INVARIANCE PRINCIPLES FOR LOCAL TIMES AT THE MAXIMUM OF RANDOM WALKS AND LÉVY PROCESSES

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We prove that when a sequence of Lévy processes $X^{(n)}$ or a normed sequence of random walks $S^{(n)}$ converges a.s. on the Skorokhod space toward a Lévy process X, the sequence $L^{(n)}$ of local times at the supremum of $X^{(n)}$ converges uniformly on compact sets in probability toward the local time at the supremum of X. A consequence of this result is that the sequence of (quadrivariate) ladder processes (both ascending and descending) converges jointly in law toward the ladder processes of X. As an application, we show that in general, the sequence $S^{(n)}$ conditioned to stay positive converges weakly, jointly with its local time at the future minimum, toward the corresponding functional for the limiting process X. From this we deduce an invariance principle for the meander which extends known results for the case of attraction to a stable law.

1. Introduction. It is well known that if a sequence of Lévy processes $X^{(n)}$ converges a.s. on the Skorokhod space to a limiting Lévy process X, then the corresponding sequence of local times at a fixed level of $X^{(n)}$ do not necessarily converge to the local time of X, whatever the definition of the local times of $X^{(n)}$ is: occupation time, crossing times, ... However, in the fluctuation theory of Lévy processes, it is the local times at extrema which play a major role, not the local times at fixed levels, so a natural and important question is whether these local times converge. A similar question can be posed about the local times at extrema of a sequence of normed random walks which converge to a Lévy process.

To our knowledge, the only known results in this vein can be found in Greenwood, Omey and Teugels [17] and in Duquesne and Le Gall [15]. The first paper deals with the "classical" case where $S^{(n)}$ is obtained by norming a fixed random walk S, the assumption being that for some norming sequence c_n $(S_{[nt]}/c_n, t \geq 0)$ converges in law to X, necessarily stable, and the conclusion being that a normed version of the bivariate ladder process of S converges in law to the bivariate ladder process of X. One can easily derive that a normed version of the local time at the maximum of S converges in law to the local time at the supremum of X. (A different proof of this result and a converse result can be found in Doney and Greenwood [13].) The second paper considers a more general scenario where each $S^{(n)}$ is

obtained by norming a different random walk, but restricts itself to the case where each random walk is downward skip-free so that the limiting Lévy process is automatically spectrally positive. (This is because the result, Theorem 2.2.1 of [15], is a tool for the study of the height process of the sequence of Galton–Watson processes related to $S^{(n)}$.) Again, convergence in law is assumed and the conclusion is again convergence in law of a normed version of the local time.

In this paper, we provide three major extensions of these results. In Theorem 1, we show that whenever $S^{(n)}$ converges in law to X, a normed version of the bivariate ladder process of S converges in law to the bivariate ladder process of X. Again, we can deduce that a normed version of the local time at the maximum of S converges in law to the local time at the supremum of X. (Our only assumption on X is that it has a continuous local time L at the supremum, but if this were to fail, a similar result could be formulated.) Next, in Theorem 2, we show that if the assumption is strengthened to a.s. convergence, then the normed sequence of local times converges to L in probability, uniformly on compacts. This result allows us to deduce, in Theorem 3, an analogous result when a sequence $X^{(n)}$ of Lévy processes converges a.s. to X. [We stress that for such a result to hold, we have to remove the ambiguity inherent in the definition of local times for Lévy processes by insisting on a standard normalization for the local times of $X^{(n)}$ and X; see (2.1).] An important corollary of these results is the convergence in law of the quadrivariate process of upgoing and downgoing ladder processes; see Corollary 2.

In the last section, we show that if a sequence $(S^{(n)}_{[nt]}, t \geq 0)$ of continuous-time random walks converges in law toward a Lévy process X, then the sequence of these processes conditioned to stay positive on the whole time interval $[0, \infty)$ converges in law toward X conditioned to stay positive. We illustrate the usefulness of the results of Section 3 by showing that this convergence also holds jointly with the local time at the future infimum. Finally, we obtain an invariance principle for the meander, that is, we show that the sequence $(S^{(n)}_{[nt]}, t \geq 0)$ conditioned to stay positive over $[0, 1]$ converges in law toward X conditioned to stay positive over $[0, 1]$. These results extend the "classical" case studied by Bolthausen [5], Doney [10] and Caravenna and Chaumont [6].

2. Preliminaries. Let X be any Lévy process for which 0 is regular for the open half-line $(0, \infty)$. Then 0 is also regular for itself for the reflected process $R := M - X$, where $M_t = \sup_{0 \leq s \leq t} X_s$, and so there exists a continuous local time for R at 0. This local time L is only specified up to multiplication by a constant, but we will assume throughout that its normalization is fixed by the requirement that

$$(2.1) \qquad \mathbb{E}\left(\int_0^\infty e^{-t} \, dL_t\right) = 1.$$

The process L will be called the *local time of X at its supremum*. It satisfies $L_\infty < \infty$ a.s. if and only if X drifts to $-\infty$.

Let us introduce the ascending bivariate ladder process (τ, H): the ladder time process is $\tau_t = \inf\{s : L_s > t\}$, with the convention that $\inf \varnothing = +\infty$, and the ladder height process is $H_t = X(\tau_t)$, if $\tau_t < \infty$, and $H_t = \infty$, if $\tau_t = \infty$. The process (τ, H) is a (possibly killed) bivariate subordinator whose Laplace exponent is given by Fridstedt's formula:

$$\kappa(\alpha, \beta) = -\log \mathbb{E}(e^{-(\alpha\tau_1 + \beta H_1)})$$
$$= \exp\left(\int_0^\infty \int_0^\infty (e^{-t} - e^{-\alpha t - \beta x})t^{-1}\mathbb{P}\{X_t \in dx\}\, dt\right)$$

for $\alpha, \beta \geq 0$, with the convention that $e^{-\infty} = 0$; see Chapter VI of [2] or Chapter 4 of [11] and note that (2.1) squares with $\kappa(1, 0) = 1$. We write q^H, δ^H and π^H, respectively, for the killing rate, the drift coefficient and the Lévy measure of H. In particular, the Laplace exponent of H is given by

$$\kappa(0, \beta) = q^H + \delta^H \beta + \int_0^\infty (1 - e^{-\beta x})\pi^H(dx).$$

Note that our assumptions imply that if $\delta^H = 0$, then $\pi^H(0, \infty) = \infty$ since, otherwise, 0 would be irregular for the open half-line $(0, \infty)$ for X.

A random walk is a discrete-time process $S = (S_k, k = 0, 1, \ldots)$ such that $S_0 = 0$ and, for $k \geq 1$, $S_k = \sum_1^k Y_r$, where $Y_1, Y_2, \ldots$ are independent and identically distributed. We define the local time at its maximum of any random walk S by $\Lambda_0 = 0$ and, for all $k \geq 1$,

$$(2.2) \qquad \Lambda_k = \#\Big\{j \in \{1, \ldots, k\} : S_j > \max_{i \leq j-1} S_i\Big\}.$$

As in continuous time, $\Lambda_\infty < \infty$, a.s. if and only if S drifts to $-\infty$. We also introduce the strict ascending ladder processes for S. The strict ascending ladder time process T of S is defined by $T_0 = 0$ and, for all $k \geq 0$,

$$T_{k+1} = \min\{j > T_k : S_j > S_{T_k}\}$$

with $\min \varnothing = \infty$. The strict ascending ladder height process is given by

$$H_k = S(T_k) \qquad \text{if } T_k < \infty \quad \text{and} \quad H_k = \infty \qquad \text{if } T_k = \infty.$$

Note that T is the inverse of Λ, that is, $\Lambda_{T_k} = k$ for all $k \leq \Lambda_\infty$. We should point out that all of the results of this paper are still valid if, in the statements, one replaces the *strict* ladder process and the *strict* local time, respectively, by the *weak* ladder process, that is, $T_0 = 0$ and, for all $k \geq 0$, $T_{k+1} = \min\{j > T_k : S_j \geq S_{T_k}\}$, and the *weak* local time, that is, $\Lambda_k = \#\{j \in \{1, \ldots, k\} : S_j \geq \max_{i \leq j-1} S_i\}$.

