The Annals of Probability
1975, Vol. 3, No. 1, 1-31

RANDOM WALKS IN A RANDOM ENVIRONMENT

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Let $\{\alpha_n\}$ be a sequence of independent, identically distributed random variables with $0 \leqq \alpha_n \leqq 1$ for all n. The random walk in a random environment on the integers is the sequence $\{X_n\}$ where $X_0 = 0$ and inductively $X_{n+1} = X_n + 1$, $(X_n - 1)$, with probability α_{X_n}, $(1 - \alpha_{X_n})$. In this paper we consider limit theorems for the random walk in a random environment. We show that "randomizing the environment" in some sense "slows down" the random walk in Section One. The remaining sections are concerned with features of this "slowing down" in some simple models.

0. Introduction. Let $\{\alpha_n\}_{-\infty}^{\infty}$ be a fixed sequence of numbers between 0 and 1; we can define a random walk on the integers Z with transition matrix $M(n, n+1) = \alpha_n$, $M(n, n-1) = \beta_n = 1 - \alpha_n$. Suppose now that $\{\alpha_n\}$ is a sequence of random variables wtih $0 \leqq \alpha_n \leqq 1$. We can still perform a random walk by first choosing a particular fixed sequence $\{\alpha_n'\}$ in accordance with the distribution of $\{\alpha_n\}$ and then using the transition matrix $M(n, n+1) = \alpha_n'$, $M(n, n-1) = \beta_n'$. The difficulty is in trying to say something about the random walk without knowing which particular "environment" $\{\alpha_n'\}$ has been chosen. If X_n denotes the position of the random walk at time n, then $\{X_n\}$ is not, in general, a Markov chain. To see that the future, given the present, is not independent of the past, consider that each time we hit and leave a given integer j increases our certainty in the value of α_j until by the law of large numbers

$$\alpha_j = P(X_n = j + 1 \mid X_{n-1} = j, X_{n-2} = i_{n-2}, \cdots, X_0 = i_0;\ \text{environment } \{\alpha_n\})$$

is completely determined as the number of hits at j approaches infinity. The complexity is somewhat alleviated by assuming the environment $\{\alpha_n\}$ to be a sequence of independent, identically distributed random variables.

We construct the process on the Cartesian product of the set of environments and the set of paths. I.e., if $N = \{0, 1, 2, \cdots\}$, then we define a probability measure on $([0, 1]^Z \times Z^N, \mathscr{F})$, where $\alpha = \{\alpha_n\} \in [0, 1]^Z$ and $\omega \in Z^N$ correspond respectively to an environment and to a path, and $\mathscr{F}$ is the σ-field generated by the cylinder sets. A natural way to define a probability on this space is to specify the probability law governing the set of paths given a fixed environment. Thus, let M_α be the Markov chain measure on Z^N determined by setting $\omega_0 = 0$ and the transition matrix $M_\alpha(n, n+1) = \alpha_n$, $M_\alpha(n, n-1) = \beta_n = 1 - \alpha_n$ where $\alpha = \{\alpha_n\}$ is an environment. On the environments $[0, 1]^Z$ let Q be a product

Received April 19, 1972; revised May 17, 1974.
AMS 1970 *subject classifications*. Primary 60.66; Secondary 60.30.
Key words and phrases. Random walk, diffusion, difference equation with random coefficients, Markov chains.

measure so that $\{\alpha_n\}$ is independent, identically distributed. Now for $A \subset$ environments $[0, 1]^Z$ and $\Omega \subset$ paths Z^N measurable with respect to the σ-fields generated by the cylinder sets let

$$P(A \times \Omega) = \int_A M_\alpha(\Omega)\, dQ(\alpha)\,.$$

A monotone class argument shows that $M_\alpha(\Omega)$ is measurable as a function of the environment α — so $P(A \times \Omega)$ is well defined. It is easily seen that P extends to a σ-additive probability on the field generated by $\{A \times \Omega : A, \Omega$ measurable$\}$. And the Caratheodory Extension Theorem shows that P extends to a probability measure on $([0, 1]^Z \times Z^N, \mathscr{F})$.

The *random walk in a random environment* is formally defined as the process $\{X_n\}_0^\infty$ defined on $([0, 1]^Z \times Z^N, \mathscr{F}, P)$ where $X_n(\alpha, \omega) = \omega_n$.

It follows directly from the construction that $\{X_n\}$ is a Markov Chain on Z which moves only to neighboring points at each step when the environment is fixed. Also, if $\{X_n\}$ has a certain property almost everywhere (a.e.) for almost every (a.e.) fixed environment, then $\{X_n\}$ has this property a.e. I.e.,

(0.1) THEOREM. *Let $\Omega \subset Z^N$ be measurable. Suppose $M_\alpha(\{X_n\} \in \Omega) = 1$ for* a.e. *environment α. Then $P(\{X_n\} \in \Omega) = 1$.*

PROOF. $P(\{X_n\} \in \Omega) = \int M_\alpha(\Omega)\, dQ(\alpha) = 1$.

This theorem provides the basis for finding the limit behavior of $\{X_n\}$ in the next section.

1. Basic results. The limit behavior of $\{X_n\}$ can be obtained by fixing the environment and considering the limit behavior of the resulting Markov chain. For those cases in which the random walk tends to $+\infty$ or to $-\infty$ a.e. we calculate the limit of the "speed" $n^{-1} \cdot X_n$. A calculation of the expected number of times the random walk hits each integer concludes this section.

When the environment $\{\alpha_n\}$ is fixed, Chung [1], page 65–71, uses systems of difference equations to derive results which we summarize in Lemmas (1.1) and (1.5). The limit behavior of the random walk in a random environment (Theorem (1.7)) will be obtained by applying fluctuation theory to Lemma (1.5).

Fix the environment $\{\alpha_n\}$ and let $\{X_n\}$ be the Markov chain on Z with transition matrix

$$M(n, n+1) = \alpha_n\,, \qquad M(n, n-1) = \beta_n = 1 - \alpha_n\,.$$

Let

$$f_{ij} = P(X_n = j, \text{ some } n > 0 \mid X_0 = i)\,,$$
$$m_{ij} = \text{mean recurrence time from } i \text{ to } j\,.$$

(1.1) LEMMA. *Fix $\{\alpha_n\}$ with $0 < \alpha_n < 1$ for all n; set $\sigma_n = \beta_n/\alpha_n$ and*

$$\begin{aligned} \rho_n &= \sigma_1 \cdots \sigma_n\,, & n > 0 \\ &= \sigma_{-1} \cdots \sigma_n\,, & n < 0\,. \end{aligned}$$

(i) *Let* $i < j$; *then*

$$
\begin{aligned}
f_{ij} &= \left(\sum_{n=-\infty}^{i} \frac{1}{\sigma_n \cdots \sigma_j}\right)\left(\sum_{n=-\infty}^{j} \frac{1}{\sigma_n \cdots \sigma_j}\right)^{-1} < 1\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \sum_{n=1}^{\infty} \frac{1}{\rho_{-n}} < \infty \\
&= 1\,, \qquad\qquad\qquad\qquad\qquad \text{if } \sum_{n=1}^{\infty} \frac{1}{\rho_{-n}} = \infty\,.
\end{aligned} \tag{1.2}
$$

(ii) *Let* $i > j$; *then*

$$
\begin{aligned}
f_{ij} &= \left(\sum_{n=i}^{\infty} \sigma_j \cdots \sigma_n\right)\left(\sum_{n=j}^{\infty} \sigma_j \cdots \sigma_n\right)^{-1} < 1\,, \qquad \text{if } \sum_{n=1}^{\infty} \rho_n < \infty \\
&= 1\,, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \sum_{n=1}^{\infty} \rho_n = \infty\,.
\end{aligned} \tag{1.3}
$$

(iii) *If* $f_{01} = 1$, *then*

$$m_{01} = (1 + \sigma_0) + \sum_{j=-\infty}^{0} (1 + \sigma_{j-1})\sigma_j \cdots \sigma_0\,. \tag{1.4}$$

(1.5) LEMMA. *Fix* $\{\alpha_n\}$ *with* $0 < \alpha_n < 1$ *for all* n. *Then*

(i) $\sum_{n=1}^{\infty} (\rho_{-n})^{-1} = \infty$, $\sum_{n=1}^{\infty} \rho_n < \infty$ *implies* $\lim_{n\to\infty} X_n = \infty$ a.e.
(ii) $\sum_{n=1}^{\infty} (\rho_{-n})^{-1} < \infty$, $\sum_{n=1}^{\infty} \rho_n = \infty$ *implies* $\lim_{n\to\infty} X_n = -\infty$ a.e.
(iii) $\sum_{n=1}^{\infty} (\rho_{-n})^{-1} = \infty = \sum_{n=1}^{\infty} \rho_n$ *implies* $\{X_n\}$ *is recurrent. In fact* $-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty$ a.e.

PROOF. (i) If

$$\sum_{n=1}^{\infty} \frac{1}{\rho_{-n}} = \infty\,, \qquad \sum_{n=1}^{\infty} \rho_n < \infty,$$

then Lemma (1.1) implies $f_{ij} = 1$ for $i < j$, but $f_{ij} < 1$ for $i > j$. Therefore $\lim_{n\to\infty} X_n = \infty$ a.e. Cases (ii) and (iii) are also clear from Lemma (1.1).

We will use Lemma (1.5) to derive the limit behavior of $\{X_n\}$ when the α_n's are independent, identically distributed random variables. This suggests that we find conditions on $\{\alpha_n\}$ under which $\sum (\rho_{-n})^{-1}$, $\sum \rho_n$ are finite. The necessary machinery is summarized in

(1.6) LEMMA. *Let* $\{Y_j\}_1^{\infty}$ *be a sequence of independent, identically distributed, nondegenerate, finite valued random variables; let* $S_n = Y_1 + \cdots + Y_n$.

(i) $\sum_{n=1}^{\infty} n^{-1}P(S_n > 0) < \infty$ *if and only if* $\lim_{n\to\infty} S_n = -\infty$ a.e. *in which case* $\sum_{n=1}^{\infty} e^{S_n} < \infty$ a.e.
(ii) $\sum_{n=1}^{\infty} n^{-1}P(S_n > 0) = \infty = \sum_{n=1}^{\infty} n^{-1}P(S_n < 0)$ *if and only if* $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty$ a.e. *in which case* $\sum_{n=1}^{\infty} e^{-S_n} = \infty = \sum_{n=1}^{\infty} e^{S_n}$ a.e.

PROOF. The "if and only if" parts follow from fluctuation theory. (See [2], Chapter 8.) The second assertion in (ii) is trivial. Thus we show that if $\lim_{n\to\infty} S_n = -\infty$ a.e., then $\sum e^{S_n} < \infty$ a.e. Stone [8] has shown that under the conditions of the lemma either

$$\limsup_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}} = \infty \quad \text{a.e.} \qquad \text{or} \qquad \lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}} = -\infty \quad \text{a.e.}$$

But $\lim_{n\to\infty} S_n = -\infty$ a.e. Therefore

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}} = -\infty \quad \text{a.e.}$$

Hence $S_n < -n^{\frac{1}{2}}$ eventually. Thus there exists an N such that

$$0 \leqq \sum_{n=N}^{\infty} e^{S_n} < \sum_{n=N}^{\infty} e^{-n^{\frac{1}{2}}} < \infty \quad \text{a.e.}$$

A complete characterization of the limit behavior of $\{X_n\}$ can now be given in terms of the random environment $\{\alpha_n\}$ by combining Lemmas (1.5) and (1.6).

(1.7) THEOREM. *Let $\{\alpha_n\}_{-\infty}^{\infty}$ be a sequence of independent, identically distributed, nondegenerate random variables with $0 \leqq \alpha_n < 1$ for all n or $0 < \alpha_n \leqq 1$ for all n.*

(i) *If* $\sum_{n=1}^{\infty} n^{-1}P(\rho_n > 1) < \infty$, *then* $\lim X_n = \infty$ a.e.
(ii) *If* $\sum_{n=1}^{\infty} n^{-1}P(\rho_n < 1) < \infty$, *then* $\lim_{n\to\infty} X_n = -\infty$ a.e.
(iii) *If* $\sum_{n=1}^{\infty} n^{-1}P(\rho_n < 1) = \infty = \sum_{n=1}^{\infty} n^{-1}P(\rho_n > 1)$, *then* $\{X_n\}$ *is recurrent; in fact* $-\infty = \liminf_{n\to\infty} X_n < \limsup_{n\to\infty} X_n = \infty$ a.e.

If $E(\ln \sigma)$ is defined (possibly $\pm\infty$), then (i), (ii), (iii) *correspond respectively to*

(i′) $E(\ln \sigma) < 0$,
(ii′) $E(\ln \sigma) > 0$,
(iii′) $E(\ln \sigma) = 0$.

Notice that the two series in (i) and (ii) cannot both converge simultaneously since $\sum n^{-1}P(\rho_n = 1) < \infty$. (See [2] Chapter 8.)

