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RIGHT INVERSES OF LÉVY PROCESSES

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We call a right-continuous increasing process K_x a *partial right inverse* (PRI) of a given Lévy process X if $X_{K_x} = x$ for at least all x in some random interval $[0, \zeta)$ of positive length. In this paper, we give a necessary and sufficient condition for the existence of a PRI in terms of the Lévy triplet.

1. Introduction and results. In this paper, a real-valued Lévy process is studied. The problem of existence of a partial right inverse (PRI) is considered and an explicit integral criterion is provided for testing whether any Lévy process possesses a PRI.

We continue work by Evans [3] and Winkel [5]. Evans has introduced the notion of a full right inverse and has defined this process K as the minimal increasing process that satisfies $X(K_x) = x$ for all $x \geq 0$; Winkel, in [5], has extended this definition to $X(K_x) = x$ on some random interval $[0, \zeta)$ of positive length and has named this process a PRI. In these two papers, it is shown that if K exists, it is a (possibly killed) subordinator.

A Lévy process $X = (X_t; t \geq 0)$ is a stochastic process which possesses stationary and independent increments, starts from zero and whose paths are a.s. right-continuous. Each Lévy process is fully characterized by its Lévy triplet (γ, σ, Π), where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and the Lévy measure Π has the property

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\Pi(dx) < \infty.$$

Also, each Lévy process X can be represented as follows:

$$X_t = \gamma t + \sigma B_t + X_t^{(1)} + \sum_{0<s\leq t} (X_s - X_{s-})\mathbf{1}_{(|X_s - X_{s-}|>1)}, \tag{1}$$

where B is a standard Brownian motion, $X^{(1)}$ is a pure jump zero mean martingale and all of the components in (1) are independent. In the class of Lévy processes, we distinguish between Lévy processes with bounded variation and Lévy processes with unbounded variation. The former are those for which $\sigma = 0$ and $\int_{-\infty}^{\infty}(1 \wedge |x|)\Pi(dx) < \infty$. In this case, X can be represented as

$$X_t = bt + X_t^+ + X_t^-, \tag{2}$$

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where b is the drift coefficient and X^+ and X^- are independent driftless subordinators (i.e., increasing Lévy processes). In our setting, as well as in many other situations, these two classes of processes exhibit quite different behaviors and need separate attention.

We write $R_t = \sup_{s \le t} X_s - X_t$. It is shown in [1], Chapter 6, that R is a strong Markov process which possesses a local time at zero, $L(t)$, and a corresponding inverse local time $L^{-1}(t) = \inf\{s : L(s) > t\}$ such that $(L^{-1}(t); X(L^{-1}(t)))$ is a bivariate subordinator: we denote its Lévy measure by $\mu^{(+)}(dt; dy)$ and we use, in particular, $\mu^{(+)}(dy) = \mu^{(+)}((0; \infty); dy)$. We also use the notation $H^+(t) := X(L^{-1}(t))$ and call H^+ the *upward ladder height process*. Similarly, we can define $Z_t = X_t - \inf_{s \le t} X_s$ and, using the same arguments, we have an associated inverse local time $L_-^{-1}(t)$ and *downward ladder height process* $H^-(t) := X(L_-^{-1}(t))$. We denote the Lévy measure of H^- by $\mu^{(-)}(dy)$. Finally, with each of the subordinators $H^+$and H^-, we associate the so-called *renewal measure*, defined as follows:

$$U_+(x) = E \int_0^\infty \mathbf{1}_{\{H_t^+ \le x\}}\, dt, \qquad U_-(x) = E \int_0^\infty \mathbf{1}_{\{H_t^- \le x\}}\, dt. \tag{3}$$

We refer to Bertoin [1] or Doney [2] for more information on Lévy processes.

