pi e^{sA} G(t)\, dw^Q(s) \tag{5.5}$$

and

$$I_\Sigma(t) := \int_0^\infty \Pi(\delta_0 - A) e^{sA} N_{\delta_0}[\Sigma(t)\, dw^B(s)]. \tag{5.6}$$

Before proceeding with the proof of Theorem 5.1, it is important to see that the two terms $I_G(t)$ and $I_\Sigma(t)$ are both well defined in $L^2(\Omega; H_\mu)$ for any $t \geq 0$.

LEMMA 5.2. *Under Hypotheses 1–3,*

$$\mathbb{E}|I_G(t)|^2_{H_\mu} < \infty, \qquad t \geq 0.$$

PROOF. Due to the invariance of μ, we have

$$I_G(t) = \sum_{k=1}^\infty \int_0^\infty \lambda_k \big(e^{sA}[G(t)e_k] - \langle G(t)e_k, \mu \rangle\big)\, d\beta_k(s)$$

so that, by proceeding as in the proof of Lemma 3.3, thanks to (2.12), we have

$$\begin{aligned}\mathbb{E}|I_G(t)|^2_{H_\mu} &= \int_0^\infty \sum_{k=1}^\infty \lambda_k^2 |e^{sA}[G(t)e_k] - \langle G(t)e_k, \mu\rangle|^2_{H_\mu}\, ds \\ &\le c\int_0^\infty \left(\sum_{k=1}^\infty \left|e^{sA}\big([G(t)e_k] - \langle G(t)e_k, \mu\rangle\big)\right|^2_{H_\mu}\right)^{1/\zeta} \\ &\qquad \times \sup_{k\in\mathbb{N}} |e^{sA}[G(t)e_k] - \langle G(t)e_k, \mu\rangle|^{2(\zeta-1)/\zeta}_{H_\mu} |e_k|^{-4/\rho}_\infty\, ds,\end{aligned} \tag{5.7}$$

where $\zeta = (\rho-2)/\rho$ and ρ is the constant appearing in (2.12). Due to (2.6) and the invariance of μ, we have

$$\left|e^{sA}\big([G(t)e_k] - \langle G(t)e_k, \mu\rangle\big)\right|^2_{H_\mu} \le e^{-\gamma s}|e^{s/2A}[G(t)e_k]|^2_{H_\mu}$$

so that, according to (3.16), we have

$$\left(\sum_{k=1}^\infty \left|e^{sA}\big([G(t)e_k] - \langle G(t)e_k, \mu\rangle\big)\right|^2_{H_\mu}\right)^{1/\zeta} \le c_t e^{-\gamma s/\zeta}\big(s^{-d/(2\zeta)} + 1\big).$$

Analogously, according to (3.17), we have

$$\sup_{k\in\mathbb{N}} |e^{sA}[G(t)e_k] - \langle G(t)e_k, \mu\rangle|^{2(\zeta-1)/\zeta}_{H_\mu} |e_k|^{-4/\rho}_\infty \le c_t e^{-\gamma(\zeta-1)s/\zeta}$$

and hence, in view of (5.7), we conclude that

$$\mathbb{E}|I_G(t)|^2_{H_\mu} \le c_t \int_0^\infty e^{-\gamma s}\big(s^{-d/(2\zeta)} + 1\big)\, ds \le c_t. \qquad \square$$

As far as I_Σ is concerned, we have the following, analogous, result.

LEMMA 5.3. *Under Hypotheses 1–3*

$$\mathbb{E}|I_\Sigma(t)|^2_{H_\mu} < \infty, \qquad t \ge 0.$$

PROOF. Due to the invariance of μ, we have

$$I_\Sigma(t) = \sum_{k=1}^\infty \int_0^\infty \theta_k\big((\delta_0 - A)e^{sA}N_{\delta_0}[\Sigma(t)f_k] - \delta_0\langle N_{\delta_0}[\Sigma(t)f_k], \mu\rangle\big)\, d\hat{\beta}_k(s).$$