In the sections which follow, $S^{(n)}$ will denote a random walk whose distribution can depend on n and $\Lambda^{(n)}$, $T^{(n)}$ and $H^{(n)}$ will denote the corresponding local

time, ladder time and ladder height process, respectively. We will say that the sequence of random walks $S^{(n)}$ converges in law (resp., almost surely) toward the Lévy process X if the sequence of continuous-time processes $(S^{(n)}_{[nt]}, t \geq 0)$ converges in law (resp., almost surely) toward X on the Skorokhod space $\mathcal{D}([0, \infty))$ of càdlàg paths. Note that, according to Theorem 2.7 of Skorokhod [18], if the process $(S^{(n)}_{[nt]}, t \geq 0)$ converges in the sense of finite-dimensional distributions, then it converges in law. If a stochastic process Y has lifetime ζ and if the $Y^{(n)}$'s have lifetimes $\zeta^{(n)}$, then we say that the sequence $Y^{(n)}$ converges toward Y in some sense if the sequence of processes $(Y^{(n)}_t \mathbb{1}_{\{t < \zeta^{(n)}\}} + Y^{(n)}_{\zeta^{(n)}-} \mathbb{1}_{\{t \geq \zeta^{(n)}\}}, t \geq 0)$ converges toward the process $(Y_t \mathbb{1}_{\{t < \zeta\}} + Y_{\zeta-} \mathbb{1}_{\{t \geq \zeta\}}, t \geq 0)$ in this sense on the space $\mathcal{D}([0, \infty))$. Also, note that convergence in law (resp., almost sure convergence) of stochastic processes on the space $\mathcal{D}([0, \infty))$ is equivalent to convergence in law (resp., almost sure convergence) on the space $\mathcal{D}([0, t])$ for all $t > 0$; see Theorem 16.7 in [4]. Convergence in law or almost sure convergence of a sequence of stochastic processes $Y^{(n)}$ toward Y will be denoted, respectively, by $Y^{(n)} \xrightarrow{\text{(law)}} Y$ and $Y^{(n)} \xrightarrow{\text{(a.s.)}} Y$.

3. Main results. The following result extends Theorem 3.2 in [17] and is the random walk counterpart of Lemma 3.4.2, page 54 in [20].

THEOREM 1. *Let X be any Lévy process such that 0 is regular for the open half-line $(0, \infty)$ and assume that some sequence of random walks $S^{(n)}$ converges in law toward X. We then have the following convergence in law*:

$$[(n^{-1} T^{(n)}_{[a_n t]}, H^{(n)}_{[a_n t]}), t \geq 0] \xrightarrow{\text{(law)}} (\tau, H)$$

as $n \to \infty$, where

$$(3.1) \qquad a_n = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} e^{-k/n} \mathbb{P}(S^{(n)}_k > 0) \right).$$

REMARK 1. Under the hypothesis of Theorem 1, that is, when 0 is regular for $(0, \infty)$, Rogozin's criterion asserts that $\int_0^1 t^{-1} \mathbb{P}(X_t > 0) \, dt = \infty$; see [2] Proposition VI.3.11. It follows from this result and weak convergence of $S^{(n)}$ toward X that in Theorem 1, we necessarily have $\lim_{n \to \infty} a_n = \infty$.

REMARK 2. The sequence $S^{(n)}$ could also be written in the form

$$S^{(n)} = \frac{1}{c_n} \widetilde{S}^{(n)}$$

and we would then recover the standard formulation for triangular arrays. However, in this case, using obvious notation, the result of Theorem 1 would become

$$[(n^{-1} \widetilde{T}^{(n)}_{[a_n t]}, c_n^{-1} \widetilde{H}^{(n)}_{[a_n t]}), t \geq 0] \xrightarrow{\text{(law)}} (\tau, H),$$

which reduces to Theorem 3.2 of [17] if the distribution of $\widetilde{S}^{(n)}$ does not depend on n.

PROOF OF THEOREM 1. We first recall Fristedt's formula for random walks; see [11], page 26. For every $\alpha > 0$ and $\beta > 0$, we have

$$1 - \mathbb{E}\big(e^{-\alpha T_1^{(n)} - \beta H_1^{(n)}}\big) = \exp\bigg(-\sum_{k=1}^{\infty} \frac{e^{-\alpha k}}{k} \mathbb{E}\big(e^{-\beta S_k^{(n)}} : S_k^{(n)} > 0\big)\bigg).$$

From this formula, we have

$$\mathbb{E}\big(e^{-\alpha n^{-1} T_{[a_n]}^{(n)} - \beta H_{[a_n]}^{(n)}}\big)$$

$$= \mathbb{E}\big(e^{-\alpha n^{-1} T_1^{(n)} - \beta H_1^{(n)}}\big)^{[a_n]}$$

$$= \bigg(1 - \exp\bigg[-\sum_{k=1}^{\infty} \frac{1}{k} e^{-\alpha n^{-1} k} \mathbb{E}\big(e^{-\beta S_k^{(n)}} : S_k^{(n)} > 0\big)\bigg]\bigg)^{[a_n]}$$

$$= \bigg(1 - \exp\bigg[-\int_1^{\infty} \frac{1}{[s]} e^{-\alpha n^{-1}[s]} \mathbb{E}\big(e^{-\beta S_{[s]}^{(n)}} : S_{[s]}^{(n)} > 0\big) ds\bigg]\bigg)^{[a_n]}$$

$$= \bigg(1 - \exp\bigg[-\int_{1/n}^{\infty} \frac{n}{[nt]} e^{-\alpha n^{-1}[nt]} \mathbb{E}\big(e^{-\beta S_{[nt]}^{(n)}} : S_{[nt]}^{(n)} > 0\big) dt\bigg]\bigg)^{[a_n]}.$$

From the assumptions and Rogozin's criterion recalled in Remark 1, we have

$$\lim_{n \to +\infty} \int_{1/n}^{\infty} \frac{n}{[nt]} e^{-\alpha n^{-1}[nt]} \mathbb{E}\big(e^{-\beta S_{[nt]}^{(n)}} : S_{[nt]}^{(n)} > 0\big) dt$$

$$= \int_0^{\infty} \frac{e^{-\alpha t}}{t} \mathbb{E}\big(e^{-\beta X_t} : X_t > 0\big) dt$$

$$= \infty,$$

hence

$$-\log \mathbb{E}\big(e^{-\alpha n^{-1} T_{[a_n]}^{(n)} - \beta H_{[a_n]}^{(n)}}\big)$$

$$\sim [a_n] \exp - \int_{1/n}^{\infty} \frac{n}{[nt]} e^{-\alpha n^{-1}[nt]} \mathbb{E}\big(e^{-\beta S_{[nt]}^{(n)}} : S_{[nt]}^{(n)} > 0\big) dt.$$

From the expression of a_n which is given in the statement of this theorem, the right-hand side of the above expression is

$$\exp\bigg(-\int_{1/n}^{\infty} \frac{n}{[nt]} e^{-\alpha n^{-1}[nt]} \mathbb{E}\big(e^{-\beta S_{[nt]}^{(n)}} : S_{[nt]}^{(n)} > 0\big) dt + \sum_{k=1}^{\infty} \frac{1}{k} e^{-k/n} \mathbb{P}\big(S_k^{(n)} > 0\big)\bigg)$$

$$= \exp \int_{1/n}^{\infty} \frac{n}{[nt]} \mathbb{E}\big(e^{-n^{-1}[nt]} - e^{-\alpha n^{-1}[nt] - \beta S_{[nt]}^{(n)}} : S_{[nt]}^{(n)} > 0\big) dt,$$

which, as n goes to ∞, converges toward

$$\exp \int_0^\infty \frac{1}{t}\mathbb{E}(e^{-t} - e^{-\alpha t - \beta X_t} : X_t > 0)\, dt = \kappa(\alpha, \beta)$$

since $X_t \neq 0$ a.s. It is clear that the process X drifts to $-\infty$ if and only if $S^{(n)}$ drifts to $-\infty$ for all n sufficiently large. First, suppose that X does not drift to $-\infty$. The above convergence proves that the sequence $[(n^{-1}T_{[a_n t]}^{(n)}, H_{[a_n t]}^{(n)}), t \geq 0]$ converges, in the sense of finite-dimensional distributions, toward (τ, H). We conclude that it converges weakly by applying Theorem 2.7 of Skorokhod [18].

If X drifts to $-\infty$, then the sequence $(T^{(n)}, H^{(n)})$ and the process (τ, H) are obtained, respectively, from a sequence of bivariate renewal processes, say $(\overline{T}^{(n)}, \overline{H}^{(n)})$, and a bivariate subordinator, say $(\overline{\tau}, \overline{H})$, all with infinite lifetime, by killing them, respectively, at independent random times. It readily follows from the convergence of the Laplace exponents which is proved above that

$$[(n^{-1}\overline{T}_{[a_n t]}^{(n)}, \overline{H}_{[a_n t]}^{(n)}), t \geq 0] \xrightarrow{\text{(law)}} (\overline{\tau}, \overline{H})$$

and that the independent killing times of $(n^{-1}\overline{T}_{[a_n t]}^{(n)}, \overline{H}_{[a_n t]}^{(n)})$ converge in law to the one of $(\overline{\tau}, \overline{H})$. As a straightforward consequence, the sequence of killed processes $(n^{-1}T_{[a_n t]}^{(n)}, H_{[a_n t]}^{(n)})$ converges to (τ, H), in the sense which is defined in the preliminary section. $\quad\square$

Since τ is an increasing process, we derive from Theorem 1 and Theorem 7.2 of [22] that when $S^{(n)}$ converges in law to X, the renormed process $(a_n^{-1}\Lambda_{[nt]}^{(n)}, t \geq 0)$ converges in law to $(L_t, t \geq 0)$. We actually establish the following, stronger, result.

