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MARTINGALE DIMENSIONS FOR FRACTALS

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We prove that the martingale dimensions for canonical diffusion processes on a class of self-similar sets including nested fractals are always one. This provides an affirmative answer to the conjecture of S. Kusuoka [*Publ. Res. Inst. Math. Sci.* **25** (1989) 659–680].

1. Introduction. The martingale dimension, which is also known as the Davis–Varaiya invariant [2] or the multiplicity of filtration, is defined for a filtration on a probability space and represents a certain index for random noises. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbf{F} = \{\mathcal{F}_t\}_{t \in [0,\infty)}$ be a filtration on it. Informally speaking, the martingale dimension for $\mathbf{F}$ is the minimal number of martingales $\{M^1, M^2, \dots\}$ with the property that an arbitrary $\mathbf{F}$-martingale X has a stochastic integral representation of the following type:

$$X_t = X_0 + \sum_i \int_0^t \varphi_s^i \, dM_s^i.$$

When $\mathbf{F}$ is provided by the standard Brownian motion on $\mathbb{R}^d$, its martingale dimension is d. Kusuoka [8] has proved a remarkable result that when $\mathbf{F}$ is induced by the canonical diffusion process on the d-dimensional Sierpinski gasket, there exists one martingale additive functional M such that every martingale additive functional with finite energy is a stochastic integral of M. We will say that the AF-martingale dimension of $\mathbf{F}$ is one. (For remarks about the connection between the Davis–Varaiya invariant and the AF-martingale dimension, see comments below Theorem 4.4.) He has also conjectured that each nested fractal has the same property. However, with the exception of a few related studies such as [9], no significant progress has been made thus far with regard to the problem of determining the martingale dimensions for concrete examples of fractals.

In this paper, we solve this problem for a class of fractals including nested fractals by proving that the AF-martingale dimensions are one. Our method of the proof is different from that of Kusuoka.

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This paper is organized as follows. In Section 2, we provide a framework for the main theorem, and a key proposition is proved in Section 3. Section 4 describes the analysis of AF-martingale dimensions and the proof of the main theorem. In Section 5, we remark on the key proposition.

2. Framework. In this section, we provide a framework of Dirichlet forms on self-similar sets according to [7]. Let K be a compact metrizable topological space, N be an integer greater than one, and set $S = \{1, 2, \dots, N\}$. Further, let $\psi_i : K \to K$ be a continuous injective map for $i \in S$. Set $\Sigma = S^{\mathbb{N}}$. For $i \in S$, we define a shift operator $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\cdots) = i\omega_1\omega_2\cdots$. Suppose that there exists a continuous surjective map $\pi : \Sigma \to K$ such that $\psi_i \circ \pi = \pi \circ \sigma_i$ for every $i \in S$. We term $\mathcal{L} = (K, S, \{\psi_i\}_{i\in S})$ a self-similar structure.

We also define $W_0 = \{\varnothing\}$, $W_m = S^m$ for $m \in \mathbb{N}$, and denote $\bigcup_{m\ge0} W_m$ by W_*. For $w = w_1w_2\cdots w_m \in W_m$, we define $\psi_w = \psi_{w_1} \circ \psi_{w_2} \circ \cdots \circ \psi_{w_m}$, $\sigma_w = \sigma_{w_1} \circ \sigma_{w_2} \circ \cdots \circ \sigma_{w_m}$, $K_w = \psi_w(K)$ and $\Sigma_w = \sigma_w(\Sigma)$. For $w = w_1w_2\cdots w_m \in W_w$ and $w' = w'_1w'_2\cdots w'_{m'} \in W_{w'}$, ww' denotes $w_1w_2\cdots w_m w'_1w'_2\cdots w'_{m'} \in W_{m+m'}$. For $\omega = \omega_1\omega_2\cdots \in \Sigma$ and $m \in \mathbb{N}$, $[\omega]_m$ denotes $\omega_1\omega_2\cdots\omega_m \in W_m$.

We set

$$\mathcal{P} = \bigcup_{m=1}^{\infty} \sigma^m\left(\pi^{-1}\left(\bigcup_{i,j\in S,\, i\ne j} (K_i \cap K_j)\right)\right) \quad\text{and}\quad V_0 = \pi(\mathcal{P}),$$

where $\sigma^m : \Sigma \to \Sigma$ is a shift operator that is defined by $\sigma^m(\omega_1\omega_2\cdots) = \omega_{m+1}\omega_{m+2}\cdots$. The set $\mathcal{P}$ is referred to as the post-critical set. In this paper, we assume that K is connected and the self-similar structure $(K, S, \{\psi_i\}_{i\in S})$ is post-critically finite, that is, $\mathcal{P}$ is a finite set.

A nested fractal [10] is a typical example of post-critically finite self-similar structures. For convenience, we explain this concept (see [7], page 117, for further comments). Let $\alpha > 1$ and ψ_i, $i \in S$, be an α-similitude in $\mathbb{R}^d$. That is, $\psi_i(x) = \alpha^{-1}(x - x_i) + x_i$ for some $x_i \in \mathbb{R}^d$. There exists a unique nonempty compact set K in $\mathbb{R}^d$ such that $K = \bigcup_{i\in S} \psi_i(K)$. We assume the following open set condition: there exists a nonempty open set U of $\mathbb{R}^d$ such that $\bigcup_{i\in S} \psi_i(U) \subset U$ and $\psi_i(U) \cap \psi_j(U) = \varnothing$ for any distinct $i, j \in S$. Let F_0 be the set of all fixed points of ψ_i's, $i \in S$. Then, $\#F_0 = N$ (see [9], Corollary 1.9). An element x of F_0 is termed an essential fixed point if there exist $i, j \in S$ and $y \in F_0$ such that $i \ne j$ and $\psi_i(x) = \psi_j(y)$. The set of all essential fixed points is denoted by F. We refer to $\psi_w(F)$ for $w \in W_n$ as an n-cell. For $x, y \in \mathbb{R}^d$ with $x \ne y$, let H_{xy} denote the hyperplane in $\mathbb{R}^d$ defined as $H_{xy} = \{z \in \mathbb{R}^d \mid |x - z| = |y - z|\}$. Let $g_{xy} : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection in H_{xy}. We call K a nested fractal if the following conditions hold:

- $\#F \ge 2$;
- (Connectivity) for any two 1-cells C and C', there exists a sequence of 1-cells C_i $(i = 0, \dots, k)$ such that $C_0 = C$, $C_k = C'$ and $C_{i-1} \cap C_i \ne \varnothing$ for all $i = 1, \dots, k$;

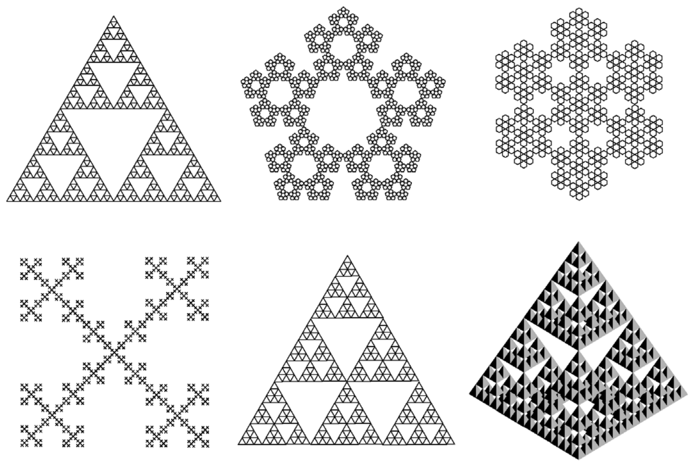

FIG. 1. *Examples of nested fractals.*

- (Symmetry) for any distinct $x, y \in F$ and $n \geq 0$, g_{xy} maps n-cells to n-cells and any n-cell that contains elements on both sides of H_{xy} to itself;
- (Nesting) for any $n \geq 1$ and distinct $w, w' \in W_n$, $\psi_w(K) \cap \psi_{w'}(K) = \psi_w(F) \cap \psi_{w'}(F)$.

Then, the triplet $(K, S, \{\psi_i\}_{i \in S})$ is a post-critically finite self-similar structure and $V_0 = F$. Figure 1 shows some typical examples of nested fractals K. The bottom right part is the three-dimensional Sierpinski gasket that is realized in $\mathbb{R}^3$, and the rest are realized in $\mathbb{R}^2$.

We resume our discussion for the general case. For a finite set V, let $l(V)$ be the space of all real-valued functions on V. We equip $l(V)$ with an inner product $(\cdot, \cdot)$ defined by $(u, v) = \sum_{p \in V} u(p)v(p)$. Let $D = (D_{pp'})_{p, p' \in V_0}$ be a symmetric linear operator on $l(V_0)$ (also considered to be a square matrix with size $\#V_0$) such that the following conditions hold:

(D1) D is nonpositive definite,
(D2) $Du = 0$ if and only if u is constant on V_0,
(D3) $D_{pp'} \geq 0$ for all $p \neq p' \in V_0$.

We define $\mathcal{E}^{(0)}(u, v) = (-Du, v)$ for $u, v \in l(V_0)$. This is a Dirichlet form on $l(V_0)$, where $l(V_0)$ is identified with the L^2 space on V_0 with the counting measure ([7], Proposition 2.1.3). Let $V_m = \bigcup_{w \in S^m} \psi_w(V_0)$ for $m \geq 1$. For $r = \{r_i\}_{i \in S}$ with $r_i > 0$, we define a bilinear form $\mathcal{E}^{(m)}$ on $l(V_m)$ as

$$(2.1)\quad \mathcal{E}^{(m)}(u, v) = \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}^{(0)}(u \circ \psi_w|_{V_0}, v \circ \psi_w|_{V_0}), \qquad u, v \in l(V_m).$$

Here, $r_w = r_{w_1} r_{w_2} \cdots r_{w_m}$ for $w = w_1 w_2 \cdots w_m$. We refer to (D, r) as a harmonic structure if $\mathcal{E}^{(0)}(u|_{V_0}, u|_{V_0}) \le \mathcal{E}^{(1)}(u, u)$ for every $u \in l(V_1)$. Then, for $m \ge 0$ and $u \in l(V_{m+1})$, we obtain $\mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) \le \mathcal{E}^{(m+1)}(u, u)$.

