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EXTREME TIME OF STOCHASTIC PROCESSES WITH STATIONARY INDEPENDENT INCREMENTS

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Let $\{S_n = \sum_{i=1}^n Y_i\}$ or $\{X_t, t \geqq 0\}$ be a stochastic process with stationary independent increments, and let $T^+(\tau)$, $T^-(\tau)$ be the times elapsed until the process has spent time τ at its maximum and minimum respectively, defined in terms of local time when necessary. Bounds in terms of moments of Y_1 or X_1 are given for $E(\min(T^+(\tau), T^-(\tau)))$. The discrete case is studied first and the result for continuous-time processes is obtained by a limiting argument. As an auxiliary it is shown that the local time at zero of a process X_t minus its maximum can be approximated uniformly in probability using the number of new maxima attained by the process observed at discrete times.

1. Introduction. By the extreme time of a process we mean the time it spends at its maxima and minima, or if these are instantaneous, the time the process spends increasing and decreasing to new extrema measured by local time. We will study stopping times for a random process defined in terms of the extreme time. Such stopping times are of interest for embeddings of processes and for those control problems in which extreme time is an important quantity.

The stopping time studied here may be compared with the crossing time of a process at a two-sided boundary. Let $\{S_n = \sum_{i=1}^n Y_i\}$ or $\{X_t, t \geqq 0, X_0 = 0\}$ be a stochastic process with stationary independent increments. In the latter case take a standard, right-continuous version. Frequently, if $b > 0$ and $T = \inf\{t > 0 : X_t > b\}$ then $ET = \infty$, whereas if $a < 0 < b$ and $T = \inf\{t > 0 : X_t \notin (a, b)\}$ then $ET < \infty$; S_n behaves similarly. If instead of boundary crossing times we consider $T^+(\tau)$ and $T^-(\tau)$, the times elapsed until the process has spent time $\tau > 0$ at its maxima and minima respectively, then frequently $E(T^+(\tau)) = \infty$, but we expect that

$$E(\min(T^+(\tau), T^-(\tau))) < \infty . \tag{1}$$

Although the maximum and minimum processes associated with S_n and X_t have been studied extensively, little is known about the joint behavior of the two processes, which are neither independent nor simply related. The primary result is the equation of Spitzer (1964) involving the transforms of the maximum and minimum processes at their increase times in the discrete case, and Fristedt's (1974) extension of this equation to the continuous time case. Here we study a particular aspect of the joint behavior, namely $\min(T^+(\tau), T^-(\tau))$.

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The quantities $T^+(\tau)$, $T^-(\tau)$ are adequately described above if X_t is a discrete-time or compound Poisson process. If X_t has continuous time parameter, is symmetric and not compound Poisson, it follows from a result of Rubinovitch (1971) that $T = \inf\{t: X(t) > 0\}$ is almost surely zero. In this case local time at zero of $X_t - M_t$, where $M_t = \sup_{s \leqq t} X_s$, exists by the construction of Blumenthal and Getoor (1968) of local time at regular points of Markov processes. Alternatively, since $X_t - M_t$ is a strong Markov process, the set of t such that $X_t - M_t = 0$ forms a regenerative set. A general study of local time on regenerative sets has been made by Maisonneuve (1971). We take $T^+(\tau)$, $T^-(\tau)$ to be the right-continuous inverse of the local time at zero, chosen in a natural way, of X_t minus its maximum, minimum process respectively.

We obtain (1) first for a random walk $S_n = \sum_{i=1}^n Y_i$, with a bound in terms of the second and fourth moments of Y_1 (Theorem 1). The result is then derived for compound Poisson processes by replacing fixed time increments with exponentially distributed ones (Theorem 2). In order to extend the result to the remaining symmetric independent increment processes we show, more generally, that whenever the local time at zero of $X_t - M_t$ exists it can be approximated uniformly by the number of new maxima attained by the process observed at times $i/2^n$, i an integer, normalized appropriately. The approximation is in the sense of convergence in probability as $n \to \infty$ (Theorem 3). This extends a similar result of Fristedt (1974) who obtained convergence in law. The approximation of local time is used to compute $ET^+(\tau)$ for processes such that $EX_1 > 0$.

Some remarks about the hypotheses, methods, and related problems are contained in the last section.

2. Results. For convenient reference the main results are stated here. Some lemmas and a corollary to Theorem 3 are stated in the next section.

THEOREM 1. *Let $S_n = \sum_{i=1}^n Y_i$, where the Y_i are independent random variables with common distribution F. Let $Y = Y_1$. Suppose that F is continuous, $EY = 0$, $EY^2 = \sigma^2$, and $EY^4 = \gamma < \infty$. Let $T_{r,s} = \min\{m$: number of new maxima of S_n is r or the number of new minima is s for $n \leqq m\}$. Let $Z = S_T$ where $T = \min\{n: S_n > 0\}$, $\tilde{Z} = -S_{\tilde{T}}$ where $\tilde{T} = \min\{n: S_n < 0\}$. Then*

$$ET_{r,s} \leqq ((EZ^2 + E\tilde{Z}^2)/EY^2)(r+s) + (((EZ)^2 + (E\tilde{Z})^2)/EY^2)(r+s)^2 . \tag{2}$$

If F is symmetric then

$$ET_{r,s} \leqq c(r+s) + (r+s)^2$$

where $c = 2EZ^2/EY^2 \leqq (2\gamma)^{\frac{1}{2}}/\sigma^2$.

