The Annals of Probability
1975, Vol. 3, No. 3, 563-568

MORE ON EQUIVALENCE OF INFINITELY DIVISIBLE DISTRIBUTIONS

BY W. N. HUDSON AND J. D. MASON

University of Utah

Any infinitely divisible distribution on R^n with infinite absolutely continuous Lévy measure and no Gaussian component has a density which is positive a.e. over its support.

1. Introduction and summary. The main result of this paper states sufficient conditions in terms of its Lévy measure for an infinitely divisible distribution μ in R^n with no Gaussian component to be equivalent to Lebesgue measure. This result extends the work of Hudson and Tucker in [3] which considers the case of infinitely divisible measures on the real line. However, the method of approach here is quite different and the topological properties of additive semigroups play a main role in our proof. Also an elementary proof of an extension of a necessary and sufficient condition for infinitely divisible measures to be continuous is indicated here. This result is originally due to Hartman and Wintner in [2] and was independently discovered later by Blum and Rosenblatt in [1].

Our main result is given by the following theorem.

THEOREM 0. *Let μ be an infinitely divisible distribution on R^n with Lévy measure ν. Assume that μ has no Gaussian component, that $\nu(R^n) = \infty$ and that ν is absolutely continuous with respect to n-dimensional Lebesgue measure λ. Then μ is equivalent to λ on its support.*

If $(x, y) = \sum_1^n x_i y_i$ and $|x| = (\sum_1^n x_i^2)^{\frac{1}{2}}$ denote the usual inner product and norm of $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in R^n, μ will have a characteristic function of the form

$$\exp\left\{i(u, \alpha) + \int \left(e^{i(u,x)} - 1 - \frac{i(u, x)}{1 + (x, x)}\right) \nu(dx)\right\},$$

where ν is a Lévy measure on R^n (i.e. $\int |x|^2/(1 + |x|^2)\nu(dx) < \infty$ and $\nu(\{0\}) = 0$). We say a measure α_1 is absolutely continuous with respect to another measure α_2 and write $\alpha_1 \ll \alpha_2$ if whenever $\alpha_2(A) = 0$, then $\alpha_1(A) = 0$. Two measures α_1 and α_2 are equivalent, $\alpha_1 \sim \alpha_2$, if $\alpha_1 \ll \alpha_2$ and $\alpha_2 \ll \alpha_1$. The support of a measure α will be denoted $S(\alpha)$ and is the smallest closed set whose complement $S(\alpha)^c$ has α-measure zero. It is well known that $S(\alpha)$ is the set of all points of increase of α (see [6]). (A point x in R^n is a point of increase of α if every neighborhood of x has positive α-measure.)

Received May 9, 1974; revised September 30, 1974.
AMS 1970 *subject classifications*. Primary 60B05; Secondary 60-00, 60-02.
Key words and phrases. Infinitely divisible distribution, support.

The second section of this paper is devoted to the study of supports of infinitely divisible distributions with infinite Lévy measures and to the proof of the theorem stated above. Such supports are of the form $G + A$ where G is an additive semigroup. It is shown below that $\lambda(\partial(G + A)) = 0$. We begin with some preliminary facts about compound Poisson distributions.

2. Supports of infinitely divisible distributions. Let μ be an infinitely divisible distribution on R^n with characteristic function $\exp\{\int (e^{i(u,x)} - 1)\nu(dx)\}$ where we assume that the Lévy measure ν satisfies the condition $\int_{|x|<1} |x|\nu(dx) < \infty$. If $\nu(R^n) < \infty$, then μ is called a compound Poisson distribution and can be written in the form

$$\mu = \sum_{k=0}^{\infty} e^{-\gamma} \frac{\gamma^k}{k!} \tilde{\nu}^k$$

where $\gamma = \nu(R^n)$, $\tilde{\nu}$ is the normalized Lévy measure on R^n, i.e. $\tilde{\nu} = \gamma^{-1}\nu$, and $\tilde{\nu}^k$ denotes the k-fold convolution of $\tilde{\nu}$ with itself. (It may make some of our results more intuitive to note that μ is the distribution of a random sum, $\sum_{i=1}^{N} X_i$, where $X_1, X_2, \cdots N$ are independent, N is a Poisson distributed integer-valued random variable with parameter $EN = \gamma$, and where $X_1, X_2, \cdots$ are identically distributed with common distribution $\tilde{\nu}$.) Here $\tilde{\nu}^0 \equiv \delta_0$ the distribution concentrated at 0. Note that the compound Poisson distribution has an atom at zero of mass $\geqq e^{-\gamma}$.