Next, we briefly discuss the definition of a PRI, that is, $K = (K_x, x \ge 0)$. We follow an approach developed in Evans [3]. Define, for each $n \ge 1$ and $k \ge 0$, the stopping times

$$T_0 = 0, \qquad T_n^{k+1} = \inf\left\{t \ge T_n^k : X_t = \frac{k+1}{2^n}\right\} \tag{4}$$

and processes

$$K_x^n = T_n^k, \qquad \frac{k}{2^n} \le x < \frac{k+1}{2^n}.$$

A pathwise argument then shows that

$$K_x = \inf_{y > x} \sup_{n \ge 0} K_y^n. \tag{5}$$

It is possible that for each $x > 0$, the definition above gives $K_x \overset{\text{a.s.}}{=} \infty$ and, in this case, we say that a PRI does not exist. The question of the existence of a PRI has been studied by Evans in [3] and Winkel in [5]. Evans has shown that for any symmetric Lévy process with $\sigma > 0$, a full right-inverse exists. Winkel [5] then showed that the same result holds for any oscillating Lévy process with $\sigma > 0$ and also described all Lévy processes with bounded variation having a PRI. Moreover, in the unbounded variation case, he provided a necessary and sufficient condition (NASC) for the existence of a PRI, but this NASC is not satisfactory since it requires knowledge about the second derivative at zero of the so-called q-potentials of the given Lévy process, which are generally unknown. Therefore, the main aim of this paper is to supply an NASC for the existence of a PRI in terms of the Lévy triplet, that is, (γ, σ, Π), in the unbounded variation case. In fact, our method, which is probabilistic in nature, also deals with the bounded variation case and gives the following result.

THEOREM 1. *Let X be a Lévy process with a Lévy measure Π such that $\Pi(\mathbb{R}) > 0$. Then*:

(i) *if X has unbounded variation, it has a partial right inverse* (*PRI*) *iff* $\sigma > 0$ *or* $\sigma = 0$, $\Pi(\mathbb{R}^-) = \infty$ *and* $J < \infty$, *where, with* $\overline{\Pi}^{(-)}(s) = \int_{-\infty}^{-s} \Pi(dx)$,

$$J = \int_0^1 \frac{x^2 \Pi(dx)}{(\int_0^x \int_y^1 \overline{\Pi}^{(-)}(s)\, ds\, dy)^2}; \tag{6}$$

(ii) *if X has bounded variation, then it has a PRI iff $\Pi(\mathbb{R}^+) < \infty$ and X has a drift coefficient $b > 0$.*

REMARK 2. If $\Pi(\mathbb{R}) = 0$, then $X_t = \gamma t + \sigma B_t$ is a continuous process and $T_x = \inf\{t : X_t = x\}$ will be a PRI on the set $\{T_x < \infty\}$. Note that, in this case, $\{T_x < \infty\}$ will be the empty set iff $\sigma = 0$ and $\gamma < 0$.

REMARK 3. A Lévy process X is said to "creep upward" if $P(X(T_x^+) = x) > 0$ for some (and then all) $x > 0$, where $T_x^+ = \inf(t > 0 : X_t > x)$. It is known that this happens iff the ladder height process H^+ has drift $\delta_+ > 0$; see, for example, Theorem 19, page 174 of [1]. Since it is always the case that $\sigma^2 = 2\delta_+\delta_-$, where δ_- is the drift of H^-, this certainly happens when $\sigma > 0$. If $\sigma = 0$ and $J < \infty$, then the integral

$$L = \int_0^1 \frac{x^2 \Pi(dx)}{\int_0^x \int_y^1 \overline{\Pi}^{(-)}(s)\, ds\, dy} \tag{7}$$

is clearly finite and it is shown in [4] that this is the NASC for $\delta_+ > 0$ in the unbounded variation case when $\sigma = 0$. (See also Section 6.4 of [2] for an alternative proof of this result.) Finally, in the bounded variation case, $b > 0$ is clearly equivalent to $\delta_+ > 0$. We therefore conclude that our theorem is consistent with the intuitively obvious claim that "upward creeping" is necessary, but not sufficient, for the existence of a PRI.