PROOF. First suppose $0 < \alpha_n < 1$ for all n. We prove (i), cases (ii) and (iii) being similar. So, suppose

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\ln \sigma_1 + \cdots + \ln \sigma_n > 0) = \sum_{n=1}^{\infty} \frac{1}{n} P(\rho_n > 1) < \infty .$$

Then Lemma (1.6) implies

$$\sum_{n=1}^{\infty} \rho_n = \sum_{n=1}^{\infty} e^{S_n} < \infty \quad \text{a.e.}$$

where $S_n = \ln \sigma_1 + \cdots + \ln \sigma_n$. Now $\rho_n = \rho_{-n}$ in distribution. Thus

$$\sum_{n=1}^{\infty} \frac{1}{\rho_{-n}} = \infty \quad \text{a.e.}$$

Hence Lemma (1.5) implies that for a.e. fixed environment $\lim_{n\to\infty} X_n = \infty$ a.e. Now randomizing the environment gives

$$P(\lim_{n\to\infty} X_n = \infty) = 1$$

as in Theorem (0.1).

If $\alpha_n = 1$, $(\alpha_n = 0)$, with positive probability, but $\alpha_n > 0$, $(\alpha_n < 1)$, for all n, then it is clear that case (i), (case ii), holds.

Suppose that $E(\ln \sigma)$ is defined. Set $S_n = \ln \sigma_1 + \cdots + \ln \sigma_n$. Then $E(\ln \sigma) < 0$ if and only if $\lim_{n\to\infty} S_n = -\infty$ a.e. if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\rho_n > 1) = \sum_{n=1}^{\infty} \frac{1}{n} P(S_n > 0) < \infty$$

by Lemma (1.6). Thus (i′) corresponds to (i). Similarly (ii′) and (iii′) correspond respectively to (ii) and (iii).

The law of large numbers or Birkhoff's Ergodic Theorem cannot be used to find $\lim_{n\to\infty} X_n/n$ directly since the sequence $\{X_j - X_{j-1}\}$ is not strictly stationary (unless $\{\alpha_n\}$ is degenerate). Suppose that $\{X_j - X_{j-1}\}$ is strictly stationary; then $P(X_1 - X_0 = 1, X_2 - X_1 = 1) = P(X_2 - X_1 = 1, X_3 - X_2 = 1)$. But it is an easy exercise to show that this is true if and only if

$$E(\alpha_n^2) = (E\alpha_n)^2$$

which is true if and only if α_n is degenerate.

However, the sequence of ladder variables τ_n is strictly stationary. Let

$$\begin{aligned} T_0 &= 0 \\ T_n &= \min\{k > 0 : X_k = n\}, &&\text{if such a } k \text{ exists} \\ &= \infty, &&\text{if no such } k \text{ exists} \\ \tau_n &= T_n - T_{n-1}, && n \geqq 1 \end{aligned}$$

and define τ_{-n} and T_{-n} similarly.

(1.8) THEOREM. *Suppose* $\limsup X_n = \infty$ a.e. *Then each* τ_n *is finite* a.e. *and the sequence* $\{\tau_n\}_1^\infty$ *is strictly stationary and ergodic.*

PROOF. If $\limsup X_n = \infty$ a.e., then trivially τ_n is finite a.e. That $\{\tau_n\}$ is strictly stationary follows from the strict stationarity of $\{\alpha_n\}$. To show $\{\tau_n\}$ is ergodic, we show that $\{\tau_n\}$ is strongly mixing—the central idea being

(1.9) LEMMA. *Let* $A_1, \cdots, A_k, B_1, \cdots, B_j \subset Z$ *be such that*

$$B_s \subset (0, m-k], \qquad s = 1, \cdots, j.$$

Then for $m > k$

$$\begin{aligned} (1.10) \qquad & P(\tau_r \in A_r, 1 \leqq r \leqq k; \tau_{m+s} \in B_s, 1 \leqq s \leqq j) \\ &= P(\tau_r \in A_r, 1 \leqq r \leqq k) \cdot P(\tau_s \in B_{s'} 1 \leqq s \leqq j). \end{aligned}$$

PROOF OF (1.9): Recalling the definition of M_α in the introduction, it is clear upon fixing the environment α that

$$\begin{aligned} (1.11) \qquad & M_\alpha(\tau_r \in A_r, 1 \leqq r \leqq k; \tau_{m+s} \in B_s, 1 \leqq s \leqq j) \\ &= M_\alpha(\tau_r \in A_r, 1 \leqq r \leqq k) \cdot M_\alpha(\tau_{m+s} \in B_s, 1 \leqq s \leqq j) \end{aligned}$$

since M_α is a Markov chain.

Now

(1.12a) $\quad M_\alpha(\tau_r \in A_r,\ 1 \leqq r \leqq k)$ is measurable with respect to $\mathscr{F}\{\alpha_n\}_{-\infty}^{k-1}$ where $\mathscr{F}\{\alpha_n\}_{-\infty}^{k-1}$ is the σ-field generated by $\{\alpha_n\}_{-\infty}^{k-1}$.

But $X_{n+1} = X_n \pm 1$ and $\tau_{m+s} \leqq m - k,\ 1 \leqq s \leqq j$ imply that $X_n \geqq k$ during the excursion from m to $(m + j)$. Therefore

(1.12b) $\quad M_\alpha(\tau_{m+s} \in B_s,\ 1 \leqq s \leqq j)$ is measurable with respect to $\{\alpha_n\}_k^\infty$ if $B_s \subset (0, m - k]$ for $1 \leqq s \leqq j$.

Combining the fact that $\{\alpha_n\}_{-\infty}^{k-1}$ and $\{\alpha_n\}_k^\infty$ are independent with (1.11), (1.12) and integrating with respect to $dQ(\alpha)$ yields (1.10).

Lemma (1.9) has an intuitive interpretation. If $m \geqq k$ and if it is certain that the random walk in its excursion from m to $(m + j)$ has not doubled back far enough to hit $(k - 1)$, then $\{\tau_1, \cdots, \tau_k\}$ and $\{\tau_{m+1}, \cdots, \tau_{m+j}\}$ are independent.

We apply Lemma (1.9) to prove the strong mixing of $\{\tau_n\}$. It suffices to show

$$(1.13) \qquad \begin{aligned} &\lim_{m\to\infty} P(\tau_r \in A_r,\ 1 \leqq r \leqq k;\ \tau_{m+s} \in B_s,\ 1 \leqq s \leqq j) \\ &\qquad = P(\tau_r \in A_r,\ 1 \leqq r \leqq k) \cdot P(\tau_s \in B_s,\ 1 \leqq s \leqq j) . \end{aligned}$$

Let $\varepsilon > 0$ and choose N such that

$$P(\tau_s \in B_s - (0, N) \text{ for at least one } s;\ 1 \leqq s \leqq j) < \varepsilon .$$

Then

$$(1.14) \qquad \begin{aligned} &P(\tau_r \in A_r,\ 1 \leqq r \leqq k;\ \tau_{m+s} \in B_s \cap (0, N],\ 1 \leqq s \leqq j) \\ &\qquad \leqq P(\tau_r \in A_r,\ 1 \leqq r \leqq k;\ \tau_{m+s} \in B_s,\ 1 \leqq s \leqq j) \\ &\qquad \leqq P(\tau_r \in A_r,\ 1 \leqq r \leqq k;\ \tau_{m+s} \in B_s \cap (0, N],\ 1 \leqq s \leqq j) + \varepsilon . \end{aligned}$$

Applying Lemma (1.9) to the extreme members of (1.14) for large m implies

$$\begin{aligned} &P(\tau_r \in A_r,\ 1 \leqq r \leqq k) \cdot P(\tau_s \in B_s \cap (0, N],\ 1 \leqq s \leqq j) \\ &\qquad \leqq \lim_{m\to\infty} P(\tau_r \in A_r,\ 1 \leqq r \leqq k;\ \tau_{m+s} \in B_s,\ 1 \leqq s \leqq j) \\ &\qquad \leqq P(\tau_r \in A_r,\ 1 \leqq r \leqq k) \cdot P(\tau_s \in B_s \cap (0, N],\ 1 \leqq s \leqq j) + \varepsilon . \end{aligned}$$

Letting $N \nearrow \infty$ and $\varepsilon \searrow 0$ implies (1.13) and thus completes the proof of Theorem (1.8).

Before applying the Ergodic Theorem to $\{\tau_n\}$ we compute $E\tau_1$.

(1.15) THEOREM.

$$\begin{aligned} E\tau_1 &= \frac{1 + E\sigma}{1 - E\sigma} , && E\sigma < 1 \\ &= \infty , && E\sigma \geqq 1 . \end{aligned}$$

Before proving the theorem it should be noted that $E\sigma < 1$ implies $X_n \to \infty$ a.e. To see this note that $P(\rho_n > 1) \leqq E\rho_n = (E\sigma)^n$. Thus $E\sigma < 1$ implies $\sum_1^\infty n^{-1}P(\rho_n > 1) < \infty$ which in turn implies that $X_n \to \infty$ a.e. by (1.7).

PROOF OF (1.15). The above remark shows that if $\limsup X_n < \infty$ a.e., then

$E\sigma \geqq 1$. But in this case $E\tau_1 = \infty$ since $P(\tau_1 = \infty) > 0$. We therefore prove (1.15) assuming $\limsup X_n = \infty$ a.e.

Fix the environment $\{\alpha_n\}$ such that $\limsup_{n\to\infty} X_n = \infty$ a.e.

$$E(\tau_1 \,|\, \{\alpha_n\} \text{ fixed}) = (1 + \sigma_0) + \textstyle\sum_{j=-\infty}^{0} (1 + \sigma_{j-1})\sigma_j \cdots \sigma_0$$

by (1.4). Taking the expectation with respect to the environment yields $E\tau_1 = \infty$ when $E\sigma \geqq 1$ and when $E\sigma < 1$

$$E\tau_1 = (1 + E\sigma)(1 + \textstyle\sum_{j=-\infty}^{0} (E\sigma)^{1-j}) = \dfrac{1 + E\sigma}{1 - E\sigma}\,.$$

(1.16) THEOREM. (i) $E\sigma < 1$ *implies*

$$\lim_{n\to\infty} \frac{T_n}{n} = \frac{1 + E\sigma}{1 - E\sigma} \quad \text{a.e.}\,, \qquad \lim_{n\to\infty} \frac{X_n}{n} = \frac{1 - E\sigma}{1 + E\sigma} \quad \text{a.e.}$$

(ii) $E(\sigma^{-1}) < 1$ *implies*

$$\lim_{n\to\infty} \frac{T_{-n}}{n} = \frac{1 + E(\sigma^{-1})}{1 - E(\sigma^{-1})} \quad \text{a.e.}\,, \qquad \lim_{n\to\infty} \frac{X_n}{n} = -\frac{1 - E(\sigma^{-1})}{1 + E(\sigma^{-1})} \quad \text{a.e.}$$

(iii) $(E\sigma)^{-1} \leqq 1 \leqq E(\sigma^{-1})$ *implies*

$$\lim_{n\to\infty} \frac{T_n}{n} = \infty = \lim_{n\to\infty} \frac{T_{-n}}{n} \quad \text{a.e.}\,, \qquad \lim_{n\to\infty} \frac{X_n}{n} = 0 \quad \text{a.e.}$$

Notice that the three cases concerning σ in this theorem are mutually exclusive and exhaustive. That this is so is a consequence of Jensen's Inequality, i.e., $(E\sigma)^{-1} \leqq E(\sigma^{-1})$.

PROOF OF (1.16). The results for T_n are a direct result of Theorems (1.8), (1.15), and Birkhoff's Ergodic Theorem. (At least this is true when $\limsup X_n = \infty$ a.e. When $\limsup X_n \neq \infty$ a.e., i.e. when $X_n \to -\infty$ a.e. by (1.7), then a.e. $T_n = \infty$ for large n. So $n^{-1}T_n \to \infty$ a.e. But in this case $E\sigma \geqq 1$ by the remark before the proof of (1.15).) The result for T_{-n} follow from those for T_n by a reversal in the roles of the positive and negative integers.

Now we will prove the results for X_n assuming the results for T_n, T_{-n}. Let k_n and l_n be the unique nonnegative integers such that

$$T_{k_n} \leqq n < T_{k_n+1}\,, \qquad T_{-l_n} \leqq n < T_{-l_n-1}\,.$$

In case (i) $X_n \to \infty$ a.e. by the remark following the statement of (1.15). Thus $k_n \to \infty$ a.e. Now $X_n < k_n + 1$; also, by time n the random walk has already hit k_n and since it moves only one integer at each step

$$k_n \leqq X_n + (n - T_{k_n})\,.$$

Thus

$$\frac{k_n}{n} - \left(1 - \frac{T_{k_n}}{n}\right) \leqq \frac{X_n}{n} < \frac{k_n + 1}{n}\,. \tag{1.17}$$

But the definition of k_n implies

$$\lim_{n\to\infty} \frac{k_n}{n} = \lim_{n\to\infty} \frac{n}{T_n}\,. \tag{1.18}$$

Thus to find the limit behavior of X_n in case (i) it suffices to combine (1.17), (1.18) with the observation that $n^{-1}T_{k_n} \to 1$ a.e.

