VOLUME GROWTH AND ESCAPE RATE OF BROWNIAN MOTION ON A COMPLETE RIEMANNIAN MANIFOLD

BY ELTON P. HSU AND GUANGNAN QIN

Northwestern University and Chinese Academy of Sciences

We give an effective upper escape rate function for Brownian motion on a complete Riemannian manifold in terms of the volume growth of the manifold. An important step in the work is estimating the small tail probability of the crossing time between two concentric geodesic spheres by reflecting Brownian motions on the larger geodesic ball.

1. Introduction. Let M be a Riemannian manifold and $p_M(t, x, y)$ the (minimal) heat kernel on M. By definition, the latter is the fundamental solution of the heat operator

$$\mathscr{L}_M = \frac{\partial}{\partial t} - \frac{1}{2}\Delta_M,$$

where Δ_M is the Laplace–Beltrami operator on M. A Riemannian manifold is stochastically complete if

$$\int_M p_M(t, x, y) \, dy = 1$$

for some, hence all, $(x, t) \in M \times (0, \infty)$. In other words, M is stochastically complete if the heat kernel is conserved. Let $\mathbb{P}_x$ be the law of Brownian motion on M starting from x and let e be the lifetime (or explosion time) of the Brownian motion. We then have

$$\mathbb{P}_x\{e > t\} = \int_M p(t, x, y) \, dy.$$

Therefore, M is stochastically complete if and only if

(1.1) $$\mathbb{P}_x\{e = \infty\} = 1,$$

that is, Brownian motion on M does not explode. Finding geometric conditions for stochastic completeness is an old geometric problem. The problem has been attacked using both analytic and probabilistic methods. Early works on this problem (Karp and Li [9], Yau [16], Hsu [6] and Varopoulos [15]) impose lower bounds on the Ricci curvature. In particular, in the last two works, it was shown that if there is

a strictly positive function $\kappa(r)$ such that the Ricci curvature of M on the geodesic ball $B(r)$ is bounded from below by $-\kappa(r)$ and

$$\int_0^\infty \frac{dr}{\sqrt{\kappa(r)}} = \infty,$$

then M is stochastically complete. In 1986, Grigor'yan [1] found the following sufficient condition for stochastic completeness solely in terms of the volume growth function of the manifold:

$$(1.2) \qquad \int_1^\infty \frac{r\,dr}{\ln|B(r)|} = \infty.$$

According to (1.1), a Riemannian manifold M is stochastically complete if Brownian motion does not escape to infinity (in the one-point compactification) in finite time. Traditionally in probability theory (see, e.g., Itô and McKean [8] and Shiga and Watanabe [12]), one often looks for upper functions for the escape rate of a diffusion process. The classical Khinchin law of iterated logarithm for one-dimensional standard Brownian motion is the most celebrated case. More generally, let $r_t = d(X_t, x)$ be the radial process of Brownian motion X on M. An increasing function $\psi(t)$ is called an upper rate function if

$$\mathbb{P}_x\{r_t \le \psi(t) \text{ for all sufficiently large } t\} = 1.$$

Finding an upper rate function is a more refined problem than proving stochastic completeness since the existence of an upper rate function implies stochastic completeness. Various explicit upper rate function for Brownian motion on a complete Riemannian manifold have been obtained under concrete volume growth assumptions (see Grigor'yan [2], Grigor'yan and Kelbert [4] and Takeda [13, 14]). More recently, Grigor'yan and Hsu [3] showed that the inverse function of the increasing function

$$\phi_1(R) = \int_1^R \frac{r\,dr}{\ln|B(r)|}$$

related to the integral test (1.2) for stochastic completeness is essentially an upper rate function for Brownian motion on M. While this result gives an upper rate function of a very general form, it was proven under the additional geometric assumption that M is a Cartan–Hadamard manifold, that is, a simply connected, geodesically complete Riemannian manifold with nonpositive sectional curvature.