THEOREM 2. *Let X_t be a compound Poisson process with characteristic exponent*

$$\log E \exp i\theta X_t = -t\lambda \int_{-\infty}^{\infty} (e^{i\theta x} - 1)\, dF(x) ,$$

where F is a continuous distribution function with second moment σ^2 and fourth moment $\gamma < \infty$. Let $L^+(t)$, $L^-(t)$ denote the time spent by X_t at maxima, minima, respectively, *up to time t. Then*

$$E(\inf t : \max(L^+(t), L^-(t)) \geqq \tau) \leqq d_1\tau + d_2 4\lambda\tau^2 ,$$

where $d_1 = 2^{\frac{3}{2}}\gamma^{\frac{1}{2}}/\sigma^2 + 2$, $d_2 = 1$ if F is symmetric. Otherwise

$$d_1 = 2(EZ^2 + E\tilde{Z}^2/\sigma^2) + 2 , \qquad d_2 = ((EZ)^2 + (E\tilde{Z})^2)/\sigma^2 .$$

THEOREM 3. *Let X_t be a process with stationary independent increments such that $X_0 = 0$ and $T = \inf\{t > 0 : X_t > 0\} = 0$, almost surely. Let $M_t = \sup_{s\leqq t} X_s$ and denote by L_t the local time at 0 of the process $X_t - M_t$, chosen so that $E\int_0^\infty e^{-s}\,dL(s) = 1$. For each positive integer n let $t^{(n)} = \min\{i/2^n : X(i/2^n) > 0\}$ and let $L_n(t) = \mathrm{card}\,\{$ascending ladder epochs of $X(i/2^n)$ before $t\} \cdot (1 - Ee^{-t^{(n)}})$. Then L_n converges to L uniformly on bounded intervals in probability.*

THEOREM 4. *Let X_t be standard symmetric process with stationary independent increments, not of Poisson type, $X_0 = 0$, $EX_1^2 = \sigma^2$, $EX_1^4 = \gamma < \infty$. Let L_t^+, L_t^- be the local time at 0 of X_t minus its maximum, minimum process, respectively. Let $T^+(\tau) = \inf\{t : L_t^+ \geqq \tau\}$, $T^-(\tau) = \inf\{t : L_t^- \geqq \tau\}$. Then*

$$E(\min(T^+(\tau), T^-(\tau))) \leqq (2^{\frac{3}{2}}\gamma^{\frac{1}{2}}/\sigma^2)\tau + 4\tau^2 .$$

THEOREM 5. *Let X_t be a standard process with stationary, independent increments, not of Poisson type, $X_0 = 0$, and $EX_1 > 0$. Then $ET^+(\tau) = \tau$.*

3. Proofs. We will adopt, with some modifications, terminology of Feller (1966). In particular, $Z = S_T$, where $T = \min(n : S_n > 0)$, and $\tilde{Z} = -S_{\tilde{T}}$, where $\tilde{T} = \min(n : S_n < 0)$ are called the ascending and descending ladder variables associated with S_n. In the symmetric case Z and $\tilde{Z}$ have the same distribution. The times at which successive new maxima and minima occur are called ladder epochs. We use the term ladder height to mean the absolute amount by which the difference of the maximum and minimum of the process increases at a ladder epoch. We note that ladder heights are distinct from ladder variables. Results of Spitzer about ladder variables used here are presented by Feller (1966) in Chapters 12 and 18.

Spitzer showed that the first moments of the ladder variables are related to the variance, σ^2, of Y_i if $EY_i = 0$ by

$$EZ = (\sigma^2/2)^{\frac{1}{2}}e^c , \qquad E\tilde{Z} = (\sigma^2/2)^{\frac{1}{2}}e^{-c} , \tag{3}$$

where $c = \sum n^{-1}(P(S_n > 0)-\frac{1}{2}) < \infty$. The following lemma gives similar *inequalities* for the second and third moments of the ladder variables. Of this information we will use only the bound for EZ^2 in the symmetric case, and the finiteness of both EZ^2 and $E\tilde{Z}^2$ in the nonsymmetric case.

LEMMA 1. *Let $S_n = \sum_{i=1}^n Y_i$, where the Y_i are independent with distibution F. Let Z, $\tilde{Z}$ be the ascending and descending ladder variables. Suppose that F is continuous with mean zero, second moment σ^2 and fourth moment $\gamma < \infty$. Then*

$EZ^2E\tilde{Z}^2 \leqq \gamma/2$ and $EZ^3E\tilde{Z}^3 \leqq \gamma^2/(2\sigma^2)$. If F is symmetric, then

$$EZ^2 \leqq (\gamma/2)^{\frac{1}{2}} \qquad \text{and} \qquad EZ^3 \leqq \gamma/(2^{\frac{1}{2}}\sigma) .$$

PROOF. Spitzer showed that

$$F = F^+ + F^- - F^+ * F^- , \tag{4}$$

where F^+ and F^- are the distributions of Z and $-\tilde{Z}$. For each positive integer m, let $S_n{}^m$ be a random walk with steps distributed like

$$\begin{aligned} W_m &= 0 \qquad \text{if} \quad |Y_1| > m \\ &= Y_1 \qquad \text{if} \quad |Y_1| \leqq m . \end{aligned}$$

Let Z_m, $\tilde{Z}_m$ be the possibly defective ladder variables of $S_n{}^m$. Relation (4) holds for the distributions of W_m, Z_m, and $-\tilde{Z}_m$. Multiply by x^4 and integrate to obtain

$$4EZ_m E\tilde{Z}_m{}^3 + 4EZ_m{}^3E\tilde{Z}_m - 6EZ_m{}^2E\tilde{Z}_m{}^2 \leqq EW_m{}^4 , \tag{5}$$

where each EX^n denotes $\int x^nG(dx)$, G possibly defective.