The next corollary illustrates how our theorem yields specific information in special cases. Here, and throughout the paper, we use the notation $f \approx g$ to denote the existence of constants $0 < c < C < \infty$ with $cg(x) \le f(x) \le Cg(x)$, for all sufficiently small x.

COROLLARY 4. *Let X be a Lévy process with $\sigma = 0$ and Lévy measure Π such that $\overline{\Pi}^+(x) = \int_x^\infty \Pi(dy) \approx x^{-\beta}$ and $\overline{\Pi}^-(x) \approx x^{-\alpha}$, where $1 \le \alpha < 2$ and $0 \le \beta < 2$. Then X has a PRI iff $\beta < 2\alpha - 2$.*

REMARK 5. This result extends Proposition 2 and Theorem 6 in [5].

2. Proofs. Recall that we denote by H^+ the ascending ladder height process of a given Lévy process X. We use δ_+ to denote the drift of H^+ and $\mu^{(+)}(dy)$ to denote its Lévy measure. We also use U_+ and U_-, which are defined in (3). We start the proof by disposing of some special cases.

Suppose, first, that $\Pi(\mathbb{R}) < \infty$. Then $V = \inf\{t > 0 : X_t - X_{t-} \neq 0\} > 0$ a.s. since it is an exponentially distributed random variable with parameter $\Pi(\mathbb{R})$ and the given process coincides up to time V with the process we get by removing all of its jumps. The resulting process will be of the form $\sigma B_t + bt$, which possesses a PRI iff $\sigma > 0$ or $\sigma = 0$ and $b > 0$, in accordance with Theorem 1. Next, suppose that $\Pi(\mathbb{R}) = \infty$, but $\Pi(\mathbb{R}^+) < \infty$. Removing all the positive jumps then gives a spectrally negative Lévy process $\tilde{X}$. If $\tilde{X}$ has unbounded variation, or has bounded variation and a positive drift b, then it passes continuously over positive levels. Then with $\tilde{T}(x) = \inf\{t > 0 : \tilde{X}_t = x\}$, we obviously have $\tilde{X}_{\tilde{T}(x)} = x$ on $\{\tilde{T}(x) < \infty\}$ and we can choose $K_x = \tilde{T}(x)$. Alternatively, $\tilde{X}$ has bounded variation and a drift $b \leq 0$, and, clearly, no PRI exists for $\tilde{X}$ or X in this case. Noting that in the unbounded variation case, we have $\int_0^1 \overline{\Pi}^{(-)}(s)\,ds = \infty$ so that, necessarily, $J < \infty$, we see that these results also accord with Theorem 1. Next, suppose that $\Pi(\mathbb{R}) = \infty$, but $\Pi(\mathbb{R}^-) < \infty$. If X has bounded variation, then removing all of the negative jumps gives us a spectrally positive process of the form $\tilde{X}_t = X_t^+ + bt$, where X^+ is a driftless subordinator. If $b \geq 0$, then $\tilde{X}$ has monotone paths and the assumption that $\Pi(\mathbb{R}^+) = \infty$ implies the existence of points $x_n \downarrow 0$ with $P(T(x_n) = \infty) = 1$, which verifies Theorem 1 in this case. Finally, if $b < 0$ or if X has unbounded variation, then the decreasing ladder height process is a pure drift, possibly killed at an exponential time, and we see that the hypothesis of Proposition 7 below holds.

The rest of our proof uses the following simple consequence of the construction of K due to Evans [3].