For infinitely divisible distributions on R^1, Hartman and Wintner have proved that μ is continuous if and only if $\nu(R^1) = \infty$. This result goes over to R^n. If ν_d and ν_c denote the discrete and continuous parts of ν, then Itô, (Chapter 0 Section 5) shows that $\nu_d(R^n) = \infty$ implies μ is continuous. A proof that $\nu_c(R^n) = \infty$ implies μ is continuous can be given which is similar to the proof of Lemma 2.3 below.

Our first lemma can be found in [7].

LEMMA 2.1. *Let ϕ and ψ be functions on R^n with ϕ integrable and ψ bounded. Then the convolution, $\phi * \psi$, is a continuous function.*

We consider first the case that $\int_{|x|<1} |x|\nu(dx) < \infty$ and specify that μ have characteristic function $\exp\{\int (e^{i(u,x)} - 1)\nu(dx)\}$. The Lévy measure ν is assumed from now on to be absolutely continuous with respect to Lebesgue measure λ on R^n and to have infinite total mass. We use A^- to denote the closure of the set A in R^n.

PROPOSITION 2.2. *The support $S(\mu)$ of μ is given by*

$$S(\mu) = \{\bigcup_{j=1}^{\infty} \sum_{i=1}^{j} S(\nu)\}^- .$$

We omit the proof. It should be intuitively clear how to proceed by taking limits of random sums.

LEMMA 2.3. $\mu \ll \lambda$.

PROOF. Suppose that $\lambda(A) = 0$ for some Borel set A. Let μ_m be the distribution whose characteristic function is $\exp\{\int_{|x|>1/m} (e^{i(u,x)} - 1)\nu(dx)\}$. Then for some distribution α_m, $\mu = \mu_m * \alpha_m$. Now μ_m in a compound Poisson distribution; hence, for $x \in R^n$

$$\mu_m(A - x) = \sum_{j=0}^{\infty} e^{-\gamma_m} \frac{(\gamma_m)^k}{k!} (\tilde{\nu}_m)^k(A - x) .$$

Since $\lambda(A) = \lambda(A - x) = 0$ and $\tilde{\nu}_m \ll \lambda$, $(\tilde{\nu}_m)^k \ll \lambda$ for $k \geqq 1$ and

$$\mu_m(A - x) = e^{-\gamma_m}\delta_0(A - x) \leqq e^{-\gamma_m} .$$

But

$$\mu(A) = \int \mu_m(A - x)\alpha_m(dx) \leqq e^{-\gamma_m}$$

and $\nu(\{x: |x| > 1/m\}) = \gamma_m \to \infty$ as $m \to \infty$. It follows that $\mu(A) = 0$. □

Now let $d\nu/d\lambda$ be the Radon–Nikodym derivative of ν with respect to λ and define $f(x) = \min (d\nu/d\lambda(x), 1)$. We denote the k-fold convolution of f with itself by f^{*k}. By Lemma 2.1, f^{*k} is continuous for $k \geqq 2$. Set $S = \bigcup_{k=2}^{\infty} [f^{*k} > 0]$. We use A^0 to denote the interior of a set A in R^n. A set A in R^n is called a *semigroup* if $A + A \subset A$.