The main purpose of the present work is to obtain an escape rate function based solely on the volume growth of the underlying manifold. We introduce the following increasing function:

$$\phi(R) = \int_6^R \frac{r\,dr}{\ln|B(r)| + \ln\ln r}.$$

We will show in Theorem 4.1 that under the sole assumption that M is a complete Riemannian manifold, the inverse function of ϕ is essentially an upper rate function for Brownian motion on M.

REMARK 1.1. The difference between the functions $\phi_1(r)$ and $\phi(r)$ is that we have introduced an extra term $\ln \ln r$ in the latter function. This addition, resulting from an attempt to remove the extraneous geometric condition, is fully justified on several grounds. First, the integral test (1.2) for stochastic completeness, which can be written as $\phi_1(\infty) = \infty$, is equivalent to the condition $\phi(\infty) = \infty$. Second, our new upper rate function implies all explicit upper rate functions existent in the literature to date (see Corollaries 4.2 and 4.3). Third, in general, the radial process of a Brownian motion on M has the form

$$r_t = \beta_t + \frac{1}{2} \int_0^t \Delta_M r(X_t) \, dt - L_t,$$

where β is a standard one-dimensional Brownian motion and L is a local time on the cut locus $C(x)$ of the point x (see Hsu [7] and Kendall [10]). In the absence of any further geometric assumptions, we do not expect to obtain an upper rate function (up to a multiplicative constant) for the process r_t better than the upper rate function of a standard Brownian motion $\psi(t) = C\sqrt{t \ln \ln t}$. This rate function cannot be achieved without the presence of the additional term $\ln \ln r$ in the function ϕ.

Our method has two key steps. In the first, we follow Hsu [6] and Grigor'yan and Hsu [3] and, by using the Borel–Cantelli lemma, reduce the problem of seeking an upper rate function to the problem of estimating the small tail probability of the crossing time between two concentric geodesic spheres (Lemma 2.1). In the second key step, instead of estimating the small tail probability by using an analytic approach, as in Grigor'yan and Hsu [3], under the assumption that the manifold in question is Cartan–Hadamard, we modify the way the Lyons–Zheng decomposition for reflecting Brownian motion is used in Takeda [13, 14]. The volume $|B(r)|$ of the geodesic ball $B(r)$ appears naturally in this step because the uniform distribution on the ball with respect to the Riemannian volume measure is the invariant measure of reflecting Brownian motion on $B(r)$. The additional term mentioned above, $\ln \ln r$, is a consequence of dealing with the Brownian motion adapted to the time-reversed filtration in the Lyons–Zheng decomposition.

2. Basic estimates on crossing times. Let M be a geodesically complete Riemannian manifold and $\mathscr{P}(M)$ the path space over M. Let X be the canonical coordinate process on the path space $\mathscr{P}(M)$ over M, that is, $X_t(\omega) = \omega_t$ for $\omega \in \mathscr{P}(M)$. If $x \in M$, then we use $\mathbb{P}_x$ to denote the law of Brownian motion on M starting from x. The radial process is $r_t = d(X_t, x)$, that is, the Riemannian distance from x to X_t, the position of Brownian motion at time t. A nonnegative increasing function $R: \mathbb{R}_+ \to \mathbb{R}_+$ is called an upper rate function for Brownian motion on M if

$$\mathbb{P}_x\{r_t \leq R(t) \text{ for all sufficiently large } t\} = 1.$$

Let $\{R_n\}$ be a strictly increasing sequence of positive numbers, to be chosen later, and define a sequence of stopping times as follows:

$$\tau_n = \inf\{t : r_t = R_n\}.$$

(We adopt the convention that $\inf \varnothing = \infty$.) Each of these is the first time the Brownian motion X reaches the corresponding geodesic sphere,

$$S(R_n) = \{x \in M : d(x, o) = R_n\}.$$

The difference $\tau_n - \tau_{n-1}$, if well defined, is the amount of time Brownian motion takes to cross from $S(R_{n-1})$ to $S(R_n)$. The basic idea of Grigor'yan and Hsu [3] (see also Hsu [6]) for controlling the rate of escape of Brownian motion is to give a good upper bound for the small tail probability $\mathbb{P}_x\{\tau_n - \tau_{n-1} \le t_n\}$ for a suitably chosen sequence $\{t_n\}$ of time steps. If the sum of these probabilities converges, then the Borel–Cantelli lemma shows that for sufficiently large n, Brownian motion X has to wait roughly until at least

$$T_n = \sum_{k=1}^{n} t_k$$

to reach the sphere $S(R_n)$ or, equivalently, $r_t \le R_n$ for all $t \le T_n$. This, after some technical manipulations (see Section 4), will give an upper escape rate function.