LEMMA 6. *Let X be an arbitrary Lévy process, and set $T_x = \inf\{t > 0 : X_t = x\}$ and $p_x = P(T_x = \infty) = P(X$ does not visit $x)$. Then*:

(i) *a PRI exists for X if*

$$\limsup_{x \downarrow 0} \frac{1 - E(e^{-\theta T_x})}{x} < \infty \qquad \text{for some } \theta > 0; \tag{8}$$

(ii) *no PRI exists for X if*

$$\lim_{x \downarrow 0} x^{-1} p_x = \infty. \tag{9}$$

PROOF. First, note that the sequence $K^{(n)} := T_n^{2^n}$, $n \geq 1$, where T_n^k are defined in (4), is monotone increasing. If we denote its limit by $\tilde{K}$, then it is immediate from (5) that $K_1 \le \tilde{K} \le K_2$. Since we know that K is a (possibly killed) subordinator, we see that existence of a PRI for X is equivalent to $P(\tilde{K} < \infty) > 0$. However, this is equivalent to

$$\lim_{n\to\infty} E\big(e^{-\theta K^{(n)}}\big) = E(e^{-\theta \tilde{K}} : \tilde{K} < \infty) > 0$$

for some (and then all) $\theta > 0$. Since $K^{(n)}$ is the sum of 2^n independent random variables distributed as $T_{2^{-n}}$, we have

$$\log E(e^{-\theta \tilde{K}} : \tilde{K} < \infty) = \lim_{n\to\infty} 2^n \log E(e^{-\theta T_{2^{-n}}})$$

and this is clearly finite for any θ for which (8) holds. Since $1 - E(e^{-\theta T_x}) \ge p_x$, we see that this limit is $-\infty$ for all $\theta > 0$ whenever (9) holds, and the result follows. □

The crux of our proof is contained in the following result, in which $\overline{\mu}^+(x) = \mu((x,\infty))$ for $x > 0$.

PROPOSITION 7. *Let X be a Lévy process having $\Pi(\mathbb{R}^+) = \infty$ and $U_-(dx) > 0$ for all small enough $x > 0$. Then X has a PRI iff $\delta_+ > 0$ and $I < \infty$, where*

$$I = \int_0^1 \overline{\mu}^+(x) U_-(dx) = \int_0^1 \mu^+(dx) U_-(x). \tag{10}$$

PROOF. Since the existence of a PRI is a local property, we can truncate the Lévy measure so that it is contained in $[-1;1]$. Indeed, the first jump of X larger than 1 in absolute value occurs after an exponential time ζ and K_x is a subordinator, therefore $K_x < \zeta$ pathwise for all x small enough. This shows that the existence of K is independent of the large jumps, so we will assume, without loss of generality, that $\Pi([1,\infty)) = \Pi((-\infty,-1])) = 0$. Moreover, the value of δ_+ is also a local property, so this is also unchanged by any alteration of the Lévy measure on closed intervals which do not contain 0. Note that our assumptions imply that $I > 0$ and that these alterations do not change the finiteness/infiniteness of I. Let us introduce some notation. For $x > 0$, we put $T_x^+ = \inf\{t > 0 : X_t > x\}$ and $T_x^- = \inf\{t > 0 : X_t < -x\}$ for the first passage times above x and below $-x$, respectively, and $O^+(x) = X_{T_x^+} - x$, $O^-(x) = x - X_{T_x^-}$ for the overshoot above x and the undershoot below $-x$, respectively. Noting that $O^+(x)$ is also the overshoot of H^+ above x, we can use Proposition 2, page 76 in [1] to deduce that for $x > 0$, $y > 0$,

$$\begin{aligned} \overline{\mu}^{(+)}(x+y)U_+(x) \le P\big(O^+(x) > y\big) &= \int_0^x \overline{\mu}^{(+)}(x+y-z)U_+(dz) \\ &\le \overline{\mu}^{(+)}(y)U_+(x). \end{aligned} \tag{11}$$