We fix a harmonic structure that is regular, namely, $0 < r_i < 1$ for all $i \in S$. Several studies have been conducted on the existence of regular harmonic structures. We only focus on the fact that all nested fractals have regular harmonic structures [7, 9, 10]; we do not go into further details in this regard.

Let μ be a Borel probability measure on K with full support. We can then define a regular local Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ as

$$\mathcal{F} = \left\{ u \in C(K) \subset L^2(K, \mu) \,\middle|\, \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty \right\},$$

$$\mathcal{E}(u, v) = \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m}), \qquad u, v \in \mathcal{F}.$$

The space $\mathcal{F}$ becomes a separable Hilbert space when it is equipped with the inner product $\langle f, g\rangle_{\mathcal{F}} = \mathcal{E}(f, g) + \int_K fg\, d\mu$. We use $\mathcal{E}(f)$ instead of $\mathcal{E}(f, f)$.

For a map $\psi : K \to K$ and a function $f : K \to \mathbb{R}$, $\psi^* f$ denotes the pullback of f by ψ, that is, $\psi^* f = f \circ \psi$. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the following self-similarity:

$$\mathcal{E}(f, g) = \sum_{i\in S} \frac{1}{r_i} \mathcal{E}(\psi_i^* f, \psi_i^* g), \qquad f, g \in \mathcal{F}. \tag{2.2}$$

For each $u \in l(V_0)$, there exists a unique function $h \in \mathcal{F}$ such that $h|_{V_0} = u$ and h attains the infimum of $\{\mathcal{E}(g) \mid g \in \mathcal{F},\ g|_{V_0} = u\}$. Such a function h is termed a harmonic function. The space of all harmonic functions is denoted by $\mathcal{H}$. By using the linear map $\iota : l(V_0) \ni u \mapsto h \in \mathcal{H}$, we can identify $\mathcal{H}$ with $l(V_0)$. In particular, $\mathcal{H}$ is a finite-dimensional subspace of $\mathcal{F}$. For each $i \in S$, we define a linear operator $A_i : l(V_0) \to l(V_0)$ as $A_i = \iota^{-1} \circ \psi_i^* \circ \iota$.

For $m \ge 0$, let $\mathcal{H}_m$ denote the set of all functions f in $\mathcal{F}$ such that $\psi_w^* f \in \mathcal{H}$ for all $w \in W_m$. Let $\mathcal{H}_* = \bigcup_{m\ge 0} \mathcal{H}_m$. The functions in $\mathcal{H}_*$ are referred to as piecewise harmonic functions.

LEMMA 2.1. *$\mathcal{H}_*$ is dense in $\mathcal{F}$.*

PROOF. Let $f \in \mathcal{F}$. For $m \in \mathbb{N}$, let f_m be a function in $\mathcal{H}_m$ such that $f_m = f$ on V_m. Then, $\mathcal{E}(f - f_m) \to 0$ as $m \to \infty$ by, for example, [7], Lemma 3.2.17. From the maximal principle ([7], Theorem 3.2.5), f_m converges uniformly to f. In particular, $f_m \to f$ in $L^2(K, \mu)$ as $m \to \infty$. Therefore, $f_m \to f$ in $\mathcal{F}$ as $m \to \infty$. □

For $f \in \mathcal{F}$, we will construct a finite measure $\lambda_{\langle f\rangle}$ on Σ as follows. For each $m \ge 0$, we define

$$\lambda_{\langle f\rangle}^{(m)}(A) = 2 \sum_{w\in A} \frac{1}{r_w} \mathcal{E}(\psi_w^* f), \qquad A \subset W_m.$$

Then, $\lambda^{(m)}_{\langle f\rangle}$ is a measure on W_m. Let $A \subset W_m$ and $A' = \{wi \in W_{m+1} \mid w \in A,\ i \in S\}$. Then,

$$\begin{aligned}\lambda^{(m+1)}_{\langle f\rangle}(A') &= 2\sum_{w\in A}\sum_{i\in S}\frac{1}{r_{wi}}\mathcal{E}(\psi^*_{wi}f)\\ &= 2\sum_{w\in A}\frac{1}{r_w}\sum_{i\in S}\frac{1}{r_i}\mathcal{E}(\psi^*_i\psi^*_w f)\\ &= 2\sum_{w\in A}\frac{1}{r_w}\mathcal{E}(\psi^*_w f)\qquad \text{[by (2.2)]}\\ &= \lambda^{(m)}_{\langle f\rangle}(A).\end{aligned}$$

Therefore, $\{\lambda^{(m)}_{\langle f\rangle}\}_{m\ge 0}$ has a consistency condition. We also note that $\lambda^{(m)}_{\langle f\rangle}(W_m) = 2\mathcal{E}(f, f) < \infty$. According to the Kolmogorov extension theorem, there exists a unique Borel finite measure $\lambda_{\langle f\rangle}$ on Σ such that $\lambda_{\langle f\rangle}(\Sigma_w) = \lambda^{(m)}_{\langle f\rangle}(\{w\})$ for every $m \ge 0$ and $w \in W_m$. For $f, g \in \mathcal{F}$, we define a signed measure $\lambda_{\langle f,g\rangle}$ on Σ by the polarization procedure; it is expressed as $\lambda_{\langle f,g\rangle} = (\lambda_{\langle f+g\rangle} - \lambda_{\langle f-g\rangle})/4$. It is easy to prove that

$$\lambda_{\langle f,g\rangle}(\Sigma_{ww'}) = r_w^{-1}\lambda_{\langle \psi^*_w f, \psi^*_w g\rangle}(\Sigma_{w'}) \tag{2.3}$$

for any $f, g \in \mathcal{F}$ and $w, w' \in W_*$.

For $f \in \mathcal{F}$, let $\mu_{\langle f\rangle}$ be the energy measure of f on K associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$. That is, $\mu_{\langle f\rangle}$ is a unique Borel measure on K satisfying

$$\int_K \varphi\, d\mu_{\langle f\rangle} = 2\mathcal{E}(f, f\varphi) - \mathcal{E}(f^2, \varphi),\qquad \varphi \in \mathcal{F} \subset C(K).$$

We define $\mu_{\langle f,g\rangle} = (\mu_{\langle f+g\rangle} - \mu_{\langle f-g\rangle})/4$ for $f, g \in \mathcal{F}$. In the same manner as the proof of [1], Theorem I.7.1.1, for every $f \in \mathcal{F}$, the image measure of $\mu_{\langle f\rangle}$ by f is proved to be absolutely continuous with respect to the one-dimensional Lebesgue measure. In particular, $\mu_{\langle f\rangle}$ has no atoms. From [4], Lemma 4.1, the image measure of $\lambda_{\langle f\rangle}$ by $\pi : \Sigma \to K$ is identical to $\mu_{\langle f\rangle}$. Since $\{x \in K \mid \#\pi^{-1}(x) > 1\}$ is a countable set, we obtain the following.

LEMMA 2.2. *For any $f \in \mathcal{F}$, $\lambda_{\langle f\rangle}(\{\omega \in \Sigma \mid \#\pi^{-1}(\pi(\omega)) > 1\}) = 0$. In particular, $(\Sigma, \lambda_{\langle f\rangle})$ is isomorphic to $(K, \mu_{\langle f\rangle})$ as a measure space by the map $\pi : \Sigma \to K$.*

Hereafter, we assume the following:

(∗) Each $p \in V_0$ is a fixed point of ψ_i for some $i \in S$ and $K \setminus \{p\}$ is connected.

This condition is a technical one; at present, we cannot remove this in order to utilize Lemma 2.3 below. A typical example that does not satisfy (∗) is Hata's tree-like set (see [7], Example 1.2.9). Every nested fractal satisfies this condition (see e.g. [7], Theorem 1.6.2 and Proposition 1.6.9 for the proof). We may and will assume that $V_0 = \{p_1, \dots, p_d\}$ and each p_i is the fixed point of ψ_i, $i \in \{1, \dots, d\} \subset S$.

Let $i \in \{1, \dots, d\}$. We recollect several facts on the eigenvalues and eigenfunctions of A_i in order to use them later. See [7], Appendix A.1, and [5] for further details. Both A_i and ${}^tA_i$ have 1 and r_i as simple eigenvalues and the modulus of any other eigenvalue is less than r_i. Let u_i be the column vector $(D_{pp_i})_{p \in V_0}$. Then, u_i is an eigenvector of ${}^tA_i$ with respect to r_i ([5], Lemma 5). We can take an eigenvector v_i of A_i with respect to r_i so that all components of v_i are nonnegative and $(u_i, v_i) = 1$. Since v_i is not a constant vector, we have $-{}^tv_i D v_i > 0$.

Let $\mathbf{1} \in l(V_0)$ be a constant function on V_0 with value 1. Let $\tilde{l}(V_0) = \{u \in l(V_0) \mid (u, \mathbf{1}) = 0\}$ and $P : l(V_0) \to l(V_0)$ be the orthogonal projection on $\tilde{l}(V_0)$. The following lemma is used in the next section.