We now use the idea of Takeda [13, 14] to estimate the small tail probability $\mathbb{P}_x\{\tau_n - \tau_{n-1} \le t_n\}$ by using the Lyons–Zheng decomposition [11] of reflecting Brownian motion starting from the uniform distribution on a geodesic ball. For an open set $B \subset M$, we denote by $\mathbb{P}_B$ the law of Brownian motion starting from the uniform distribution on B, that is,

$$\mathbb{P}_B = \frac{1}{|B|} \int_B \mathbb{P}_x \, dx.$$

Likewise, we use $\mathbb{Q}_B$ to denote the law of reflecting Brownian motion on B starting from the same uniform distribution. Let $B_n = B(R_n)$ be the geodesic ball of radius R_n centered at x. In order to take advantage of the volume growth condition, we consider the probability $\mathbb{P}_{B_1}\{\tau_n - \tau_{n-1} \le t_n\}$ instead of $\mathbb{P}_x\{\tau_n - \tau_{n-1} \le t_n\}$. Recall that τ_n is the first time the process X reaches the boundary $S(R_n)$ of the geodesic ball $B(R_n)$. Before reaching the boundary, Brownian motion and reflecting Brownian motion have the same law. Therefore, if $C \in \mathscr{B}_{\tau_n}$ is an event which is measurable up to time τ_n, then $\mathbb{P}_{B_n}(C) = \mathbb{Q}_{B_n}(C)$. From

$$\mathbb{P}_{B_1}(C) = \frac{1}{|B_1|} \int_{B_1} \mathbb{P}_x(C) \, dx \le \frac{1}{|B_1|} \int_{B_n} \mathbb{P}_x(C) \, dx = \frac{|B_n|}{|B_1|} \mathbb{P}_{B_n}(C),$$

we have

(2.1) $$\mathbb{P}_{B_1}(C) \le \frac{|B_n|}{|B_1|} \mathbb{Q}_{B_n}(C), \qquad C \in \mathscr{B}_{\tau_n}.$$

We apply this inequality to the event

(2.2) $$C = \{\tau_n - \tau_{n-1} \le t_n\}.$$

Now, according to the Lyons–Zheng decomposition [11], on a fixed time horizon $[0, T_n]$, the radial process can be decomposed as the difference

(2.3) $$r_t - r_0 = \frac{B_t}{2} - \frac{\tilde{B}_{T_n} - \tilde{B}_t}{2},$$

where B is a standard Brownian motion adapted to the natural filtration $msb_* = \mathscr{B}(\mathscr{P}(M))_*$ of the path space $\mathscr{P}(M)$ and $\tilde{B}$ is also a standard Brownian motion, but adapted to the reversed filtration $\tilde{\mathscr{B}}_*$ defined by

$$\tilde{\mathscr{B}}_t = \sigma\{X_{T_n - s} : 0 \le s \le t\}, \qquad 0 \le t \le T_n.$$

The advantage of such a decomposition is obvious, for we have eliminated from consideration the bounded variation component of the radial process, which can be rather complicated. The price is that we have to deal with a Brownian motion not adapted to the original filtration. Another complication is that the decomposition cannot be applied directly to the event (2.2) because it may go beyond the fixed time horizon $[0, T_n]$. In order to remedy this situation, we will use a slightly modified event

$$C_n = \{\tau_n - \tau_{n-1} \le t_n, \tau_n \le T_n\}$$

in the inequality (2.1). Fortunately, this additional restriction $\{\tau_n \le T_n\}$ will not be an obstacle for us, as shown by the following simple observation.