To prove the result in one direction, we alter the Lévy measure by adding a mass at $\{1\}$, if necessary, to make X drift to $+\infty$. We then have the estimate

$$\begin{aligned} p_x &\geq P(O_x^+ > 0, \text{ and } X \text{ stays above } x) \\ &= \int_0^1 P\big(O^+(x) \in dy\big) P(T_y^- = \infty) \\ &= c \int_0^1 P\big(O^+(x) \in dy\big) U_-(y) \\ &= c \int_0^1 P\big(O^+(x) > y\big) U_-(dy), \end{aligned}$$

where the fact that $P(T_y^- = \infty) = cU_-(y)$ comes from Proposition 17, page 172 of [1]. [It is obvious that, in fact, $c = 1/U_-(\infty)$ since $P(T_y^- = \infty) \to 1$ as $y \to \infty$.] From (11), it then follows that

$$\begin{aligned} \liminf_{x \downarrow 0} x^{-1} p_x &\geq c \liminf_{x \downarrow 0} x^{-1} U_+(x) \int_0^1 \overline{\mu}^{(+)}(x+y) U_-(dy) \\ &\geq cI \liminf_{x \downarrow 0} x^{-1} U_+(x). \end{aligned}$$

Finally, we recall from Proposition 1, page 74 in [1] that $U_+(x) \approx x/(\delta_+ + \int_0^x \overline{\mu}^{(+)}(y)\,dy)$ so that $x^{-1}U_+(x) \approx 1/\delta_+$ as $x \downarrow 0$, and thus (9) holds and no PRI exists, whenever $\delta_+ = 0$, or $\delta_+ > 0$ and $I = \infty$. To argue in the other direction, we assume that $\delta_+ > 0$ and $I < \infty$. Then, without loss of generality, we can take $\delta_+ = 1$. Next, we denote by P^θ the law of this process killed at an independent exponential time τ with parameter θ and note that

$$p_x^\theta := P^\theta(T_x = \infty) = P(T_x > \tau) = 1 - E(e^{-\theta T_x}).$$

Our aim is to show that there exists some $\theta > 0$ such that

$$\limsup_{x \downarrow 0} x^{-1} p_x^\theta < \infty \tag{12}$$

since then the existence of a PRI for X will follow from Lemma 6. We decompose p_x^θ according to the number of upcrossings and downcrossings of level x that occur. To do so, we denote by $T^+(x,n)$ the time of nth crossing above x, by $T^-(x,n)$ the time of nth crossing below x and for $n \geq 1$, we put

$$\begin{aligned} p_x^\theta(n) &= P^\theta\big\{T_x = \infty, T^+(x,n) < \infty, T^-(x,n) = \infty\big), \\ q_x^\theta(n) &= P^\theta\big\{T_x = \infty, T^-(x,n) < \infty, T^+(x,n+1) = \infty\big). \end{aligned}$$

Since X creeps upward, it is then easy to see that

$$p_x^\theta = P^\theta(T_x^+ = \infty) + \sum_1^\infty p_x^\theta(n) + \sum_1^\infty q_x^\theta(n). \tag{13}$$

We start by noting that

$$P^\theta(T_x^+ = \infty) = c^+(\theta)U_+^\theta(x) \qquad \text{where } c^+(\theta) = \frac{1}{U_+^\theta(\infty)},$$

and $U_+^\theta(x)$ is the renewal function of the ladder height process H^+ under P^θ. Of course, under P^θ, H^+ is killed at some rate $k^+(\theta) > 0$ and has Lévy measure $\mu^+(\theta, dx) \le \mu^+(dx)$. However, as we have mentioned, its drift is unchanged and equals 1. Using a version of Erickson's bound for killed subordinators, which can be found in [4], we therefore have

$$U_+^\theta(x) \le \frac{c_0 x}{1 + \int_0^x \overline{\mu}^+(y, \theta)\, dy + xk^+(\theta)} \le c_0 x, \tag{14}$$

where c_0 is an absolute constant. Also,

$$U_+^\theta(\infty) = \lim_{y\to\infty} \int_0^\infty e^{-tk^+(\theta)} P(H_t^+ \le y)\, dt = \frac{1}{k^+(\theta)}$$