Let $a = (E\tilde{Z}_m EZ_m{}^3)^{\frac{1}{2}}$, $b = (E\tilde{Z}_m{}^3EZ_m)^{\frac{1}{2}}$. From the moment inequalities

$$EZ_m{}^2 \leqq (EZ_m EZ_m{}^3)^{\frac{1}{2}} \qquad \text{and} \qquad E\tilde{Z}_m{}^2 \leqq (E\tilde{Z}_m E\tilde{Z}_m{}^3)^{\frac{1}{2}}$$

we have

$$EZ_m{}^2E\tilde{Z}_m{}^2 \leqq ab ,$$

and from (5)

$$4(a^2 + b^2) - 6ab \leqq EW_m{}^4 ,$$

or

$$ab/2 \leqq (a - b)^2 + ab/2 \leqq EW_m{}^4/4 .$$

Let $m \to \infty$ and use (3) to obtain

$$\begin{aligned} &\liminf EZ_m{}^2 \liminf E\tilde{Z}_m{}^2 \leqq \gamma/2 , \\ &\liminf EZ_m{}^3 \liminf E\tilde{Z}_m{}^3 \leqq \gamma^2/(2\sigma^2) . \end{aligned}$$

For each sample function, if m is sufficiently large, Z_m and $\tilde{Z}_m$ are defined and equal Z and $\tilde{Z}$. Fatou's lemma completes the proof.

PROOF OF THEOREM 1. If the number of ladder epochs of S_n, $n \leqq n_0$, is at least $r + s$ then either at least r ascending or at least s descending ladder epochs have occurred before n_0. We consider the stopping time $T_u = \min(m\colon$ at least u ladder epochs occur in $(n \leqq m))$.

Let t_i be the time from the $(i - 1)$st to the ith ladder epoch. Then $ET_u = \sum_{i=1}^u Et_i$. Let H_i be the absolute size of the ith ladder height. Let

$$W_i = \textstyle\sum_{j=1}^i H_j = \max_{j=1,\cdots,T_i} S_j - \min_{j=1,\cdots,T_i} S_j .$$

To show that Et_i is finite we look at the process starting at T_{i-1}. The time until the next new minimum or maximum is the time it takes the process S_n to escape from the random interval of width $W = W_{i-1}$ starting from one endpoint. For the present let the process be symmetric. In this calculation we may assume $S_{T_{i-1}}$ to be a minimum and the endpoint in question to be a lower endpoint.

For w fixed let $N_w = \min\{n : S_n \notin [0, w]\}$. We write $N_w = N$, remembering that N always depends on w. For each w, $EN = ES_N^2/EX_1^2$, and $Et_i = E_W[ES_N^2/EX_1^2]$, where E_W denotes integration with respect to the distribution of W. We will show that Et_i is bounded in terms of moments of the ladder variable Z. Using the identity $S_N^2 = (S_N - w)^2 + 2w(S_N - w) + w^2$, we have

$$\begin{aligned} E_W ES_N^2 = E_W[E((S_N - w)^2 S_N > w) + E(S_N^2, S_N < 0)] \\ + 2E_W[wP(S_N > w)E(S_N - w \mid S_N > w)] \\ + E_W[w^2 P(S_N > w)] . \end{aligned} \tag{6}$$

For fixed w the Wald equation

$$ES_N = P(S_N > w)E(S_N \mid S_N > w) + E(S_N, S_N < 0) = 0$$

and the observation that S_N is the same as $-\tilde{Z}$ on the set $\{S_N < 0\}$ give

$$\begin{aligned} P(S_N > w) &\leqq E\tilde{Z}/E(S_N \mid S_N > w) \\ &= E\tilde{Z}/(E(S_N - w \mid S_N > w) + w) . \end{aligned}$$

Under the continued assumption that the process is symmetric,

$$wP(S_N > w) \leqq EZ .$$

From the same observation, $E(S_N^2, S_N < 0) \leqq EZ^2$ for every w. A similar argument will give bounds for the other terms of (6), which can now be viewed as

$$\begin{aligned} E_W ES_N^2 \leqq E_W E((S_N - w)^2, S_N > w) + 2E_W E(S_N - w \mid S_N > w)EZ \\ + EWEZ + EZ^2 . \end{aligned} \tag{7}$$

If H_i is ith ladder height,

$$\begin{aligned} E_W E((S_N - w)^2, S_N > w) &= E(H_i^2, H_i \text{ has opposite sense from } H_{i-1}) \\ &\leqq EH_i^2 . \end{aligned}$$

Similarly, $E_W E(S_N - w \mid S_N > w) \leqq EH_i/q_i$ where $q_i = P(H_i$ has opposite sense from $H_{i-1}) > 0$. Also, $EW = EW_{i-1} = \sum_{j=1}^{i-1} EH_j$. We see that (7) is finite if for each ladder height H_i, EH_i and EH_i^2 are finite. But H_i, in each of its possible rôles, is a restriction of some ladder variable Z_j:

$$\begin{aligned} EH_i &= \textstyle\sum_{j=1}^{i} E(H_i, H_i \text{ is the } j\text{th ascending ladder height}) \\ &\qquad + E(H_i, H_i \text{ is the } j\text{th descending ladder height}) \\ &\leqq 2iEZ , \end{aligned}$$

and

$$EH_i^2 \leqq 2iEZ^2 .$$

This establishes that Et_i is finite in the symmetric case. If F is not symmetric we calculate $E_W ES_N^2$ by first conditioning with respect to the two events that T_{i-1} is an ascending, decending ladder epoch. The inequalities remain valid if EZ and EZ^2 are replaced by $\max(EZ, E\tilde{Z})$ and $\max(EZ^2, E\tilde{Z}^2)$ which are finite under our hypotheses according to Lemma 1.