The proof in case (ii) is similar, but uses l_n instead of k_n. In case (iii) note that

$$\frac{T_{k_n}}{k_n} \leqq \frac{n}{k_n}\,, \qquad \frac{T_{-l_n}}{l_n} \leqq \frac{n}{l_n}$$

so that $k_n/n \to 0$ a.e. and $l_n/n \to 0$ a.e.
But

$$X_n < k_n + 1\,, \quad -l_n - 1 < X_n$$

by definition of k_n, l_n.
So

$$\frac{-l_n - 1}{n} < \frac{X_n}{n} < \frac{k_n + 1}{n}$$

which completes case (iii) since the extreme members of this inequality tend to 0 a.e.

When $\{X_n\}$ is recurrent, it is reasonable to conjecture that $\lim_{n\to\infty} n^{-1} \cdot X_n = 0$ a.e. This conjecture is, of course, correct since Theorem (1.16) implies that $\lim_{n\to\infty} n^{-1} \cdot X_n$ exists a.e. and so this limit must be 0 in the recurrent case.

An interesting feature of Theorem (1.16) is that the random environment "slows down" the random walk in certain cases. That is, if each α_n is replaced by $E\alpha$, then the law of large numbers implies

$$\lim_{n\to\infty} \frac{X_n}{n} = E\alpha - E\beta \quad \text{a.e.}$$

But Jensen's Inequality implies

$$\left(E\frac{1}{\alpha}\right)^{-1} \leqq E\alpha\,.$$

Hence

$$\begin{aligned} \frac{1 - E\sigma}{1 + E\sigma} &= \frac{2 - E\left(\frac{1}{\alpha}\right)}{E\left(\frac{1}{\alpha}\right)}\,, \qquad \text{since } \sigma = \frac{\beta}{\alpha} \\ &\leqq 2E\alpha - 1 \\ &= E\alpha - E\beta\,. \end{aligned}$$

and similarly

$$-\frac{1 - E(\sigma^{-1})}{1 + E(\sigma^{-1})} \geqq E\alpha - E\beta\,.$$

I.e., in cases (i) and (ii) of Theorem (1.16) the limit of the distance covered per unit time is less in the random environment than in the fixed environment with each $\alpha_n = E\alpha$.

Even in case (iii) it is easy to see examples in which the random environment "slows down" the random walk in some sense. For example, if we choose a distribution for α such that $(E\sigma)^{-1} \leqq 1 \leqq E(\sigma^{-1})$ and $E\alpha \neq \frac{1}{2}$, then $n^{-1}X_n$ tends to 0 a.e. in this environment while $n^{-1}X_n$ tends to a nonzero constant a.e. in the fixed environment with each $\alpha_n = E\alpha$. (An example of a random variable α satisfying these requirements is given by

$$\sigma = \frac{1-\alpha}{\alpha}$$

$$\begin{aligned} \sigma &= \theta^{q}\,, && \text{with probability } p \\ &= \theta^{-p}\,, && \text{with probability } q \end{aligned}$$

where $\theta > 0$ but $\theta \neq 1$, $p = 1 - q$ but $p \neq 0, \frac{1}{2}$, or 1. It is an easy exercise to show that α or σ defined in this way satisfies $(E\sigma)^{-1} \leqq 1 \leqq E(\sigma^{-1})$, $E\alpha \neq \frac{1}{2}$).

An interesting consequence of Theorems (1.7) and (1.16) is that when $E(\ln\sigma) < 0$ and $E\sigma \geqq 1$, even though $X_n \to \infty$ a.e., we still have $n^{-1}\cdot X_n \to 0$ a.e. (Notice that $E(\ln\sigma) < 0$ and $E\sigma \geqq 1$ automatically implies that case (iii) or (1.16) holds since $E(\ln\sigma) < 0$ implies $E\sigma^{-1} > 1$ by Jensen's Inequality.) Section Two is devoted to certain examples of this. The next theorem may help to explain another aspect of this phenomenon.

(1.19) THEOREM. *Suppose* $\limsup_{n\to\infty} X_n = \infty$ a.e. *Let* $G_j = E$ *(number of* $k \geqq 0$ *such that* $X_k = j \mid X_0 = 0$).

(i) *If* $E\sigma < 1$, *then*

$$\begin{aligned} G_j &= \frac{1+E\sigma}{1-E\sigma}\cdot(E\sigma)^{-j}\,, && j \leqq -1 \\ &= \frac{1+E\sigma}{1-E\sigma}\,, && j \geqq 0 \end{aligned}$$

(ii) *If* $E\sigma \geqq 1$, *then* $G_j = \infty$, *for all* j.

PROOF. Case (ii) clearly holds when $-\infty = \liminf X_n < \limsup X_n = \infty$ a.e. So suppose $\lim_{n\to\infty} X_n = \infty$ a.e. Let

$$G_{ij} = G^{\alpha}_{ij} = E(\text{number of } k \geqq 0 \text{ such that } X_k = j \mid X_0 = i,\ \alpha = \{\alpha_n\} \text{ fixed})$$
$$f_{ij} = f^{\alpha}_{ij} = P(X_n = j,\ \text{some } n > 0 \mid X_0 = i,\ \alpha = \{\alpha_n\} \text{ fixed})\,.$$

Then (see [1] page 53)

$$(1.20) \qquad G_{ij} = \delta_{ij} + f_{ij}G_{jj}\,, \qquad \text{where } \delta_{ij} = 1,\ i = j\,;\ \ \delta_{ij} = 0,\ i \neq j\,.$$

Now a.e.

$$\begin{aligned} f_{ij} &= 1,\ i < j \\ &= \left(\textstyle\sum_{n=i}^{\infty}\sigma_j\cdots\sigma_n\right)\left(\textstyle\sum_{n=j}^{\infty}\sigma_j\cdots\sigma_n\right)^{-1} < 1\,, && i > j \end{aligned}$$

by (1.2), (1.3). So

$$f_{jj} = \alpha_j f_{j+1,j} + \beta_j f_{j-1,j}$$
$$= (\beta_j \sigma_j + \textstyle\sum_{n=j+1}^{\infty} \sigma_j \cdots \sigma_n)(\sum_{n=j}^{\infty} \sigma_j \cdots \sigma_n)^{-1} .$$

Using (1.20) with $i \neq j$ gives

$$G_{ij} = f_{ij} G_{jj} = \frac{1}{\beta_j} \cdot \textstyle\sum_{n=j}^{\infty} \sigma_j \cdots \sigma_n , \qquad i < j$$
$$= \frac{1}{\beta_j} \cdot \textstyle\sum_{n=i}^{\infty} \sigma_j \cdots \sigma_n , \qquad i > j .$$

I.e.,

$$G_{ij} = \frac{1}{\beta_j} \cdot \textstyle\sum_{n=i \vee j}^{\infty} \sigma_j \cdots \sigma_n$$

where $i \vee j = \max\{i, j\}$. Taking expectations with respect to the random environment completes the theorem.

Notice that when $\sum n^{-1} P(\rho_n > 1) < \infty$ and $E(\sigma) \geqq 1$ an unusual situation occurs: Although the random walk originating at zero has zero probability of hitting -1 infinitely often, the number of times it does so has infinite expectation.

2. The slow approach to infinity: an example. In this section we consider the case $\lim_{n\to\infty} X_n = \infty$ a.e., but $\lim_{n\to\infty} n^{-1} X_n = 0$ a.e. We seek norming constants $f(n)$ such that $f(n) \cdot X_n$ tends to limiting districutions along subsequences. The problem in its complete generality involves difference equations with random coefficients (2.1) which we cannot solve; hence we simplify the problem by considering the case where σ_j can take only two possible values, $\sigma_j = 0$ or θ. The points j such that $\sigma_j = 0$ are barriers reflecting to the right. The principal advantage of this scheme is that the random walk can be decomposed into the independent excursions from one barrier to the next. There are two justifications for this simplification: The phenomena discussed in Theorem (1.16) are still apparent; this new model has a natural analogue in the diffusion process of Section 3.

The following notation will be used throughout this section. If x is a real number, $[x]$ denotes the largest integer less than or equal to x; $\{x\}$ is the fractional part of x, i.e., $\{x\} = x - [x]$; $\log_x y$ denotes the logarithm of y in the base x. Also, E_α will denote the expectation with the environment $\alpha = \{\alpha_n\}$ fixed, i.e., E_α is the expectation corresponding to the probability M_α of the introduction.

Throughout this section assume that

$$\sigma_j = 0 \quad \text{with probability} \quad (1 - \gamma)$$
$$= \theta \quad \text{with probability} \quad \gamma , \qquad 0 < \theta < \infty \ \text{fixed.}$$

So α_j is 1, $(1 + \theta)^{-1}$ respectively when $\sigma_j = 0, \theta$. Then each integer j such that $\sigma_j = 0$ is a "one way mirror" reflecting to the right. Now $E(\ln \sigma) = -\infty$ so that $\lim_n X_n = \infty$ a.e. When $E(\sigma) = \gamma\theta < 1$, then

$$\lim_n \frac{T_n}{n} = \frac{1+\gamma\theta}{1-\gamma\theta} \quad \text{a.e.}$$

but when $\gamma\theta \geqq 1$, $\lim_n n^{-1} \cdot T_n = \infty$ a.e. by Theorem (1.16). It is the case $\gamma\theta \geqq 1$ to which the present section is devoted. In particular we assume $\theta > 1$ throughout.

Set $V_0 = 0$ and inductively $V_n = \min\{k > V_{n-1}\colon \alpha_k = 1\}$ for $n \geqq 1$. V_n is the position of the nth "one way mirror" to the right of zero. Since V_n is finite a.e., so is T_{V_n}; also, it is clear that the excursions between times T_{V_n} and $T_{V_{n+1}}$ form a sequence of independent, identically distributed random variables for $n \geqq 1$. We first calculate the Laplace transform of the distribution of $T_{V_2} - T_{V_1}$, use this to obtain limit distributions for T_{V_n} suitably normalized, and then use the law of large numbers to obtain limit distributions for T_n suitably normalized.

LEMMA. *Fix $\alpha = \{\alpha_n\}$ so that* $\limsup_{n\to\infty} X_n = \infty$ a.e. *Let $F_j(u) = E_\alpha \exp(-uT_j)$ be the Laplace Transform of T_j. Then*

$$\frac{1}{F_j(u)} = \frac{\alpha_j e^{-u}}{F_{j+1}(u)} + \frac{\beta_j e^{-u}}{F_{j-1}(u)}, \qquad j \geqq 1. \tag{2.1}$$

PROOF. For $\alpha = \{\alpha_n\}$ fixed, $\tau_1, \tau_2, \cdots$ are independent. Now

$$\begin{aligned} M_\alpha(\tau_{j+1} = k) &= \alpha_j, & k &= 1 \\ &= \beta_j \cdot M_\alpha(\tau_{j+1} + \tau_j = k-1), & k &> 1. \end{aligned}$$

Hence if $\varphi_j(u) = E_\alpha \exp(-u\tau_j)$, i.e., if $\varphi_j(u)$ is the Laplace transform of τ_j given a fixed environment $\{\alpha_n\}$, then

$$\varphi_{j+1}(u) = \alpha_j e^{-u} + \beta_j e^{-u}\varphi_j(u)\varphi_{j+1}(u), \qquad j \geqq 0.$$

But

$$F_j(u) = \varphi_1(u) \cdots \varphi_j(u), \qquad \text{i.e.,} \quad \varphi_j(u) = \frac{F_j(u)}{F_{j-1}(u)}, \qquad j \geqq 1$$

when the environment is fixed. Thus

$$\frac{F_{j+1}(u)}{F_j(u)} = \alpha_j e^{-u} + \beta_j e^{-u} \cdot \frac{F_{j+1}(u)}{F_{j-1}(u)}, \qquad j \geqq 1.$$

Dividing by $F_{j+1}(u)$ yields (2.1).