LEMMA 2.3 ([5], Lemmas 6 and 7). *Let $i \in \{1, \dots, d\}$ and $u \in l(V_0)$. Then,*

(1) $\lim_{n\to\infty} r_i^{-n} P A_i^n u = (u_i, u) P v_i$,
(2) $\lim_{n\to\infty} r_i^{-n} \lambda_{\langle \iota(u)\rangle}(\Sigma_{\underbrace{i\cdots i}_{n}}) = -2(u_i, u)^2\, {}^t v_i D v_i$.

3. Properties of measures on the shift space. Let I be a finite set $\{1, \dots, N_0\}$ or a countable infinite set $\mathbb{N}$. Take a sequence $\{e_i\}_{i\in I}$ of piecewise harmonic functions such that $2\mathcal{E}(e_i) = 1$ for all $i \in I$. A real sequence $\{a_i\}_{i\in I}$ is fixed such that $a_i > 0$ for every $i \in I$ and $\sum_{i\in I} a_i = 1$. We define $\lambda = \sum_{i=1}^{\infty} a_i \lambda_{\langle e_i\rangle}$, which is a probability measure on Σ. For $i, j \in I$, it is easy to see that $\lambda_{\langle e_i, e_j\rangle}$ is absolutely continuous with respect to λ. The Radon–Nikodym derivative $d\lambda_{\langle e_i, e_j\rangle}/d\lambda$ is denoted by $Z^{i,j}$. It is evident that $\sum_{i\in I} a_i Z^{i,i}(\omega) = 1$ λ-a.s. ω. We may assume that this identity holds for all ω. For $n \in \mathbb{N}$, let $\mathcal{B}_n$ be a σ-field on Σ generated by $\{\Sigma_w \mid w \in W_n\}$. We define a function $Z_n^{i,j}$ on Σ as $Z_n^{i,j}(\omega) = \lambda_{\langle e_i, e_j\rangle}(\Sigma_{[\omega]_n})/\lambda(\Sigma_{[\omega]_n})$. Then, $Z_n^{i,j}$ is the conditional expectation of $Z^{i,j}$ given $\mathcal{B}_n$ with respect to λ. According to the martingale convergence theorem, $\lambda(\Sigma') = 1$, where

$$\Sigma' = \left\{\omega \in \Sigma \,\middle|\, \lim_{n\to\infty} Z_n^{i,j}(\omega) = Z^{i,j}(\omega) \text{ for all } i, j \in I\right\}.$$

We define

$$\mathcal{K} = \left\{f \in \mathcal{H} \,\middle|\, \int_K f\, d\mu = 0,\ 2\mathcal{E}(f) = 1\right\}.$$

$\mathcal{K}$ is a compact set in $\mathcal{F}$. For $f \in \mathcal{H}$ we set

$$\gamma(f) = \max\{|(u_i, f|_{V_0})|;\ i = 1, \dots, d\}.$$

Here, $(\cdot, \cdot)$ denotes the inner product on $\ell(V_0)$. When f is not constant,

$$D(f|_{V_0}) = \begin{pmatrix} (u_1, f|_{V_0}) \\ \vdots \\ (u_d, f|_{V_0}) \end{pmatrix}$$

is not a zero vector therefore $\gamma(f) > 0$. Due to the compactness of $\mathcal{K}$ and the continuity of γ, $\delta := \min_{f \in \mathcal{K}} \gamma(f)$ is greater than 0. For $f \in \mathcal{H}$, we set

$$\eta(f) = \min\{i = 1, \dots, d;\ |(u_i, f|_{V_0})| = \gamma(f)\}.$$

The map $\mathcal{H} \ni f \mapsto \eta(f) \in \{1, \dots, d\}$ is Borel measurable.

LEMMA 3.1. *For $k \in \mathbb{N}$, there exists $c_k \in (0, 1]$ such that for any $n \geq m \geq 1$ and $e \in \mathcal{H}_m$, $w \in W_n$,*

$$\begin{aligned} &\lambda_{\langle e \rangle}\big(\{\omega_1\omega_2 \cdots \in \Sigma_w \mid \omega_{n+j} = \eta(\psi_w^* e) \text{ for all } j = 1, \dots, k\}\big) \\ &\quad \geq c_k \lambda_{\langle e \rangle}(\Sigma_w). \end{aligned} \tag{3.1}$$

REMARK 3.2. When $\lambda_{\langle e \rangle}$ is a probability measure, (3.1) for all $w \in W_n$ is equivalent to

$$\lambda_{\langle e \rangle}[\{\omega_1\omega_2 \cdots \in \Sigma \mid \omega_{n+j} = \eta(\psi_w^* e) \text{ for all } j = 1, \dots, k\} \mid \mathcal{B}_n] \geq c_k \quad \lambda_{\langle e \rangle}\text{-a.s.},$$

where $\lambda_{\langle e \rangle}[\,\cdot \mid \mathcal{B}_n]$ denotes the conditional probability of $\lambda_{\langle e \rangle}$ given $\mathcal{B}_n$.

PROOF OF LEMMA 3.1. Equation (3.1) is equivalent to

$$\lambda_{\langle \psi_w^* e \rangle}\big(\{\omega_1\omega_2 \cdots \in \Sigma \mid \omega_j = \eta(\psi_w^* e) \text{ for all } j = 1, \dots, k\}\big) \geq c_k \lambda_{\langle \psi_w^* e \rangle}(\Sigma).$$

Therefore, it is sufficient to prove that for $f \in \mathcal{K}$,

$$\lambda_{\langle f \rangle}\big(\{\omega_1\omega_2 \cdots \in \Sigma \mid \omega_j = \eta(f) \text{ for all } j = 1, \dots, k\}\big) \geq c_k.$$

Let $i \in \{1, \dots, d\}$. From Lemma 2.3(2), for any $f \in \mathcal{H}$,

$$\lim_{n \to \infty} r_i^{-n} \lambda_{\langle f \rangle}\Big(\Sigma_{\underbrace{i \cdots i}_{n}}\Big) = -2(u_i, f|_{V_0})^2\, {}^t v_i D v_i.$$

Therefore, if $(u_i, f|_{V_0}) \neq 0$ for $f \in \mathcal{H}$, we obtain $\lambda_{\langle f \rangle}(\Sigma_{\underbrace{i \cdots i}_{n}}) > 0$ for sufficiently large n ($\geq k$). In particular, $\lambda_{\langle f \rangle}(\Sigma_{\underbrace{i \cdots i}_{k}}) \geq \lambda_{\langle f \rangle}(\Sigma_{\underbrace{i \cdots i}_{n}}) > 0$. Let $\mathcal{K}_i = \{f \in \mathcal{K} \mid$

$|(u_i, f|_{V_0})| \geq \delta\}$. Since $\mathcal{K}_i$ is compact and the map $\mathcal{K}_i \ni f \mapsto \lambda_{\langle f\rangle}(\Sigma_{\underbrace{i\cdots i}_k}) \in \mathbb{R}$ is continuous, $\varepsilon_i := \min_{f\in\mathcal{K}_i} \lambda_{\langle f\rangle}(\Sigma_{\underbrace{i\cdots i}_k})$ is strictly positive. We define $c_k = \min\{\varepsilon_i \mid i = 1, \dots, d\}$. For $f \in \mathcal{K}$, let $i = \eta(f)$. Since $f \in \mathcal{K}_i$, we have $\lambda_{\langle f\rangle}(\Sigma_{\underbrace{i\cdots i}_k}) \geq \varepsilon_i \geq c_k$. This completes the proof. $\square$

We fix $k \in \mathbb{N}$, $n \geq m \geq 1$, and $e \in \mathcal{H}'_m$. For $j \geq n$, let

$$\Omega_j = \left\{\omega \in \Sigma \,\middle|\, \sigma^{jk}(\omega) \notin \Sigma_{\underbrace{i\cdots i}_k} \text{ where } i = \eta(\psi^*_{[\omega]_{jk}} e)\right\}.$$

Let $M > n$. Since $\bigcap_{j=n}^{M-1} \Omega_j$ is $\mathcal{B}_{Mk}$-measurable, an application of Lemma 3.1 yields $\lambda_{\langle e\rangle}(\bigcap_{j=n}^{M} \Omega_j) \leq (1 - c_k)\lambda_{\langle e\rangle}(\bigcap_{j=n}^{M-1} \Omega_j)$. By repeating this procedure, we obtain $\lambda_{\langle e\rangle}(\bigcap_{j=n}^{M} \Omega_j) \leq (1 - c_k)^{M-n}\lambda_{\langle e\rangle}(\Omega_n)$. Letting $M \to \infty$, we have $\lambda_{\langle e\rangle}(\bigcap_{j=n}^{\infty} \Omega_j) = 0$. Therefore, when we set

$$\Xi(e) = \left\{\omega \in \Sigma \,\middle|\, \begin{array}{l}\text{for all } k \in \mathbb{N}, \text{ infinitely often } j, \\ \sigma^{jk}(\omega) \in \Sigma_{\underbrace{i\cdots i}_k}, \text{ where } i = \eta(\psi^*_{[\omega]_{jk}} e)\end{array}\right\}$$

for $e \in \mathcal{H}$ then $\lambda_{\langle e\rangle}(\Sigma \setminus \Xi(e)) = 0$.

For $\omega \in \Sigma$, we set

$$M(\omega) = \min\left\{i \in I \,\middle|\, a_i Z^{i,i}(\omega) = \max_{j\in I} a_j Z^{j,j}(\omega)\right\}.$$

Clearly, $Z^{M(\omega),M(\omega)}(\omega) > 0$ and the map $\Sigma \ni \omega \mapsto M(\omega) \in I$ is measurable.