Having shown that ET_u is finite, we use Wald's equation,

$$ET_u = ES^2_{T_u}/EY^2\,, \tag{8}$$

to obtain a bound. The quantity S_{T_u} is a sum of u ladder heights of which a random proportion are ascending. The component sets of ascending and descending ladder variables can be described in terms of random times $n(\omega)$, but these, we note, are not stopping times for S_n. Let Z_i, $i = 1, \cdots, r$ be the first r ascending ladder heights, which are independent and distributed like Z, and let $\tilde{Z}_i$, $i = 1, \cdots, r$ be the magnitudes of the first r descending ladder heights, another independent family of random variables each distributed like $\tilde{Z}$. For any nonnegative X, Y, $E(X - Y)^2 \leqq EX^2 + EY^2$. Similarly,

$$\begin{aligned} ES^2_{T_u} &= E(\textstyle\sum_{i=1}^{n(\omega)} Z_i - \sum_{j=1}^{u-n(\omega)} \tilde{Z}_j)^2 \\ &\leqq E(\textstyle\sum_{i=1}^{u} Z_i)^2 + E(\sum_{j=1}^{u} \tilde{Z}_j)^2\,. \end{aligned} \tag{9}$$

Recalling the first paragraph of this proof, we replace T_u by $T_{r,s}$ and u by $r + s$ in (8) and (9) to obtain (2). If F is symmetric, Z and $\tilde{Z}$ have the same distribution. From (9),

$$ET_{r,s} \leqq 2\,\frac{EZ^2}{EY^2}\,(r + s) + 2\,\frac{(EZ)^2}{EY^2}\,(r + s)(r + s - 1)\,.$$

Spitzer's equation (3) and Lemma 1 allow us to replace $(EZ)^2/EY^2$ by $\frac{1}{2}$ and EZ^2/EY^2 by $(\gamma/2\sigma^2)^{\frac{1}{2}}$.

Proof of Theorem 2. The Poisson-type process remains in each state for an exponentially distributed time with parameter λ. The number of states required to accumulate a total time of at least τ is a Poisson random variable with parameter $\lambda\tau$, independent of the identity of the states involved. The number of maximal states visited by X_t before $L^+(t) \geqq \tau$ and the number of minimal states visited before $L^-(t) \geqq \tau$ are independent Poisson variables Y^+ and Y^-, each with parameter $\lambda\tau$. Let $S_{a,b} = \inf(t > 0 : X_s$ has attained a maximal states or b minimal states, $s \leqq t)$. Then $S_{a,b}$ is a sum of $T_{a,b}$ exponentially distributed independent summands R_i, where $T_{a,b}$ is defined as in Theorem 1 in terms of the discrete-time process S_n with jump distribution F. Using Theorem 1, we have

$$\begin{aligned} &E\inf(t > 0 : \max(L^+(t), L^-(t) \geqq \tau) \\ &\quad = \textstyle\sum_{a,b=1}^{\infty} P((Y^+, Y^-) = (a, b)) \\ &\qquad \times E(S_{a,b}\,|\,(Y^+, Y^-) = (a, b)) \\ &\quad = \textstyle\sum_{a,b=1}^{\infty} P((Y^+, Y^-) = (a, b)) \\ &\qquad \times \textstyle\sum_c E(S_{a,b}\,|\,(Y^+, Y^-) = (a, b), T_{a,b} = c) \\ &\qquad \times P(T_{a,b} = c\,|\,(Y^+, Y^-) = (a, b))\,. \end{aligned} \tag{10}$$

Now $T_{a,b}$ is independent of (Y^+, Y^-), and given that $T_{a,b} = c$, $S_{a,b}$ is the sum of c exponentially distributed waiting times. That $(Y^+, Y^-) = (a, b)$ means that two particular subsets of a and b waiting times each total $\leqq \tau$ and one of these totals τ. Expression (10) is bounded by

$$\begin{aligned}
&\textstyle\sum_{a,b=1}^{\infty} P((Y^+, Y^-) = (a, b)) \sum_{c=a+b+1}^{\infty} E \sum_{i=1}^{c-(a+b)} R_i P(T_{a,b} = c) + 2\tau \\
&\quad = \sum_{a,b=1}^{\infty} P((Y^+, Y^-) = (a, b)) \sum_{c=a+b+1}^{\infty} \frac{c-(a+b)}{\lambda} P(T_{a,b} = c) + 2\tau \\
&\quad = \frac{1}{\lambda} \sum_{a,b=1}^{\infty} P((Y^+, Y^-) = (a, b))(ET_{a,b} - (a+b)) + 2\tau\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } F \text{ is symmetric} \\
&\quad \leqq \frac{1}{\lambda} \sum_{a,b=1}^{\infty} P((Y^+, Y^-) = (a, b))((c_1 - 1)(a+b) + (a+b)^2) + 2\tau \\
&\quad = \frac{1}{\lambda} [(c_1 - 1)2\lambda\tau + 4(\lambda\tau)^2 + 2\lambda\tau] + 2\tau \\
&\quad = d_1 + 4\lambda\tau^2\,,
\end{aligned}$$

where $d_1 = 2^{\frac{3}{2}}\gamma^{\frac{1}{2}}/\sigma^2 + 2$.

If F is not symmetric use (2) for $ET_{a,b}$.