LEMMA 2.1.   *Let C_n be defined as above. Suppose that $\sum_{n=1}^{\infty} \mathbb{P}(C_n) < \infty$. Then, with probability 1, there exists T_{-1} such that $\tau_n \ge T_n - T_{-1}$ for all n.*

PROOF.   By the Borel–Cantelli lemma, the probability that the events $\{C_n\}$ happen infinitely often is 0. Therefore, with probability 1, there exists n_0 such that for all $n \ge n_0$, either $\tau_n - \tau_{n-1} \ge t_n$ or $\tau_n \ge T_n$. We show, by induction, that $\tau_n \ge T_n - T_{n_0}$ holds for all n. If $1 \le n \le n_0$, then $\tau_n \ge 0 \ge T_n - T_{n_0}$. Suppose that $\tau_n \ge T_n - T_0$ for an $n \ge n_0$. If $\tau_{n+1} \ge T_{n+1}$, then, trivially, $\tau_{n+1} \ge T_{n+1} - T_{n_0}$. Otherwise, $\tau_{n+1} - \tau_n \ge t_{n+1}$ and

$$\tau_{n+1} = \tau_{n+1} - \tau_n + \tau_n \ge t_{n+1} + T_n - T_{n_0} = T_{n+1} - T_{n_0}.$$

This completes the proof.   □

We now prove the main estimate for the crossing time $\tau_n - \tau_{n-1}$.

PROPOSITION 2.2.   *Let τ_n be the first hitting time of the sphere $S(R_n)$ and $r_n = R_n - R_{n-1}$. There then exists a constant C such that*

$$\mathbb{P}_{B_1}\{\tau_n - \tau_{n-1} \le t_n, \tau_n \le T_n\} \le \frac{|B_n|}{|B_1|} \frac{C}{\sqrt{\pi t_n}} \frac{T_n}{r_n} e^{-r_n^2 / 8t_n}.$$

PROOF.   The event $\{\tau_n - \tau_{n-1} \leq t_n\}$ implies the event

(2.4) $$\left\{ \sup_{0 \leq s \leq t_n} (r_{\tau_{n-1}+s} - r_{\tau_{n-1}}) \geq r_n \right\}.$$

Now, from the decomposition (2.3), we have

$$2(r_{\tau_{n-1}+s} - r_{\tau_{n-1}}) = B_{\tau_{n-1}+s} - B_{\tau_{n-1}} + \tilde{B}_{\tau_{n-1}+s} - \tilde{B}_{\tau_{n-1}}.$$

Since τ_{n-1} is a stopping time with respect to the natural filtration $\mathscr{B}_*$, the first term on the right-hand side is a Brownian motion in time s starting from 0. This is not so for the second term because τ_{n-1} is not a stopping time with respect to the filtration $\tilde{\mathscr{B}}_*$ of the reversed process. However, for any $s \leq t_n$ such that $\tau_{n-1} \leq T_n$, taking k such that $(k-1)t_n \leq \tau_{n-1} \leq kt_n$, we see that both τ_{n-1} and $\tau_{n-1} + s$ lie in the interval $[(k-1)t_n, (k+1)t_n]$. From

$$r_{\tau_{n-1}+s} - r_{\tau_{n-1}} = r_{\tau_{n-1}+s} - r_{kt_n} + r_{kt_n} - r_{\tau_{n-1}},$$

the event (2.4) is contained in the union of the $[T_n/t_n] + 1$ events

$$\left\{ \sup_{|s| \leq t_n} |r_{kt_n+s} - r_{kt_n}| \geq \frac{r_n}{2} \right\}, \qquad 1 \leq k \leq \left[\frac{T_n}{t_n} \right] + 1.$$