LEMMA 2.4. *The set S is an open semigroup and $S^- = S(\mu)$.*

PROOF. Since f^{*k} is continuous for $k \geqq 2$, S must be open. Suppose x and y are any two points in S. Then for some integers j and $k \geqq 2$, $f^{*j}(x) > 0$ and $f^{*k}(y) > 0$. This implies that $f^{*(k+j)}(x + y) > 0$ and $x + y \in S$. Thus S is a semigroup. Now $\nu(R^n) = \infty$ so $0 \in S(\nu)$. Hence $S(\nu) \subset S(\nu) + S(\nu)$. Let $\tilde{\nu}(A) = \int_A f\,d\lambda$. Clearly $\nu \sim \tilde{\nu}$ and $S(\tilde{\nu}^k) = [f^{*k} > 0]^- = S(\nu^k)$. Thus

$$S(\mu) = (\textstyle\bigcup_{j=1}^{\infty} \sum_{i=1}^{j} S(\nu))^- = (\bigcup_{j=2}^{\infty} S(\tilde{\nu}^j))^- = (\bigcup_{j=2}^{\infty} [f^{*j} > 0])^- = S^- . \qquad \square$$

LEMMA 2.5. *Let G be an open semigroup such that $0 \in G^0$. If A is any set in R^n then $(G + A)^{-0} = G + A$.*

PROOF. The case $G + A = G$ is in Hille and Phillips (Theorem 8.7.2, page 266). A similar proof works here. □

From Lemma 2.5, it follows that $\partial((G + A)^-) = (G + A)^- - (G + A)^0 = \partial(G + A)$, where $\partial(B)$ denotes the boundary of B.

We now show that if G is an open semigroup in R^n such that $0 \in G^-$, then $\lambda(\partial(G + A)) = 0$ for an arbitrary set A. Suppose that $n = 1$. Then Lemma 2.6 will show that either $[0, \infty)$ or $(-\infty, 0]$ is contained in G^-. Consequently, $(G + A)^-$ must be an interval of the form $(-\infty, a]$, $[a, \infty)$, or $(-\infty, \infty)$. In this case it is clear that $\lambda(\partial(G + A)) = 0$. Consequently, Lemma 2.7 will be proved for $n \geqq 2$.

LEMMA 2.6. *If G is an open semigroup in R^n and if $0 \in G^-$, then there is a vector x in R^n such that $|x| = 1$ and $\{tx: t \geqq 0\} \subset G^-$.*

PROOF. This is Theorem 8.7.4. of Hille and Phillips. □

LEMMA 2.7. *Let G be an open semigroup in R^n such that $0 \in G^-$. If A is any set, then $\lambda(\partial(A + G)) = 0$.*

PROOF. The proof will proceed by contradiction. Temporarily, we denote n-dimensional Lebesgue measure by λ_n.

Since G is an open semigroup, we have

$$(1) \qquad (G + A)^- + G \subset (G + A)^{-0}.$$

Now suppose that $\lambda_n(\partial(G + A)) \neq 0$. Then $\partial(G + A)$ has a point of density β (see [5] page 288). It follows that there exists a cube J centered at β with sides of length $2l$ such that $\lambda_n[\partial(G + A) \cap J] > .9\lambda_n(J)$. Without loss of generality we may and do assume that $\{(t, 0, \cdots, 0) : t \geqq 0\} \subset G^-$ and that the sides of J are parallel to the coordinate axes. Let $B_x = \{(x_2, \cdots, x_n) : (x, x_2, \cdots, x_n) \in B\}$ denote the x cross section of B. Then there is a point x in $[B - l, B - .6l]$ such that

$$(2) \qquad \lambda_{n-1}[(G + A)^- \cap J)_x] \geqq .5(2l)^{n-1}.$$

(Otherwise, $\lambda_n[(G + A)^{-c} \cap J] > .5(2l)^{n-1}(.4l) = .1\lambda_n(J)$ contrary to the choice of J.) Since $0 \in G^-$ we can choose a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ in G of length $|\alpha| < \min\{(.1)n^{-1}(2l), x - \beta + l\}$. From (1) and Lemma 2.5 it follows that $G + ((G + A)^- \cap J) \subset G + A$. Now roughly speaking, if we translate $(G + A)^- \cap J$ by a small vector $\alpha \in G$, we cannot move many points of $(G + A)^-$ into or out of J. More precisely, it is not difficult to see that

$$(3) \qquad \lambda_{n-1}[((G + A) \cap J)_{x+\alpha_1}] \geqq (.4)(2l)^{n-1}.$$