THEOREM 2. *Let X be as in Theorem 1 and assume that*

$$(3.2) \qquad\qquad (S_{[nt]}^{(n)}, t \geq 0) \xrightarrow{(a.s.)} (X_t, t \geq 0).$$

Let $\Lambda^{(n)}$ be the local time at its maximum of $S^{(n)}$. A normed version of $\Lambda^{(n)}$ then converges uniformly in probability on compacts sets toward L. More specifically, for all $t \geq 0$ and $\varepsilon > 0$,

$$(3.3) \qquad\qquad \lim_{n\to\infty} \mathbb{P}\Big(\sup_{s\in[0,t]} |a_n^{-1}\Lambda_{[ns]}^{(n)} - L_s| > \varepsilon \Big) = 0,$$

where a_n is defined by expression (3.1).

The proof of this theorem requires the two following lemmas. We denote by π^τ and π^H the Lévy measures of τ and H, respectively.

LEMMA 1. *The Lévy measure π^H has no atom whenever X is not a compound Poisson process. If, moreover, 0 is regular for $(-\infty, 0)$, then the Lévy measure π^τ has a density with respect to the Lebesgue measure.*

PROOF. Let us introduce some notation: we call $(\widehat{\tau}, \widehat{H})$ the *ladder process* associated to $\widehat{X} = -X$ and we call $\widehat{U}$ the *renewal measure* of this bivariate subordinator. The renewal measure of $\widehat{H}$ is denoted by $U^{\widehat{H}}$ and the Lévy measure of X is denoted by Π.

From Vigon's "équation amicale inversée" (see Vigon [21] or [20], page 71), we have that for all $x > 0$ and $0 < h < x$,

$$\pi^H(x - h, x] = \int_0^\infty U^{\widehat{H}}(dy)\Pi(x + y - h, x + y].$$

By monotone convergence, we get

$$\pi^H(\{x\}) = \int_0^\infty U^{\widehat{H}}(dy)\Pi(\{x + y\})$$

and this is zero because there are countably many atoms of Π and $U^{\widehat{H}}$ is diffuse when X is not a compound Poisson process; see Proposition 1.15, [2]. This proves the first assertion.

Corollary 6, page 50 of [11], asserts that whenever X is not a compound Poisson process, the Lévy measure π of (τ, H) is given by

$$\pi(dt, dh) = \int_{[0,\infty)} \widehat{U}(dt, dx)\Pi(dh + x).$$

Then, from Theorem 5 of [1], under the additional assumption that 0 is regular for $(-\infty, 0)$, we have, for all $t > 0$, that

$$q_t(dx)\, dt = c\widehat{U}(dt, dx),$$

where c is a constant and $q_t(dx)$ is the entrance law of the measure of the excursions away from 0 of the process X reflected at its supremum. The second assertion is thus proved. □

The second lemma follows from Theorem 1, Lemma 1 and a standard criterion on convergence of sums of independent random variables (see, e.g., [16]), so we omit its proof.

LEMMA 2. *Define, for $0 < a < b \le \infty$, $0 < c < \infty$ and $n \ge 1$,*

$$\pi_n^{a,b} = \mathbb{P}(H_1^{(n)} \in (a, b]), \qquad m_1^{n,a} = \mathbb{E}(H_1^{(n)} : H_1^{(n)} \le a),$$

$$m_2^{n,a} = \mathbb{E}((H_1^{(n)})^2 : H_1^{(n)} \le a), \qquad v_n^c = \mathbb{P}(n^{-1}T_1^{(n)} > c).$$

Under the assumptions of Theorem 1, *the following asymptotics hold*:

$$\lim_{n\to\infty} a_n \pi_n^{a,b} = \pi^H(a,b], \qquad \lim_{n\to\infty} a_n m_1^{n,a} = \delta^H + \int_0^a x\pi^H(dx)$$

and

$$\lim_{n\to\infty} a_n m_2^{n,a} = \int_0^a x^2 \pi^H(dx).$$

If, moreover, 0 *is regular for* $(-\infty, 0)$, *then*

$$\lim_{n\to+\infty} a_n v_n^c = \pi^\tau(c,\infty).$$

PROOF OF THEOREM 2. We first observe that since $(a_n^{-1}\Lambda_{[nt]}^{(n)}, t \geq 0)$ is a sequence of nondecreasing processes which converges toward the continuous process L, in order to prove the uniform convergence in (3.3), it suffices to establish pointwise convergence in probability, that is, for all $t \geq 0$,

(3.4) $$\lim_{n\to\infty} \mathbb{P}(|a_n^{-1}\Lambda_{[nt]}^{(n)} - L_t| > \varepsilon) = 0.$$

We first treat the case where $\pi^H[0,\infty) < \infty$. Since we have assumed that 0 is regular for $(0,\infty)$, we necessarily have $\delta^H > 0$ and then

(3.5) $$\delta^H L_t = \lambda(M_s : s \leq t),$$

where λ is the Lebesgue measure. Let $M_k^{(n)} = \max_{0\leq j\leq k} S_j^{(n)}$, $k \geq 0$, and for $a > 0$, define the truncated past maxima of $S^{(n)}$ and X, respectively, as

$$M_{[nt]}^{n,a} = M_{[nt]}^{(n)} - \sum_{s\in[0,t]} \Delta M_s^{(n)} \mathbb{1}_{\{\Delta M_s^{(n)} > a\}} \quad \text{and} \quad M_t^a = M_t - \sum_{s\in[0,t]} \Delta M_s \mathbb{1}_{\{\Delta M_s > a\}}.$$

Since, in this case, M has only a finite number of jumps in each interval $[0, t]$, we have the almost sure convergence

(3.6) $$\lim_{n\to\infty} M_{[nt]}^{n,a} = M_t^a \qquad \text{a.s.}$$

Moreover, for the same reason and by (3.5), for all a small enough, we have

(3.7) $$\delta^H L_t = M_t^a.$$

From (3.6) and (3.7), it then suffices to prove that

(3.8) $$\lim_{a\downarrow 0} \limsup_{n\to\infty} \mathbb{P}\left(\left| M_{[nt]}^{n,a} - \frac{\delta^H}{a_n}\Lambda_{[nt]}^{(n)} \right| > \varepsilon \right) = 0.$$

Note that for all k, $M^{n,a}(T_k^{(n)})$ is the sum of k i.i.d. random variables with mean $m_1^{n,a}$ and second moment $m_2^{n,a}$ defined in Lemma 2. Hence, for all $K > 0$ and $\varepsilon > 0$, from Kolmogorov's inequality,

$$\mathbb{P}\left(\max_{0\leq j\leq T_{[Ka_n]}^{(n)}} |M_j^{n,a} - m_1^{n,a}\Lambda_j^{(n)}| > \varepsilon \right) \leq \frac{Ka_n m_2^{n,a}}{\varepsilon^2}.$$

Now, write the inequality

$$\mathbb{P}\left(\max_{0 \le j \le T^{(n)}_{[Ka_n]}} \left| M^{n,a}_j - \frac{\delta^H}{a_n} \Lambda^{(n)}_j \right| > 2\varepsilon\right)$$

$$\le \mathbb{P}\left(\max_{0 \le j \le T^{(n)}_{[Ka_n]}} \left| \frac{\delta^H}{a_n} \Lambda^{(n)}_j - m^{n,a}_1 \Lambda^{(n)}_j \right| > \varepsilon\right) + \frac{Ka_n m^{n,a}_2}{\varepsilon^2}.$$

Then observe that the first term on the right-hand side is nothing but $\mathbb{1}_{\{|K\delta^H - m^{n,a}_1 Ka_n| > \varepsilon\}}$ and, from Lemma 2, $\lim_{a \to 0} \lim_{n \to \infty} \mathbb{1}_{\{|K\delta^H - m^{a,n}_1 Ka_n| > \varepsilon\}} = 0$. From the same lemma, we have, for the second term, $\lim_{a \to 0} \lim_{n \to \infty} a_n m^{n,a}_2 = 0$. Hence,

$$\lim_{a \to 0} \lim_{n \to \infty} \mathbb{P}\left(\max_{0 \le j \le T^{(n)}_{[Ka_n]}} \left| M^{n,a}_j - \frac{\delta^H}{a_n} \Lambda^{(n)}_j \right| > 2\varepsilon\right) = 0.$$

Finally, write

$$\mathbb{P}\left(\left| M^{n,a}_n - \frac{\delta^H}{a_n} \Lambda^{(n)}_n \right| > \varepsilon\right)$$

$$\le \mathbb{P}\left(\max_{0 \le j \le T^{(n)}_{[Ka_n]}} \left| M^{n,a}_j - \frac{\delta^H}{a_n} \Lambda^{(n)}_j \right| > \varepsilon\right) + \mathbb{P}(T^{(n)}_{[Ka_n]} < n).$$

But, from Theorem 1, we have $\lim_{K \to +\infty} \lim_{n \to \infty} \mathbb{P}(T^{(n)}_{[Ka_n]} < n) = 0$ and (3.8) follows for $t = 1$. (This proof can readily be extended to any time $t \ge 0$.) So, we have proven (3.4) in the case $\pi^H[0, \infty) < \infty$.