Using

$$r_{kt_n+s} - r_{kt_n} = \frac{B_{kt_n+s} - B_{kt_n}}{2} + \frac{\tilde{B}_{kt_n+s} - \tilde{B}_{kt_n}}{2},$$

we see that the event $\{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\}$ is also contained in the union of the following $2[T_n/t_n] + 2$ events:

$$\left\{ \sup_{|s| \leq t_n} |B_{kt_n+s} - B_{kt_n}| \geq \frac{r_n}{2} \right\}$$

and

$$\left\{ \sup_{|s| \leq t_n} |\tilde{B}_{kt_n+s} - \tilde{B}_{kt_n}| \geq \frac{r_n}{2} \right\}$$

for $1 \leq k \leq [T_n/t_n] + 1$. Under the probability $\mathbb{Q}_{B_n}$, these events have the same probability,

$$\mathbb{P}\left\{ \sup_{|s| \leq t_n} |B_{t_n+s} - B_{t_n}| \geq \frac{r_n}{2} \right\} \leq 2\mathbb{P}\left\{ \sup_{0 \leq s \leq t_n} |B_s| \geq \frac{r_n}{2} \right\} \leq \frac{C\sqrt{t_n}}{r_n} e^{-r_n^2/8t_n}.$$

The probability $\mathbb{Q}_{B_n}\{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\}$ is bounded from above by $2[T_n/t_n] + 2 \leq 4T_n/t_n$ times the above probability. The desired inequality now follows immediately from this and the inequality [see (2.1)]

$$\mathbb{P}_{B_1}\{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\} \leq \frac{|B_n|}{|B_1|} \mathbb{Q}_{B_n}\{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\}. \qquad \square$$

3. Total crossing time. In the preceding section, we have found an upper bound for the probability $\mathbb{P}_{B_1}\{\tau_n - \tau_{n-1} \le t_n, \tau_n \le T_n\}$. We are still free to choose the upper bounds t_n of the crossing times $\tau_n - \tau_{n-1}$ and the radii R_n of the expanding geodesic balls $B(R_n)$. We need to choose them so that the series

$$\sum_{n=1}^{\infty} \mathbb{P}_{B_1}\{\tau_n - \tau_{n-1} \le t_n, \tau_n \le T_n\}$$

$$\le \frac{C}{|B_1|} \sum_{n=1}^{\infty} \frac{T_n}{\sqrt{t_n} r_n} \exp\left[\ln|B(R_n)| - \frac{r_n^2}{8t_n}\right]$$

$$\le \frac{C_1}{|B_1|} \sum_{n=1}^{\infty} \frac{T_n}{r_n^2} \exp\left[\ln|B(R_n)| - \frac{r_n^2}{16t_n}\right]$$

converges and the Borel–Cantelli lemma can be applied. The obvious choice is for t_n equal to a small multiple of $r_n^2/\ln|B(R_n)|$, as was adopted in Grigor'yan and Hsu [3]. However, this choice will not enable us to eliminate the extra factor T_n/r_n^2, whose presence can be traced back to the Brownian motion $\tilde{B}$ adapted to the reverse filtration $\tilde{\mathscr{B}}_*$ in the Lyons–Zheng decomposition (2.3). We diminish the obvious choice by letting

$$(3.1) \qquad t_n = \frac{1}{32} \frac{r_n^2}{\ln|B(R_n)| + h(R_n)}$$

with a strictly increasing function h to be determined. If we assume, without loss of generality, that $B(R_1) \ge 1$ and $h(R_1) \ge 1$, then $t_n \le r_n^2/32$. If we further assume that the sequence $\{r_n\}$ is increasing, then there is an obvious bound

$$32T_n \le \sum_{k=1}^{n} r_k^2 \le \sum_{k=1}^{n} r_k r_n = R_n r_n.$$

It follows that

$$\mathbb{P}_{B_1}\{\tau_n - \tau_{n-1} \le t_n, \tau_n \le T_n\} \le \frac{C_2}{|B_1|^2} \frac{R_n}{r_n} e^{-2h(R_n)}.$$