Using the same arguments used in Lemma 4.4, due to (2.6) and the invariance of μ, we have

$$\begin{aligned}&|(\delta_0 - A)e^{sA}N_{\delta_0}[\Sigma(t)f_k] - \delta_0\langle N_{\delta_0}[\Sigma(t)f_k], \mu\rangle|^2_{H_\mu} \\ &\qquad \le ce^{-\gamma s}|(\delta_0 - A)e^{s/2A}N_{\delta_0}[\Sigma(t)f_k]|^2_{H_\mu}\end{aligned}$$

and then, as in the proof of Lemma 3.1, due to (2.13), we get

$$\mathbb{E}|I_\Sigma(t)|^2_{H_\mu} \le c\int_0^\infty e^{-\gamma s}\left(\sum_{k=1}^\infty |(\delta_0 - A)e^{s/2A}N_{\delta_0}[\Sigma(t)f_k]|^2_{H_\mu}\right)^{1/\zeta}$$
$$\times \sup_{k\in\mathbb{N}} |(\delta_0 - A)e^{s/2A}N_{\delta_0}[\Sigma(t)f_k]|^{2(\zeta-1)/\zeta}_{H_\mu}\,ds.$$

By using (3.10) and (3.11), this allows us to conclude that for some $\bar\rho > 0$ such that $(d+\zeta)/2\zeta + \bar\rho/2 < 1$,

$$\mathbb{E}|I_\Sigma(t)|^2_{H_\mu} \le c_t\int_0^\infty e^{-\gamma s}\big(s^{-(d+\zeta)/(2\zeta)+\bar\rho/2} + 1\big)\,ds < +\infty. \qquad \square$$

5.1. *Proof of Theorem* 5.1. It is immediate to check that for any $t \ge 0$,

$$z_\varepsilon(t) = \int_0^t D\hat F(s, v(s))z_\varepsilon(s)\,ds + R_\varepsilon(t) + I_\varepsilon(t),$$

where

$$\begin{aligned} R_\varepsilon(t) :=& \frac{1}{\sqrt\varepsilon}(e^{t/\varepsilon A}u_0 - \langle u_0, \mu\rangle) \\ &+ \frac{1}{\sqrt\varepsilon}\int_0^t \big(e^{(t-s)A/\varepsilon}F(s, u_\varepsilon(s)) - \hat F(s, u_\varepsilon(s))\big)\,ds \\ &+ \int_0^t\int_0^1 \big[D\hat F\big(s, v(s) + \theta\big(u_\varepsilon(s) - v(s)\big)\big) - D\hat F(s, v(s))\big]z_\varepsilon(s)\,ds\,d\theta \\ =:& \sum_{i=1}^3 R_{\varepsilon,i}(t) \end{aligned}$$

and

$$I_\varepsilon(t) := \frac{1}{\sqrt\varepsilon}\big(w^\varepsilon_{A,Q}(t) - \hat w_{A,Q}(t)\big) + \frac{1}{\sqrt\varepsilon}\big(w^\varepsilon_{A,B}(t) - \hat w_{A,B}(t)\big). \tag{5.8}$$

Due to (1.2), we have

$$|R_{\varepsilon,1}(t)|_{H_\mu} \le \frac{c}{\sqrt\varepsilon}e^{-\gamma t/\varepsilon}|u_0|_{H_\mu}. \tag{5.9}$$

For $R_{\varepsilon,2}(t)$, with a change of variables, due to (2.6), we have, for any $t \in [0, T]$,

$$\begin{aligned} |R_{\varepsilon,2}(t)|_{H_\mu} &\le \frac{c}{\sqrt\varepsilon}\int_0^t e^{-\gamma(t-s)/\varepsilon}|F(s, u_\varepsilon(s))|_{H_\mu}\,ds \\ &\le \frac{c_t}{\sqrt\varepsilon}\Big(1 + \sup_{s\in[0,t]}|u_\varepsilon(s)|_{H_\mu}\Big)\int_0^t e^{-\gamma s/\varepsilon}\,ds \\ &\le c_t\sqrt\varepsilon\Big(1 + \sup_{s\in[0,t]}|u_\varepsilon(s)|_{H_\mu}\Big) \end{aligned}$$

and then, thanks to (3.18), we get

$$\mathbb{E} \sup_{t\in[0,T]} |R_{\varepsilon,2}(t)|_{H_\mu} \le c_T\sqrt{\varepsilon}(1+|u_0|_{H_\mu}). \tag{5.10}$$