Now, let us suppose that $\pi^H[0, \infty) = \infty$ and, for $0 < a < b < \infty$, define the following approximations of the local times L and $\Lambda^{(n)}$:

$$L^{a,b}_t = \#\{s \le t : \Delta M_s \in (a, b]\}$$

and

$$\Lambda^{n,a,b}_k = \#\{j \le k : M^{(n)}_j + a < S^{(n)}_{j+1} \le M^{(n)}_j + b\}.$$

Since $L^{a,b}_t$ is a finite integer, it readily follows from the almost sure convergence of $S^{(n)}_{[n\cdot]}$ toward X that

$$(3.9) \qquad \lim_{n \to +\infty} \Lambda^{n,a,b}_{[nt]} = L^{a,b}_t \qquad \text{a.s.}$$

On the other hand, observe that $(L^{a,b}_{\tau_t}, t \ge 0)$ is a Poisson process with intensity $\pi^H(a, b)$. Moreover, from the hypothesis, we have $\lim_{a \downarrow 0} \pi^H(a, b) = +\infty$. It therefore follows from the law of large numbers that for all $t > 0$,

$\lim_{a\to 0} \pi^H(a, b]^{-1} L_{\tau_t}^{a,b} = t$ a.s. From monotonicity, this convergence can be strengthened to uniform convergence: for all $u > 0$,

$$\lim_{a\to 0} \sup_{t\in[0,u]} |\pi^H(a, b]^{-1} L_{\tau_t}^{a,b} - t| = 0 \qquad \text{a.s.}$$

Fix $\varepsilon > 0$. For all $\eta > 0$, we can choose u sufficiently large that $\mathbb{P}(\tau_u < 1) < \eta/2$ and a sufficiently small that $\mathbb{P}(\sup_{t\in[0,u]} |\pi^H(a, b]^{-1} L_{\tau_t}^{a,b} - t| > \varepsilon) < \eta/2$. The inequality

$$\mathbb{P}\Big(\sup_{t\in[0,1]} |\pi^H(a, b]^{-1} L_t^{a,b} - L_t| > \varepsilon \Big)$$

$$\leq \mathbb{P}\Big(\sup_{t\in[0,\tau_u]} |\pi^H(a, b]^{-1} L_t^{a,b} - L_t| > \varepsilon \Big) + \mathbb{P}(\tau_u < 1),$$

then allows us to obtain

$$(3.10) \qquad \lim_{a\to 0} \mathbb{P}\Big(\sup_{t\in[0,1]} |\pi^H(a, b]^{-1} L_t^{a,b} - L_t| > \varepsilon \Big) = 0.$$

Note that, for all k, $\Lambda^{n,a,b}(T_k^{(n)})$ is the sum of k independent Bernoulli random variables with mean $\pi_n^{a,b}$ defined in Lemma 2. Hence, for all $K > 0$, from Kolmogorov's inequality, we have

$$\mathbb{P}\Big(\max_{0\leq j\leq T_{[Ka_n]}^{(n)}} \frac{1}{\pi^H(a, b]} |\Lambda_j^{a,b,n} - \pi_n^{a,b}\Lambda_j^{(n)}| > \varepsilon \Big) \leq \frac{Ka_n\pi_n^{a,b}}{\pi^H(a, b]^2\varepsilon^2}.$$

Now, write the inequality

$$\mathbb{P}\Big(\max_{0\leq j\leq T_{[Ka_n]}^{(n)}} \Big| \frac{1}{\pi^H(a, b]} \Lambda_j^{a,b,n} - \frac{1}{a_n}\Lambda_j^{(n)} \Big| > 2\varepsilon \Big)$$

$$\leq \mathbb{P}\Big(\max_{0\leq j\leq T_{[Ka_n]}^{(n)}} \Big| \frac{1}{a_n} \Lambda_j^{(n)} - \frac{\pi_n^{a,b}}{\pi^H(a, b]}\Lambda_j^{(n)} \Big| > \varepsilon \Big) + \frac{Ka_n\pi_n^{a,b}}{\pi^H(a, b]^2\varepsilon^2}.$$

The first term of the right-hand side is $\mathbb{1}_{\{|K-Ka_n\pi_n^{a,b}/\pi^H(a,b]|>\varepsilon\}}$ and from Lemma 2, $\lim_{n\to\infty} a_n\pi_n^{a,b} = \pi^H(a, b]$, so this term converges to 0 for all a and b. The second term converges to $K/\varepsilon^2\pi^H(a, b]$ as n tends to ∞. Since, from the hypothesis, we have $\lim_{a\to 0} \pi^H(a, b] = \infty$ for all b, we conclude that

$$\lim_{a\to 0}\lim_{n\to\infty} \mathbb{P}\Big(\max_{0\leq j\leq T_{[Ka_n]}^{(n)}} \Big| \frac{1}{\pi^H(a, b]} \Lambda_j^{a,b,n} - \frac{1}{a_n}\Lambda_j^{(n)} \Big| > 2\varepsilon \Big) = 0.$$

Finally, we write

$$\mathbb{P}\Big(\sup_{t\in[0,1]} \Big| \frac{1}{\pi^H(a, b]} \Lambda_{[nt]}^{a,b,n} - \frac{1}{a_n}\Lambda_{[nt]}^{(n)} \Big| > \varepsilon \Big)$$

$$\leq \mathbb{P}\Big(\max_{0\leq j\leq T_{[Ka_n]}^{(n)}} \Big| \frac{1}{\pi^H(a, b]} \Lambda_j^{a,b,n} - \frac{1}{a_n}\Lambda_j^{(n)} \Big| > \varepsilon \Big) + \mathbb{P}(T_{[Ka_n]}^{(n)} < n).$$

Therefore, (3.4) for $t = 1$ follows from (3.9), (3.10) and the fact that

$$\lim_{K \to \infty} \lim_{n \to \infty} \mathbb{P}(T^{(n)}_{[Ka_n]} < n) = 0,$$

which is a consequence of Theorem 1. Again, the proof can readily be extended to any time t. Hence, (3.4) is proved in this case. □

When 0 is regular for $(-\infty, 0)$, we may also define the local time at the infimum of X to be the local time at the supremum of $-X$. Let us denote this process by $\widehat{L}$ and denote by $\widehat{\Lambda}^{(n)}$ the local time at the maximum of the approximating sequence $-S^{(n)}$. A straightforward consequence of the previous theorem is the following result.

COROLLARY 1. *Under the hypotheses of Theorem* 2, *the sequence*

$$\left[\left(S^{(n)}_{[nt]}, \frac{1}{a_n} \Lambda^{(n)}_{[nt]}\right), t \geq 0\right]$$

converges in probability to $[(X_t, L_t), t \geq 0]$. *If, in addition*, 0 *is regular for* $(-\infty, 0)$, *then the sequence*

$$\left[\left(S^{(n)}_{[nt]}, \frac{1}{a_n} \Lambda^{(n)}_{[nt]}, \frac{1}{\widehat{a}_n} \widehat{\Lambda}^{(n)}_{[nt]}\right), t \geq 0\right]$$

converges in probability to $[(X_t, L_t, \widehat{L}_t), t \geq 0]$, *where* $\widehat{a}_n = \exp(\sum_{k=1}^{\infty} \frac{1}{k} e^{-k/n} \times \mathbb{P}(S^{(n)}_k < 0))$.

In Corollary 1, convergence in probability means that each coordinate converges in probability with respect to some distance which generates the Skorokhod topology on the space $\mathcal{D}([0, \infty))$. But, more particularly, the first coordinate converges almost surely, whereas the second coordinate converges uniformly in probability on compact sets, in the sense which was defined in Theorem 2.

When 0 is regular for $(-\infty, 0)$, we denote by $(\widehat{\tau}, \widehat{H})$ the strict ascending ladder process of $-X$ and, for the sequence of random walks $S^{(n)}$, we denote by $(\widehat{T}^{(n)}, \widehat{H}^{(n)})$ the strict ascending ladder height process of $-S$. Another consequence of Theorem 2 is the following invariance principle for both the ascending and descending ladder processes jointly.

COROLLARY 2. *Let X be any Lévy process such that* 0 *is regular for both of the open half-lines* $(0, \infty)$ *and* $(-\infty, 0)$, *and assume that some sequence of random walks $S^{(n)}$ converges in law toward X. The process*

$$\left[\left(n^{-1} T^{(n)}_{[a_n t]}, H^{(n)}_{[a_n t]}, n^{-1} \widehat{T}^{(n)}_{[\widehat{a}_n t]}, \widehat{H}^{(n)}_{[\widehat{a}_n t]}\right), t \geq 0\right]$$

then converges toward the process

$$[(\tau_t, H_t, \widehat{\tau}_t, \widehat{H}_t), t \geq 0]$$

in the sense of finite-dimensional distributions, as $n \to \infty$, where a_n and $\widehat{a}_n$ are defined, respectively, in Theorem 1 *and Corollary* 1.