For $\alpha \in I$, we set

$$\Sigma(\alpha) = \{\omega \in \Sigma' \mid M(\omega) = \alpha\} \cap \Xi(e_\alpha).$$

Then, $\lambda_{\langle e_\alpha\rangle}(\{M(\omega) = \alpha\} \setminus \Sigma(\alpha)) = 0$. Since $\lambda_{\langle e_\alpha\rangle}(d\omega) = Z^{\alpha,\alpha}(\omega)\lambda(d\omega)$ and $Z^{\alpha,\alpha}(\omega) > 0$ on $\{M(\omega) = \alpha\}$, $\lambda_{\langle e_\alpha\rangle}$ is equivalent to λ on $\{M(\omega) = \alpha\}$. Thus, $\lambda(\{M(\omega) = \alpha\} \setminus \Sigma(\alpha)) = 0$. Therefore, we obtain the following.

LEMMA 3.3. $\lambda(\Sigma \setminus \bigcup_{\alpha\in I} \Sigma(\alpha)) = 0$.

PROOF. It is sufficient to notice that $\Sigma \setminus \bigcup_{\alpha\in I} \Sigma(\alpha) = \bigcup_{\alpha\in I}(\{M(\omega) = \alpha\} \setminus \Sigma(\alpha))$. $\square$

We fix $\alpha \in I$ and $\omega \in \Sigma(\alpha)$. It is noteworthy that $\lambda_{\langle e_\alpha\rangle}(\Sigma_{[\omega]_n}) > 0$ for all n. Indeed, if $\lambda_{\langle e_\alpha\rangle}(\Sigma_{[\omega]_n}) = 0$ for some n, then $\lambda_{\langle e_\alpha\rangle}(\Sigma_{[\omega]_m}) = 0$ for all $m \geq n$, which implies that $Z^{\alpha,\alpha}(\omega)=0$, thereby resulting in a contradiction. In particular, $\psi^*_{[\omega]_n}e_\alpha$ is not constant for an arbitrary n.

Take an increasing sequence $\{n(k)\}\uparrow\infty$ of natural numbers such that $e_\alpha\in\mathcal{H}_{n(1)}$, and for every k,

$$\sigma^{n(k)}\omega\in\Sigma_{\underbrace{\eta(\psi^*_{[\omega]_{n(k)}}e_\alpha)\cdots\eta(\psi^*_{[\omega]_{n(k)}}e_\alpha)}_{k}}.$$

By noting that $\eta(\psi^*_{[\omega]_{n(k)}}e_\alpha)$ belongs to $\{1,\dots,d\}$, there exists $\beta\in\{1,\dots,d\}$ such that $\{k\in\mathbb{N}\mid\eta(\psi^*_{[\omega]_{n(k)}}e_\alpha)=\beta\}$ is an infinite set. Take a subsequence $\{n(k')\}$ of $\{n(k)\}$ such that $\eta(\psi^*_{[\omega]_{n(k')}}e_\alpha)=\beta$ for all k'. For $f\in\mathcal{F}$, we set

$$\xi(f)=\begin{cases}\left(f-\int_K f\,d\mu\right)\Big/\sqrt{2\mathcal{E}(f)}, & \text{if } f \text{ is not constant},\\ 0, & \text{if } f \text{ is constant}.\end{cases}$$

For $i\in I$, if k' is sufficiently large so that $e_i\in\mathcal{H}_{n(k')}$, then $\xi(\psi^*_{[\omega]_{n(k')}}e_i)\in\mathcal{K}\cup\{0\}$. By using the diagonal argument if necessary, we can take a subsequence $\{n(k(l))\}$ of $\{n(k')\}$ such that $\xi(\psi^*_{[\omega]_{n(k(l))}}e_i)$ converges in $\mathcal{F}$ as $l\to\infty$ for every $i\in I$. For notational conveniences, we denote $\xi(\psi^*_{[\omega]_{n(k(l))}}e_i)$ by f_l^i and its limit by f^i, which belongs to $\mathcal{K}\cup\{0\}$. Since $f_l^\alpha\in\mathcal{K}$ for every l, we have $|(u_\beta,f_l^\alpha|_{V_0})|\ge\delta$ for every l, hence $|(u_\beta,f^\alpha|_{V_0})|\ge\delta$.

From Lemma 2.3(1),

$$\lim_{k\to\infty}r_\beta^{-k}PA_\beta^k u=(u_\beta,u)Pv_\beta$$

for any $u\in l(V_0)$. In particular, the operator norms of $r_\beta^{-k}PA_\beta^k$ are bounded in k. Therefore, since $f_l^i|_{V_0}\to f^i|_{V_0}$ as $l\to\infty$, we obtain

$$\lim_{l\to\infty}r_\beta^{-k(l)}PA_\beta^{k(l)}f_l^i|_{V_0}=(u_\beta,f^i|_{V_0})Pv_\beta, \tag{3.2}$$

which implies that

$$\begin{aligned}\lim_{l\to\infty}&r_\beta^{-2k(l)}\mathcal{E}\big((\psi_\beta^*)^{k(l)}f_l^i\big)\\ &=\lim_{l\to\infty}-r_\beta^{-2k(l)}\,{}^t\big(PA_\beta^{k(l)}f_l^i|_{V_0}\big)D\big(PA_\beta^{k(l)}f_l^i|_{V_0}\big)\\ &=-(u_\beta,f^i|_{V_0})^2\,{}^tv_\beta PDPv_\beta\\ &=-(u_\beta,f^i|_{V_0})^2\,{}^tv_\beta Dv_\beta.\end{aligned} \tag{3.3}$$

It should be noted that the right-hand side of (3.3) does not vanish when $i=\alpha$, since $|(u_\beta,f^\alpha|_{V_0})|\ge\delta$.

For $i \in I$ and $n \in \mathbb{N}$, define $y_n^i = \lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_n})/\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_n})$. Since

$$y_n^i = \frac{\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_n})}{\lambda(\Sigma_{[\omega]_n})} \bigg/ \frac{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_n})}{\lambda(\Sigma_{[\omega]_n})},$$

y_n^i converges to $Z^{i,i}(\omega)/Z^{\alpha,\alpha}(\omega) \in [0,\infty)$ as n tends to ∞. We denote $y^i = Z^{i,i}(\omega)/Z^{\alpha,\alpha}(\omega)$. Clearly, $y^\alpha = 1$.

Suppose $y^i = 0$. Then, for any $j \in I$,

$$\begin{aligned}\left|\frac{Z^{i,j}(\omega)}{Z^{\alpha,\alpha}(\omega)}\right| &= \lim_{n\to\infty}\left|\frac{\lambda_{\langle e_i,e_j \rangle}(\Sigma_{[\omega]_n})}{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_n})}\right| \\ &\le \limsup_{n\to\infty}\left(\frac{\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_n})}{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_n})}\right)^{1/2}\left(\frac{\lambda_{\langle e_j \rangle}(\Sigma_{[\omega]_n})}{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_n})}\right)^{1/2} \\ &= \sqrt{y^i y^j} = 0.\end{aligned}$$

Thus, $Z^{i,j}(\omega) = 0$. We set $\tau_i = 1$ for later use.

Next, suppose $y^i > 0$. Note that $\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_n}) > 0$ for any n. [Indeed, if $\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_n}) = 0$ for some n, then $Z^{i,i}(\omega) = 0$, which implies that $y^i = 0$.] In particular, $\psi^*_{[\omega]_n} e_i$ is not a constant function for an arbitrary n. Take a sufficiently large l_0 such that $e_i \in \mathcal{H}_{n(k(l_0))}$ and $y_n^i > 0$ for all $n \ge n(k(l_0))$. For $m \ge l_0$, we define

$$x_m^i = y^i_{n(k(m))+k(m)}/y^i_{n(k(m))}.$$

Since $\log y_n^i$ converges as $n \to \infty$, $\log x_m^i$ converges to 0 as $m \to \infty$. In other words, $\lim_{m\to\infty} x_m^i = 1$. On the other hand, we have

$$\begin{aligned} x_m^i &= \frac{\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_{n(k(m))+k(m)}})}{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_{n(k(m))+k(m)}})} \bigg/ \frac{\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_{n(k(m))}})}{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_{n(k(m))}})} \\ &= \frac{\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_{n(k(m))+k(m)}})}{\lambda_{\langle e_i \rangle}(\Sigma_{[\omega]_{n(k(m))}})} \bigg/ \frac{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_{n(k(m)+k(m))}})}{\lambda_{\langle e_\alpha \rangle}(\Sigma_{[\omega]_{n(k(m))}})} \\ &= \frac{r^{-1}_{[\omega]_{n(k(m))}} \lambda_{\langle \psi^*_{[\omega]_{n(k(m))}} e_i \rangle}(\Sigma_{\underbrace{\beta\cdots\beta}_{k(m)}})}{r^{-1}_{[\omega]_{n(k(m))}} \lambda_{\langle \psi^*_{[\omega]_{n(k(m))}} e_i \rangle}(\Sigma)} \bigg/ \frac{r^{-1}_{[\omega]_{n(k(m))}} \lambda_{\langle \psi^*_{[\omega]_{n(k(m))}} e_\alpha \rangle}(\Sigma_{\underbrace{\beta\cdots\beta}_{k(m)}})}{r^{-1}_{[\omega]_{n(k(m))}} \lambda_{\langle \psi^*_{[\omega]_{n(k(m))}} e_\alpha \rangle}(\Sigma)} \\ &\qquad\qquad \text{[from (2.3)]} \\ &= \frac{2r_\beta^{-k(m)} \mathcal{E}((\psi^*_\beta)^{k(m)} \psi^*_{[\omega]_{n(k(m))}} e_i)}{2\mathcal{E}(\psi^*_{[\omega]_{n(k(m))}} e_i)} \bigg/ \frac{2r_\beta^{-k(m)} \mathcal{E}((\psi^*_\beta)^{k(m)} \psi^*_{[\omega]_{n(k(m))}} e_\alpha)}{2\mathcal{E}(\psi^*_{[\omega]_{n(k(m))}} e_\alpha)} \\ &= \frac{\mathcal{E}((\psi^*_\beta)^{k(m)} f_m^i)}{\mathcal{E}((\psi^*_\beta)^{k(m)} f_m^\alpha)} \end{aligned}$$

$$\xrightarrow{m\to\infty} \frac{-(u_\beta, f^i|_{V_0})^2\, {}^t v_\beta D v_\beta}{-(u_\beta, f^\alpha|_{V_0})^2\, {}^t v_\beta D v_\beta} \qquad \text{[from (3.3)]}$$
$$= \frac{(u_\beta, f^i|_{V_0})^2}{(u_\beta, f^\alpha|_{V_0})^2}.$$