Finally, for $R_{\varepsilon,3}(t)$, due to the Lipschitz continuity of $D\hat{F}(s,\cdot): H_\mu \to H$, uniform with respect to $s \in [0,t]$, and estimate (4.19) with $p=2$ and $\theta \in (1/2,1)$, we get

$$\begin{aligned}\mathbb{E}|R_{\varepsilon,3}(t)|_{H_\mu} &\le \frac{c_t}{\sqrt{\varepsilon}}\int_0^T \mathbb{E}|u_\varepsilon(s)-v(s)|^2_{H_\mu}\,ds \\ &\le c_T\varepsilon^{\theta-1/2}(1+|u_0|^2).\end{aligned} \tag{5.11}$$

Therefore, collecting together (5.9), (5.10) and (5.11), we can conclude that for any $T>0$ and $\varepsilon \in (0,1]$,

$$\mathbb{E}|R_\varepsilon(t)|_{H_\mu} \le \frac{c}{\sqrt{\varepsilon}}e^{-\gamma t/\varepsilon}|u_0|_{H_\mu} + c_T(1+|u_0|^2_{H_\mu})\varepsilon^{\theta-1/2}, \qquad t\in[0,T]. \tag{5.12}$$

Next, for any $\varepsilon>0$, we introduce the problem

$$\zeta(t) = \int_0^t D\hat{F}(s,v(s))\zeta(s)\,ds + I_\varepsilon(t),$$

where $I_\varepsilon(t)$ is the process introduced in (5.8). For any $\varepsilon>0$, we denote by ζ_ε its unique solution.

We have the following result, whose proof is postponed to the end of this section.

LEMMA 5.4. *Under Hypotheses 1–4, for any $t>0$, we have*

$$\zeta_\varepsilon(t) \rightharpoonup I_0(t), \qquad \varepsilon \downarrow 0,$$

in H_μ, where $I_0(t)$ is the H_μ-valued Gaussian vector field defined in (5.4).

Now, for any $\varepsilon>0$ and $t>0$, we define $\rho_\varepsilon(t) := z_\varepsilon(t) - \zeta_\varepsilon(t)$. We have

$$\rho_\varepsilon(t) = \int_0^t D\hat{F}(s,v(s))\rho_\varepsilon(s)\,ds + R_\varepsilon(t)$$

so that

$$\mathbb{E}|\rho_\varepsilon(t)|_{H_\mu} \le c_T\int_0^t \mathbb{E}|\rho_\varepsilon(s)|_{H_\mu} + \mathbb{E}|R_\varepsilon(t)|_{H_\mu}.$$

By comparison, we get

$$\mathbb{E}|\rho_\varepsilon(t)|_{H_\mu} \le c_T\mathbb{E}|R_\varepsilon(t)|_{H_\mu} + c_T\int_0^t \mathbb{E}|R_\varepsilon(s)|_{H_\mu}\,ds$$

and, thanks to (5.12), this implies that

$$\mathbb{E}|\rho_\varepsilon(t)|_{H_\mu} \le \frac{c_T}{\sqrt{\varepsilon}} e^{-\gamma t/\varepsilon} |u_0|_{H_\mu} + c_T(1 + |u_0|^2_{H_\mu})\varepsilon^{\theta - 1/2} + \frac{c_T}{\sqrt{\varepsilon}} \int_0^t e^{-\gamma s/\varepsilon}\, ds\, |u_0|_{H_\mu}.$$

Hence, we can conclude that for any $t > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{E}|z_\varepsilon(t) - \zeta_\varepsilon(t)|_{H_\mu} = \lim_{\varepsilon \to 0} \mathbb{E}|\rho_\varepsilon(t)|_{H_\mu} = 0$$

so that, in view of Lemma 5.4, Theorem 5.1 is proved.

5.1.1. *Proof of Lemma* 5.4. For any $x \in D$ and $t > 0$, we have

$$\zeta_\varepsilon(t, x) = \int_0^t \int_D \frac{\partial f}{\partial \xi}(s, y, v(s))\zeta_\varepsilon(s, y)\mu(dy)\, ds + I_\varepsilon(t, x).$$

Then, if we multiply both sides above by $\partial f/\partial \xi(t, x, v(t))$ and integrate in x with respect to the measure μ, we get

$$\Psi_\varepsilon(t) = H(t) \int_0^t \Psi_\varepsilon(s)\, ds + K_\varepsilon(t),$$

where

$$\Psi_\varepsilon(t) := \int_D \frac{\partial f}{\partial \xi}(t, x, v(t))\zeta_\varepsilon(t, x)\mu(dx)$$

and

$$H(t) := \int_D \frac{\partial f}{\partial \xi}(t, x, v(t))\mu(dx),$$

$$K_\varepsilon(t) := \int_D \frac{\partial f}{\partial \xi}(t, x, v(t)) I_\varepsilon(t, x)\mu(dx).$$