It remains to choose the radii R_n and the function h such that

$$(3.2) \qquad \sum_{n=1}^{\infty} t_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{R_n}{r_n} e^{-2h(R_n)} < \infty$$

under the integral condition

$$(3.3) \qquad \int_1^{\infty} \frac{r\,dr}{\ln|B(r)|} = \infty.$$

The divergence of the above integral is to be linked to the divergence of the total crossing time in (3.2). This leads to the natural requirement that

$$r_n^2 \ge C R_n(R_{n+1} - R_n).$$

This requirement can be fulfilled by setting $R_n = 2^n$ with $C = 1/4$. From (3.1), we have

$$(3.4) \quad T_n = \sum_{k=1}^{n} t_k = \frac{1}{128} \sum_{k=1}^{n} \frac{R_k(R_{k+1} - R_k)}{\ln|B(R_k)| + h(R_k)} \geq \frac{1}{256} \int_{R_1}^{R_{n+1}} \frac{r\,dr}{\ln|B(r)| + h(r)},$$

which seems to fall slightly short of the condition (3.3). The apparently disadvantageous situation can be salvaged by first looking at a typical candidate for the function h. From our choice of $R_n = 2^n$, we have $R_n/r_n = 2$ and the convergence of the total probability in (3.2) becomes

$$\sum_{n=1}^{\infty} e^{-2h(2^n)} < \infty.$$

This leads to the choice $h(R) = \ln\ln R$. We have the following simple observation.

LEMMA 3.1. *Let f be a positive, nondecreasing and continuous function on* $[0, +\infty)$ *such that*

$$(3.5) \qquad \int_3^{\infty} \frac{r\,dr}{f(r)} = \infty.$$

Then

$$\int_3^{\infty} \frac{r\,dr}{f(r) + \ln\ln r} = \infty.$$

PROOF. Divide the integral into the sum of the integrals over the intervals $[n-1, n]$ for $n \geq 4$. Since f is increasing, we have

$$\int_3^{\infty} \frac{r\,dr}{f(r) + \ln\ln r} \geq \sum_{n=4}^{\infty} \frac{n-1}{f(n) + \ln\ln n}$$

$$\geq \frac{1}{2} \sum_{f(n) \geq \ln\ln n} \frac{n-1}{f(n)} + \frac{1}{2} \sum_{f(n) < \ln\ln n} \frac{n-1}{\ln\ln n}.$$

Since $(n-1)/\ln\ln n \geq 1$ for all sufficiently large n, if the second sum has infinitely many terms, then it is clearly diverges; otherwise, $f(n) \geq \ln\ln n$ for all sufficiently large n and we have, for some n_0,

$$\int_3^{\infty} \frac{r\,dr}{f(r) + \ln\ln r} \geq \frac{1}{2} \sum_{n=n_0}^{\infty} \frac{n-1}{f(n)}$$

$$\geq \frac{1}{2} \sum_{n=n_0}^{\infty} \frac{n-1}{n+1} \int_n^{n+1} \frac{r\,dr}{f(r)}$$

$$\geq \frac{1}{2} \frac{n_0 - 1}{n_0 + 1} \int_{n_0}^{\infty} \frac{r\,dr}{f(r)}.$$

This completes the proof.   □

We are now in a position to bound the range of Brownian motion on a finite time interval.

PROPOSITION 3.2.   *Let $R_n = 2^n$ and*

$$T_n = \frac{1}{128} \sum_{k=1}^{n} \frac{R_k (R_{k+1} - R_k)}{\ln|B(R_k)| + h(R_k)}.$$

Then, with probability 1, there exists T_{-1} such that $\sup_{0 \leq t \leq T_n - T_{-1}} r_t \leq 2^n$ for all n.

PROOF.   By our choice of R_n,

$$\sum_{n=1}^{\infty} \mathbb{P}_{B_1} \{\tau_n - \tau_{n-1} \leq t_n, \tau_n \leq T_n\} < \infty.$$

By the Borel–Cantelli lemma and Lemma 2.1, with probability 1, there exists T_{-1} such that $\tau_n \geq T_n - T_{-1}$. However, τ_n is the hitting time of the sphere $S(2^n)$, hence $\sup_{0 \leq t \leq T_n - T_{-1}} r_t \leq 2^n$ for sufficiently large n.   □

An easy consequence of the above result is a probabilistic proof of Grigor'yan's criterion for stochastic completeness.