But $(t, 0, \cdots, 0) \in G^-$ for $t \geqq 0$ and so $(G + A) \cap J + (t, 0, \cdots, 0) \subset G$ for $t \geqq 0$. Since $\beta - l \leqq x + \alpha_1 \leqq \beta - .4l$, it follows that

$$(4) \qquad [\beta - .4l, \beta + l] \times ((G + A) \cap J)_{x+\alpha_1} \subset (G + A) \cap J$$

and hence

$$(5) \qquad \lambda_n[(G + A) \cap J] \geqq (.7)(2l)(.4)(2l)^{n-1} > .1\lambda_n(J).$$

Since G is open, $G + A$ is disjoint from $\partial(G + A)$. Therefore (5) implies that $\lambda_n(\partial(G + A) \cap J) < .9\lambda(J)$ which contradicts the choice of J. □

We summarize our results for the first case in the following theorem.

THEOREM. 1. *Let ν be a Lévy measure on R^n such that $\int_{|x|<1} |x|\nu(dx) < \infty$. If μ is the infinitely divisible distribution whose characteristic function is $\exp\{\int (e^{i(u,x)} - 1)\nu(dx)\}$, then $S(\mu)$, the support of μ, is a semigroup and $\mu \sim \lambda$ on $S(\mu)$, where λ is n-dimensional Lebesgue measure.*

PROOF. We have already shown that $S(\mu)$ is a semigroup in Lemma 2.4. Since $\lambda(\partial S(\mu)) = 0$ and $\mu \ll \lambda$, it suffices to show that $\lambda \ll \mu$ on $S = S(\mu)^0$. Let $\nu'(A) = \int_A f\nu(dx)$ again and let μ' have the characteristic function $\exp\{\int (e^{i(u,x)} - 1)\nu'(dx)$. Since $\nu'(R^n) < \infty$, μ' is a compound Poisson distribution. Since

$\nu' \leqq \nu$, $\nu - \nu'$ is also a Lévy measure and

$$\exp\{\int (e^{i(u,x)} - 1)\nu(dx)\}$$
$$= \exp\{\int (e^{i(u,x)} - 1)\nu'(dx)\} \exp\{\int (e^{i(u,x)} - 1)(\nu - \nu')(dx)\} .$$

It follows that $\mu = \mu' * \alpha$ for some distribution α. Now suppose A is a Borel set in S such that $\mu(A) = 0$. Since

$$0 = \mu(A) = \int \mu'(A - x)\alpha(dx) ,$$

we have

$$\mu'(A - x) = 0 \qquad \text{for } \alpha\text{—a.e.—}x .$$

In fact, $\mu'(A - x) = 0$ for all x in some dense subset of $S(\alpha)$. Let $\{x_d : d \in D\}$ be a countable dense subset of $S(\alpha)$ such that $\mu'(A - x_d) = 0$ for all $d \in D$. Since

$$\mu' = \sum_{k=0}^{\infty} e^{-\gamma'} \frac{(\gamma')^k}{k!} \left(\frac{\nu'}{\gamma'}\right)^k ,$$

$(\nu')^k(A - x_d) = 0$ for $k \geqq 2$. But $S(\nu')^k = [f^{*k} > 0]^-$ and hence $(\nu')^k \sim \lambda$ on $[f^{*k} > 0]$. Thus $\lambda((A - x_d) \cap S) = 0$ for $d \in D$. Since λ is translation invariant,

$$\lambda(A \cap (S + x_d)) = 0 \qquad d \in D .$$

But $S + S(\alpha) \subset \bigcup_{d \in D} (S + x_d)$. To see this, let $x \in S$ and $y \in S(\alpha)$. Since S is open and $\{x_d : d \in D\}$ is dense in $S(\alpha)$, we can choose an x_d such that $x + (y - x_d) \in S$. Then $x + y = x + (y - x_d) + x_d \in \bigcup_{d \in D} (S + x_d)$. Now $(\nu - \nu')(R^n) = \infty$ so $0 \in S(\alpha)$. Thus $S \subset S + S(\alpha)$. Then