Therefore, $(u_\beta, f^i|_{V_0}) = \tau_i (u_\beta, f^\alpha|_{V_0})$ where $\tau_i = 1$ or -1. Then, from (3.2),

$$\lim_{l\to\infty} r_\beta^{-k(l)} P A_\beta^{k(l)} f_l^i|_{V_0} = \tau_i (u_\beta, f^\alpha|_{V_0}) P v_\beta. \tag{3.4}$$

Now, let $i, j \in I$ and suppose $y^i > 0$ and $y^j > 0$. Then,

$$\lim_{m\to\infty} \frac{\lambda_{\langle e_i, e_j\rangle}(\Sigma_{[\omega]_{n(k(m))+k(m)}})}{\lambda_{\langle e_\alpha\rangle}(\Sigma_{[\omega]_{n(k(m))+k(m)}})} = \frac{Z^{i,j}(\omega)}{Z^{\alpha,\alpha}(\omega)}.$$

On the other hand, for sufficiently large m, we have

$$\begin{aligned}
&\frac{\lambda_{\langle e_i, e_j\rangle}(\Sigma_{[\omega]_{n(k(m))+k(m)}})}{\lambda_{\langle e_\alpha\rangle}(\Sigma_{[\omega]_{n(k(m))+k(m)}})}\\
&\quad = \frac{r_{[\omega]_{n(k(m))}}^{-1} \lambda_{\langle \psi^*_{[\omega]_{n(k(m))}} e_i, \psi^*_{[\omega]_{n(k(m))}} e_j\rangle}(\Sigma_{\underbrace{\beta\cdots\beta}_{k(m)}})}{r_{[\omega]_{n(k(m))}}^{-1} \lambda_{\langle \psi^*_{[\omega]_{n(k(m))}} e_\alpha\rangle}(\Sigma_{\underbrace{\beta\cdots\beta}_{k(m)}})} \qquad \text{[from (2.3)]}\\
&\quad = \frac{\sqrt{2\mathcal{E}(\psi^*_{[\omega]_{n(k(m))}} e_i)}\sqrt{2\mathcal{E}(\psi^*_{[\omega]_{n(k(m))}} e_j)}\lambda_{\langle f_m^i, f_m^j\rangle}(\Sigma_{\underbrace{\beta\cdots\beta}_{k(m)}})}{2\mathcal{E}(\psi^*_{[\omega]_{n(k(m))}} e_\alpha)\lambda_{\langle f_m^\alpha\rangle}(\Sigma_{\underbrace{\beta\cdots\beta}_{k(m)}})}\\
&\quad = \frac{\sqrt{\lambda_{\langle e_i\rangle}(\Sigma_{[\omega]_{n(k(m))}})\lambda_{\langle e_j\rangle}(\Sigma_{[\omega]_{n(k(m))}})}}{\lambda_{\langle e_\alpha\rangle}(\Sigma_{[\omega]_{n(k(m))}})} \cdot \frac{2r_\beta^{-k(m)}\mathcal{E}((\psi_\beta^*)^{k(m)} f_m^i, (\psi_\beta^*)^{k(m)} f_m^j)}{2r_\beta^{-k(m)}\mathcal{E}((\psi_\beta^*)^{k(m)} f_m^\alpha)}\\
&\quad = \sqrt{y^i_{n(k(m))} y^j_{n(k(m))}} \cdot \frac{-{}^t(r_\beta^{-k(m)} P A_\beta^{k(m)} f_m^i|_{V_0}) D (r_\beta^{-k(m)} P A_\beta^{k(m)} f_m^j|_{V_0})}{-{}^t(r_\beta^{-k(m)} P A_\beta^{k(m)} f_m^\alpha|_{V_0}) D (r_\beta^{-k(m)} P A_\beta^{k(m)} f_m^\alpha|_{V_0})}\\
&\quad \xrightarrow{m\to\infty} \sqrt{y^i y^j} \cdot \frac{\tau_i (u_\beta, f^\alpha|_{V_0}) \cdot \tau_j (u_\beta, f^\alpha|_{V_0})\, {}^t v_\beta D v_\beta}{(u_\beta, f^\alpha|_{V_0})^2\, {}^t v_\beta D v_\beta} \qquad \text{[from (3.4)]}\\
&\quad = \sqrt{y^i y^j}\,\tau_i \tau_j.
\end{aligned}$$

Therefore, $\sqrt{y^i y^j}\,\tau_i\tau_j = Z^{i,j}(\omega)/Z^{\alpha,\alpha}(\omega)$. This relation is valid even when $y^i = 0$ or $y^j = 0$.

For $i \in I$, we define

$$\zeta_i = \frac{Z^{i,\alpha}(\omega)}{\sqrt{Z^{\alpha,\alpha}(\omega)}}. \tag{3.5}$$

Then,

$$\begin{aligned}\zeta_i\zeta_j &= \frac{Z^{i,\alpha}(\omega)}{Z^{\alpha,\alpha}(\omega)} \cdot \frac{Z^{j,\alpha}(\omega)}{Z^{\alpha,\alpha}(\omega)} \cdot Z^{\alpha,\alpha}(\omega)\\ &= \sqrt{y^i y^\alpha}\tau_i\tau_\alpha \cdot \sqrt{y^j y^\alpha}\tau_j\tau_\alpha \cdot Z^{\alpha,\alpha}(\omega)\\ &= \sqrt{y^i y^j}\tau_i\tau_j \cdot Z^{\alpha,\alpha}(\omega)\\ &= Z^{i,j}(\omega).\end{aligned}$$

Along with Lemma 3.3, we have proved the following key proposition.

PROPOSITION 3.4. *There exist measurable functions $\{\zeta_i\}_{i\in I}$ on Σ such that for every $i, j \in I$, $Z^{i,j}(\omega) = \zeta_i(\omega)\zeta_j(\omega)$ λ-a.s. ω.*

REMARK 3.5. According to this proposition, when I is a finite set, the matrix $(Z^{i,j}(\omega))_{i,j\in I}$ has a rank one λ-a.s. ω. In particular, the proposition implies that the matrix $Z(\omega)$ defined in [8, 9] has rank one a.s. ω.

4. AF-martingale dimension. We use the same notation as those used in Sections 2 and 3. We take $I = \mathbb{N}$ and a sequence $\{e_i\}_{i\in I}$ of piecewise harmonic functions so that the linear span of $\{e_i\}_{i\in I}$ is dense in $\mathcal{F}$. Let ν denote the induced measure of λ by $\pi : \Sigma \to K$. From Lemma 2.2, (K, ν) and (Σ, λ) are isomorphic as measure spaces. For each $i \in \mathbb{N}$, take a Borel function ρ_i on K such that $\zeta_i = \rho_i \circ \pi$ λ-a.s., where ζ_i appeared in Proposition 3.4. For $i, j \in \mathbb{N}$, let $z^{i,j}$ be the Radon–Nikodym derivative $d\mu_{\langle e_i, e_j\rangle}/d\nu$, which is a function on K. Then, $Z^{i,j} = z^{i,j} \circ \pi$ λ-a.s. and the result of Proposition 3.4 can be rewritten as

$$z^{i,j} = \rho_i\rho_j \qquad \nu\text{-a.s.} \tag{4.1}$$

Let $\mathcal{L}^2(Z)$ be a set of all Borel measurable maps $g = (g_i)_{i\in\mathbb{N}}$ from K to $\mathbb{R}^{\mathbb{N}}$ such that $\int_K (\sum_{i=1}^\infty |g_i(x)\rho_i(x)|)^2\, \nu(dx) < \infty$. We define a preinner product $\langle\cdot,\cdot\rangle_Z$ on $\mathcal{L}^2(Z)$ by

$$\langle g, g'\rangle_Z = \frac{1}{2}\int_K \left(\sum_{i=1}^\infty g_i(x)\rho_i(x)\right)\left(\sum_{i=1}^\infty g_i'(x)\rho_i(x)\right)\nu(dx), \qquad g, g' \in \mathcal{L}^2(Z).$$

For $g, g' \in \mathcal{L}^2(Z)$, we write $g \sim g'$ if $\sum_{i=1}^\infty (g_i(x) - g_i'(x))\rho_i(x) = 0$ ν-a.s. x. Then, $\sim$ is an equivalence relation. We denote $\mathcal{L}^2(Z)/\sim$ by $L^2(Z)$, which becomes a Hilbert space with inner product $\langle\cdot,\cdot\rangle_Z$ by abuse of notation. We identify a function in $\mathcal{L}^2(Z)$ with its equivalence class. It should be noted that $L^2(Z)$

is isomorphic to $L^2(K \to \mathbb{R}, \nu)$ by the map $\{g_i\}_{i\in\mathbb{N}} \mapsto 2^{-1/2}\sum_{i=1}^{\infty} g_i\rho_i$, since $\sum_{i=1}^{\infty}|\rho_i(x)| > 0$ ν-a.s. x.