and this gives the bound

$$P^\theta(T_x^+ = \infty) \le c_0 k^+(\theta) x. \tag{15}$$

Next, using a similar notation, we see that

$$\begin{aligned} p_x^{(\theta)}(1) &= \int_0^1 P^\theta\bigl(O_+(x) \in dy\bigr) P^\theta(T_y^- = \infty) \\ &= c^-(\theta) \int_0^1 P^\theta\bigl(O_+(x) \in dy\bigr) U_-^\theta(y) \\ &= c^-(\theta) \int_0^1 P^\theta\bigl(O_+(x) > y\bigr) U_-^\theta(dy) \\ &\le c^-(\theta) U_+^\theta(x) \int_0^1 \overline{\mu}^+(\theta, y) U_-^\theta(dy) \\ &:= c^-(\theta) I(\theta) U_+^\theta(x), \end{aligned}$$

where we have used the P^θ version of (11). Using (14) again gives the bound

$$p_x^{(\theta)}(1) \le x c_0 c^-(\theta) I(\theta). \tag{16}$$

Writing $O_\pm(n, x)$ for the successive overshoots upward and downward over level x, we then have

$$p_x^{(\theta)}(n) = \int_0^1 P^\theta\bigl(O_-(n-1, x) \in dz\bigr) p_z^{(\theta)}(1) \le c_0 c^-(\theta) I(\theta) E^\theta\bigl(O_-(n-1, x)\bigr).$$

Also, Wald's identity gives $E^\theta(O^-(y)) \le m_-^\theta U_-^\theta(y)$, where $m_-^\theta = E^\theta(H_1^-)$, and so we have

$$\begin{aligned}
&E^\theta\bigl(O_-(n-1,x)|O_-(n-2,x)=y\bigr)\\
&\quad = E^\theta(O^-(y)) \le m_-(\theta)\int_0^1 P^\theta(O_y^+\in dz)U_-^\theta(z)\\
&\quad = m_-(\theta)\int_0^1 P^\theta(O_y^+ > z)U_-^\theta(dz)\\
&\quad \le m_-(\theta)U_+^\theta(y)\int_0^1 U_-^\theta(dz)\overline{\mu}^+(\theta,z)\\
&\quad \le c_0 m_-(\theta)I(\theta)y,
\end{aligned}$$

where we have again used (11). Iterating this gives

$$E^\theta\bigl(O_-(n-1,x)\bigr) \le \{c_1(\theta)\}^{n-1}x, \tag{17}$$

where $c_1(\theta) = c_0 m_-(\theta)I(\theta)$, and thus

$$p_x^{(\theta)}(n) \le c_0 c^-(\theta)I(\theta)\{c_1(\theta)\}^{n-1}x, \qquad n \ge 1.$$

Moreover, using (15) and (17), we get the bound

$$\begin{aligned}
q_x^{(\theta)}(n) &= \int_0^1 P^\theta\bigl(O_-(n,x)\in dz\bigr)P^\theta(T_z^+ = \infty)\\
&\le c_0 k^+(\theta)E^\theta(O_-(n,x)) \le c_0 k^+(\theta)\{c_1(\theta)\}^{n-1}x.
\end{aligned}$$

So, (12) will follow, provided that θ can be chosen such that

$$c_1(\theta) = c_0 m_-(\theta)I(\theta) < 1. \tag{18}$$

To see this, we need to note first that $m_-(\theta) \le E(H_1^-)$. Also, provided that $k^-(\theta)\to\infty$, by applying bound (14) to H^-, we get $U_-^\theta(z)\to 0$ for each $z\in(0,1]$ as $\theta\to\infty$, and since $U_-^\theta(z)\le U_-(z)$ and $I<\infty$, dominated convergence will give

$$I(\theta) = \int_0^1 U_-^\theta(z)\mu^+(\theta,dz) \le \int_0^1 U_-^\theta(z)\mu^+(dz) \to 0 \qquad \text{as } \theta\to\infty.$$