REMARK 3. Note that, in this case, we cannot conclude that weak convergence holds by using Skorokhod's theorem as in Theorem 1 since the quadrivariate processes which are involved in Corollary 2 do not have independent increments.

PROOF OF COROLLARY 2. By virtue of the Skorokhod representation theorem, there exists a sequence $\widetilde{S}^{(n)}$ (possibly defined on an enlarged probability space) such that for each n, $\widetilde{S}^{(n)} \stackrel{(d)}{=} S^{(n)}$ and such that $\widetilde{S}^{(n)}$ converges almost surely toward X. Let $(\mathcal{T}^{(n)}, \mathcal{H}^{(n)})$ and $(\widehat{\mathcal{T}}^{(n)}, \widehat{\mathcal{H}}^{(n)})$ be, respectively, the strict ascending and the strict descending ladder processes of $\widetilde{S}^{(n)}$.

Recall that if a sequence of stochastic processes converges almost surely on the Skorokhod space, then the sequence defined by the first passage time processes converges at all continuity points of the limit process; see the remark after Theorem 7.1 in [22]. Moreover, it is clear that the subordinators τ and $\widehat{\tau}$ are a.s. continuous at each $t \geq 0$. Therefore, from Theorem 2 applied to $\widetilde{S}^{(n)}$ and $-\widetilde{S}^{(n)}$, for fixed $t \geq 0$, there exists a subsequence (k_n) such that $k_n^{-1} \mathcal{T}_{[a_{k_n} t]}^{(k_n)}$ and $k_n^{-1} \widehat{\mathcal{T}}_{[\widehat{a}_{k_n} t]}^{(k_n)}$ converge almost surely toward τ_t and $\widehat{\tau}_t$, respectively.

Since τ_t and $\widehat{\tau}_t$ are announceable stopping times in the filtration generated by X, it follows from the quasi-left continuity of X that this process is a.s. continuous at times τ_t and $\widehat{\tau}_t$; see Example 3, Chapter I in [2].

We deduce from the almost sure convergence of $\widetilde{S}^{(k_n)}$ toward X that for every (possibly random) continuity point u of X, $\widetilde{S}_{[k_n u]}^{(k_n)}$ converges almost surely to X_u; see [4], page 112. Therefore, the sequence

$$\left(k_n^{-1} \mathcal{T}_{[a_{k_n} t]}^{(k_n)}, \mathcal{H}_{[a_{k_n} t]}^{(k_n)}, k_n^{-1} \widehat{\mathcal{T}}_{[\widehat{a}_{k_n} t]}^{(k_n)}, \widehat{\mathcal{H}}_{[\widehat{a}_{k_n} t]}^{(k_n)}\right)$$

$$= \left(k_n^{-1} \mathcal{T}_{[a_{k_n} t]}^{(k_n)}, \widetilde{S}^{(k_n)}\big(\mathcal{T}_{[a_{k_n} t]}^{(k_n)}\big), k_n^{-1} \widehat{\mathcal{T}}_{[\widehat{a}_{k_n} t]}^{(k_n)}, -\widetilde{S}^{(k_n)}\big(\widehat{\mathcal{T}}_{[\widehat{a}_{k_n} t]}^{(k_n)}\big)\right)$$

converges almost surely toward $(\tau_t, X(\tau_t), \widehat{\tau}_t, -X(\widehat{\tau}_t)) = (\tau_t, H_t, \widehat{\tau}_t, \widehat{H}_t)$ as $n \to \infty$. This almost sure convergence is easily extended to the multidimensional case, that is, there is a subsequence (k_n') such that it holds simultaneously at any sequence of times $t_1, \ldots, t_j$. So, we have proven that the variables $\{n^{-1} \mathcal{T}_{[a_n t_i]}^{(n)}, \mathcal{H}_{[a_n t_i]}^{(n)}, n^{-1} \widehat{\mathcal{T}}_{[\widehat{a}_n t_i]}^{(n)}, \widehat{\mathcal{H}}_{[\widehat{a}_n t_i]}^{(n)}, i = 1, \ldots, j\}$ converge in probability and we conclude the proof using the identity in law

$$\left(\mathcal{T}^{(n)}, \mathcal{H}^{(n)}, \widehat{\mathcal{T}}^{(n)}, \widehat{\mathcal{H}}^{(n)}\right) \stackrel{(d)}{=} \left(T^{(n)}, H^{(n)}, \widehat{T}^{(n)}, \widehat{H}^{(n)}\right),$$

which holds for each n, as a consequence of the identity $\widetilde{S}^{(n)} \stackrel{(d)}{=} S^{(n)}$. □

We now suppose that there is a sequence of Lévy processes $X^{(n)}$, all of which satisfy the same hypothesis as X, that is, 0 is regular for $(0, \infty)$. Call $L^{(n)}$ the version of the local time of $X^{(n)}$ at its supremum, as defined in Section 2.

THEOREM 3.   *Suppose that, as n tends to* ∞,

$$X^{(n)} \xrightarrow{\text{a.s.}} X.$$

The sequence of local times $L^{(n)}$ *then converges uniformly on compact sets in probability toward* L, *that is, for all* $t > 0$ *and* $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\Big(\sup_{s \in [0,t]} |L_t^{(n)} - L_t| > \varepsilon \Big) = 0.$$

PROOF.   For each n, we define a sequence of random walks $(S^{n,k}, k \geq 0)$ whose paths are embedded in those of $X^{(n)}$ as follows:

$$S_j^{n,k} = X_{j/k}^{(n)}, \qquad j \geq 0.$$

We may then readily check that for each n, as k tends to ∞,

$$\big(S_{[kt]}^{n,k}, t \geq 0\big) \xrightarrow{\text{a.s.}} X^{(n)}.$$

We will call $\Lambda^{n,k}$ the local time at its maximum of $S^{n,k}$, as defined for $S^{(n)}$ in (2.2). From Theorem 2, we have, for all $n \geq 1$, $t \geq 0$ and $\varepsilon > 0$,

$$\lim_{k \to \infty} \mathbb{P}\Big(\sup_{s \in [0,t]} \Big| \frac{1}{a_k^n} \Lambda_{[ns]}^{n,k} - L_s^{(n)} \Big| > \varepsilon \Big) = 0,$$

where $\log a_k^n = \sum_{j=1}^{\infty} \frac{1}{j} e^{-j/k} \mathbb{P}(S_j^{n,k} > 0)$. We can choose a sequence of integers $(k_n)_{n \geq 1}$ such that

$$\lim_{n \to \infty} \mathbb{P}\Big(\sup_{s \in [0,t]} \Big| \frac{1}{a_{k_n}^n} \Lambda_{[ns]}^{n,k_n} - L_s^{(n)} \Big| > \varepsilon \Big) = 0$$

and, as n goes to ∞,

$$\big(S_{[k_n t]}^{n,k_n}, t \geq 0\big) \xrightarrow{\text{a.s.}} X.$$

Hence by again applying Theorem 2, we have

$$\lim_{n \to \infty} \mathbb{P}\Big(\sup_{s \in [0,t]} \Big| \frac{1}{a_{k_n}^n} \Lambda_{[ns]}^{n,k_n} - L_s \Big| > \varepsilon \Big) = 0,$$

which allows us to conclude the proof. $\square$

It is clear that the equivalent results to Corollaries 1 and 2 are also valid in the setting of Theorem 3, that is, replacing the approximating sequence $S^{(n)}$ by the sequence $X^{(n)}$.

4. Applications to conditioned processes. In this section, we will prove that if a sequence $S^{(n)}$ of random walks converges weakly toward a Lévy process X, then the sequence $S^{(n)}$ conditioned to stay positive also converges weakly toward X conditioned to stay positive. For simplicity, in the statements and proofs, we will always suppose that $S^{(n)}$ and X do not drift to $-\infty$ and that for X, the state 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$.