Proof of Theorem 3. We assume that $(X_t, \mathscr{F}_t)$ is a standard (Hunt) process on a probability space $(\Omega, \mathscr{F}, P)$. Let $A_n(t) = \int_0^t e^{-s}\, dL_n(s)$, $A(t) = \int_0^t e^{-s}\, dL(s)$. It suffices to show $A_n \to A$ uniformly in probability. Let $t_{i,n}$ be the time from the $(i-1)$st to the ith ladder epoch of $X^{(n)}(j) = X(j/2^n)$, $t^{(n)} = t_{1,n}$, and $a_n = 1 - Ee^{-t^{(n)}}$. Then

$$\begin{aligned}
EA_n(\infty) &= a_n E \textstyle\sum_{l=1}^{\infty} \exp(-\sum_{i=1}^{l} t_{i,n}) \\
&= a_n \textstyle\sum_{l=1}^{\infty} (E(e^{-t^{(n)}}))^l = Ee^{-t^{(n)}}\,.
\end{aligned}$$

Also, $EA(\infty) = E \int_0^\infty e^{-s}\, dL(s) = 1$ by our choice of local time.

For each $t > 0$ let $[t] = \inf(s \geqq t\colon (X - M)(s) = 0)$. Then $\bigcup \mathscr{F}_{[t]} = \bigcup \mathscr{F}_t$ and

$$e_n(t) = E[A_n(\infty) \mid \mathscr{F}_{[t]}]$$

and

$$e(t) = E[A(\infty) \mid \mathscr{F}_{[t]}]$$

are martingales with respect to the family of σ-fields $\mathscr{F}_{[t]}$. Let $[t]_n = \min(i/2^n \geqq [t]\colon X(j/2^n) < X(i/2^n), \text{all } j < i)$. For fixed t and n, that $P([t] = [t]_n) = 0$ follows from the fact that $P(X_s = a) = 0$ for any fixed s and a and any initial distribution. Since $[t]_n$ is a stopping time for L_n which does not increase between $[t]$ and $[t]_n$,

$$\begin{aligned}
e_n(t) &= A_n([t]) + E[\textstyle\int_{[t]+}^{\infty} e^{-s}\, dL_n(s) \mid \mathscr{F}_{[t]}] \\
&= A_n([t]) + a_n E(e^{-[t]_n} \mid \mathscr{F}_{[t]}) + E[\textstyle\int_0^\infty e^{-(s+[t]_n)}\, dL_n(s + [t]_n) \mid \mathscr{F}_{[t]}] \qquad (11)\\
&= A_n([t]) + (a_n + E \textstyle\int_0^\infty e^{-s}\, dL_n(s)) E[e^{-[t]_n} \mid \mathscr{F}_{[t]}] \\
&= A_n([t]) + E[e^{-[t]_n} \mid \mathscr{F}_{[t]}]\,.
\end{aligned}$$

Similarly, since L does not increase between t and $[t]$,

$$e(t) = A[t] + e^{-[t]} E \textstyle\int_0^\infty e^{-s}\, dL(s) = A(t) + e^{-[t]}\,.$$

To show that A_n converges to A uniformly in probability, it suffices to show that, as $n \to \infty$:

(i) $Ee^{-t^{(n)}} \to Ee^{-T} = 1$, i.e., $a_n \to 0$;
(ii) $E[e^{-[t]_n} \mid \mathscr{F}_{[t]}] \to e^{-[t]}$ uniformly, almost surely;
(iii) $P(|e_n(t) - e(t)| > \delta) \to 0$ uniformly;
(iv) $A_n[t] - A_n(t) \to 0$ uniformly.

PROOF OF (i). Clearly $t^{(n)} \geqq T$ and $t^{(n)} \downarrow T$ almost surely since X_t is right continuous.

PROOF OF (ii). Since X_t begins anew at $[t]$, there are arbitrarily small $\varepsilon > 0$ such that $X([t] + \varepsilon) > X(s)$, all $s \leqq [t]$. By the right continuity of X_t at $[t] + \varepsilon$, there is an $i/2^n$ such that $X(i/2^n) > X(s)$, all $s \leqq [t]$, and $i/2^n < [t] + 2\varepsilon$. This n depends only on the path X_t, $t > [t]$. Hence

$$E[e^{-[t]_n} \mid \mathscr{F}_{[t]}] = e^{-[t]}E[e^{-([t]_n-[t])} \mid \mathscr{F}_{[t]}] \to e^{-[t]}$$

uniformly in t, almost surely.

PROOF OF (iii). Apply Doob's inequality to the submartingale $|e_n(t) - e(t)|$:

$$\begin{aligned} P(\sup_t |e_n(t) - e(t)| \geqq \delta) & \\ &\leqq \frac{1}{\delta^2} E(A_n(\infty) - A(\infty))^2 \\ &= \frac{1}{\delta^2} E[\textstyle\int_0^\infty e^{-s}(dL_n - dL)]^2 \\ &= \frac{2}{\delta^2} E \textstyle\int_0^\infty e^{-s}(dL_n - dL) \int_s^\infty e^{-t}(dL_n - dL) \\ &= \frac{2}{\delta^2} E \textstyle\int_0^\infty e^{-s}(dL_n - dL)\, E(\int_s^\infty e^{-t}(dL_n - dL) \mid \mathscr{F}_s) \\ &\leqq \frac{2}{\delta^2} E(\sup_s |E(g_n(s) \mid \mathscr{F}_s)|\, |\textstyle\int_0^\infty e^{-s}(dL_n - dL)|)\,, \end{aligned}$$

where

$$\begin{aligned} g_n(s) &= \textstyle\int_s^\infty e^{-t}(dL_n - dL)\,, \\ &\leqq \frac{2}{\delta^2} (E \sup_s |E(g_n(s) \mid \mathscr{F}_s)|^2)^{\frac12}(E(\textstyle\int_0^\infty e^{-s}\, dL_n - dL)^2)^{\frac12}\,. \end{aligned}$$