It is then immediate to check that

$$\int_0^t \Psi_\varepsilon(s)\, ds = \int_0^t \exp\biggl(\int_s^t H(r)\, dr\biggr) K_\varepsilon(s)\, ds$$

so that

$$\zeta_\varepsilon(t, x) = \int_0^t H(t, s) K_\varepsilon(s)\, ds + I_\varepsilon(t, x),$$

where

$$H(t, s) := \exp\biggl(\int_s^t H(r)\, dr\biggr).$$

Step 1. We show that for any $t \geq 0$,

$$\lim_{\varepsilon \to 0} \mathbb{E}\left|\int_0^t H(t,s)K_\varepsilon(s)\,ds\right|^2 = 0. \tag{5.13}$$

Due to (5.8) and the stochastic Fubini theorem, we have

$$\begin{aligned}
&\int_0^t H(t,s)K_\varepsilon(s)\,ds \\
&\quad = \frac{1}{\sqrt{\varepsilon}} \sum_{k=0}^{\infty} \int_0^t \int_\sigma^t H(t,s)\Big\langle \frac{\partial f}{\partial \xi}(s,\cdot,v(s)), \\
&\qquad\qquad e^{(s-\sigma)A/\varepsilon}\Pi[G(\sigma)Qe_k]\Big\rangle_{H_\mu} ds\,d\beta_k(\sigma) \\
&\qquad + \frac{1}{\sqrt{\varepsilon}} \sum_{k=0}^{\infty} \int_0^t \int_\sigma^t H(t,s)\Big\langle \frac{\partial f}{\partial \xi}(s,\cdot,v(s)), \\
&\qquad\qquad (\delta_0 - A)e^{(s-\sigma)A/\varepsilon} \\
&\qquad\qquad \times \Pi[N_{\delta_0}(\Sigma(\sigma)Bf_k)]\Big\rangle_{H_\mu} ds\,d\hat{\beta}_k(\sigma).
\end{aligned}$$

Then, as w^Q and w^B are independent and $\partial f/\partial \xi$ is uniformly bounded, we get

$$\begin{aligned}
&\mathbb{E}\left|\int_0^t H(t,s)K_\varepsilon(s)\,ds\right|^2 \\
&\quad \leq \frac{\kappa_t}{\varepsilon} \int_0^t \sum_{k=0}^{\infty} \left(\int_\sigma^t e^{\kappa_t(t-s)} \left|e^{(s-\sigma)A/\varepsilon}\Pi[G(\sigma)Qe_k]\right|_{H_\mu} ds\right)^2 d\sigma \\
&\qquad + \frac{\kappa_t}{\varepsilon} \int_0^t \sum_{k=0}^{\infty} \left(\int_\sigma^t e^{\kappa_t(t-s)} \left|(\delta_0 - A)e^{(s-\sigma)A/\varepsilon}\right.\right. \\
&\qquad\qquad \left.\left. \times \Pi[N_{\delta_0}(\Sigma(\sigma)Bf_k)]\right|_{H_\mu} ds\right)^2 d\sigma \\
&\quad =: \frac{\kappa_t}{\varepsilon} \int_0^t \big(J_{\varepsilon,1}(t,\sigma) + J_{\varepsilon,2}(t,\sigma)\big)\,d\sigma.
\end{aligned}$$

For the first term $J_{\varepsilon,1}$, in view of (2.6), for any $\alpha \in (0,2)$, we have

$$\begin{aligned}
&J_{\varepsilon,1}(t,\sigma) \\
&\quad \leq \sum_{k=0}^{\infty} \lambda_k^2 \left(\int_\sigma^t e^{\kappa_t(t-s)} e^{-\gamma(s-\sigma)/(2\varepsilon)} \left|e^{(s-\sigma)A/(2\varepsilon)}\Pi[G(\sigma)e_k]\right|_{H_\mu} ds\right)^2 \\
&\quad \leq \left(\int_\sigma^t e^{\kappa_t(2-\alpha)(t-s)} e^{-\gamma(2-\alpha)(s-\sigma)/(2\varepsilon)}\,ds\right)^{2/(2-\alpha)}
\end{aligned}$$

$$\times \sum_{k=0}^{\infty} \lambda_k^2 \Big(\int_\sigma^t |e^{(s-\sigma)A/(2\varepsilon)}[G(\sigma)e_k]|_{H_\mu}^{(2-\alpha)/(1-\alpha)}\, ds \Big)^{2(1-\alpha)/(2-\alpha)}$$

$$\le c_t \varepsilon^{2/(2-\alpha)} \sum_{k=0}^{\infty} \lambda_k^2 \Big(\int_\sigma^t |e^{(s-\sigma)A/(2\varepsilon)}[G(\sigma)e_k]|_{H_\mu}^{(2-\alpha)/(1-\alpha)}\, ds \Big)^{2(1-\alpha)/(2-\alpha)}.$$