COROLLARY 3.3 (Grigor'yan [1]).   *Suppose that M is a complete Riemannian manifold and $B(R)$ its geodesic ball of radius R centered at a fixed point. If*

$$\int_{1}^{\infty} \frac{r \, dr}{\ln|B(r)|} = \infty,$$

then M is stochastically complete.

PROOF.   By Lemma 3.1, under the assumption, we have

$$T_n \geq \frac{1}{256} \int_{R_1}^{R_{n+1}} \frac{r \, dr}{\ln|B(r)| + h(r)} \to \infty$$

as $n \to \infty$. By the above proposition, $\sup_{t \leq T} r_t < \infty$ for all finite T. Hence, Brownian motion does not explode and M is stochastic complete.   □

4. Upper rate function.   Proposition 3.2 allows us to obtain an upper rate function in terms of the volume growth function $|B(r)|$, as was similarly done in Grigor'yan and Hsu [3].

Let

$$\phi(R) = \int_{6}^{R} \frac{r \, dr}{\ln|B(r)| + \ln \ln r}.$$

From (3.4), we have $(1/256)\phi(2^{n+1}) \leq T_n$. Proposition 3.2 then gives

$$\sup_{t \leq (1/256)\phi(2^{n+1}) - T_{-1}} r_t \leq 2^n$$

for all $n \geq 1$. This implies that

$$\sup_{t \leq (1/256)\phi(R) - T_{-1}} r_t \leq 2R$$

for all $R \geq 0$. Denote by ψ the unique inverse function of ϕ. Letting $R = \psi(256(T + T_{-1}))$ in the above inequality, we have

$$\sup_{t \leq T} r_t \leq 2\psi\big(256(T + T_{-1})\big) \leq 512\psi(512T)$$

for all sufficiently large T. This shows that $512\psi(512t)$ is an upper rate function of Brownian motion on M under the probability $\mathbb{P}_{B_1}$. The technical point of passing from the average probability $\mathbb{P}_{B_1}$ to the pointwise probability $\mathbb{P}_x$ is taken care of in the proof of the our main theorem below.

THEOREM 4.1.   *Let M be a complete Riemannian manifold and let $x \in M$. Let $B(R)$ be the geodesic ball on M of radius R and centered at z. Define*

$$\phi(R) = \int_6^R \frac{r \, dr}{\ln|B(r)| + \ln\ln r}$$

and let ψ be the inverse function of ϕ. There then exists a constant C such that $C\psi(Ct)$ is an upper rate function of Brownian motion X on M, that is,

$$\mathbb{P}_x\{d(X_t, x) \leq C\psi(Ct) \text{ for all sufficiently large } t\} = 1.$$

PROOF.   Let

$$H = \{d(X_t, X_0) \leq C\psi(Ct) \text{ for all sufficiently large } t\}.$$

We have shown that $\mathbb{P}_{B_1}(H) = 1$. This shows that $C\psi(Ct)$ is an upper rate function for Brownian motion on M starting from the uniform distribution on the geodesic ball B_1. Passing to a single starting point is easy. Let

$$h(z) = \mathbb{P}_z(H).$$

Let $\theta_t : \mathscr{P}(M) \to \mathscr{P}(M)$ be the shift operator defined by

$$(\theta_t \omega)(s) = \omega(s + t).$$

By the definition of the event H, it is clear that for any stopping time τ, we have $\omega \in H$ if and only if $\theta_\tau \omega \in H$; in other words, $I_H \circ \theta_\tau = I_H$. It follows that $h(z) = \mathbb{P}_z(H) = \mathbb{E}_z I_H$ is a harmonic function on M. On the other hand, we have $0 \leq h \leq 1$ and

$$\frac{1}{|B_1|} \int_{B_1} h(z) \, dz = 1.$$

By the maximum principle for harmonic functions, we see that h must be identically equal to 1.   □

The following special cases have all appeared in the literature (see the references cited in Section 1). They now follow from our main Theorem 4.1 and all are now valid without any geometric restrictions.