The last factor squared, by the same calculation as above, is

$$\begin{aligned} &\leqq 2E \textstyle\int_0^\infty e^{-s}(dL_n - dL) \int_s^\infty e^{-t}(dL_n - dL) \\ &\leqq 2(E \textstyle\int_0^\infty e^{-s}(dL_n + dL))^2 \\ &= 2(Ee^{-t^{(n)}} + e^{-T})^2 \leqq 8\,. \end{aligned}$$

A computation similar to (11) gives

$$E(g_n(s) \mid \mathscr{F}_s) = E(e^{-s_n} \mid \mathscr{F}_s) - E(e^{-[s]} \mid \mathscr{F}_s)\,.$$

Each term is bounded by 1. Therefore

$$\sup_s |E(g_n(s) \mid \mathscr{F}_s)|^2 \leqq 4 .$$

We will show that $\sup_s E(|e^{-s_n} - e^{-[s]}| \mid \mathscr{F}_s) \to 0$ almost surely, and conclude from the dominated convergence theorem that $E \sup_s |E(g_n(s) \mid \mathscr{F}_s)|^2 \to 0$.

For each s let $B_n(s) = \{\omega : [s] \leqq s_n\}$. Recall that $[s] = \inf\{t > s : (X - M)(t) = 0\}$, $s_n =$ the 1st ladder epoch of $X(i/2^n)$ after s. We write

$$E(|e^{-s_n} - e^{-[s]}| \mid \mathscr{F}_s) = E(e^{-[s]} - e^{-s_n}, B_n(s) \mid \mathscr{F}_s) + E(e^{-s_n} - e^{-[s]}, B_n'(s) \mid \mathscr{F}_s) .$$

For $\omega \in B_n(s)$, we have $[s] \leqq s_n \leqq [s]_n$, so that

$$E[e^{-[s]} - e^{-s_n}, B_n(s) \mid \mathscr{F}_s) \leqq E(e^{-[s]} - e^{-[s]_n} \mid \mathscr{F}_s) .$$

This goes to 0 uniformly in s by the argument used to prove (ii).

Fix ω. Let s_0 be any 0 of $X - M$. Then

$$\lim_{s \downarrow s_0} E(e^{-[s]} \mid \mathscr{F}_s) = \lim_{s \downarrow s_0} E(\textstyle\int_s^\infty e^{-t}\, dL(t) \mid \mathscr{F}_s) = e^{-s_0}$$

for this ω, since $L(t)$ is continuous and the path is right continuous. Given $\varepsilon > 0$, we can find $\varepsilon_0 > 0$ such that

$$E(e^{-s_n} - e^{-[s]}, B_n'(s) \mid \mathscr{F}_s) < \varepsilon \qquad \text{for all} \quad s_0 < s < s_0 + \varepsilon_0 ,$$

and for all n, since $s_0 < s_n < [s]$ and $E(e^{-[s]} \mid \mathscr{F}_s)$ is near e^{-s_0}. We note that ε_0 depends on ω and s_0. Let

$$C(s) = \{\omega : s_0 < s < s_0 + \varepsilon_0 \ \text{ where } \ s_0 \ \text{ is any } \ 0 \ \text{ of } \ X - M\} .$$

We have $E\{|e^{-s_n} - e^{-[s]}|, B_n(s) \cup C(s) \mid \mathscr{F}_s\} < \varepsilon$ for large n, uniformly in s.

The complement of $B_n(s) \cup C(s)$, denoted by $(B_n(s) \cup C(s))'$, may contain some ω for which $0 < [s] - s_n < \varepsilon$. Call this set $D_n(s)$. Since $|e^{-s_n} - e^{-[s]}| < \varepsilon$ on $D_n(s)$, we have $E(|e^{-s_n} - e^{-[s]}|, D_n(s) \mid \mathscr{F}_s) < \varepsilon$. Choose t_0 so that $e^{-t_0} < \varepsilon$. The proof will be finished once we see that there is a null set N not depending on s such that $\bigcup_{s<t_0} (B_n(s) \cup C(s) \cup D_n(s) \cup N)' \to \phi$ as $n \to \infty$. In a moment we will see what N should be.

Suppose that $\omega \in \bigcup_{s \leqq t_0} (B_n(s) \cup C(s) \cup D_n(s) \cup N)'$ for infinitely many n. Then for each of these n's there is an $s < t_0$ such that $s_0 + \varepsilon_0 < s \leqq s_n < [s] - \varepsilon$ where $s_0 = \inf\{t > 0 : M(t) = M(s)\}$, a 0 of $X - M$. There are finitely many intervals $(s_0, [s])$, $s < t_0$, of length greater than ε, so there are infinitely many of the pairs s, s_n in one particular such interval. There exists a $\delta > 0$ such that $(X - M)(s_n) < -\delta$ for all n. Otherwise a subsequence $\bar{s}_n$ of the s_n would converge to an $\bar{s}$ where $(X - M)(\bar{s}_n) \to 0$ but $(X - M)(\bar{s}) \neq 0$. But this happens only with probability 0, as is shown in the proof of Theorem 9.1 of Fristedt (1974). Put such ω into N. Since $X - M$ is right continuous, for all large enough n there is a point $i/2^n$ in $(s_0, s_0 + \varepsilon_0)$ such that $(X - M)(i/2^n) > -\delta$. This contradicts our assumption about ω.