We define

$$\mathcal{C} = \left\{ g = (g_i)_{i\in\mathbb{N}} \in \mathcal{L}^2(Z) \;\middle|\; \begin{array}{l} g_i \text{ is a bounded Borel function on } K \\ \text{and there exists some } n \in \mathbb{N} \text{ such that} \\ g_i = 0 \text{ for all } i \geq n \end{array} \right\}$$

and let $\tilde{\mathcal{C}}$ be the equivalence class of $\mathcal{C}$ in $L^2(Z)$.

LEMMA 4.1. *Set $\tilde{\mathcal{C}}$ is dense in $L^2(Z)$.*

PROOF. Consider $g = (g_i)_{i\in\mathbb{N}} \in \mathcal{L}^2(Z)$. For $m \in \mathbb{N}$, let $g_i^{(m)}(\omega) = ((-m) \vee g_i(\omega)) \wedge m$ when $i \leq m$ and $g_i^{(m)}(\omega) = 0$ when $i > m$. Then, $g^{(m)} = (g_i^{(m)})_{i\in\mathbb{N}}$ belongs to $\mathcal{C}$ and converges to g in $L^2(Z)$. Therefore, $\tilde{\mathcal{C}}$ is dense in $L^2(Z)$. □

Let us review the theory of additive functionals associated with local and conservative regular Dirichlet forms $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ (see [3], Chapter 5, for details). From the general theory of regular Dirichlet forms, we can construct a conservative diffusion process $\{X_t\}$ on K defined on a filtered probability space $(\Omega, \mathcal{F}, P, \{P_x\}_{x\in K}, \{\mathcal{F}_t\}_{t\in[0,\infty)})$ associated with $(\mathcal{E}, \mathcal{F})$. Let E_x denote the expectation with respect to P_x. Under the framework of this paper, the following is a basic fact in the analysis of post-critically finite self-similar sets with regular harmonic structures. We provide a proof for readers' convenience.

PROPOSITION 4.2. *The capacity derived from $(\mathcal{E}, \mathcal{F})$ of any nonempty set in K is uniformly positive.*

PROOF. Since the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is constructed by a regular harmonic structure (D, r), we can utilize [7], Theorem 3.4, to assure that there exists $C > 0$ such that

$$\left(\max_{x\in K} f(x) - \min_{x\in K} f(x)\right)^2 \leq C\mathcal{E}(f, f), \qquad f \in \mathcal{F} \subset C(K). \tag{4.2}$$

Let U be an arbitrary nonempty open set of K. Let f be a function in $\mathcal{F} \subset C(K)$ such that $f \geq 1$ μ-a.e. on U. If $\min_{x\in K} f(x) \leq 1/2$, then from (4.2), $\mathcal{E}(f, f) \geq 1/(4C)$. Otherwise, since $f > 1/2$ on K, $\|f\|^2_{L^2(\mu)} \geq 1/4$. Therefore, the capacity of U is not less than $\min\{1/(4C), 1/4\}$. This completes the proof. □

From this proposition, the concept of a "quasi-every point" is identical to that of "every point." We may assume that for each $t \in [0, \infty)$, there exists a shift operator $\theta_t : \Omega \to \Omega$ that satisfies $X_s \circ \theta_t = X_{s+t}$ for all $s \geq 0$. A real-valued function $A_t(\omega)$, $t \in [0, \infty)$, $\omega \in \Omega$, is referred to as an additive functional if the following conditions hold:

- $A_t(\cdot)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$.
- There exists a set $\Lambda \in \sigma(\mathcal{F}_t; t \geq 0)$ such that $P_x(\Lambda) = 1$ for all $x \in K$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$; moreover, for each $\omega \in \Lambda$, $A_{\cdot}(\omega)$ is right continuous and has the left limit on $[0, \infty)$, $A_0(\omega) = 0$, and

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega), \qquad t, s \geq 0.$$

A continuous additive functional is defined as an additive functional such that $A_{\cdot}(\omega)$ is continuous on $[0, \infty)$ on Λ. A $[0, \infty)$-valued continuous additive functional is referred to as a positive continuous additive functional. From [3], Theorem 5.1.3, for each positive continuous additive functional A of $\{X_t\}$, there exists a measure μ_A (termed the Revuz measure of A) such that the following identity holds for any $t > 0$ and nonnegative Borel functions f and h on K:

$$\int_K E_x\left[\int_0^t f(X_s)\, dA_s\right] h(x)\mu(dx) = \int_0^t \int_K E_x[h(X_s)] f(x) \mu_A(dx)\, ds. \tag{4.3}$$

Let P_μ be a probability measure on Ω defined as $P_\mu(\cdot) = \int_K P_x(\cdot)\, \mu(dx)$. Let E_μ denote the expectation with respect to P_μ. We define the energy $e(A)$ of the additive functional A_t as

$$e(A) = \lim_{t \to 0} (2t)^{-1} E_\mu(A_t^2)$$

if the limit exists.

Let $\mathcal{M}$ be the space of martingale additive functionals of $\{X_t\}$ that is defined as

$$\mathcal{M} = \left\{ M \,\middle|\, \begin{array}{l} M \text{ is an additive functional such that for each } t > 0, \\ E_x(M_t^2) < \infty \text{ and } E_x(M_t) = 0 \text{ for every } x \in K \end{array} \right\}.$$

Due to the (strong) local property of $(\mathcal{E}, \mathcal{F})$, any $M \in \mathcal{M}$ is a continuous additive functional ([3], Lemma 5.5.1(ii)).

Each $M \in \mathcal{M}$ admits a positive continuous additive functional $\langle M \rangle$ referred to as the quadratic variation associated with M that satisfies

$$E_x[\langle M \rangle_t] = E_x[M_t^2], \qquad t > 0 \text{ for every } x \in K,$$

and the following equation holds

$$e(M) = \tfrac{1}{2} \mu_{\langle M \rangle}(K). \tag{4.4}$$

We set $\overset{\circ}{\mathcal{M}} = \{M \in \mathcal{M} \mid e(M) < \infty\}$. Then, $\overset{\circ}{\mathcal{M}}$ is a Hilbert space with an inner product $e(M, M') := (e(M + M') - e(M - M'))/4$ ([3], Theorem 5.2.1).

The space $\mathcal{N}_c$ of the continuous additive functionals of zero energy is defined as

$$\mathcal{N}_c = \left\{ N \,\middle|\, \begin{array}{l} N \text{ is a continuous additive functional,} \\ e(N) = 0,\ E_x[|N_t|] < \infty \text{ for all } x \in K \text{ and } t > 0 \end{array} \right\}.$$

For each $u \in \mathcal{F} \subset C(K)$, there exists a unique expression as

$$u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \qquad M^{[u]} \in \overset{\circ}{\mathcal{M}},\ N^{[u]} \in \mathcal{N}_c.$$

(See [3], Theorem 5.2.2.) From [3], Theorem 5.2.3, $\mu_{\langle M^{[u]}\rangle}$ equals $\mu_{\langle u\rangle}$.

For $M \in \overset{\circ}{\mathcal{M}}$ and $f \in L^2(K, \mu_{\langle M\rangle})$, we can define the stochastic integral $f \bullet M$ ([3], Theorem 5.6.1), which is a unique element in $\overset{\circ}{\mathcal{M}}$ such that

$$e(f \bullet M, L) = \frac{1}{2}\int_K f(x)\mu_{\langle M,L\rangle}(dx) \qquad \text{for all } L \in \overset{\circ}{\mathcal{M}}.$$

Here, $\mu_{\langle M,L\rangle} = (\mu_{\langle M+L\rangle} - \mu_{\langle M-L\rangle})/4$. We may write $\int_0^{\cdot} f(X_t)\,dM_t$ for $f \bullet M$ since $(f \bullet M)_t = \int_0^t f(X_s)\,dM_s$, $t > 0$, P_x-a.s. for all $x \in K$ as long as f is a continuous function on K ([3], Lemma 5.6.2). We follow the standard textbook [3] and use the notation $f \bullet M$ to denote the stochastic integral with respect to martingale additive functionals. From [3], Lemma 5.6.2, we also have

$$d\mu_{\langle f\bullet M,L\rangle} = f \cdot d\mu_{\langle M,L\rangle}, \qquad L \in \overset{\circ}{\mathcal{M}}. \tag{4.5}$$

Now, for $g = (g_i)_{i\in\mathbb{N}} \in \mathcal{C}$, we define

$$\chi(g) = \sum_{i=1}^{\infty} g_i \bullet M^{[e_i]} \in \overset{\circ}{\mathcal{M}}. \tag{4.6}$$

We note that the sum is in fact a finite sum.

LEMMA 4.3. *The map $\chi : \mathcal{C} \to \overset{\circ}{\mathcal{M}}$ preserves the* (*pre-*)*inner products*.

PROOF. Take $g = (g_i)_{i\in\mathbb{N}} \in \mathcal{C}$ and $g' = (g'_i)_{i\in\mathbb{N}} \in \mathcal{C}$. Since $\mu_{\langle e_i,e_j\rangle}$ is equal to $\mu_{\langle M^{[e_i]},M^{[e_j]}\rangle}$, we obtain

$$\begin{aligned}
\langle g, g'\rangle_Z &= \frac{1}{2}\int_K \left(\sum_{i=1}^{\infty} g_i(x)\rho_i(x)\right)\left(\sum_{j=1}^{\infty} g'_j(x)\rho_j(x)\right)\nu(dx)\\
&= \frac{1}{2}\sum_{i,j}\int_K g_i(x)g'_j(x)z^{i,j}(x)\nu(dx) \qquad [\text{by } (4.1)]\\
&= \frac{1}{2}\sum_{i,j}\int_K g_i(x)g'_j(x)\,\mu_{\langle e_i,e_j\rangle}(dx)\\
&= \frac{1}{2}\sum_{i,j}\int_K g_i(x)g'_j(x)\,\mu_{\langle M^{[e_i]},M^{[e_j]}\rangle}(dx)\\
&= \sum_{i,j} e\big(g_i \bullet M^{[e_i]}, g'_j \bullet M^{[e_j]}\big)\\
&= e(\chi(g), \chi(g')).
\end{aligned}$$

This completes the proof. □

By virtue of [3], Lemma 5.6.3, and the fact that the linear span of $\{e_i\}_{i\in\mathbb{N}}$ is dense in $\mathcal{F}$, $\chi(\mathcal{C})$ is dense in $\mathring{\mathcal{M}}$. Therefore, along with Lemma 4.1 and Lemma 4.3, χ can extend to an isomorphism from $L^2(Z)$ to $\mathring{\mathcal{M}}$. By the routine argument, we can prove (4.6) for all $g \in L^2(Z)$, where the infinite sum is considered in the topology of $\mathring{\mathcal{M}}$.