We find the Laplace transform of the distribution of $T_{V_{n+1}} - T_{V_n}$ under the assumption

$$\begin{aligned} \alpha_j &= 1, & &\text{with probability } (1-\gamma) \\ &= \frac{1}{1+\theta}, & &\text{with probability } \gamma. \end{aligned}$$

Again fix $\{\alpha_n)$ and let $V_n \leqq k \leqq V_{n+1}$ and $G_k(u) = E_\alpha(-u(T_k - T_{V_n}))$; (2.1) together with $T_{V_n+1} - T_{V_n} \equiv 1$ imply

$$\frac{1}{G_k(u)} = \left(\frac{1}{\theta+1}\right)\frac{e^{-u}}{G_{k+1}(u)} + \left(\frac{\theta}{\theta+1}\right)\frac{e^{-u}}{G_{k-1}(u)}, \qquad V_n < k < V_{n+1} \tag{2.2}$$

$$G_{V_n} \equiv 1, \qquad G_{V_n+1}(u) = e^{-u}.$$

It is easy to verify that the unique solution to (2.2) is

$$G_k(u) = \frac{c(u)\beta(u)^{k-V_n}}{a(u) + b(u)(\theta\beta^2(u))^{k-V_n}} \tag{2.3}$$

where

$$\lambda_1(u) = \frac{e^u}{2}(\theta + 1 + [(\theta+1)^2 - 4\theta e^{-2u}]^{\frac{1}{2}}),$$

$$\lambda_2(u) = \frac{e^u}{2}(\theta + 1 - [(\theta+1)^2 - 4\theta e^{-2u}]^{\frac{1}{2}})$$

and

$$\begin{aligned} a(u) &= \lambda_1(u) - e^u = \theta - 1 + O(u) \searrow \theta - 1, \\ b(u) &= e^u - \lambda_2(u) = \frac{2\theta}{\theta - 1}u + O(u^2), \\ c(u) &= \lambda_1(u) - \lambda_2(u) = \theta - 1 + O(u) \searrow \theta - 1, \\ \beta(u) &= \theta^{-1}\lambda_1(u) = 1 + O(u) \searrow 1 \end{aligned} \tag{2.4}$$

as $u \searrow 0$. Now let $\varphi(u) = E\exp(-u(T_{V_{n+1}} - T_{V_n}))$ with the environment randomized. Then

$$\begin{aligned} \varphi(u) &= \textstyle\sum_{j=1}^{\infty} G_{j+V_n}(u) \cdot P(V_{n+1} - V_n = j) \\ &= \frac{1-\gamma}{\gamma} \textstyle\sum_{j=1}^{\infty} \gamma^j G_{j+V_n}(u). \end{aligned}$$

Combining this with (2.3) gives the following:

(2.5) LEMMA. *Let $\varphi(u) = E\exp(-u(T_{V_{n+1}} - T_{V_n}))$ for $n \geqq 1$. Then*

$$\varphi(u) = \frac{1-\gamma}{\gamma} \cdot \sum_{j=1}^{\infty} \frac{c(u)(\gamma\beta(u))^j}{a(u) + b(u)(\theta\beta^2(u))^j}.$$

Before calculating the limit distributions of T_{V_n} suitably normalized we show that φ may be replaced by ψ where

$$\psi(u) = \frac{1-\gamma}{\gamma} \cdot \sum_{j=1}^{\infty} \frac{\gamma^j}{1 + \nu u\theta^j}, \qquad \nu = \frac{2\theta}{(\theta-1)^2}.$$

(2.6) LEMMA. *As $u \searrow 0$*

$$\varphi(u) - \psi(u) = O(u).$$

PROOF. Let

$$A_1(u) = \varphi(u) - \frac{1-\gamma}{\gamma} \cdot \sum_{j=1}^{\infty} \frac{(\theta-1)\gamma^j}{a + b(\theta\beta^2)^j},$$

$$A_2(u) = \frac{1-\gamma}{\gamma} \cdot \sum_{j=1}^{\infty} (\theta-1)\gamma^j\left(\frac{1}{a + b(\theta\beta^2)^j} - \frac{1}{a + b\theta^j}\right)$$

$$A_3(u) = \frac{1-\gamma}{\gamma} \cdot \sum_{j=1}^{\infty} \frac{(\theta-1)\gamma^j}{a + b\theta^j} - \psi(u).$$

Then $\varphi - \psi = A_1 + A_2 + A_3$; thus it suffices to show $A_j(u) = O(u)$ as $u \searrow 0$ for $j = 1, 2, 3$.

Now

$$
\begin{aligned}
|A_1(u)| &= \frac{1-\gamma}{\gamma} \cdot \textstyle\sum_{j=1}^{\infty} \dfrac{(c\beta^j - (\theta-1))\gamma^j}{a + b(\theta\beta^2)^j} \\
&\leqq \frac{1-\gamma}{a\gamma} \cdot \textstyle\sum_{j=1}^{\infty} (c\beta^j - (\theta-1))\gamma^j \\
&\leqq \frac{1-\gamma}{a\gamma} \cdot \left(\frac{c}{1-\beta\gamma} - \frac{\theta-1}{1-\gamma}\right), && \text{for small } u \\
&= 0(u), && \text{by (2.4)}. \\
|A_2(u)| &\leqq \frac{(1-\gamma)(\theta-1)}{\gamma a} \cdot \textstyle\sum_{j=1}^{\infty} ((\gamma\beta^2)^j - \gamma^j) \\
&= 0(u), && \text{by (2.4)}. \\
|A_3(u)| &\leqq \frac{1-\gamma}{\gamma} \cdot \textstyle\sum_{j=1}^{\infty} \dfrac{(a - (\theta-1) + |b - \nu(\theta-1)u|\theta^j)\gamma^j}{(a + b\theta^j)(1 + \nu u\theta^j)} \\
&\leqq (a - (\theta-1))M + \frac{|b - \nu(\theta-1)u|}{u} \cdot M' \\
&\leqq O(u), && \text{by (2.4)}
\end{aligned}
$$

where M and M' are finite constants.

(2.7) LEMMA. *For each* $u > 0$

(i) $\gamma\theta = 1$ *implies*

$$\lim_{y\to\infty} y\left(1 - \varphi\left(\frac{u}{y \ln y}\right)\right) = \frac{2\theta}{(\theta-1)\ln\theta} \cdot u. \tag{2.8}$$

(ii) $\gamma\theta > 1$ *implies*

$$\lim_{y\to\infty}\left(y\left(1 - \varphi\left(\frac{u}{y^\rho}\right)\right) - Ku \cdot \textstyle\sum_{j=-\infty}^{\infty} \dfrac{(\gamma\theta)^{j-\{\omega(y)\}}}{1 + \nu u\theta^{j-\{\omega(y)\}}}\right) = 0 \tag{2.9}$$

where $\nu = 2\theta/(\theta-1)^2$, $K = (1-\gamma)/\gamma \cdot \nu$, $\omega(y) = \log_{1/\gamma} y$, $\rho = \log_{1/\gamma}\theta > 1$.

PROOF. Since $y = o(y \ln y)$ and $y = o(y^\rho)$, Lemma (2.6) shows that it is enough to prove (2.8), (2.9) for ϕ replacing φ. Now set $\sigma = \sigma(x) = \log_\theta x$. Then

$$
\begin{aligned}
1 - \phi\left(\frac{u}{x}\right) &= K\left(\frac{u}{x}\right) \cdot \textstyle\sum_{j=1}^{\infty} \dfrac{(\gamma\theta)^j}{1 + \nu\left(\dfrac{u}{x}\right)\theta^j} \\
&= K\left(\frac{u}{x}\right)(\gamma\theta)^\sigma \cdot \textstyle\sum_{j=1}^{\infty} \dfrac{(\gamma\theta)^{j-\sigma}}{1 + \nu u\theta^{j-\sigma}} \\
&= Ku\gamma^\sigma \cdot \textstyle\sum_{j=1-[\sigma]}^{\infty} \dfrac{(\gamma\theta)^{j-\{\sigma\}}}{1 + \nu u\theta^{j-\{\sigma\}}}.
\end{aligned}
$$

(i) If $\gamma\theta = 1$, then $\gamma^\sigma = x^{-1}$. Thus

$$1 - \phi\left(\frac{u}{x}\right) = K\left(\frac{u}{x}\right)\left(\sum_{j=1-[\sigma]}^{0} \frac{1}{1 + \nu u\theta^{j-\{\sigma\}}} + O(1)\right)$$
$$= K\left(\frac{u}{x}\right)([\sigma] + O(1))$$
$$= K\left(\frac{u}{x}\right)([\log_\theta x] + O(1)) .$$

Therefore

$$\lim_{x\to\infty} \frac{x}{\ln x}\left(1 - \phi\left(\frac{u}{x}\right)\right) = \frac{Ku}{\ln\theta} = \frac{2\theta u}{(\theta - 1)\ln\theta}, \qquad \text{for } \gamma\theta = 1 .$$

Now let $x = y\ln y$; then

$$\frac{y\ln y}{\ln y + \ln\ln y}\left(1 - \phi\left(\frac{u}{y\ln y}\right)\right) \to \frac{2\theta u}{(\theta - 1)\ln\theta} \qquad \text{as } y \to \infty$$

which implies (2.8).

(ii) If $\gamma\theta > 1$, then $\gamma^\sigma = x^{-1/\rho}$ implies

$$x^{1/\rho} \cdot \left(1 - \phi\left(\frac{u}{x}\right)\right) = Ku \cdot \sum_{j=1-[\sigma]}^{\infty} \frac{(\gamma\theta)^{j-\{\sigma\}}}{1 + \nu u\theta^{j-\{\sigma\}}} .$$

Therefore

$$\lim_{x\to\infty}\left(x^{1/\rho} \cdot \left(1 - \phi\left(\frac{u}{x}\right)\right) - Ku \cdot \sum_{j=-\infty}^{\infty} \frac{(\gamma\theta)^{j-\{\sigma\}}}{1 + \nu u\theta^{j-\{\sigma\}}}\right) = 0 .$$

(2.9) is now apparent by letting $y = x^{1/\rho}$ and noting

$$\sigma(x) = \sigma(y^\rho) = \log_\theta y^\rho = \log_{1/\gamma} y = \omega(y) .$$

(2.10) LEMMA. (i) $\gamma\theta = 1$ *implies*

$$\text{(2.11)} \qquad \mathrm{p} - \lim_{n\to\infty} \frac{T_{V_n}}{2n\log_\theta n} = \frac{\theta}{\theta - 1}$$

where $\mathrm{p} - \lim_{n\to\infty}$ *indicates the limit in probability.*

(ii) *If* $\gamma\theta > 1$ *and* $\{n_k\}$ *is a sequence of integers tending to* ∞ *such that*

$$\lim_{k\to\infty} \{\log_{1/\gamma} n_k\} = \varepsilon ,$$

then $n_k^{-\rho} \cdot T_{V_{n_k}}$ *approaches in law, as* $k \to \infty$, *the probability distribution function with Laplace transform*

$$\text{(2.12)} \qquad \exp\left(-Ku \sum_{j=-\infty}^{\infty} \frac{(\gamma\theta)^{j-\varepsilon}}{1 + \nu u\theta^{j-\varepsilon}}\right)$$

where $\nu = 2\theta/(\theta - 1)^2$, $K = (1 - \gamma)/\gamma \cdot \nu$, $\rho = \log_{1/\gamma}\theta > 1$.

PROOF. Since $\{T_{V_j} - T_{V_{j-1}}\}_{j=2}^{\infty}$ is a sequence of independent, identically distributed random variables each with Laplace transform φ,

$$(2.13)\qquad \lim_{n\to\infty} E\exp\left(-u\frac{T_{V_n}}{f(n)}\right) = \lim_{n\to\infty}\varphi\left(\frac{u}{f(n)}\right)^{n-1}\cdot E\exp\left(-u\frac{T_{V_1}}{f(n)}\right) = \lim_{n\to\infty}\varphi\left(\frac{u}{f(n)}\right)^{n}$$

if $f(n)\to\infty$ as $n\to\infty$.

(i) If $\gamma\theta = 1$, then by (2.8)

$$\lim_{n\to\infty}\varphi\left(\frac{u}{n\ln n}\right)^n = \lim_{n\to\infty}\left(1-\frac{n\left(1-\varphi\left(\frac{u}{n\ln n}\right)\right)}{n}\right)^n = \lim_{n\to\infty}\left(1-\frac{1}{n}\left(\frac{2\theta u}{(\theta-1)\ln\theta}+o(1)\right)\right)^n = \exp\left(\frac{-2\theta u}{(\theta-1)\ln\theta}\right).$$

Therefore (2.13) implies as $n\to\infty$

$$\frac{T_{V_n}}{n\ln n}\to\frac{2\theta}{(\theta-1)\ln\theta}$$

in distribution and hence also in probability. Thus

$$\mathrm{p}-\lim_{n\to\infty}\frac{T_{V_n}}{n\log_\theta n} = \frac{2\theta}{\theta-1}.$$

(ii) Let $\gamma\theta > 1$ and let $\{n_k\}$ be a sequence of integers such that

$$\lim_{k\to\infty} n_k = \infty\,,\qquad \lim_{k\to\infty}\{\log_{1/\gamma} n_k\} = \varepsilon\,.$$

Then by (2.9)

$$\lim_{k\to\infty}\varphi\left(\frac{u}{n_k{}^\rho}\right)^{n_k} = \lim_{k\to\infty}\left(1-\frac{n_k\left(1-\varphi\left(\frac{u}{n_k{}^\rho}\right)\right)}{n_k}\right)^{n_k} = \lim_{k\to\infty}\left(1-\frac{1}{n_k}\left(Ku\sum_{j=-\infty}^{\infty}\frac{(\gamma\theta)^{j-\varepsilon}}{1+\nu u\theta^{j-\varepsilon}}+o(1)\right)\right)^{n_k} = \exp\left(-Ku\sum_{j=-\infty}^{\infty}\frac{(\gamma\theta)^{j-\varepsilon}}{1+\nu u\theta^{j-\varepsilon}}\right).$$

Therefore by (2.13) $n_k{}^{-\rho}T_{V_{n_k}}$ approaches in law the distribution with Laplace transform (2.12). (See [5] page 408). To complete the proof it is only necessary to observe that as $u\searrow 0$ the Laplace transform in (2.12) approaches 1 and hence is the Laplace transform of a probability distribution.