PROOF OF (iv). To show that $A_n[t]$ is uniformly near $A_n(t)$ for large n in the sense of convergence in probability, we write

$$A_n(t) = E(A_n(t) \mid \mathscr{F}_t) = E(A_n(\infty) \mid \mathscr{F}_t) - E(\textstyle\int_t^\infty e^{-s}\, dL_n(s) \mid \mathscr{F}_t)$$

and

$$A_n[t] = E(A_n[t] \mid \mathscr{F}_{[t]}) = e_n(t) - E(\textstyle\int_{[t]}^\infty e^{-s}\, dL_n(s) \mid \mathscr{F}_{[t]}) .$$

Then

$$A_n[t] - A_n(t) = e_n(t) - E(A_n(\infty) \mid \mathscr{F}_t) - E(e^{-[t]_n} \mid \mathscr{F}_{[t]}) + E(e^{-t_n} \mid \mathscr{F}_t) ,$$

by an argument familiar from the proof of (iii). The processes $E(A_n(\infty) \mid \mathscr{F}_t)$ and $E(A(\infty) \mid \mathscr{F}_t)$ are martingales with respect to the σ-fields $\mathscr{F}_t$. Doob's inequality gives

$$P(\sup_t |E(A_n(\infty) \mid \mathscr{F}_t) - E(A(\infty) \mid \mathscr{F}_t)| > \delta) \leqq \frac{1}{\delta^2} E(A_n(\infty) - A(\infty))^2$$

which is shown to approach 0 as $n \to \infty$ in the proof of (iii). We know also from (ii) and (iii) that $E(e^{-[t]_n} \mid \mathscr{F}_{[t]}) \to e^{-[t]}$ and $E(e^{-t_n} \mid \mathscr{F}_t) \to E(e^{-[t]} \mid \mathscr{F}_t)$ uniformly. Combining these gives us

$$\lim_{n\to\infty} A_n[t] - A_n(t) = e(t) - E(A(\infty) \mid \mathscr{F}_t) + E(e^{-[t]} \mid \mathscr{F}_t) - e^{-[t]} .$$

A closer look at the first two terms on the right gives

$$\begin{aligned} e(t) &= A[t] + E(\textstyle\int_{[t]}^\infty e^{-s}\, dL(s) \mid \mathscr{F}_{[t]}) \\ &= A[t] + e^{-[t]} E \textstyle\int_0^\infty e^{-s}\, dL(s) , \\ E(A(\infty) \mid \mathscr{F}_t) &= A(t) + E(e^{-[t]} \mid \mathscr{F}_t) E \textstyle\int_0^\infty e^{-s}\, dL(s) . \end{aligned}$$

Since $A[t] = A(t)$ and $E \int_0^\infty e^{-s}\, dL(s) = 1$ we conclude that

$$\lim_{n\to\infty} A_n[t] - A_n(t) = 0 ,$$

in the sense of uniform convergence in probability.

COROLLARY. *Let $T(\tau) = \inf\{t > 0 : L(t) \geqq \tau\}$, $T_n(\tau) = \inf\{t > 0 : L_n(t) \geqq \tau\}$. There exists a subsequence n_k such that $T_{n_k}(\tau) \to T(\tau)$ almost surely.*

PROOF. Since $P(\sup_{0<t<t_0} |L(t) - L_n(t)| > \delta) \to 0$, there exists a subsequence n_k such that $\sup_{0<t<t_0} |L(t) - L_{n_k}(t)| \to 0$ almost surely. By taking $t_0 \to \infty$ through a countable sequence, constructing successive subsequences of n_k, and using a diagonalization argument we find a subsequence which we again call n_k such that $\sup |L(t) - L_{n_k}(t)| \to 0$ almost surely.

Fix τ. For any t_1 and almost any sample path such that $L(t_1) > \tau$, $L_{n_k}(t_1) \to L(t_1)$ so that for large n_k, $L_{n_k}(t_1) > \tau$ and $T_{n_k}(\tau) \leqq t_1$. Similarly, if $L(t) < \tau$, for large n_k, $T_{n_k}(\tau) \geqq t_1$. Thus $T_{n_k}(\tau)$ is eventually arbitrarily near the set $\{t : L(t) = \tau\}$. But since $T(s)$ is a subordinator (see e.g., Fristedt) and almost surely continuous at $s = \tau$, the set $\{t : L(t) = \tau\} = T(\tau)$.

The proof of Theorem 4 will utilize the corollary just proved and an additional lemma which tells how fast the moments of the ladder variables $Z^{(n)}$ for the discrete processes $X(i/n)$ decrease with increasing n.

LEMMA 2. *Let X_t be a symmetric process with stationary independent increments, $EX_1^4 = \gamma < \infty$ and $EX_1^2 = \sigma^2$. Let $X^{(n)}(i)$ denote the process $X(i/n)$ and $Z^{(n)}$ the corresponding ladder variable. Then $EZ^{(n)} = (\sigma^2/2n)^{\frac{1}{2}}$, and $E(Z^{(n)2}) \leqq (\gamma/2n)^{\frac{1}{2}}$.*