The AF-martingale dimension associated with $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ is one in the following sense, which is our main theorem.

THEOREM 4.4. *There exists $M^1 \in \mathring{\mathcal{M}}$ such that, for any $M \in \mathring{\mathcal{M}}$, there exists $f \in L^2(K, \mu_{\langle M^1\rangle})$ that satisfies $M = f \bullet M^1$. Moreover, we can take M^1 so that $\mu_{\langle M^1\rangle} = \nu$.*

This theorem states that every martingale additive functional with finite energy is expressed by a stochastic integral with respect to only one fixed martingale additive functional. Note that considering martingale additive functionals instead of pure martingales under some P_x seems more natural in the framework of time-homogeneous Markov processes (or of the theory of Dirichlet forms). Of course, every martingale additive functional is a martingale under P_x for every x, but it is doubtful whether a pure martingale (under some P_x) is derived from a certain martingale additive functional. Therefore, the connection between AF-martingale dimensions and the Davis–Varaiya invariants is not straightforward. A general theory of AF-martingale dimensions has been discussed by Motoo and Watanabe [11], which is prior to the work by Davis and Varaiya [2]. Clarifying the connection between AF-martingale dimensions and the Davis–Varaiya invariants (whose definition seems "too general" from our standpoint) should be discussed elsewhere, in a more general framework.

PROOF OF THEOREM 4.4. First, by taking (3.5) into consideration, we note that

$$0 < \sum_{j=1}^{\infty} a_j \rho_j(x)^2 = \sum_{j=1}^{\infty} a_j \frac{z^{j,\alpha}(x)^2}{z^{\alpha,\alpha}(x)} \le \sum_{j=1}^{\infty} a_j z^{j,j}(x) = 1.$$

For each $i \in \mathbb{N}$, we define

$$h_i = a_i \rho_i \Bigg/ \left(\sum_{k=1}^{\infty} a_k \rho_k^2 \right).$$

Since $h_i \rho_i \ge 0$ and $\sum_{i=1}^{\infty} h_i \rho_i = 1$, $h = (h_i)_{i\in\mathbb{N}}$ belongs to $\mathcal{L}^2(Z)$. We also define $M^1 \in \mathring{\mathcal{M}}$ as the image of the equivalence class of h in $L^2(Z)$ by χ. Then, at least formally,

$$
\begin{aligned}
d\mu_{\langle M^1\rangle} &= d\mu_{\langle \sum_{i=1}^\infty a_i\rho_i(\sum_{k=1}^\infty a_k\rho_k^2)^{-1}\bullet M^{[e_i]}, \sum_{j=1}^\infty a_j\rho_j(\sum_{k=1}^\infty a_k\rho_k^2)^{-1}\bullet M^{[e_j]}\rangle} \\
&= \left(\sum_{k=1}^\infty a_k\rho_k^2\right)^{-2} \sum_{i,j} a_i\rho_i a_j\rho_j\, d\mu_{\langle M^{[e_i]}, M^{[e_j]}\rangle} \qquad \text{[by (4.5)]} \\
&= \left(\sum_{k=1}^\infty a_k\rho_k^2\right)^{-2} \sum_{i,j} a_i\rho_i a_j\rho_j z^{i,j}\, d\nu \\
&= \left(\sum_{k=1}^\infty a_k\rho_k^2\right)^{-2} \sum_{i,j} a_i\rho_i^2 a_j\rho_j^2\, d\nu \qquad \text{[by (4.1)]} \\
&= d\nu.
\end{aligned}
$$

This calculation is justified by approximating h by the elements in $\mathcal{C}$ and performing a similar calculation.

Let $g = (g_i)_{i\in\mathbb{N}} \in \mathcal{L}^2(Z)$ such that $\sum_{j=1}^\infty |g_j\rho_j|$ is a bounded function. We define $f = \sum_{j=1}^\infty g_j\rho_j$ and $g_i' = fh_i$ for $i \in \mathbb{N}$. We have $\sum_{i=1}^\infty |g_i'\rho_i| = |f|$ and $\sum_{i=1}^\infty g_i'\rho_i = f$, which imply that $g' = (g_i')_{i\in\mathbb{N}}$ belongs to $\mathcal{L}^2(Z)$ and $g \sim g'$. Then, $f \bullet M^1 = \sum_{i=1}^\infty f \bullet (h_i \bullet M^{[e_i]})$ and $\chi(g') = \sum_{i=1}^\infty (fh_i) \bullet M^{[e_i]}$. According to [3], Corollary 5.6.1, these two additive functionals coincide. In other words, $\chi(g) = f \bullet M^1$. We also have

$$
\langle g, g\rangle_Z = e(f \bullet M^1) = \int_K f^2\, d\mu_{\langle M^1\rangle} = \int_K f^2\, d\nu.
$$

Based on the approximation argument using this relation, for general $g \in L^2(Z)$, we can take some $f \in L^2(K, \mu_{\langle M^1\rangle})$ such that $\chi(g) = f \bullet M^1$. This completes the proof. □

REMARK 4.5. (1) The underlying measure μ on K does not play an important role in this paper.

(2) In "nondegenerate" examples of fractals, only a finite number of harmonic functions $\{e_i\}$ are required for the argument in this section. Such cases are treated in [8, 9]. However, in order to include "degenerate" examples such as the Vicsek set (example in the bottom left part of Figure 1), it is not sufficient to consider only harmonic functions.

5. Concluding remarks. In this section, we remark on Proposition 3.4. In Section 3, the functions $\{e_i\}_{i\in I}$ were considered to be piecewise harmonic functions such that $2\mathcal{E}(e_i) = 1$. In fact, Proposition 3.4 is true for an arbitrary choice of $\{e_i\}$ in $\mathcal{F}$. More precisely, let J be a finite set $\{1, \dots, N_0\}$ or an infinite set $\mathbb{N}$. Let $\{f_i\}_{i\in J}$ be a sequence in $\mathcal{F}$. Take a real sequence $\{b_i\}_{i\in J}$ such that $b_i > 0$ for every

$i \in J$ and $\hat{\lambda} := \sum_{i \in J} b_i \lambda_{\langle f_i \rangle}$ is a probability measure on Σ. For $i, j \in J$, we denote the Radon–Nikodym derivative $\lambda_{\langle f_i, f_j \rangle}/d\hat{\lambda}$ by $\hat{Z}^{i,j}$ and obtain the following.

PROPOSITION 5.1. *There exist measurable functions $\{\hat{\zeta}_i\}_{i \in J}$ on Σ such that, for every $i, j \in J$, $\hat{Z}^{i,j}(\omega) = \hat{\zeta}_i(\omega)\hat{\zeta}_j(\omega)$ $\hat{\lambda}$-a.s. ω.*

PROOF. We may assume that $\int_K f_i \, d\mu = 0$ for every $i \in J$ without loss of generality. In the setting of Section 3, take $I = \mathbb{N}$ and $\{e_i\}_{i \in I}$ so that $\{e_i\}_{i \in I}$ is dense in $\{f \in \mathcal{F} \mid \int_K f \, d\mu = 0, \, 2\mathcal{E}(f) = 1\}$ in the topology of $\mathcal{F}$. The definitions of $\{a_i\}_{i \in \mathbb{N}}$, λ, $Z^{i,j}$ and ζ_i are the same as those in Section 3. First, we prove the following.

LEMMA 5.2. *For any $f \in \mathcal{F}$, $\lambda_{\langle f \rangle}$ is absolutely continuous with respect to λ.*

PROOF. It should be noted that for any measurable set A of Σ and $g \in \mathcal{F}$,

$$|\lambda_{\langle f \rangle}(A)^{1/2} - \lambda_{\langle g \rangle}(A)^{1/2}|^2 \le \lambda_{\langle f-g \rangle}(A). \tag{5.1}$$

Indeed, this is proved from the inequalities $\lambda_{\langle sf-tg \rangle}(A) \ge 0$ for all $s, t \in \mathbb{R}$.

For the proof of the claim, we may assume $\int_K f \, d\mu = 0$. Take $c \ge 0$ and a sequence of natural numbers $\{n(k)\}_{k=1}^{\infty}$ such that $g_k := c e_{n(k)}$ converges to f in $\mathcal{F}$ as $k \to \infty$. Suppose $\lambda(A) = 0$. Then, $\lambda_{\langle g_k \rangle}(A) = 0$ for all $k \in \mathbb{N}$. From (5.1),

$$|\lambda_{\langle f \rangle}(A)^{1/2}|^2 \le \lambda_{\langle f-g_k \rangle}(A) \le 2\mathcal{E}(f - g_k, f - g_k) \to 0 \qquad \text{as } k \to \infty.$$

Therefore, $\lambda_{\langle f \rangle}(A) = 0$ and we have $\lambda_{\langle f \rangle} \ll \lambda$. □

In particular, $\hat{\lambda} \ll \lambda$ according to this lemma. Next, we prove the following.