REMARK. Let $\{Y_j\}$ be independent random variables with distribution functions

$$\exp\left(-\frac{K\gamma^{j-\varepsilon}}{\nu}\right)\cdot\sum_{n=0}^{\infty}\frac{(\nu^{-1}K\gamma^{j-\varepsilon})^n}{n!}F_j^{n*}(x)$$

where $F_j(x) = 1 - \exp(-x/\nu\theta^{j-\varepsilon})$ is an exponential distribution and F_j^{n*} is the n-fold convolution of F_j. The Laplace transform of Y_j is

$$\exp\left(-K(\gamma\theta)^{j-\varepsilon} \cdot \frac{u}{1 + \nu u\theta^{j-\varepsilon}}\right).$$

Thus (2.12) is the Laplace transform of $\sum_{-\infty}^{\infty} Y_j$. The Lévy measure of $\sum_{-\infty}^{\infty} Y_j$ is (see [5] page 427)

$$R(dx) = \frac{K}{\nu^2} \sum_{j=-\infty}^{\infty} \left(\frac{\nu}{\theta}\right)^{j-\varepsilon} \exp\left(-\frac{x}{\nu\theta^{j-\varepsilon}}\right) dx .$$

Thus $R(x, \infty) = K/\nu \sum_{j=-\infty}^{\infty} \gamma^{j-\varepsilon} \exp(-(x/\nu\theta^{j-\varepsilon}))$. So $R(0, \infty) = \lim_{x\searrow 0} R(x, \infty) = \infty$ which implies that the distribution function whose Laplace transform is (2.12) has no atoms. (See [6] page 11.) And we conclude that this probability distribution is continuous and concentrated on $(0, \infty)$ since we already know that it is concentrated on $[0, \infty)$.

The immediate use of the next lemma will be to obtain the limit distributions of T_n suitably normalized.

(2.14) LEMMA. *Let $\{V_x\}$, $\{T_x\}$, $0 \leqq x < \infty$, be non-decreasing families of non-negative random variables such that*

$$\mathrm{p} - \lim_{x\to\infty} \frac{V_x}{x} = \frac{1}{\beta}, \qquad 0 < \beta < \infty .$$

Then for each $0 < \delta < \beta$ and any sequence $\{y = y_k\}$ tending to ∞

$$\liminf_{y\to\infty} \mathscr{L}\left(\frac{T_{V_{y(\beta+\delta)}}}{f(y)}\right) \leqq \liminf_{y\to\infty} \mathscr{L}\left(\frac{T_y}{f(y)}\right), \tag{2.15}$$

$$\limsup_{y\to\infty} \mathscr{L}\left(\frac{T_y}{f(y)}\right) \leqq \limsup_{y\to\infty} \mathscr{L}\left(\frac{T_{V_{y(\beta-\delta)}}}{f(y)}\right) \tag{2.16}$$

where $\mathscr{L}(X)$ denotes the Laplace transform of the distribution of X and f is a strictly positive function.

PROOF. Let $0 < \delta < \beta$ and set

$$A = A_{x,\delta} = \left\{y : \beta - \delta < \frac{x}{y} < \beta + \delta\right\}.$$

Now

$$\mathscr{L}\left(f\left(\frac{x}{\beta-\delta}\right)^{-1} \cdot T_{x/(\beta-\delta)}\right)$$
$$= \int_A + \int_{A^c} \mathscr{L}\left(f\left(\frac{x}{\beta-\delta}\right)^{-1} \cdot T_{x/(\beta-\delta)} \,\middle|\, V_x = y\right) P(V_x = dy) .$$

Since T_y increases with y

$$(2.17)\quad \mathscr{L}\left(f\left(\frac{x}{\beta-\delta}\right)^{-1}\cdot T_{x/(\beta-\delta)}\right)$$
$$\leqq \int_A \mathscr{L}\left(f\left(\frac{x}{\beta-\delta}\right)^{-1}\cdot T_{V_x}\,\middle|\, V_x = y\right) P(V_x = dy) + P(V_x \in A^c)$$
$$\leqq \mathscr{L}\left(f\left(\frac{x}{\beta-\delta}\right)^{-1}\cdot T_{V_x}\right) + P(V_x \in A^c)\,.$$

Let $\{y = y_k\}$ be a sequence tending to ∞ and set $x = (\beta-\delta)y$. Then (2.17) implies (2.16). (Note that $\mathrm{p}-\lim_{x\to\infty} x/V_x = \beta$ implies $\lim_{x\to\infty} P(V_x \in A^c) = 0$.)

(2.15) is proved in a similar way.

The limit distributions of T_n suitably normalized are now easily obtained by applying Lemma (2.14) to Lemma (2.10).

(2.18) THEOREM. (i) *$\gamma\theta = 1$ implies*

$$\mathrm{p}-\lim_{n\to\infty} \frac{T_n}{2n\log_\theta n} = 1\,.$$

(ii) *If $\gamma\theta > 1$ and $\{n_k\}$ is a sequence of integers tending to ∞ such that*

$$\lim_{k\to\infty}\{\log_{1/\gamma} n_k\} = \varepsilon\,,$$

then $n_k^{-\rho}\cdot T_{n_k}$ approaches in law, as $k\to\infty$, the probability distribution function with Laplace transform

$$\exp\left(-Lu\sum_{j=-\infty}^{\infty}\frac{(\gamma\theta)^{j-\eta}}{1+\mu u\theta^{j-\eta}}\right)$$

where $\mu = 2\theta/(\theta-1)^2\cdot(1-\gamma)^\rho$, $L = (1-\gamma)/\gamma\cdot\mu$, $\eta = \varepsilon + \log_{1/\gamma}(1-\gamma)$, $\rho = \log_{1/\gamma}\theta > 1$.

PROOF. Let $T_x = T_{[x]}$ and $V_x = V_{[x]}$. Then

$$\lim_{x\to\infty}\frac{V_x}{x} = \frac{1}{\beta} = \frac{1}{1-\gamma}\quad\text{a.e.}$$

(i) If $\gamma\theta = 1$, then (2.11) implies

$$\mathscr{L}\left(\frac{T_{V_{n(\beta\pm\delta)}}}{2n\log_\theta n}\right) = \mathscr{L}\left(\frac{T_{V_{n(\beta\pm\delta)}}}{2n(\beta\pm\delta)\log_\theta n(\beta\pm\delta)}\cdot\frac{(B\pm\delta)\log_\theta n(\beta\pm\delta)}{\log_\theta n}\right)$$
$$\to \exp\left(-\frac{\theta(\beta\pm\delta)u}{\theta-1}\right).$$

Therefore Lemma (2.14) implies for each $0<\delta$ small

$$\exp\left(-\frac{\theta(\beta+\delta)u}{\theta-1}\right) \leqq \liminf_{n\to\infty} E\exp\left(-u\cdot\frac{T_n}{2n\log_\theta n}\right)$$
$$\limsup_{n\to\infty} E\exp\left(-u\cdot\frac{T_n}{2n\log_\theta n}\right) \leqq \exp\left(-\frac{\theta(\beta-\delta)u}{\theta-1}\right).$$

Therefore (since $\beta = 1 - \gamma = (\theta - 1)/\theta$) as $n \to \infty$

$$\frac{T_n}{2n \log_\theta n} \to 1$$

in distribution and hence also in probability.

(ii) Let $\{n = n_k\}$ be a sequence of integers such that

$$\lim_{k\to\infty} n = \infty\,, \qquad \lim_{k\to\infty} \{\log_{1/\gamma} n\} = \varepsilon\,.$$

Then

$$\{\log_{1/\gamma} [n(\beta \pm \delta)]\} \equiv \{\log_{1/\gamma} n\} + \{\log_{1/\gamma} (\beta \pm \delta)\} + \left\{\log_{1/\gamma} \frac{[n(\beta \pm \delta)]}{n(\beta \pm \delta)}\right\} \mod 1$$

$$\to \varepsilon + \{\log_{1/\gamma} (\beta \pm \delta)\} \mod 1$$

as $n \to \infty$. Therefore

$$\mathscr{L}\left(\frac{T_{V_{n(\beta\pm\delta)}}}{n^\rho}\right) = \mathscr{L}\left(\frac{T_{V_{n(\beta\pm\delta)}}}{(n(\beta \pm \delta))^\rho} \cdot (\beta \pm \delta)^\rho\right)$$

$$\to \exp\left(-Ku(\beta \pm \delta)^\rho \sum_{j=-\infty}^{\infty} \frac{(\gamma\theta)^{j-\eta}}{1 + \nu u(\beta \pm \delta)^\rho \theta^{j-\eta}}\right), \quad \text{as } k \to \infty\,,$$

by Lemma (2.10), where $\eta = \varepsilon + \{\log_{1/\gamma} (\beta \pm \delta)\}$. Combining this with Lemma (2.14) implies case (ii) of the theorem.

Although the following corollary is weaker than the theorem, it is more concise.

COROLLARY. *If* $\gamma\theta > 1$, *then*

$$\text{p} - \lim_{n\to\infty} \frac{\ln T_n}{\ln n} = \log_{1/\gamma} \theta\,. \tag{2.19}$$

PROOF. Let $\{n_k\}$ be any increasing sequence of integers. To prove (2.19) it suffices to show that there exists a subsequence $\{m_k\} \subset \{n_k\}$ such that (2.19) is valid along $\{m_k\}$. Thus extract $\{m = m_k\} \subset \{n_k\}$ such that

$$\lim_{k\to\infty} m = \infty\,, \qquad \lim_{k\to\infty} \{\log_{1/\gamma} m\} = \varepsilon\,.$$

Then T_m/m^ρ tends in law to a limit distribution by Theorem (2.18) and this distribution is concentrated on $(0, \infty)$ by the remark preceding Lemma (2.14). Therefore if $\delta > 0$, then

$$\begin{aligned} 1 &= \lim_{k\to\infty} P\left(\frac{1}{m^\delta} < \frac{T_m}{m^\rho} < m^\delta\right) \\ &= \lim_{k\to\infty} P((\rho - \delta) \ln m < \ln T_m < (\rho + \delta) \ln m) \\ &= \lim_{k\to\infty} P\left(\left|\frac{\ln T_m}{\ln m} - \rho\right| < \delta\right). \end{aligned}$$

I.e., $\text{p} - \lim \ln T_m/\ln m = \rho = \log_{1/\gamma} \theta$ which completes the proof.

We can now obtain limit distributions of X_n suitably normalized.

(2.20) THEOREM. (i) *If* $\gamma\theta = 1$, *then*

$$\mathrm{p} - \lim_{n\to\infty} \frac{X_n}{n/\log_\theta n} = \tfrac{1}{2}\,.$$

(ii) *If* $\gamma\theta > 1$ *and* $\{n_k\}$ *is a sequence of integers such that*

$$\lim_{k\to\infty} n_k = \infty\,, \qquad \lim_{k\to\infty} \{\log_\theta n_k\} = \varepsilon\,,$$

then for each $x > 0$

$$\lim_{k\to\infty} P(n_k{}^{-1/\rho} \cdot X_{n_k} < x) = 1 - F_{\lambda(x)}(((1-\gamma)x)^{-\rho})$$

where $\rho = \log_{1/\gamma}\theta > 1$, $\lambda(x) = \varepsilon + \log_{1/\gamma}((1-\gamma)x)$, *and* F_ε *is the distribution whose Laplace transform is given by* (2.12).

PROOF. Let $N(n)$ be the unique integer such that

$$V_{N(n)} \leqq X_n < V_{N(n)+1}$$

i.e., $T_{V_{N(n)}} \leqq n < T_{V_{N(n)+1}}$.
Thus

$$1 \leqq \frac{X_n}{V_{N(n)}} \leqq \frac{V_{N(n)+1}}{V_{N(n)}} \longrightarrow 1 \quad \text{a.e.} \tag{2.21}$$

since $n^{-1}V_n \to (1-\gamma)^{-1}$ a.e.
Notice also that

$$P(N(n) \geqq y) = P(T_{V_{]y[}} \leqq n) \tag{2.22}$$

where $]y[$ is the least integer greater than or equal to y.