We first define $S^{(n)}$ and X conditioned to stay positive on the whole time interval $[0, \infty)$. Let $V^{(n)}(x) = \sum_{k \geq 0} \mathbb{P}(\widehat{H}_k^{(n)} \leq x)$, $x \geq 0$, be the renewal function of $\widehat{H}^{(n)}$ and let $S^{(n)*}$ be the process $S^{(n)}$ killed when it enters the negative half-line. The Markovian transition function, which is given by

$$q^\uparrow(x, dy) = \frac{V^{(n)}(y)}{V^{(n)}(x)} \mathbb{P}(S_{k+1}^{(n)*} \in dy | S_k^{(n)*} = x)$$

for $x > 0$, $y > 0$ if $k \geq 1$, and for $x \geq 0$, $y > 0$ if $k = 0$, characterizes the law of an h-process of $S^{(n)*}$, which is called *the law of $S^{(n)}$ conditioned to stay positive*. If we suppose that 0 is regular for $(-\infty, 0)$ and let $h(x)$ be the renewal function of the subordinator $\widehat{H}$, that is, $h(x) = \mathbb{E}(\int_0^\infty \mathbb{1}_{\{\widehat{H}_t \leq x\}} dt)$, then the Markovian semigroup

$$p_t^\uparrow(x, dy) = \frac{h(y)}{h(x)} \mathbb{P}(X_{t+s}^* \in dy | X_s^* = x) \qquad \text{for } x, y > 0 \text{ and } s, t > 0$$

is that of the Lévy process X *conditioned to stay positive*. For $x = 0$, this semigroup admits a unique entrance law which is specified in terms of the measure of the excursions above the infimum of the process X. We refer to [3, 7] and [8] for a more complete account on random walks and Lévy processes conditioned to stay positive.

The proof of our invariance principle makes use of a pathwise construction of $S^{(n)}$ and X conditioned to stay positive which is due to Tanaka and Doney; see [12, 19] and [11], page 91. Let us briefly recall it, both in discrete time and in continuous time. For $k \geq 0$, denote by $e^{(k)}$ the kth excursion of the reflected process $M^{(n)} - S^{(n)}$,

$$e_i^{(k)} = \left(M^{(n)} - S^{(n)} \right)_{T_k^{(n)} + i}, \qquad 0 \leq i \leq T_{k+1}^{(n)} - T_k^{(n)},$$

and denote by $\widehat{e}^{(k)}$ the time reversal of $e^{(k)}$, that is,

$$\widehat{e}_i^{(k)} = H_{k+1}^{(n)} - S^{(n)}(T_{k+1}^{(n)} - i), \qquad 0 \leq i \leq T_{k+1}^{(n)} - T_k^{(n)}.$$

The process $S^{(n)\uparrow}$, which is obtained from the concatenation of $\widehat{e}^{(0)}, \widehat{e}^{(1)}, \ldots$, that is,

(4.1) $$S_i^{(n)\uparrow} = H_k^{(n)} + \widehat{e}_{i-T_k^{(n)}}^{(k)} \qquad \text{if } T_k^{(n)} \leq i \leq T_{k+1}^{(n)}$$

has the law of $S^{(n)}$ conditioned to stay positive. A similar construction in continuous time was obtained in [11]: for $t > 0$, if we let

$$g(t) = \sup\{s < t : X_s = M_s\} \quad \text{and} \quad d(t) = \inf\{s > t : X_s = M_s\},$$

then the process defined by

(4.2)
$$X_0^\uparrow = 0 \quad \text{and}$$
$$X_t^\uparrow = M_{d(t)} + \mathbb{1}_{\{d(t) > g(t)\}} (M - X)_{(d(t)+g(t)-t)-}, \qquad t > 0,$$

has the law of X conditioned to stay positive.

Let us also define the local time at the future minimum of $S^{(n)\uparrow}$ and $X^\uparrow$. The first of these processes is simply the counting process defined by $\underline{\underline{\Lambda}}_0^{(n)} = 0$ and, for $k \geq 1$,

$$\underline{\underline{\Lambda}}_k^{(n)} = \#\left\{ j \in \{1, \ldots, k\} : S_{j-1}^{(n)\uparrow} < \min_{i \geq j} S_i^{(n)\uparrow} \right\}.$$

Recall that in continuous time, the set $\{t : X_t^\uparrow = \inf_{s \geq t} X_s^\uparrow\}$ is regenerative so that we may define on this set a local time $\underline{L}$; see [7], page 44. This local time is unique up to a normalizing constant and we will normalize it by $\mathbb{E}(\int_0^\infty e^{-t} d\underline{L}_t) = 1$. One easily derives from the above pathwise constructions the identities

$$\left\{ j \geq 1 : S_{j-1}^{(n)\uparrow} < \min_{i \geq j} S_i^{(n)\uparrow} \right\} = \left\{ j \geq 1 : S_j > \max_{i \leq j-1} S_i \right\}$$

and $\{t : X_t^\uparrow = \inf_{s \geq t} X_s^\uparrow\} = \{t : X_t = \sup_{s \leq t} X_s\}$. In particular, we have

(4.3)
$$\underline{\underline{\Lambda}}^{(n)} = \Lambda^{(n)} \quad \text{and} \quad \underline{L} = L \qquad \text{a.s.}$$

The following theorem has been partially obtained in the particular setting of stable processes in [6]; see Theorem 1.1.

THEOREM 4. *Suppose that some sequence of random walks $S^{(n)}$ converges almost surely toward X. Recall the definition of a_n from Theorem* 1. *Then:*

1. *the sequence of processes $(S_{[nt]}^{(n)\uparrow}, t \geq 0)$ converges almost surely toward $X^\uparrow$;*
2. *the sequence $[(S_{[nt]}^{(n)\uparrow}, a_n^{-1} \underline{\underline{\Lambda}}_{[nt]}^{(n)}), t \geq 0]$ converges in probability toward $(X^\uparrow,$*
   $\underline{L})$.

Consequently, if some sequence of random walks $S^{(n)}$ converges weakly toward X, then the sequence $[(S_{[nt]}^{(n)\uparrow}, a_n^{-1} \underline{\underline{\Lambda}}_{[nt]}^{(n)}), t \geq 0]$ converges weakly toward $(X^\uparrow, \underline{L})$.

Actually, the result displayed in Theorem 4 holds in the very general case, although, as stated at the beginning of this section, for simplicity in its statement and proof, we restrict ourself to the case where 0 is regular for both half-lines $(-\infty, 0)$ and $(0, \infty)$.

The time-reversal relationships between X and $X^\uparrow$ and between $S^{(n)}$ and $S^{(n)\uparrow}$ which are presented below, in Theorem 5 and Lemma 3, are required for the proof

of Theorem 4. Let us denote by $U^{(n)}$ and σ, respectively, the inverses of $\underline{\underline{\Lambda}}^{(n)}$ and $\underline{L}$, that is,

$$U_k^{(n)} = \min\{i : \underline{\underline{\Lambda}}_i^{(n)} = k\}, \qquad k \geq 0 \quad \text{and} \quad \sigma_t = \inf\{s : \underline{L}_s > t\}, \qquad t \geq 0.$$

We also set

$$G_k^{(n)\uparrow} = \max\left\{ j \leq k : S_j^{(n)\uparrow} = \inf_{i \geq j} S_i^{(n)\uparrow} \right\}, \qquad g_t^\uparrow = \sup\left\{ s \leq t : X_s^\uparrow = \inf_{u \geq s} X_u^\uparrow \right\}$$

and $G_k^{(n)} = \max\{ j \leq k : M_j^{(n)} = S_j^{(n)} \}$.

THEOREM 5. *The following time-reversal relationships hold between X and $X^\uparrow$:*

1. *for all $t > 0$, the law of the process $[(X_{\tau_t} - X_{(\tau_t - s)-}, L_{\tau_t} - L_{\tau_t - s}), 0 \leq s < \tau_t]$ is the same as that of the process $[(X_s^\uparrow, \underline{\underline{L}}_s), 0 \leq s < \sigma_t]$;*
2. *for all $t > 0$, the law of the process $[(X_{g(t)} - X_{(g(t)-s)-}, L_{g(t)} - L_{g(t)-s}), 0 \leq s \leq g(t)]$ (with the convention that $X_{0-} = X_0$) is the same as that of the process $[(X_s^\uparrow, \underline{\underline{L}}_s), 0 \leq s \leq g_t^\uparrow]$.*

Note that in the above statement, we have $X_{g(t)} = M_t$ and $L_{\tau_t} = t$ almost surely. Part 1 of this theorem is Lemma 4.3 of Duquesne [14]. The case where these processes have no positive jumps is treated in Theorem VII.18 of [2]. It generalizes a well-known transformation between Brownian motion and the three-dimensional Bessel process due to Williams. Here, we show that this result can easily be derived from simple arguments involving the Tanaka–Doney transformation. Our next lemma states the discrete-time counterpart of Theorem 5. Its proof is very similar to that of Theorem 5, hence we will only prove the discrete-time case.