To see that $k^-(\theta)\to\infty$, note that the killing time of H^- under P^θ is the same as that of the ladder time subordinator L_-^{-1} and this has the distribution of $L_-(\tau)$, which is $\exp(\kappa_-(\theta))$, where κ_- is the Laplace exponent of L_- under P. The assumption that $U_-(dx)>0$ for all small $x>0$ implies that L_- is not a compound Poisson process so, by Corollary 3, page 17 of [1], $\kappa_-(\infty)=\infty$ and, thus, if we choose θ large enough, (18) will hold and the proof is complete. □

PROPOSITION 8. (i) *Let X be a Lévy process having $\Pi(\mathbb{R}^+)=\infty$ and $\sigma>0$. Then a PRI exists.*

(ii) *Let X be a Lévy process having $\sigma=0$, $\Pi(\mathbb{R}^+)=\infty$ and $\Pi(\mathbb{R}^-)<\infty$. Then no PRI exists.*

PROOF. (i) Here, $\delta_+>0$ and $\delta_->0$, so $U_-(x)\backsim x/\delta_-$ and since $\int_0^1 x\mu^+(dx)$ is automatically finite, we have $I<\infty$.

(ii) By the argument preceding Lemma 6, we can take $\Pi(\mathbb{R}^-)=0$ and assume that $\delta_->0$, so that, again, I is necessarily finite. However, $\sigma=0$ and $\delta_->0$ imply $\delta_+=0$, so the result follows. $\square$

To deal with the remaining situations, we need the following lemma.

LEMMA 9. *Let X be an oscillating Lévy process whose Lévy measure is supported by $[-1,1]$ and satisfies $\Pi([-1,0))=\Pi((0,1]))=\infty$. Suppose, additionally, that $\sigma=0$ and $\delta^+>0$. Then $I=\int_0^1\overline{\mu}^+(x)U_-(dx)<\infty$ iff*

$$J=\int_0^1\frac{x^2\Pi(dx)}{(\int_0^x\int_y^1\overline{\Pi}^{(-)}(s)\,ds\,dy)^2}<\infty. \tag{19}$$

PROOF. We use Vigons' "équation amicale inversée" (see [4]), which, since our Lévy measure lives on $[-1;1]$, takes the form

$$\overline{\mu}^+(x)=\int_0^\infty\overline{\Pi}^+(x+y)U_-(dy)=\int_x^1 U_-(y-x)\Pi(dy).$$

We then use this in the following computation:

$$\begin{aligned}I&=\int_0^1\overline{\mu}^+(x)U_-(dx)<\infty=\int_0^1\int_x^1 U_-(y-x)\Pi(dy)U_-(dx)\\&=\int_0^1\Pi(dy)\int_0^y U_-(y-x)U_-(dx)=\int_0^1 U_-^{*2}(y)\Pi(dy).\end{aligned}$$

Next, we recall that the potential function $U_-(x)$ is increasing in x. This is enough to show that

$$\big(U_-(y/2)\big)^2\le U_-^{*2}(y)=\int_0^y U_-(y-x)U_-(dx)\le(U_-(y))^2.$$

Moreover, since X oscillates, H_- is an unkilled subordinator with zero drift and we have that $U_-(y)\approx y/A(y)$, where $A(y)=\int_0^y\overline{\mu}^{(-)}(s)\,ds$ satisfies $A(y)/2\le A(y/2)\le A(y)$. This implies that $U_-(y)\approx U_-(y/2)$ and therefore that $U_-^{*2}(y)\approx(U_-(y))^2$. We therefore conclude that

$$I=\int_0^1 U_-^{*2}(y)\Pi(dy)<\infty\quad\Longleftrightarrow\quad\int_0^1\frac{y^2\Pi(dy)}{A(y)^2}<\infty.$$

Next, we need the "équation amicale intégrée" of Vigon (see [4]), which, in our case, takes the form

$$\overline{\overline{\Pi}}^{(-)}(x) = \int_x^1 \overline{\Pi}^{(-)}(y)\,dy = \int_0^1 \overline{\mu}^{(+)}(y)\overline{\mu}^{(-)}(x+y)\,dy + \delta_+\overline{\mu}^{(-)}(x).$$