Then, if we set $\zeta = \rho/(\rho-2)$, by using the Hölder inequality for infinite series, we get

$$\begin{aligned} &J_{\varepsilon,1}(t,\sigma) \\ &\quad \le c_t \varepsilon^{2/(2-\alpha)} \kappa_Q^{2/\rho} \\ &\qquad \times \Bigg(\int_\sigma^t \Bigg(\sum_{k=0}^{\infty} |e^{(s-\sigma)A/(2\varepsilon)}[G(\sigma)e_k]|_H^{2\zeta} \\ &\qquad\qquad \times |e_k|_\infty^{-4/(\rho-2)} \Bigg)^{1/\zeta(2-\alpha)/(2(1-\alpha))} ds \Bigg)^{2(1-\alpha)/(2-\alpha)} \end{aligned}$$

and, by proceeding as in the proof of Lemma 3.3, we conclude that for $\varepsilon \in (0,1]$,

$$J_{\varepsilon,1}(t,\sigma) \le c_t \varepsilon^{2/(2-\alpha)} \kappa_Q^{2/\rho} \Big(\int_\sigma^t \big((s-\sigma)^{-d/(2\zeta)(2-\alpha)/(2(1-\alpha))} + 1 \big)\, ds \Big)^{2(1-\alpha)/(2-\alpha)}.$$

Now, in view of Hypothesis 3, we have $d/2\zeta < 1$ and can fix $\bar\alpha_1 > 0$ such that

$$\frac{d}{2\zeta} \frac{2-\bar\alpha_1}{2(1-\bar\alpha_1)} < 1$$

and then

$$\frac{\kappa_t}{\varepsilon} \int_0^t J_{\varepsilon,1}(t,\sigma)\, d\sigma \le c_{t,\bar\alpha_1} \varepsilon^{\bar\alpha_1/(2-\bar\alpha_1)}, \qquad \varepsilon \in (0,1], t \ge 0. \tag{5.14}$$

The same arguments can be repeated for the term $J_{\varepsilon,2}$, so we can find some $\bar\alpha_2 > 0$ such that

$$\frac{\kappa_t}{\varepsilon} \int_0^t J_{\varepsilon,2}(t,\sigma)\, d\sigma \le c_{t,\bar\alpha_2} \varepsilon^{\bar\alpha_2/(2-\bar\alpha_2)}, \qquad \varepsilon \in (0,1], t \ge 0.$$

This, together with (5.14), implies that

$$\mathbb{E} \Big| \int_0^t H(t,s) K_\varepsilon(s)\, ds \Big|^2 \le c_t \varepsilon^\gamma, \qquad \varepsilon \in (0,1], t \ge 0,$$

where

$$\gamma = \frac{\bar\alpha_1 \wedge \bar\alpha_2}{2 - \bar\alpha_1 \wedge \bar\alpha_2},$$

so (5.13) follows.

Step 2. We show that for any fixed $t > 0$,

$$I_\varepsilon(t) \rightharpoonup I_0(t), \qquad \varepsilon \downarrow 0. \tag{5.15}$$

With a change of variable, we have

$$\begin{aligned} I_\varepsilon(t) &= \frac{1}{\sqrt{\varepsilon}}\Big(\int_0^t e^{(t-s)A/\varepsilon}\Pi[G(s)\,dw^Q(s)] \\ &\qquad + \int_0^t (\delta_0 - A)e^{(t-s)A/\varepsilon}\Pi[N_{\delta_0}(\Sigma(s)\,dw^B(s))]\Big) \\ &= \int_0^{t/\varepsilon} e^{rA}\Pi[G(t-\varepsilon r)\,dw^Q_{\varepsilon,t}(r)] \\ &\qquad + \int_0^{t/\varepsilon} (\delta_0 - A)e^{rA}\Pi\big[N_{\delta_0}\big(\Sigma(t-\varepsilon r)\,dw^B_{\varepsilon,t}(r)\big)\big], \end{aligned}$$

where

$$w^Q_{\varepsilon,t}(r) = \frac{1}{\sqrt{\varepsilon}}\big(w^Q(t) - w^Q(t-\varepsilon r)\big), \qquad w^B_{\varepsilon,t}(r) = \frac{1}{\sqrt{\varepsilon}}\big(w^B(t) - w^B(t-\varepsilon r)\big).$$