COROLLARY 4.2.   *Let M be a complete Riemannian manifold. Under the following volume growth conditions, ψ is an upper rate function for Brownian motion on M:*

(1) $|B(r)| \leq Cr^D$ *and* $\psi(t) = C_1\sqrt{t \ln t}$;
(2) $|B(r)| \leq e^{Cr^\alpha}$ $(0 < \alpha < 2)$ *and* $\psi(t) = C_1 t^{1/(2-\alpha)}$;
(3) $|B(r)| \leq e^{Cr^2}$ *and* $\psi(t) = C_1 \exp(C_1 t^2 \ln t)$;
(4) $|B(r)| \leq e^{Cr^2 \ln r}$ *and* $\psi(t) = \exp(\exp(C_1 t))$.

PROOF.   These upper rate functions follow directly from the main theorem. Since the volume grows faster than the additional term $\ln \ln r$ in the function ϕ, these rate functions are the same as if the additional term were not there.   □

Riemannian manifolds with slow volume growth are interesting test cases for our main result Theorem 4.1. Although, in general, slow volume growth corresponds to slow upper rate functions, our result will not yield upper rate functions better than $\sqrt{t \ln \ln t}$ once $|B(r)| \leq (\ln r)^\gamma$ or some $\gamma > 0$ (see Remark 1.1). It also explains the condition $v(r) \geq (\ln r)^\gamma$ in the following result, due to Grigor'yan [2].

COROLLARY 4.3.   *Let M be a complete manifold such that $|B(r)| \leq v(r)$ for an increasing function $v(r) \geq (\ln r)^\gamma$ with some $\gamma > 0$. Define $R(t)$ by*

$$\frac{R(t)^2}{\ln v(R(t))} = t.$$

Then $CR(Ct)$ is an upper rate function for Brownian motion on M. In particular, if M has finite volume, then $C\sqrt{t \ln \ln t}$ is an upper rate function.

PROOF.   With the lower bound for $v(r)$, we have

$$\phi(R) = \int_6^R \frac{r\,dr}{\ln|B(r)| + \ln \ln r} \geq C_1 \int_6^R \frac{r\,dr}{\ln v(r)} \geq \frac{C_2 R^2}{\ln v(R)}.$$

Therefore, the inverse function $\psi(t) \leq C_3 R(C_3 t)$. By Theorem 4.1, $CR(Ct)$ is an upper rate function for some C.   □

REMARK 4.4.   In all of the concrete cases we have mentioned thus far, upper rate functions are determined up to multiplicative constants. The question naturally

arises as to whether we could have been more careful in our computations so as to recover the best constants in some cases, for instance, $\psi(t) = \sqrt{(2 + \varepsilon)t \ln \ln t}$ for the standard one-dimensional Brownian motion. Such a precise upper rate function is impossible without further geometric assumptions other than the volume growth. This can be explained by means of manifolds with power volume growth $|B(r)| \leq Cr^D$. According to Corollary 4.2(1), the corresponding rate function is $\psi(t) = C_1\sqrt{t \ln t}$. By comparison with a Euclidean Brownian motion, we would expect a double logarithm instead of a single one. However, there are known examples showing that the above rate function with a single logarithm is indeed sharp up to a multiplicative constant (see Grigor'yan and Kelbert [5]). This is the reason why we have been somewhat cavalier about multiplicative constants in our proofs. It should be pointed out that these constants, denoted by C with or without subscripts, are universal; they do not depend on the manifold M (not even on its dimension).

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DEPARTMENT OF MATHEMATICS
NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS 60208
USA
E-MAIL: ehsu@math.northwestern.edu

INSTITUTE OF APPLIED MATHEMATICS
ACADEMY OF MATHEMATICS
 AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
BEIJING
CHINA
E-MAIL: gn_qin@amss.ac.cn