LEMMA 5.3. *For any $f, g \in \mathcal{F}$,*

$$\left(\sqrt{\frac{d\lambda_{\langle f \rangle}}{d\lambda}} - \sqrt{\frac{d\lambda_{\langle g \rangle}}{d\lambda}}\right)^2 \le \frac{d\lambda_{\langle f-g \rangle}}{d\lambda} \qquad \lambda\text{-a.s.} \tag{5.2}$$

PROOF. Since $d\lambda_{\langle sf-tg \rangle}/d\lambda \ge 0$ λ-a.s. for all $s, t \in \mathbb{R}$, we have, for λ-a.s., for all $s, t \in \mathbb{Q}$,

$$s^2 \frac{d\lambda_{\langle f \rangle}}{d\lambda} - 2st \frac{d\lambda_{\langle f,g \rangle}}{d\lambda} + t^2 \frac{d\lambda_{\langle g \rangle}}{d\lambda} \ge 0.$$

Therefore,

$$\left(\frac{d\lambda_{\langle f,g \rangle}}{d\lambda}\right)^2 \le \frac{d\lambda_{\langle f \rangle}}{d\lambda} \cdot \frac{d\lambda_{\langle g \rangle}}{d\lambda} \qquad \lambda\text{-a.s.}$$

Equation (5.2) is derived from this inequality. □

For each $i \in J$, take $c_i \geq 0$ and a sequence of natural numbers $\{n_i(k)\}_{k=1}^{\infty}$ such that $c_i e_{n_i(k)}$ converges to f_i in $\mathcal{F}$ as $k \to \infty$. Let $g_{i,k} = c_i e_{n_i(k)}$.

Let $i, j \in J$ and $\sigma \in \{0, \pm 1\}$. From Lemma 5.3, we have

$$\begin{aligned}
&\int_\Sigma \left(\sqrt{\frac{d\lambda_{\langle f_i + \sigma f_j \rangle}}{d\lambda}} - \sqrt{\frac{d\lambda_{\langle g_{i,k} + \sigma g_{j,k} \rangle}}{d\lambda}} \right)^2 d\lambda \\
&\quad \leq \int_\Sigma \frac{d\lambda_{\langle (f_i - g_{i,k}) + \sigma (f_j - g_{j,k}) \rangle}}{d\lambda} \, d\lambda \\
&\quad = 2\mathcal{E}\big((f_i - g_{i,k}) + \sigma(f_j - g_{j,k}), (f_i - g_{i,k}) + \sigma(f_j - g_{j,k})\big) \\
&\quad \to 0 \qquad \text{as } k \to \infty.
\end{aligned}$$

Since $d\lambda_{\langle g_{i,k} + \sigma g_{j,k} \rangle}/d\lambda = (c_i \zeta_{n_i(k)} + \sigma c_j \zeta_{n_j(k)})^2$ from Proposition 3.4, $|c_i \zeta_{n_i(k)} + \sigma c_j \zeta_{n_j(k)}|$ converges to $\sqrt{d\lambda_{\langle f_i + \sigma f_j \rangle}/d\lambda}$ in $L^2(\lambda)$ as $k \to \infty$. By the diagonal argument, we may assume that $|c_i \zeta_{n_i(k)} + \sigma c_j \zeta_{n_j(k)}|$ converges λ-a.s. as $k \to \infty$ for all $i, j \in J$ and $\sigma \in \{0, \pm 1\}$. In particular, $|c_i \zeta_{n_i(k)}|$ converges to $\sqrt{d\lambda_{\langle f_i \rangle}/d\lambda}$ λ-a.s. Moreover, for $i, j \in J$, λ-a.s.,

$$\begin{aligned}
c_i \zeta_{n_i(k)} c_j \zeta_{n_j(k)} &= \frac{1}{4}\big\{\big(c_i \zeta_{n_i(k)} + c_j \zeta_{n_j(k)}\big)^2 - \big(c_i \zeta_{n_i(k)} - c_j \zeta_{n_j(k)}\big)^2\big\} \\
&\xrightarrow{k \to \infty} \frac{1}{4}\left(\frac{d\lambda_{\langle f_i + f_j \rangle}}{d\lambda} - \frac{d\lambda_{\langle f_i - f_j \rangle}}{d\lambda} \right) = \frac{d\lambda_{\langle f_i, f_j \rangle}}{d\lambda}.
\end{aligned} \tag{5.3}$$

For $\alpha \in J$, we define

$$\Omega(\alpha) = \left\{ \omega \in \Sigma \,\middle|\, \frac{d\lambda_{\langle f_i \rangle}}{d\lambda}(\omega) = 0 \ (i = 1, \ldots, \alpha - 1) \text{ and } \frac{d\lambda_{\langle f_\alpha \rangle}}{d\lambda}(\omega) > 0 \right\}.$$

Clearly, $\lambda(\{\frac{d\hat{\lambda}}{d\lambda}(\omega) > 0\} \setminus \bigcup_{\alpha \in J} \Omega(\alpha)) = 0$. Let $\alpha \in J$ and $\omega \in \Omega(\alpha)$. For $k \in \mathbb{N}$, we define

$$\tau_k(\omega) = \begin{cases} 1, & \text{if } \zeta_{n_\alpha(k)}(\omega) \geq 0 \\ -1, & \text{otherwise.} \end{cases}$$

Then, $\tau_k(\omega) c_\alpha \zeta_{n_\alpha(k)}(\omega)$ converges to $\sqrt{d\lambda_{\langle f_\alpha \rangle}/d\lambda(\omega)} > 0$ λ-a.s. on $\Omega(\alpha)$. By combining this with (5.3) with $j = \alpha$, $\tau_k(\omega) c_i \zeta_{n_i(k)}(\omega)$ converges λ-a.s. on $\Omega(\alpha)$. We denote the limit by $\tilde{\zeta}_i(\omega)$. Then, from (5.3) again, we have $d\lambda_{\langle f_i, f_j \rangle}/d\lambda = \tilde{\zeta}_i \tilde{\zeta}_j$ λ-a.s. on $\Omega(\alpha)$, for every $i, j \in J$. Therefore, by defining

$$\hat{\zeta}_i(\omega) = \begin{cases} \tilde{\zeta}_i(\omega) \Big/ \sqrt{\dfrac{d\hat{\lambda}}{d\lambda}(\omega)}, & \text{if } \dfrac{d\hat{\lambda}}{d\lambda}(\omega) > 0, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the claim of the proposition. $\square$

We will turn to the next remark. Take $f_1, \dots, f_n \in \mathcal{F}$ and consider the map $\Phi : K \ni x \mapsto (f_1(x), \dots, f_n(x)) \in \mathbb{R}^n$. Suppose that Φ is injective. Then, since Φ is continuous, K and $\Phi(K)$ are homeomorphic. (For example, when K is a d-dimensional Sierpinski gasket, $n = d - 1$, and f_i ($i \in \{1, \dots, d-1\}$) is a harmonic function with $f_i(p_j) = \delta_{ij}$, $j \in \{1, \dots, d\}$, this is true by [6], Theorem 3.6.) Take $a_i > 0$, $i = 1, \dots, n$, such that $\nu := \sum_{i=1}^n a_i \mu_{\langle f_i \rangle}$ is a probability measure on K. We denote the Radon–Nikodym derivative $d\mu_{\langle f_i, f_j \rangle}/d\nu$ by $z^{i,j}$, $i, j = 1, \dots, n$. Let $z(x) = (z^{i,j}(x))_{i,j=1}^n$.

Let G be a C^1-class function on $\mathbb{R}^n$. We define $g(x) = G(f_1(x), \dots, f_n(x))$. Then, $g \in \mathcal{F}$, and from the chain rule of energy measures of conservative local Dirichlet forms ([3], Theorem 3.2.2),

$$\begin{aligned} d\mu_{\langle g \rangle} &= d\mu_{\langle G(f_1,\dots,f_n), G(f_1,\dots,f_n) \rangle} \\ &= \sum_{i,j=1}^n \frac{\partial G}{dx_i}(f_1, \dots, f_n) \frac{\partial G}{dx_j}(f_1, \dots, f_n)\, d\mu_{\langle f_i, f_j \rangle} \\ &= \sum_{i,j=1}^n \frac{\partial G}{dx_i}(f_1, \dots, f_n) \frac{\partial G}{dx_j}(f_1, \dots, f_n) z^{i,j}\, d\nu \\ &= ((\nabla G)(f_1, \dots, f_n), z(\nabla G)(f_1, \dots, f_n))_{\mathbb{R}^n}\, d\nu. \end{aligned}$$

In particular,

$$\mathcal{E}(g, g) = \frac{1}{2} \int_K ((\nabla G)(f_1, \dots, f_n), z(\nabla G)(f_1, \dots, f_n))_{\mathbb{R}^n}\, d\nu. \tag{5.4}$$

If we set $\mathcal{E}'(G, G) = \mathcal{E}(g, g)$, $z' = z \circ \Phi^{-1}$, and $\nu' = \nu \circ \Phi^{-1}$, (5.4) is rewritten as

$$\mathcal{E}'(G, G) = \frac{1}{2} \int_{\Phi(K)} ((\nabla G)(y), z'(y)(\nabla G)(y))_{\mathbb{R}^n}\, \nu'(dy).$$

Since the rank of z' is one ν'-a.s., z' can be regarded as a "Riemannian metric" on $\Phi(K)$ and $\Phi(K)$ is considered to be a one-dimensional "measure-theoretical Riemannian submanifold" in $\mathbb{R}^n$. This observation has been stated in [6] in the case of Sierpinski gaskets.

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