(i) $\gamma\theta = 1$: Let $g(t) = t\log_\theta t$. Then for $s > 0$

$$\begin{aligned} P\left(\frac{g^{-1}(n)}{N(n)} \leqq s\right) &= P\left(N(n) \geqq \frac{g^{-1}(n)}{s}\right) \\ &= P(T_{V_{]g^{-1}(n)/s[}} \leqq n) \qquad \text{by (2.22)} \\ &= P\left(\frac{T_{V_{]g^{-1}(n)/s[}}}{\frac{g^{-1}(n)}{s}\log_\theta \frac{g^{-1}(n)}{s}} \leqq \frac{n}{\frac{g^{-1}(n)}{s}\log_\theta \frac{g^{-1}(n)}{s}}\right). \end{aligned}$$

Now

$$\frac{n}{\frac{g^{-1}(n)}{s}\log_\theta \frac{g^{-1}(n)}{s}} = \frac{s}{1 - n^{-1}g^{-1}(n)\log_\theta s}$$

and $g^{-1}(n)/n \to 0$ as $n \to \infty$. Combining these observations with (2.11) implies

$$\mathrm{p} - \lim_{n\to\infty} \frac{g^{-1}(n)}{N(n)} = \frac{2\theta}{\theta - 1}\,.$$

But $\lim_{j\to\infty} V_j/j = (1-\gamma)^{-1}$ a.e. So $\mathrm{p} - \lim_{n\to\infty} g^{-1}(n)/V_{N(n)} = 2\theta(1-\gamma)/(\theta-1) = 2$ in case (i). Thus $\mathrm{p} - \lim X_n/g^{-1}(n) = \frac{1}{2}$ by (2.21). Noting that $g^{-1}(n) \sim n/\log_\theta n$ now completes the proof of case (i).

(ii) Let $\{n = n_k\}$ be any sequence of integers such that $n \to \infty$, $\{\log_\theta n\} \to \varepsilon$. Then for $x > 0$

$$P\left(\frac{N(n)}{n^{1/\rho}} \geqq x\right) = P\left(T_{V]xn^{1/\rho}[} \leqq n\right), \qquad \text{by (2.22)}$$

$$= P\left(\frac{T_{V]xn^{1/\rho}[}}{]xn^{1/\rho}[^\rho} \leqq \frac{1}{x^\rho} \cdot \frac{x^\rho n}{]xn^{1/\rho}[^\rho}\right) \to F_{\lambda(x/1-\gamma)}(x^{-\rho})$$

by (2.10) since

$$\{\log_{1/\gamma}[xn^{1/\rho}]\} \to \log_{1/\gamma} x + \lim\{\log_\theta n\} \mod 1$$

$$= \log_{1/\gamma} x + \varepsilon = \lambda(x/1-\gamma)$$

and since each F_ε is continuous by the remark preceding Lemma (2.14). Now

$$\frac{V_n}{n} \to \frac{1}{1-\gamma} \quad \text{a.e.}$$

So

$$P\left(\frac{V_{N(n)}}{n^{1/\rho}} \geqq x\right) = P\left(\frac{V_{N(n)}}{N(n)} \cdot \frac{N(n)}{n^{1/\rho}} \geqq x\right) \to F_{\lambda(x)}(((1-\gamma)x)^{-\rho}).$$

Combining this with (2.21) completes the proof of case (ii).

COROLLARY. *If $\gamma\theta > 1$, then*

$$\mathrm{p} - \lim_{n\to\infty} \frac{\ln X_n}{\ln n} \log_\theta\left(\frac{1}{\gamma}\right).$$

PROOF. In the same way that Corollary (2.19) follows from Theorem (2.18), the previous theorem implies this result. However, we must verify that if

$$G(x) = 1 - F_{\lambda(x)}(((1-\gamma)x)^{-\rho})$$

is the limit distribution in case (ii) of (2.20), then G is concentrated on $(0, \infty)$. We show that $G(x) \searrow 0$ as $x \searrow 0$; the proof that $G(x) \nearrow 1$ as $x \nearrow \infty$ is similar. Now F_ε is concentrated on $(0, \infty)$ by the remark preceding Lemma (2.14). Given $\delta > 0$, choose x_0 so small that $F_\varepsilon(((1-\gamma)x)^{-\rho}) > 1 - \delta$ for $0 < x < x_0$. Choose $0 < x_1 < x_0$ so that $\lambda(x_1) \equiv \varepsilon \mod 1$. Then

$$F_{\lambda(x_1)}(((1-\gamma)x_1)^{-\rho}) = F_\varepsilon(((1-\gamma)x_1)^{-\rho}) > 1 - \delta$$

where the equality follows from the fact that F_ε is periodic in ε of period 1. So $G(x_1) < \delta$. And $G(x_1) < \delta$ for $0 < x < x_1$ since G, by its expression as the limit in case (ii) of (2.20), decreases as x decreases. Thus $\lim_{x\searrow 0} G(x) = 0$.

3. A diffusion in a random Environment. In the random walk model of the previous section we showed that $n^{-\rho} \cdot T_n$ tends to limiting distribution only along subsequences. The present section shows that this behavior is the result of the discrete nature of the model. We do this by considering a diffusion $\{X_t\}$ with a drift and randomly placed reflecting barriers; in this process $X_t \to \infty$ a.e. and, depending on the drift rate,

(i) $\lim_{x\to\infty} T_x/x =$ nonzero constant a.e.,
(ii) $\mathrm{p} - \lim_{x\to\infty} T_x/x \ln x =$ nonzero constant, or
(iii) $\mathscr{L} - \lim_{x\to\infty} T_x/x^\rho =$ stable distribution

where T_x is the first passage time from 0 to x and $\mathscr{L} - \lim$ indicates the limit in distribution.

We first construct a diffusion on $[r_0, r_1]$ where r_0 and r_1 are both finite. The diffusion will have a constant drift in (r_0, r_1), a reflecting barrier at r_0, and an adhesive boundary at r_1. Let $C[r_0, r_1]$ denote the set of continuous, real functions on $[r_0, r_1]$ and $C^2[r_0, r_1]$ denote the set of twice differentiable functions u on $[r_0, r_1]$ such that $u'' \in C[r_0, r_1]$. Define $A : \mathscr{D}(A) \to C[r_0, r_1]$ by

$$\begin{aligned} Au(x) &= \tfrac{1}{2}u''(r_0)\,, && x = r_0 \\ &= \tfrac{1}{2}u''(x) - \beta u'(x)\,, && r_0 < x < r_1 \\ &= 0\,, && x = r_1 \end{aligned} \tag{3.1}$$

where

$$\mathscr{D}(A) = \{u \in C^2[r_0, r_1] : \lim_{x\nearrow r_1} \tfrac{1}{2}u''(x) - \beta u'(x) = 0,\ \lim_{x\searrow r_0} u'(x) = 0\} \tag{3.2}$$

and β is a finite constant.

(3.3) LEMMA. *The operator A is the infinitesimal generator of a contraction semi-group $\{p_t(x, dy)\}$ on $C[r_0, r_1]$. There exists a Markov process $\{Y_t\}_0^\infty$ with continuous sample paths and with transition semi-group $\{p_t(x, dy)\}$.*

PROOF. Writing the operator A in the form

$$D_m D_p = \tfrac{1}{2}e^{2\beta x} \cdot \frac{d}{dx}\left(e^{-2\beta x} \cdot \frac{d}{dx}\right)$$

where $p(x) = (2\beta)^{-1} \cdot (e^{2\beta x} - 1)$ and $m(x) = \beta^{-1} \cdot (1 - e^{-2\beta x})$ we see that both r_0 and r_1 are accessible, regular boundaries for the operator $D_m D_p$. Hence Lemma 1 of [7] page 54 implies (3.3).

(Suppose $r_0 = 0$. Perhaps the most intuitive way to see that 0 is a reflecting boundary for $\{Y_t\}$ is by considering the process $\{Y_t'\}$ defined by the infinitesimal generator

$$\begin{aligned} A'u(x) &= \tfrac{1}{2}u''(x) - \beta u'(x)\,, && 0 < x < r_1 \\ &= \tfrac{1}{2}u''(x) + \beta u'(x)\,, && -r_1 < x < 0 \\ &= 0\,, && x = \pm r_1\,. \end{aligned}$$

Then $\{Y_t'\}$ is a diffusion with a drift toward zero (at least on $(-r_1, r_1)$). $\{|Y_t'|\}$ can be shown to have generator defined by (3.1), (3.2). (See [4] pages 325–329). But $\{|Y_t'|\}$ is a diffusion on $[0, r_1]$ with a reflecting barrier at zero.)

We derive one more result about the process $\{Y_t\}$ which will be needed—the distribution of the first passage time from r_0 to $r_0 + x \leqq r_1$. Let $\{p_t(x, dy)\}$ be the transition semi-group of $\{Y_t\}$ and R_α the resolvent operator of $\{p_t(x, dy)\}$. Since a passage from x to $z > y > x$ must first be accompanied by a first passage from x to y,

$$\int_0^\infty e^{-\alpha t} \cdot p_t(x, dz)\, dt = \varphi_\alpha(x, y) \cdot \int_0^\infty e^{-\alpha t} \cdot p_t(y, dz)\, dt$$

where $\varphi_\alpha(x, y)$ is the Laplace transform for the distribution of the first passage time from x to y. Thus

$$R_\alpha f(x) = \varphi_\alpha(x, y) \cdot R_\alpha f(y) \tag{3.4}$$

for $f \in C[r_0, r_1]$ vanishing on $[r_0, y]$. Now fix y and apply $(\alpha - A)$ to both sides of (3.4). If f vanishes on $[r_0, y]$, then

$$f(x) = 0 = (\alpha - A)\varphi_\alpha(\cdot, y)(x) \cdot R_\alpha f(y)\,.$$

Or $(\alpha - A)\varphi_\alpha(\cdot, y) = 0$. Setting $u(x) = \varphi_\alpha(x, y)$ and using the definition of A we thus have

$$\begin{aligned} \alpha u(x) + \beta u'(x) - \tfrac{1}{2}u''(x) &= 0\,, \qquad r_0 < x < y\,, \\ \lim_{x \searrow r_0} u'(x) &= 0\,, \\ \lim_{x \nearrow y} u(x) &= 1\,, \end{aligned} \tag{3.5}$$

where $u'(x) \to 0$ as $x \to r_0$ since u is $R_\alpha f(\cdot)/R_\alpha f(y)$ on $[r_0, y]$ and this function is in $\mathscr{D}(A)$; also $u(y) = 1$ by definition. The solution to (3.5) is easily obtained and implies the following

(3.6) LEMMA. *Let $\varphi_x(u)$ be the Laplace transform for a first passage from r_0 to $r_0 + x \leqq r_1$. Then*

$$\varphi_x(u) = (h + \beta + (h - \beta)e^{2h\cdot x})^{-1} \cdot 2he^{(h-\beta)x} \tag{3.7}$$

where $h = h(u) = (\beta^2 + 2u)^{\frac{1}{2}}$.

Now let us construct a Markov process with randomly placed reflecting barriers to the right. Independent copies of the process $\{Y_t\}$ constructed above can be pieced together to give a process with reflecting barriers at points a_j. More specifically let $0 = a_0 < a_1 < a_2 < \cdots$ and let $\{{}_jY_t\}$ be the process constructed in Lemma (3.3) with the reflecting barrier at $r_0 = a_j$ and the adhesive barrier at $r_1 = a_{j+1}$. Set ${}_jY_0 \equiv a_j$ and let

$$V_{j+1} = \inf\{t \geqq 0 : {}_jY_t = a_{j+1}\}\,, \qquad j \geqq 0\,.$$

Set

$$\begin{aligned} X_t &= {}_0Y_t\,, & t &< V_1 \\ &= {}_1Y_{t-V_1}\,, & V_1 &\leqq t < V_1 + V_2 \\ &= \cdots \\ &= {}_nY_{t-(V_1+\cdots+V_n)}\,, & V_1 + \cdots + V_n &\leqq t < V_1 + \cdots + V_{n+1} \\ &= \cdots\,. \end{aligned}$$

Then $\{X_t\}$ is a Markov process with continuous sample paths.

The process in which we are interested consists of $\{X_t\}$ with the a_j's randomized so that $\{a_{j+1} - a_j\}$ is a sequence of independent, identically distributed random variables each exponentially distributed with mean γ^{-1}. The construction of the process is, except for notation, the same as the construction of the random walk in a random environment in the introduction.

We find the limiting distributions of T_x = the first passage time from 0 to x in the same way as in Section 2. Thus the Laplace transform for the distribution of the first passage time from 0 to the first reflecting barrier at a_1 is

$$\begin{aligned} \varphi(u) &= \int E(e^{-uT_x} \mid a_1 = x) \cdot P(a_1 = dx) \\ &= \gamma \int_0^\infty \varphi_x(u) e^{-\gamma x}\, dx \end{aligned} \tag{3.8}$$

where $\varphi_x(u)$ is defined in (3.6).