This means that for any $\varepsilon > 0$ and $t > 0$,

$$\mathcal{L}(I_\varepsilon(t)) = \mathcal{L}(\hat{I}_\varepsilon(t)),$$

where

$$\begin{aligned} \hat{I}_\varepsilon(t) &:= \int_0^{t/\varepsilon} e^{rA}\Pi[G(t-\varepsilon r)\,dw^Q(r)] \\ &\qquad + \int_0^{t/\varepsilon} (\delta_0 - A)e^{rA}\Pi\big[N_{\delta_0}\big(\Sigma(t-\varepsilon r)\,dw^B(r)\big)\big]. \end{aligned}$$

Thus, in order to obtain (5.15), it is sufficient to prove

$$\lim_{\varepsilon\to 0} \mathbb{E}|\hat{I}_\varepsilon(t) - I_0(t)|^2_{H_\mu} = 0. \tag{5.16}$$

We have

$$\begin{aligned} \hat{I}_\varepsilon(t) - I_0(t) &= \int_0^{t/\varepsilon} e^{rA}\Pi\big[\big(G(t-\varepsilon r) - G(t)\big)\,dw^Q(r)\big] \\ &\qquad + \int_0^{t/\varepsilon} (\delta_0 - A)e^{rA}\Pi\big[N_{\delta_0}\big(\big(\Sigma(t-\varepsilon r) - \Sigma(t)\big)\,dw^B(r)\big)\big] \\ &\qquad - \int_{t/\varepsilon}^\infty e^{rA}\Pi[G(t)\,dw^Q(r)] \\ &\qquad - \int_{t/\varepsilon}^\infty (\delta_0 - A)e^{rA}\Pi[N_{\delta_0}(\Sigma(t)\,dw^B(r))] \\ &=: \sum_{i=1}^4 J_{\varepsilon,i}(t). \end{aligned}$$

With the same arguments used several times throughout the paper, we have

$$\mathbb{E}|J_{\varepsilon,1}(t)|^2_{H_\mu} \le c\int_0^{t/\varepsilon} e^{-\gamma s}\big(s^{-d/(2\zeta)}+1\big)|g(t-\varepsilon s,\cdot)-g(t,\cdot)|^2_{H_\mu}\,ds.$$

Then, due to Hypothesis 4, we have

$$\mathbb{E}|J_{\varepsilon,1}(t)|^2_{H_\mu} \le c_t\varepsilon^{2\alpha}\int_0^{\infty} e^{-\gamma s}\big(s^{-d/(2\zeta)}+1\big)s^{2\alpha}\,ds \le c_t\varepsilon^{2\alpha}. \tag{5.17}$$

Analogously, we have

$$\mathbb{E}|J_{\varepsilon,2}(t)|^2_{H_\mu} \le c_t\varepsilon^{2\alpha}. \tag{5.18}$$

Concerning $J_{\varepsilon,3}(t)$, we have

$$\begin{aligned}\mathbb{E}|J_{\varepsilon,3}(t)|^2_{H_\mu} &\le c\int_{t/\varepsilon}^{\infty} e^{-\gamma s}\big(s^{-d/(2\zeta)}+1\big)\,ds\,|g(t,\cdot)|^2_{H_\mu}\\ &\le c_t\int_{t/\varepsilon}^{\infty} e^{-\gamma s}\big(s^{-d/(2\zeta)}+1\big)\,ds\end{aligned}$$

so that

$$\lim_{\varepsilon\to 0}\mathbb{E}|J_{\varepsilon,3}(t)|^2_{H_\mu} = 0. \tag{5.19}$$

In an identical way, we can show that

$$\lim_{\varepsilon\to 0}\sup_{t\in[\delta,T]}\mathbb{E}|J_{\varepsilon,4}(t)|^2_{H_\mu} = 0$$

and this, together with (5.17), (5.18) and (5.19), implies (5.16).

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