LEMMA 3. *For any $k \geq 1$:*

1. *the law of the process $[(S_{T_k^{(n)}}^{(n)} - S_{T_k^{(n)}-i}^{(n)}, k - \Lambda^{(n)}(T_k^{(n)} - i)), 0 \leq i \leq T_k^{(n)}]$ is the same as that of the process $[(S_i^{(n)\uparrow}, \underline{\underline{\Lambda}}_i^{(n)}), 0 \leq i \leq U_k^{(n)})]$;*
2. *the law of the process $[(S_{G_k^{(n)}}^{(n)} - S_{G_k^{(n)}-i}^{(n)}, \Lambda_{G_k^{(n)}}^{(n)} - \Lambda_{G_k^{(n)}-i}^{(n)}), 0 \leq i \leq G_k^{(n)}]$ is the same as that of the process $[(S_i^{(n)\uparrow}, \underline{\underline{\Lambda}}_i^{(n)}), 0 \leq i \leq G_k^{(n)\uparrow}]$.*

PROOF. From the transformation which is recalled in (4.1), the process $S^{(n)\uparrow}$ is the concatenation of the time-reversed excursions $\widehat{e}^{(0)}, \widehat{e}^{(1)}, \ldots$. It is clear that the times where this process reaches its future minimum occur at the end of each of these reversed excursions. Therefore, $T_k^{(n)} = U_k^{(n)}$ a.s. and the concatenation of the k excursions $\widehat{e}^{(0)}, \widehat{e}^{(1)}, \ldots, \widehat{e}^{(k)}$ is the process $(S_i^{(n)\uparrow}, 0 \leq i \leq U_k^{(n)})$.

From the Markov property, these excursions are i.i.d. so that the concatenation of $\widehat{e}^{(0)}, \widehat{e}^{(1)}, \ldots, \widehat{e}^{(k)}$ has the same law as the concatenation of $\widehat{e}^{(k)}, \widehat{e}^{(k-1)}, \ldots, \widehat{e}^{(1)}$. However, the latter concatenation is precisely the process $(S^{(n)}_{T^{(n)}_k} - S^{(n)}_{T^{(n)}_k - i}, 0 \le i \le T^{(n)}_k)$. The same reasoning justifies the identity on the second coordinate.

The second part of the statement follows from the same arguments, together with the identity $G^{(n)}_k = G^{(n)\uparrow}_k$ which holds for all $k \ge 0$. $\quad\square$

Actually, in the proof of Theorem 4, we will only use the second part of Theorem 5, which says that the returned pre-supremum part of X before time t has the same law as $X^\uparrow$ up to its last passage time at the future infimum before t. However, in order to avoid the need to justify an invariance principle for returned processes, we will reformulate this identity in law in terms of the post-infimum process.

PROOF OF THEOREM 4.   From identity (4.3) and Theorem 2, we only have to prove part 1 of Theorem 4. Define

$$K^{(n)}_j = \max\left\{i \le j : S^{(n)}_i = \min_{l \le i} S^{(n)}_l\right\} \quad \text{and} \quad k(t) = \sup\left\{s \le t : X_s = \inf_{u \le s} X_u\right\}.$$

From time-reversal properties of $S^{(n)}$ and X, we have

$$\left(S^{(n)}_{K^{(n)}_k + i} - S^{(n)}_{K^{(n)}_k}, 0 \le i \le k - K^{(n)}_k\right) \stackrel{(d)}{=} \left(S^{(n)}_{G^{(n)}_k} - S^{(n)}_{G^{(n)}_k - i}, 0 \le i \le G^{(n)}_k\right)$$

and

$$\left(X_{k(t)+s} - X_{k(t)}, 0 \le s \le t - k(t)\right) \stackrel{(d)}{=} \left(X_{g(t)} - X_{(g(t)-s)-}, 0 \le s \le g(t)\right).$$

(Recall the convention that $X_{0-} = X_0$.) Since 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$, the time $k(t)$ is a continuity point of X, hence the almost sure convergence of $S^{(n)}$ toward X implies that $\lim_n n^{-1} K^{(n)}_{[nt]} = k(t)$, a.s. for all $t \ge 0$. Then recall from the preliminary section our definition of the a.s. convergence of stochastic processes with finite lifetime. We clearly have the almost sure convergence of the sequence of processes

$$Y^{(n)} = \left(S^{(n)}_{K^{(n)}_{[nt]} + [ns]} - S^{(n)}_{K^{(n)}_{[nt]}}, 0 \le s \le n^{-1}([nt] - K^{(n)}_{[nt]})\right)$$

toward the process $(X_{k(t)+s}, 0 \le s \le t - k(t))$. From Lemma 3 and the time-reversal property of $S^{(n)}$, the sequence $Y^{(n)}$, $n \ge 0$, has the same law as the sequence

$$\left(S^{(n)\uparrow}_{[ns]}, 0 \le s \le n^{-1} G^{(n)\uparrow}_{[nt]}\right).$$

Therefore, the sequence $(S^{(n)\uparrow}_{[ns]}, 0 \le s \le n^{-1} G^{(n)\uparrow}_{[nt]})$ converges almost surely toward the process $(X^\uparrow_s, 0 \le s \le g^\uparrow_t)$.

Let (t_k) be an increasing sequence of positive reals which tends to ∞. We deduce from the above convergence that for each k, $\lim_{n\to\infty} n^{-1} G_{[nt_k]}^{(n)\uparrow} = g^{\uparrow}(t_k)$ a.s. and, more generally,

$$\left(S_{[ns]}^{(n)\uparrow} \mathbb{1}_{\{n^{-1} G_{[nt_k]}^{(n)\uparrow} \leq 1 < n^{-1} G_{[nt_{k+1}]}^{(n)\uparrow}\}}, 0 \leq s \leq 1\right)$$

converges a.s. toward $(X_s^{\uparrow} \mathbb{1}_{\{g^{\uparrow}(t_k) \leq 1 < g^{\uparrow}(t_{k+1})\}}, 0 \leq s \leq 1)$. Since all processes $S^{(n)\uparrow}$ and $X^{\uparrow}$ drift to $+\infty$, we have $\lim_{k\to\infty} G_{[nt_k]}^{(n)\uparrow} = \infty$ and $\lim_{k\to\infty} g^{\uparrow}(t_k) = +\infty$ a.s. so that, with $t_0 = 0$, we have

$$\left(S_{[ns]}^{(n)\uparrow}, 0 \leq s \leq 1\right) = \left(\sum_{k \geq 0} S_{[ns]}^{(n)\uparrow} \mathbb{1}_{\{n^{-1} G_{[nt_k]}^{(n)\uparrow} \leq 1 < n^{-1} G_{[nt_{k+1}]}^{(n)\uparrow}\}}, 0 \leq s \leq 1\right)$$

and

$$\left(X_s^{\uparrow}, 0 \leq s \leq 1\right) = \left(\sum_{k \geq 0} X_s^{\uparrow} \mathbb{1}_{\{g^{\uparrow}(t_k) \leq 1 < g^{\uparrow}(t_{k+1})\}}, 0 \leq s \leq 1\right).$$

However, almost surely there exist k and n_0 such that for all $n \geq n_0$, the processes on the right-hand sides of the two equalities above are, respectively, equal to

$$\left(S_{[ns]}^{(n)\uparrow} \mathbb{1}_{\{n^{-1} G_{[nt_k]}^{(n)\uparrow} \leq 1 < n^{-1} G_{[nt_{k+1}]}^{(n)\uparrow}\}}, 0 \leq s \leq 1\right)$$

and $(X_s^{\uparrow} \mathbb{1}_{\{g^{\uparrow}(t_k) \leq 1 < g^{\uparrow}(t_{k+1})\}}, 0 \leq s \leq 1)$. Therefore, $(S_{[ns]}^{(n)\uparrow}, 0 \leq s \leq 1)$ converges toward $(X_s^{\uparrow}, 0 \leq s \leq 1)$ on the space $\mathcal{D}([0, 1])$. The same argument holds on each space $\mathcal{D}([0, t])$, $t > 0$, so we deduce the convergence on $\mathcal{D}([0, \infty))$ from Theorem 16.7 in [4], as recalled in Section 2. $\square$

We now define $S^{(n)}$ and X conditioned to stay positive, respectively, on $\{0, 1, \ldots, k\}$ and $[0, t]$, where k and t are deterministic. Letting $\mathcal{C}_k^{(n)} = \{S_1^{(n)} \geq 0, \ldots, S_k^{(n)} \geq 0\}$, we denote by $S^{(n,k)}$ a process whose law is defined on $\{0, 1, \ldots, k\}$ by $S_0^{(n,k)} = 0$ and

$$\mathbb{P}(S_1^{(n,k)} \in dx_1, \ldots, S_k^{(n,k)} \in dx_k)$$
$$= \mathbb{P}(S_1^{(n)} \in dx_1, \ldots, S_k^{(n)} \in dx_k | \mathcal{C}_k^{(n)}).$$

It clearly follows from the definitions that this law is absolutely continuous with respect to the law of $S^{(n)\uparrow}$: for $x_1 > 0, \ldots, x_k > 0$,

$$\begin{aligned} (4.4) \qquad & \mathbb{P}(S_1^{(n,k)} \in dx_1, \ldots, S_k^{(n,k)} \in dx_k) \\ & = \frac{1}{\mathbb{P}(\mathcal{C}_k^{(n)}) V^{(n)}(x_k)} \mathbb{P}(S_1^{(n)\uparrow} \in dx_1, \ldots, S_k^{(n)\uparrow} \in dx_k); \end{aligned}$$

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see also (3.2) in [6]. The process $S^{(n,k)}$ is called the (discrete-time) *meander* with length k.