THEOREM.

$$\begin{aligned} E(T_{a_1}) &= \frac{2}{\gamma(\gamma - 2\beta)}\,, & \gamma > 2\beta \\ &= \infty\,, & \gamma \leqq 2\beta\,. \end{aligned} \tag{3.6}$$

PROOF. It suffices to show $-\lim_{u\searrow 0} \varphi'(u)$ has the form (3.9). But

$$\varphi'(u) = \lim_{\varepsilon\searrow 0} \gamma \int_0^\infty \frac{\varphi_x(u+\varepsilon) - \varphi_x(u)}{\varepsilon} e^{-\gamma x}\, dx$$

by (3.8). Now $\varphi_x(u)$ is convex as a function of u. Therefore monotone convergence implies

$$\varphi'(u) = \gamma \int_0^\infty \varphi_x'(u) e^{-\gamma x}\, dx$$

where $\varphi_x'(u)$ is the derivative of $\varphi_x(u)$ with respect to u. φ_x is a Laplace transform; thus $\varphi_x'(u)$ decreases as u decreases; so monotone convergence implies

$$-\lim_{u\searrow 0} \varphi'(u) = \gamma \int_0^\infty (-\lim_{u\searrow 0} \varphi_x'(u)) e^{-\gamma x}\, dx\,.$$

Direct calculation from (3.7) shows

$$\begin{aligned} \lim_{u\searrow 0} \varphi_x'(u) &= \frac{1}{2\beta^2} \cdot (1 + 2\beta x - e^{2\beta x})\,, & \beta \neq 0 \\ &= -x^2\,, & \beta = 0\,. \end{aligned}$$

And another calculation shows $-\lim_{u\searrow 0} \varphi'(u)$ has the form (3.9).

(3.10) THEOREM. (i) *If* $\gamma > 2\beta$, *then*

$$\lim_{x\to\infty} \frac{T_x}{x} = \frac{2}{\gamma - 2\beta} \quad \text{a.e.}\,, \qquad \lim_{t\to\infty} \frac{X_t}{t} = \frac{\gamma - 2\beta}{2} \quad \text{a.e.}$$

(ii) *If* $\gamma \leqq 2\beta$, *then*

$$\lim_{x\to\infty} \frac{T_x}{x} = \infty \quad \text{a.e.}\,, \qquad \lim_{t\to\infty} \frac{X_t}{t} = 0 \quad \text{a.e.}$$

PROOF. Noting that $\{T_{a_j} - T_{a_{j-1}}\}_{j=1}^\infty$ is a sequence of independent, identically distributed random variables, the law of large numbers implies

$$\lim_{n\to\infty} \frac{T_{a_n}}{n} = E(T_{a_1}) \quad \text{a.e.}, \qquad \lim_{n\to\infty} \frac{a_n}{n} = \frac{1}{\gamma} \quad \text{a.e.}$$

Hence

$$\lim_{n\to\infty} \frac{T_{a_n}}{a_n} = \gamma E(T_{a_1}) \quad \text{a.e.}$$

Let m_x be the unique integer such that $a_{m_x} \leqq x < a_{m_x+1}$. So

$$\frac{T_{a_{m_x}}}{a_{m_x+1}} \leqq \frac{T_x}{x} \leqq \frac{T_{a_{m_x+1}}}{a_{m_x}};$$

therefore

$$\begin{aligned} \lim_{x\to\infty} \frac{T_x}{x} &= \gamma E(T_{a_1}) \quad \text{a.e.} \\ &= \frac{2}{\gamma - 2\beta}, && \gamma > 2\beta \\ &= \infty, && \gamma \leqq 2\beta . \end{aligned}$$

For each $t \geqq 0$ choose the unique integer $n = n_t$ such that $a_n \leqq X_t < a_{n+1}$. Then

$$\frac{T_{a_n}}{n} \leqq \frac{t}{n} < \frac{T_{a_{n+1}}}{n}$$

and

$$\frac{a_n}{n} \cdot \frac{n}{t} \leqq \frac{X_t}{t} < \frac{a_{n+1}}{n} \cdot \frac{n}{t} .$$

Thus

$$\begin{aligned} \lim_{t\to\infty} \frac{X_t}{t} &= \lim_{n\to\infty} \frac{a_n}{n} \cdot \lim_{n\to\infty} \frac{n}{T_{a_n}} \\ &= \frac{\gamma - 2\beta}{2}, && \gamma > 2\beta \\ &= 0, && \gamma \leqq 2\beta . \end{aligned}$$

Before finding the limiting distribution of T_x suitably normalized when $\gamma \leqq 2\beta$, we prove the following result to simplify subsequent calculations. Assume for the remainder of this section that $\gamma \leqq 2\beta$ so that, in particular, $\beta > 0$. The proofs of the following results leading to the limit behavior of T_x are substantially the same as the proofs in Section 2; we therefore delete the details.

(3.11) LEMMA. *Let*

$$\psi(u) = \gamma \cdot \int_0^\infty \frac{e^{-\gamma x}\, dx}{1 + (2\beta^2)^{-1} \cdot u e^{2\beta x}} .$$

Then as $u \searrow 0$

$$\varphi(u) - \psi(u) = O(u) .$$

For ease of notation set $\rho = 2\beta/\gamma$ and $r = \rho^{-1}$ throughout the rest of this section.

(3.12) LEMMA. (i) *If $\gamma = 2\beta$, then for each $u \geqq 0$*

$$\lim_{n\to\infty} n\left(1 - \varphi\left(\frac{u}{n \ln n}\right)\right) = \frac{u}{2\beta^2}\,.$$

(ii) *If $\gamma < 2\beta$, then*

$$\lim_{u\searrow 0} u^{-r}\cdot(1-\varphi(u)) = C(\beta,\gamma) = \gamma\cdot\int_{-\infty}^{\infty}\frac{e^{(2\beta-\gamma)y}}{2\beta^2+e^{2\beta y}}\,dy\,.$$

We use Lemma (3.12) to find the limiting distributions of T_x suitably normalized when $\gamma \leqq 2\beta$. Before doing so, however, it should be noted that two results are immediate from the lemma. The first of these concerns the asymptotic behavior of $P(T_{a_1} > t)$. (Recall that T_{a_1} is the first passage time from 0 to the first reflecting barrier and that φ is the Laplace transform of the distribution of T_{a_1}.)

(3.13) THEOREM. *Let $\gamma < 2\beta$. Then*

$$\lim_{t\to\infty} t^r\cdot P(T_{a_1} > t) = \frac{C}{\Gamma(1-r)}$$

where Γ is the gamma function and $C = C(\beta,\gamma)$ is defined in the previous lemma.

PROOF. Let $\lambda(u) = (1-\varphi(u))/u$. Then

$$\lambda(u) = \int_0^\infty e^{-ut}P(T_{a_1} > t)\,dt\,.$$

Hence the theorem follows from a Tauberian theorem. (See [5] page 423, Theorem 4.)

Also of interest are the asymptotic behaviors of the residual waiting time until the next reflecting barrier is hit and the spent waiting time from the last reflecting barrier hit. Let n_t be the unique integer such that

$$T_{a_{n_t}} \leqq t < T_{a_{n_t+1}}$$

and let Y_t and Z_t be respectively the spent and the residual waiting time, i.e.,

$$Y_t = t - T_{a_{n_t}}\,, \qquad Z_t = T_{a_{n_t+1}} - t\,.$$

If $\gamma < 2\beta$, then $t^{-1}\cdot Y_t$ and $t^{-1}\cdot Z_t$ tend to generalized arc sine distributions. In fact (see Dynkin [3] or pages 445–447 of Feller [5]). We have the following result which follows from Theorem (3.13).

(3.14) THEOREM. *Let $\gamma < 2\beta$. Then*

$$\lim_{t\to\infty} P\left(\frac{Y_t}{t} > y, \frac{Z_t}{t} > z\right) = \frac{\sin \pi r}{\pi}\int_y^1 (z+u)^{-r}\cdot(1-u)^{r-1}\,du$$

for $0 \leqq z < \infty$ and $0 \leqq y < 1$.

Returning now to the calculation of the limiting distributions of T_x suitably normalized, we first find the limiting distributions of T_{a_n} normalized.

(3.15) LEMMA. (i). *If* $\gamma = 2\beta$, *then*

$$\lim_{n\to\infty} E \exp\left(-u \cdot \frac{T_{a_n}}{n \ln n}\right) = \exp\left(-\frac{u}{2\beta^2}\right).$$

(ii) *If* $\gamma < 2\beta$, *then*

$$\lim_{n\to\infty} E \exp\left(-u \cdot \frac{T_{a_n}}{n^\rho}\right) = e^{-Cu^r},$$

where $C = C(\beta, \gamma)$ *is defined in Lemma* (3.12).

The main result concerning the asymptotic behavior of T_x for large x when $\gamma \leqq 2\beta$ can now be stated.

(3.16) THEOREM. (i) *If* $\gamma = 2\beta$, *then*

$$\text{p} - \lim_{x\to\infty} \frac{T_x}{x \ln x} = \frac{1}{\beta}.$$

(ii) *Let* $\gamma < 2\beta$ *and set* $r = \rho^{-1} = \gamma/2\beta$. *Let* G_r *be the stable distribution concentrated on* $(0, \infty)$ *of index* r *with Laplace transform* e^{-u^r}. *Then*

$$\mathscr{L} - \lim_{x\to\infty} \frac{T_x}{(Kx)^\rho} = G_r$$

where

$$K = K(\beta, \gamma) = \gamma C(\beta, \gamma) = \gamma^2 \int_{-\infty}^{\infty} \frac{e^{(2\beta-\gamma)y}}{2\beta^2 + e^{2\beta y}}\, dy.$$

PROOF. The proof of this theorem is an application of Lemma (2.14) to the previous lemma. We let $V_x = a_{[x]}$ where $[x]$ denotes the largest integer less than or equal to x. The proof is similar to the proof of Theorem (2.18).

Limiting distributions of X_t suitably normalized can now be found and a proof similar to the proof of Theorem (2.20) implies

(3.17) THEOREM. (i) *If* $\gamma = 2\beta$, *then*

$$\text{p} - \lim_{t\to\infty} \frac{X_t \ln X_t}{t} = \beta.$$

(ii) *Let* $\gamma < 2\beta$ *and set* $r = \rho^{-1} = \gamma/2\beta$. *Then*

$$\mathscr{L} - \lim_{t\to\infty} \frac{t}{(KX_t)^\rho} = G_r \tag{3.18}$$

where

$$K = K(\beta, \gamma) = \gamma^2 \cdot \int_{-\infty}^{\infty} \frac{e^{(2\beta-\gamma)y}}{2\beta^2 + e^{2\beta y}}\, dy.$$

When $\beta \leqq \gamma < 2\beta$ the theory of stable distributions allows us to calculate the limit distributions of $t^{-r} \cdot X_t$. Let $Z_{\alpha,\gamma}$ be a random variable with characteristic function

$$E\exp(i\lambda Z_{\alpha,\gamma}) = \exp(-|\lambda|^\alpha \cdot e^{\pm i\pi\gamma/2})$$

where the sign in the exponent is positive if $\lambda > 0$ and negative if $\lambda < 0$. Then (see [5] page 548–549). $Z_{(\alpha,\gamma)}$ is stable if $0 < \alpha < 1$ or $1 < \alpha < 2$ and

$$\begin{aligned} |\gamma| &\leqq \alpha\,, && 0 < \alpha < 1 \\ &\leqq 2 - \alpha\,, && 1 < \alpha < 2\,. \end{aligned}$$

(3.19) THEOREM. *Let* $r = \rho^{-1} = \gamma/2\beta$. *If* $\beta \leqq \gamma < 2\beta$, *then*

$$\lim_{t\to\infty} P\left(\frac{X_t}{t^r} \leqq x\right) = P(Z_{(\rho,\rho-2)} \leqq K \cdot x \mid Z_{(\rho,\rho-2)} > 0)\,.$$

In particular, if $\beta = \gamma$, *then*

$$\lim_{t\to\infty} P\left(\frac{X_t}{t^{\frac{1}{2}}} \leqq x\right) = 2N\left(\frac{Kx}{2^{\frac{1}{2}}}\right) - 1\,,$$

where N *is the normal distribution with mean* 0 *and variance* 1 *and* $K = K(\beta, \gamma)$ *is defined in the previous theorem.*

PROOF. First let $\beta < \gamma < 2\beta$ so that $\frac{1}{2} < r < 1$. By (3.18)

$$\lim_{t\to\infty} E\exp\left(-u \cdot \frac{t}{(KX_t)^\rho}\right) = e^{-u^r}\,.$$

Therefore

$$\lim_{t\to\infty} E\exp\left(i\lambda \cdot \frac{t}{(KX_t)^\rho}\right) = \exp(-|\lambda|^r \cdot e^{\pm(i\pi r/2)})\,.$$

I.e.,

$$\mathscr{L} - \lim_{t\to\infty} \frac{t}{(K \cdot X_t)^\rho} = Z_{(r,-r)}\,.$$

Let $p(x, \alpha, \gamma)$ be the density of $Z_{\alpha,\gamma}$. Feller ([5] page 549) showed that

$$y^{-1-\rho} \cdot p(y^{-\rho}, r, -r) = p(y, \rho, \rho - 2)$$

for $y > 0$ and $\frac{1}{2} < r < 1$. Thus for $x > 0$

$$\begin{aligned} \lim_{t\to\infty} P\left(K \cdot \frac{X_t}{t^r} \leqq x\right) &= \lim_{t\to\infty} P\left(\frac{t}{(KX_t)^\rho} \geqq x^{-\rho}\right) \\ &= \textstyle\int_{x^{-\rho}}^\infty p(y, r, -r)\,dy \\ &= \rho \cdot \textstyle\int_0^x y^{-1-\rho} \cdot p(y^{-\rho}, r, -r)\,dy \\ &= \rho \cdot \textstyle\int_0^x p(y, \rho, \rho - 2)\,dy \\ &= P(Z_{(\rho,\rho-2)} \leqq x \mid Z_{(\rho,\rho-2)} > 0)\,. \end{aligned}$$

The case with $\beta = \gamma$ is similar.