Our assumptions imply that $\overline{\overline{\Pi}}^{(-)}(0+) > 0$. If $\overline{\overline{\Pi}}^{(-)}(0+) < \infty$, then it is obvious that $0 < \overline{\mu}^{(-)}(0+) < \infty$, and if $\overline{\overline{\Pi}}^{(-)}(0+) = \infty$, it is easy to deduce that $\overline{\mu}^{(-)}(0+) = \infty$. Then, from dominated convergence, it follows that

$$\lim_{x\downarrow 0} \frac{\overline{\overline{\Pi}}^{(-)}(x)}{\overline{\mu}^{(-)}(x)} = \delta_+.$$

Thus, in both cases, $A(y) \approx \int_0^y \overline{\overline{\Pi}}^{(-)}(z)\,dz$ and the result follows. □

PROOF OF THEOREM 1. We have already covered all cases except those having $\sigma = 0$ and $\Pi(\mathbb{R}^+) = \Pi(\mathbb{R}^-) = \infty$. By the standard argument, we can find another process, $\tilde{X}$, which oscillates and whose Lévy measure $\tilde{\Pi}$ agrees with Π on $(-1, 1)$ and is supported by $[-1, 1]$, and is such that a PRI exists for X iff a PRI exists for $\tilde{X}$. Note that $\tilde{\Pi}([-1, 0)) = \tilde{\Pi}((0, 1])) = \infty$ and that, in the obvious notation, $\tilde{J} < \infty$ iff $J < \infty$. Proposition 7 and Lemma 9 then apply and show that a PRI exists iff $\delta_+ > 0$ and $J < \infty$. If X has bounded variation, then $\overline{\overline{\Pi}}^{(-)}(0+) \in (0, \infty)$, and $J = \infty$ is then automatic. If X has unbounded variation, then, as previously noted, $J < \infty$ implies $\delta_+ > 0$ and this completes the proof. □

PROOF OF COROLLARY 4. Since $\overline{\Pi}^-(x) \approx x^{-\alpha}$, where $1 < \alpha < 2$, we are in the unbounded variation case and we need only check the value of the integral (6). Clearly, $\int_0^x \int_y^1 \overline{\Pi}^{(-)}(s)\,ds\,dy \approx x^{2-\alpha}$, so this reduces to checking whether

$$\int_0^1 x^{2\alpha-2}\Pi(dx) = (2\alpha - 2)\int_0^1 x^{2\alpha-3}\overline{\Pi}^+(x)\,dx < \infty$$

and this holds iff $\beta < 2\alpha - 2$. □

REMARK 10. A similar calculation for the integral L in (7) shows that in this example, X creeps upward iff $\beta < \alpha$.

3. The excursion measure. Evans [3] and Winkel [5] both observed that we can associate an excursion theory with K.

They introduced $\Lambda_t = \inf\{x : K_x > t\}$, $Z = X - \Lambda$ and showed that Z is a strong Markov process with Λ as a local time at zero. It is clear that excursions away from 0 of Z evolve in the same way as excursions away from 0 of X, namely, they have the same semigroup, but their entrance laws will be different. For example, if $X = B$, then all excursions of Z are negative and the characteristic measure n^Z is n^X restricted to negative excursion paths.

Winkel showed that when $\sigma > 0$, n^Z is the restriction of n^X to the set of excursion paths which *start* negative. (To do this, he had to demonstrate that all excursion paths either start negative or start positive, that is, cannot leave 0 in an oscillatory fashion.) Therefore, n^Z is absolutely continuous w.r.t. n^X.

However, this depends on both δ_+ and δ_- being positive. When $\sigma = 0$ and $\delta_+ > 0$, we have $\delta_- = 0$, which means that excursions of X have to return to 0 from below. By time reversal, this means that they must start positive and since excursions of Z start negative, the two measures must be mutually singular whenever $\sigma = 0$. We believe that the problem of describing the excursion measure n^Z in this case is both interesting and difficult.

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