$$\lambda(A) \leqq \sum_{d \in D} \lambda(A \cap (S + x_d)) = 0$$

and $\lambda \ll \mu$. □

We now consider the general case where ν is any Lévy measure such that $\nu(R^n) = \infty$ and that $\nu \ll \lambda$. Then μ may have a characteristic function of the form $\exp\{\int (e^{i(u,x)} - 1 - i(u, x)/(1 + |x|^2))\nu(dx) + i(u, \alpha)\}$. We now prove the theorem stated in the introduction. Define $\nu_1(A) = \nu(A \cap \{|x| > 1\}) + \int_{A \cap \{|x| \leq 1\}} |x|\nu(dx)$ and let μ_1 have characteristic function $\exp\{\int (e^{i(u,x)} - 1)\nu_1(dx)\}$. Since $\nu_1 \leqq \nu$, it follows from the Lévy representations of the characteristic functions that μ_1 is a factor of μ; that is, there is an (infinitely divisible) distribution α on R^n such that $\mu = \mu_1 * \alpha$. Now by Lemma 2.3 $\mu_1 \ll \lambda$ and hence $\mu \ll \lambda$. It remains to show that $\lambda \ll \mu$ on $S(\mu)$. It is easy to see that $S(\mu) = (S(\mu_1) + S(\alpha))^-$. Suppose that A is a Borel subset of $S(\mu)^0$ such that $\mu(A) = 0$. Then $\int \mu_1(A - x)\alpha(dx) = 0$ and so for every x in some countable dense subset $\{x_d : d \in D\}$ of $S(\alpha)$ $\mu_1(A - x) = 0$. Let $S = S(\mu_1)^0$; then S is an open semigroup such that $0 \in S^-$, (Proposition 2.2 and Lemma 2.4). By Theorem 1, $\mu_1 \sim \lambda$ on $S(\mu_1)$, so

$$\lambda\{(A - x_d) \cap S(\mu_1)\} = 0 , \qquad d \in D ,$$

or

$$\lambda[A \cap (S(\mu_1) + x_d)] = 0 , \qquad d \in D .$$

According to Lemma 3.3, $S(\mu)^0 = S + S(\alpha)$ and since $S + S(\alpha) \subset \bigcup_{d\in D}(S + x_d)$, we have

$$\mu(A) = \mu(A \cap S(\mu)^0) \leqq \sum_{d\in D} \mu(A \cap (S + x_d)) = 0 .$$

Thus $\lambda \ll \mu$ on $S(\mu)^0$. But according to Lemma 2.6, $\lambda(\partial S(\mu)) = 0$ and hence $\lambda \ll \mu$ on $S(\mu)$ which proves the theorem.

The question arises as to what sets are possible supports of infinitely divisible distributions. In the case of R^1, if $\int_{|x|<1} |x|\nu(dx) = \infty$, $\nu \ll \lambda$, Hudson and Tucker [3], show that the support must be the whole real line. It is easy to see that $S(\mu) = R^n$ whenever $S(\nu)$ contains a neighborhood of 0. In this case, the smallest semigroup containing $S(\nu)$ is R^n and so μ has a factor with support equal to R^n.

Acknowledgments. The authors would like to express their gratitude to Professor Howard G. Tucker for suggesting this problem and providing encouragement.

REFERENCES

[1] BLUM, J. R. and ROSENBLATT, M. (1959). On the structure of infinitely divisible distributions. *Pacific J. Math.* **9** 1–7.

[2] HARTMAN, P. and WINTNER, A. (1942). On the infinitesimal generators of integral convolutions. *Amer. J. Math.* **64** 273–298.

[3] HILLE, E. and PHILLIPS R. S. (1957). *Functional Analysis and Semigroups.* A.M.S. Colloq. Pub. Vol. XXXI, Providence.

[4] HUDSON, W. N. and TUCKER, H. G. (1975). Equivalence of infinitely divisible distributions. *Ann. Probability* **3** 70–79.

[5] ITÔ, K. (1969). *Stochastic Processes.* Aarhus Universitet (Lecture notes series No. 16).

[6] MUNROE, M. E. (1953). *Introduction to Measure and Integration.* Addison-Wesley, Reading.

[7] PARTHASARATHY, K. R. (1967). *Probability Measures on Metric Spaces.* Academic Press, New York.

[8] RUDIN, W. (1960). *Fourier Analysis on Groups.* Interscience Publishers, New York.

DEPARTMENT OF MATHEMATICS
205 MATHEMATICS BUILDING
THE UNIVERSITY OF UTAH
SALT LAKE CITY, UTAH 84112