4. A system of difference equations. Consider the system of difference equations

$$(4.1)\qquad \begin{aligned} Z_0 &= 0 \\ Z_n &= \sigma_n(1 + Z_{n-1}), \qquad n \geqq 1 \end{aligned}$$

where $\{\sigma_n\}$ is a sequence of independent, identically distributed nonnegative random variables. Our principal interest is in finding limits of $Z_1 + \cdots + Z_n$ suitably normalized.

To see the application of this system to the random walk model first fix the sequence of transition probabilities $\{\alpha_n, \beta_n\}$ with $\alpha_n > 0$ for all n and let μ_n be the mean first passage time from n to $(n + 1)$. Then

$$\mu_n = \alpha_n + \beta_n(1 + \mu_{n-1} + \mu_n).$$

I.e.,

$$\mu_n = \sigma_n \mu_{n-1} + (\sigma_n + 1).$$

If we set $\alpha_0 = 1$ so that zero is reflecting to the right, then

$$\mu_0 = 1, \qquad \mu_n = \sigma_n \mu_{n-1} + (\sigma_n + 1), \qquad n \geqq 1.$$

Now set $Z_n = \frac{1}{2}(\mu_n - 1)$. Then $\{Z_n\}$ satisfies (4.1) with $\{\sigma_n\}$ fixed. I.e., if $\{Z_n\}$ satisfies {4.1}, then $2Z_n + 1$ is the conditional mean first passage time from n to $n + 1$ given the environment. For simplicity we assume $Z_0 = 0$, i.e., $\alpha_0 \equiv 1$.

(4.2) LEMMMA. *The solution of* (4.1) *is*

$$(4.3)\qquad Z_n = \textstyle\sum_{j=1}^n \sigma_j \cdots \sigma_n, \qquad n \geqq 1.$$

(i) *If* $\sum_{n=1}^\infty n^{-1}P(\rho_n > 1) < \infty$, *then* $\mathscr{L} - \lim_{n\to\infty} Z_n = \sum_{j=1}^\infty \rho_j < \infty$ a.e. *where* $\rho_j = \sigma_1 \cdots \sigma_j$.

(ii) *If* $\sum_{n=1}^\infty n^{-1}P(\rho_n > 1) = \infty$, *then* p $- \lim_{n\to\infty} Z_n = \infty$. *When* $E(\ln \sigma)$ *is defined* (*possibly infinite*) (i) *and* (ii) *correspond respectively to* (i′) $E(\ln \sigma) < 0$, (ii′) $E(\ln \sigma) \geqq 0$.

PROOF. (4.3) is trivial. The rest of the lemma follows from Lemma (1.6) and the observation

$$Z_n = \textstyle\sum_{j=1}^n \sigma_j \cdots \sigma_n =_{\mathscr{L}} \sum_{j=1}^n \rho_j.$$

Using the fact that $\{\sigma_n\}$ is independent, identically distributed we obtain

(4.4) THEOREM. *Let* $\nu = E(\sigma)$. *Then* a.e.

$$\begin{aligned} \lim_{n\to\infty} \frac{Z_1 + \cdots + Z_n}{n} &= \frac{\nu}{1 - \nu}, \qquad \nu < 1 \\ &= \infty, \qquad \nu \geqq 1. \end{aligned}$$

PROOF. Let

$$Y_k^n = \textstyle\sum_{j=1}^{n-k+1} \prod_{i=j}^{j+k-1} \sigma_i.$$

It is clear that $Z_1 + \cdots Z_n = Y_1^n + \cdots Y_n^n$. Now Y_k^n is the $(n - k + 1)$st partial sum of random variables that are strictly stationary and ergodic, each with expectation ν^k. Thus for k fixed

$$\lim_{n\to\infty} \frac{Y_k^n}{n} = \lim_{n\to\infty} \frac{n-k+1}{n} \cdot \frac{Y_k^n}{n-k+1} = \nu^k \quad \text{a.e.}$$

Thus

$$\begin{aligned}\liminf_{n\to\infty} \frac{Z_1 + \cdots + Z_n}{n} &= \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^n Y_k^n \\ &\geqq \sum_{k=1}^\infty \liminf_{n\to\infty} \frac{Y_k^n}{n} \\ &= \sum_{k=1}^\infty \nu^k\end{aligned}$$

where the second line follows from Fatou's Lemma.
This proves (4.4) when $\nu \geqq 1$.

Let $\nu < 1$; it suffices to show

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^n Z_k \leqq \frac{\nu}{1-\nu} \quad \text{a.e.}$$

Without loss of generality we may assume that σ_j is the jth coordinate function on the probability space $([0, \infty]^Z, \mathscr{F}, P)$ where $\mathscr{F}$ is the σ-field generated by the cylinder sets and P is the product measure. Now

$$Z_n(\sigma) = \sum_{j=1}^n \sigma_j \cdots \sigma_n \leqq \sum_{j=-\infty}^n \sigma_j \cdots \sigma_n = S_n(\sigma) .$$

(S_n is finite a.e. since $\nu < 1$). But $S_n = S_1 \cdot T^{n-1}$ where T is the shift

$$(T\sigma)_j = \sigma_{j+1} .$$

Hence Birkhoff's Ergodic Theorem applies to S_n and we obtain

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^n Z_k &\leqq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n S_k \\ &= E(S_1) \quad \text{a.e.} \\ &= E\left(\sum_{j=-\infty}^1 \sigma_j \cdots \sigma_1\right) \\ &= \frac{\nu}{1-\nu}\end{aligned}$$

which completes the proof.

We now consider the case $n^{-1}(Z_1 + \cdots Z_n) \to \infty$. As in Section 2 in a special case we find norming constants $f(n)$ so that $f(n)(Z_1 + \cdots Z_n)$ tends in law to nontrivial limiting distributions along subsequences.

Throughout the remainder of this section assume

$$\begin{aligned}\sigma_j &= 0 , \quad \text{with probability} \quad (1-\gamma) \\ &= \theta , \quad \text{with probability} \quad \gamma\end{aligned}$$

where $0 < \gamma < 1$, $\theta > 0$. As in Section 2 we are assuming the existence of only two types of transition probabilities—one of which is the condition for a "one way mirror" reflecting to the right. Assume also that $\gamma\theta = E(\sigma) \geqq 1$—the case $\gamma\theta < 1$ being covered by Theorem (4.4). In particular we assume $\theta > 1$ throughout this section. Let $V_0 = 0$ and inductively

$$V_n = \min\{k > V_{n-1} : \sigma_k = 0\}$$

and set

$$W_{n+1} = Z_{V_n+1} + \cdots + Z_{V_{n+1}}, \qquad n \geqq 0 .$$

Then $\{W_n\}_1^\infty$ is a sequence of independent, identically distributed random variables. Now (4.3) implies $W_1 = 0$ with probability $(1-\gamma)$ and

$$W_1 = \sum_{k=1}^n \sum_{j=1}^k \theta^{k-j+1}$$

with probability $P(V_1 = n+1) = (1-\gamma)\gamma^n$, $n \geqq 1$. Thus

$$W_1 = \frac{\theta}{\theta-1}\left[\frac{\theta}{\theta-1}(\theta^n - 1) - n\right]$$

with probability $(1-\gamma)\gamma^n$, $n \geqq 0$. Hence we have shown

(4.5) LEMMA. *Let $W_{n+1} = Z_{V_n+1} + \cdots + Z_{V_{n+1}}$, $n \geqq 0$. Then $\{W_n\}_1^\infty$ is a sequence of independent, identically distributed random variables. If $\varphi(u) = E(e^{-uW_1})$, then*

$$\varphi(u) = (1-\gamma)\sum_{n=0}^\infty \gamma^n \exp\left[-u\left(\frac{\theta}{\theta-1}\right)\left(\frac{\theta}{\theta-1}(\theta^n-1) - n\right)\right].$$

The proofs of the following results leading up the limit behavior of $Z_1 + \cdots + Z_n$ are so similar to the corresponding proofs in Section 2 that we delete them and state the results as an outlline.

Let

$$\psi(u) = (1-\gamma)\sum_{n=0}^\infty \gamma^n \exp\left(-u\left(\frac{\theta}{\theta-1}\right)^2\theta^n\right).$$

(4.6) LEMMA. *As $u \searrow 0$*

$$\varphi(u) - \psi(u) = O(u) .$$

(4.7) LEMMA. (i) *$\gamma\theta = 1$ implies*

$$\lim_{x\to\infty} x\left(1 - \varphi\left(\frac{u}{x\ln x}\right)\right) = \frac{\theta u}{(\theta-1)\ln\theta}$$

(ii) *If $\gamma\theta > 1$, then*

$$\lim_{x\to\infty} x\left(1 - \varphi\left(\frac{u}{x^\rho}\right)\right) - (1-\gamma)\sum_{n=-\infty}^\infty \gamma^{n-\{\omega\}}(1 - e^{-u\mu\theta^{n-\{\omega\}}}) = 0$$

where $\omega = \omega(x) = \log_{1/\gamma} x$, $\{\omega\}$ is the fractional part of ω, $\mu = (\theta/(\theta-1))^2$, and $\rho = \log_{1/\gamma}\theta > 1$.

(4.8) THEOREM. (i) *$\gamma\theta = 1$ implies*

$$\mathrm{p} - \lim_{n\to\infty} \frac{W_1 + \cdots + W_n}{n\log_\theta n} = \frac{\theta}{\theta-1} .$$

(ii) *If* $\gamma\theta > 1$ *and* $\{n_k\}$ *is a sequence of integers tending to infinity such that*

$$\lim_{k\to\infty} \{\log_{1/\gamma} n_k\} = \varepsilon ,$$

then

$$\lim_{k\to\infty} \mathscr{L}\left(\frac{W_1 + \cdots + W_{n_k}}{n_k \rho}\right)(u) = \exp(-(1-\gamma) \textstyle\sum_{n=-\infty}^{\infty} \gamma^{n-\varepsilon}(1 - e^{-u\mu\theta^{n-\varepsilon}}))$$

where $\mu = (\theta/(\theta-1))^2$ *and* $\rho = \log_{1/\gamma}\theta > 1$.

Applying Lemma (2.14) to the previous theorem gives a limit theorem for the limit distributions of Z_n suitably normalized.

(4.9) THEORM. (i) $\gamma\theta = 1$ *implies*

$$\mathrm{p} - \lim_{n\to\infty} \frac{Z_1 + \cdots + Z_n}{n \log_\theta n} = 1 .$$

(ii) *If* $\gamma\theta > 1$ *and* $\{n_k\}$ *is a sequence of integers tending to infinity such that*

$$\lim_{k\to\infty} \{\log_{1/\gamma} (n_k)\} = \varepsilon .$$

Then

$$\lim_{k\to\infty} E \exp\left(-u \frac{Z_1 + \cdots + Z_{n_k}}{n_k \rho}\right)$$
$$= \exp(-(1-\gamma) \textstyle\sum_{n=-\infty}^{\infty} \gamma^{n-\eta}(1 - e^{-u\mu\theta^{n-\eta}}))$$

where $\rho = \log_{1/\gamma}(\theta)$, $\mu = (\theta/(\theta-1))^2(1-\gamma)^\rho$, $\eta = \varepsilon + \log_{1/\gamma}(1-\gamma)$.

Acknowledgment. This paper was completed in partial fulfillment of the Ph. D. degree at Cornell University. The author wishes to thank his thesis advisor, Frank Spitzer, whose insight was invaluable. Thanks also extend to the referee who pointed out several serious flaws in the first draft.

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