The definition of the analogous conditional law in continuous time requires some care since the set $\{X_t \geq 0 : t \in [0, 1]\}$ always has probability 0 when 0 is regular for $(-\infty, 0)$.

LEMMA 4. *For $x_1 > 0, \ldots, x_j > 0$ and $t_1, \ldots, t_j \in [0, 1]$, we have*

$$\lim_{x \downarrow 0} \mathbb{P}_x(X_{t_1} \in dx_1, \ldots, X_{t_j} \in dx_j | X_t > 0, t \in [0, 1])$$

$$= \frac{1}{\beta h(x_j)} \mathbb{P}(X_{t_1}^{\uparrow} \in dx_1, \ldots, X_{t_j}^{\uparrow} \in dx_j),$$

where $\beta = \mathbb{E}(h(X_1^{\uparrow})^{-1})$.

PROOF. This is a direct application of Corollary 1 in [8]; see also [9]. □

Clearly, the weak limit obtained in this lemma defines a unique probability measure on the space $\mathcal{D}([0, 1])$. We will denote by X^+ a process with this law, that is, for $x_1 > 0, \ldots, x_j > 0$ and $t_1, \ldots, t_j \in [0, 1]$,

$$(4.5) \quad \mathbb{P}(X_{t_1}^+ \in dx_1, \ldots, X_{t_j}^+ \in x_j) = \frac{1}{\beta h(x_j)} \mathbb{P}(X_{t_1}^{\uparrow} \in dx_1, \ldots, X_{t_j}^{\uparrow} \in dx_j).$$

This process is called the *meander* with length 1.

LEMMA 5. *Assume that $S^{(n)}$ converges weakly to X. Recall the definition of the renewal function $V^{(n)}(x) = \sum_{k \geq 0} \mathbb{P}(\widehat{H}_k^{(n)} \leq x)$ for $x \geq 0$.*

1. *If we let $\pi^{\widehat{\tau}}$ denote the Lévy measure of the ladder time process $\widehat{\tau}$, then*

$$\lim_{n \to +\infty} \widehat{a}_n \mathbb{P}(\mathcal{C}_n^{(n)}) = \pi^{\widehat{\tau}}(1, \infty).$$

2. *The sequence of functions $\mathbb{P}(\mathcal{C}_n^{(n)}) V^{(n)}(x)$ converges uniformly on compact sets toward $\gamma h(x) = \gamma \mathbb{E}(\int_0^\infty \mathbb{1}_{\{\widehat{H}_t \leq x\}} dt)$ with $\gamma = \pi^{\widehat{\tau}}(1, \infty)$.*

PROOF. To prove the first part, it suffices to note that $\mathbb{P}(\mathcal{C}_n^{(n)}) = \mathbb{P}(n^{-1}\widehat{T}_1^{(n)} > 1)$ and to apply Lemma 2. To prove the second part, observe that from the hypothesis, Theorem 1 and dominated convergence, we have, for every $x \geq 0$,

$$\lim_{n \to \infty} \int_0^\infty \mathbb{P}(\widehat{H}_{[\widehat{a}_n t]}^{(n)} \leq x) dt = \lim_{n \to \infty} \widehat{a}_n^{-1} V^{(n)}(x) = h(x).$$

The result then follows from part 1, the fact that $V^{(n)}(x)$ is a sequence of increasing functions and the continuity of h. □

The following invariance principle for the meander has been obtained in the case where all $S^{(n)}$ have the same law (in particular, X is stable) in [5] and [10].

THEOREM 6. *Suppose that some sequence of random walks $S^{(n)}$ converges weakly toward X. The sequence of discrete meanders $(S^{(n,n)}_{[nt]}, 0 \leq t \leq 1)$ then converges weakly toward the meander X^+.*

PROOF. We will prove that for all continuous and bounded functionals F on $\mathcal{D}([0, 1])$,

$$\mathbb{E}\big(F(S^{(n,n)}_{[nt]}, 0 \leq t \leq 1)\big) \longrightarrow \mathbb{E}\big(F(X^+_t, 0 \leq t \leq 1)\big) \qquad \text{as } n \to \infty.$$

From the absolute continuity relations (4.4) and (4.5), it suffices to prove that

$$\mathbb{E}\bigg(\frac{1}{\mathbb{P}(\mathcal{C}^{(n)}_n)V^{(n)}(S^{(n)\uparrow}_n)} F(S^{(n)\uparrow}_{[nt]}, 0 \leq t \leq 1)\bigg)$$

$$\longrightarrow \mathbb{E}\bigg(\frac{1}{\beta h(X^\uparrow_1)} F(X^\uparrow_t, 0 \leq t \leq 1)\bigg) \qquad \text{as } n \to \infty.$$

For $\eta > 0$, we write

$$\bigg|\mathbb{E}\bigg(\frac{1}{\mathbb{P}(\mathcal{C}^{(n)}_n)V^{(n)}(S^{(n)\uparrow}_n)} F(S^{(n)\uparrow}_{[nt]}, 0 \leq t \leq 1)\bigg) - \mathbb{E}\bigg(\frac{1}{\gamma h(X^\uparrow_1)} F(X^\uparrow_t, 0 \leq t \leq 1)\bigg)\bigg|$$

$$\leq \bigg|\mathbb{E}\bigg(\frac{1}{\mathbb{P}(\mathcal{C}^{(n)}_n)V^{(n)}(S^{(n)\uparrow}_n)} \mathbb{1}_{\{S^{(n)\uparrow}_n \geq \eta\}} F(S^{(n)\uparrow}_{[nt]}, 0 \leq t \leq 1)\bigg)$$

$$- \mathbb{E}\bigg(\frac{1}{\gamma h(X^\uparrow_1)} \mathbb{1}_{\{X^\uparrow_1 \geq \eta\}} F(X^\uparrow_t, 0 \leq t \leq 1)\bigg)\bigg|$$

$$+ \mathbb{E}\bigg(\frac{1}{\mathbb{P}(\mathcal{C}^{(n)}_n)V^{(n)}(S^{(n)\uparrow}_n)} \mathbb{1}_{\{S^{(n)\uparrow}_n < \eta\}} F(S^{(n)\uparrow}_{[nt]}, 0 \leq t \leq 1)\bigg)$$

$$+ \mathbb{E}\bigg(\frac{1}{\gamma h(X^\uparrow_1)} \mathbb{1}_{\{X^\uparrow_1 < \eta\}} F(X^\uparrow_t, 0 \leq t \leq 1)\bigg).$$

Since F is bounded by a constant, say B and

$$(4.6) \qquad \mathbb{E}\bigg(\frac{1}{\mathbb{P}(\mathcal{C}^{(n)}_n)V^{(n)}(S^{(n)\uparrow}_n)}\bigg) = 1 \quad \text{and} \quad \mathbb{E}\bigg(\frac{1}{\gamma h(X^\uparrow_1)}\bigg) = \beta/\gamma,$$

it follows from Hölder's inequality that the two last terms of the right-hand side of the above inequality are bounded above by, respectively, $B\mathbb{P}(S^{(n)\uparrow}_n < \eta)$ and $B\mathbb{P}(X^\uparrow_1 < \eta)\beta/\gamma$. From the assumption of convergence and the fact that $\mathbb{P}(X^\uparrow_1 > 0) = 1$, for every $\varepsilon > 0$, there exist n_0 and $\eta > 0$ such that for all $n \geq n_0$,

$B\mathbb{P}(S_n^{(n)\uparrow} < \eta) < \varepsilon$ and $B\mathbb{P}(X_1^{\uparrow} < \eta)\beta/\gamma < \varepsilon$. Finally, note that from the hypothesis of convergence and Lemma 5, we easily derive that for all $\eta > 0$,

$$\mathbb{E}\left(\frac{1}{\mathbb{P}(\mathcal{C}_n^{(n)})V^{(n)}(S_n^{(n)\uparrow})}\mathbb{1}_{\{S_n^{(n)\uparrow}\geq\eta\}}F\big(S_{[nt]}^{(n)\uparrow}, 0 \leq t \leq 1\big)\right)$$

$$\longrightarrow \mathbb{E}\left(\frac{1}{\gamma h(X_1^{\uparrow})}\mathbb{1}_{\{X_1^{\uparrow}\geq\eta\}}F(X_t^{\uparrow}, 0 \leq t \leq 1)\right) \qquad \text{as } n \to \infty.$$

We have then proven that

$$\mathbb{E}\left(\frac{1}{\mathbb{P}(\mathcal{C}_n^{(n)})V^{(n)}(S_n^{(n)\uparrow})}F\big(S_{[nt]}^{(n)\uparrow}, 0 \leq t \leq 1\big)\right)$$

$$\longrightarrow \mathbb{E}\left(\frac{1}{\gamma h(X_1^{\uparrow})}F(X_t^{\uparrow}, 0 \leq t \leq 1)\right) \qquad \text{as } n \to \infty.$$

Taking $F \equiv 1$ in this relation and comparing with (4.6), we obtain $\beta = \gamma$, which proves the result. $\square$

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