PROOF. For each n write $X_1 = \sum_{i=1}^n Y_i$ where $Y_i = X(i/n) - X((i-1)/n)$. Then $EY_i^2 = \sigma^2/n$, and

$$\gamma = E(\sum_{i=1}^n Y_i)^4 = E\sum_{i=1}^n Y_i^4 + \sum_{i,j=1;i\neq j}^n EY_i^3Y_j + \sum_{i,j=1;i\neq j}^n EY_i^2Y_i^2$$
$$= nEY_1^4 + n(n-1)\sigma^4/n^2 ,$$

and

$$EY_1^4 = \gamma/n - (n-1)\sigma^4/n^2 < \gamma/n .$$

From Lemma 1,

$$EZ^{(n)} = (EY_1^2/2)^{\frac{1}{2}} = (\sigma^2/2n)^{\frac{1}{2}}$$

and

$$EZ^{(n)2} \leqq (EY_1^4/2)^{\frac{1}{2}} < (\gamma/2n)^{\frac{1}{2}} .$$

PROOF OF THEOREM 4. If X_t is symmetric and not of Poisson type then $P(T > 0) = 0$ where $T = \inf\{t > 0 : X_t > 0\}$, as shown by Rubinovitch (1971). Let $X^{(n)}(i) = X(i/n)$. Theorem 3 says that on the subsequence $m = 2^n$, $L_m^+(t) \to L^+(t)$ and $L_m^-(t) \to L^-(t)$ uniformly on bounded intervals in probability. We use $+$ and $-$ here to denote the processes L_n and L arising from the maximum and minimum processes respectively. Let $t^{(n)}$ denote the first ascending ladder epoch of $X^{(n)}$ divided by n. A relation due to Sparre-Anderson (see Feller, Chapter 12) is

$$\log 1/(1 - Ee^{-t^{(n)}}) = \sum_{i=1}^\infty \frac{e^{-i/n}}{i} P(X(i/n) > 0) .$$

In our case X is symmetric, $P(X(i/n) > 0) = \frac{1}{2}$, and $1 - Ee^{-t^{(n)}} \sim n^{-\frac{1}{2}}$.

Let $T_n^\bullet(\tau) = \inf\{t > 0 : L_n^\bullet(t) = \tau\}$ and $T^\bullet(\tau) = \inf\{t > 0 : L^\bullet(t) \geqq \tau\}$, where $^\bullet$ is $+$ or $-$. According to the above corollary, there is a subsequence n_k of the integers such that almost surely $T_{n_k}^\bullet(\tau) \to T^\bullet(\tau)$ as $k \to \infty$, where $^\bullet$ is $+$ or $-$. By renumbering, we replace $\{n_k\}$ by $\{n\}$. Let n be large enough so that $(1 - Ee^{-t^{(n)}})n^{\frac{1}{2}}$ is near 1, and let $\tau n^{\frac{1}{2}}$ denote its own integral part. Then $T_n^+(\tau) \simeq \inf(t > 0 : \text{card}\,\{\text{ascending ladder epochs before } t\} \geqq \tau n^{\frac{1}{2}})$, and a similar description gives $T_n^-(\tau)$. Let $t_{i,n}^+$ denote the time between the $(i-1)$st and ith ascending ladder epochs of $X(i/n)$, similarly $t_{i,n}^-$. Then $T_n^+(\tau) \simeq \sum_{i=1}^{\tau n^{\frac{1}{2}}} t_{i,n}^+$ and $T_n^-(\tau) \simeq \sum_{i=1}^{\tau n^{\frac{1}{2}}} t_{i,n}^-$. Let $B_n = \{\omega : T_n^+(\tau) < T_n^-(\tau)\}$ and let $B = \{\omega : T^+(\tau) < T^-(\tau)\}$. Except for a set of probability zero, $\omega \in B$ implies $\omega \in B_n$ for all large enough n. Hence, by Fatou's lemma,

$$\begin{aligned} E(T^+(\tau) \wedge T^-(\tau)) &= E(T^+(\tau), B) + E(T^-(\tau), B') \\ &\leqq \liminf_{n\to\infty} (E(\textstyle\sum_{i=1}^{\tau n^{\frac{1}{2}}} t_{i,n}^+, B_n) + E(\sum_{i=1}^{\tau n^{\frac{1}{2}}} t_{i,n}^-, B_n')) \\ &= \liminf_{n\to\infty} E(\textstyle\sum_{i=1}^{\tau n^{\frac{1}{2}}} t_{i,n}^+ \wedge \sum_{i=1}^{\tau n^{\frac{1}{2}}} t_{i,n}^-) \\ &\leqq \liminf_{n\to\infty} E\textstyle\sum_{i=1}^{2\tau n^{\frac{1}{2}}} t_{i,n} , \end{aligned}$$

where $t_{i,n}$ denotes the time between the $(i-1)$st and ith ladder epochs of any type. The last inequality is obtained by reasoning that when $2\tau n^{\frac{1}{2}}$ ladder epochs have been attained, among them are at least $\tau n^{\frac{1}{2}}$ of one type or the other. We apply Theorem 1 with $r+s$ replaced by $2\tau n^{\frac{1}{2}}$, and note that the discrete process $X(i/n)$ proceeds by time units of size $1/n$ to obtain

$$\begin{aligned} E \textstyle\sum_{i=1}^{2\tau n^{\frac{1}{2}}} t_{i,n} &\leqq (2EZ^{(n)2}/(EX(1/n)^2)2\tau n^{\frac{1}{2}} + 4\tau^2 n)n^{-1} \\ &\leqq (2(\gamma/2n)^{\frac{1}{2}}/(\sigma^2/n))2n^{\frac{1}{2}-1}\tau + 4\tau^2 \\ &= \frac{\gamma^{\frac{1}{2}}}{\sigma^2} 2^{\frac{3}{2}}\tau + 4\tau^2 . \end{aligned}$$

Lemma 3 was used to evaluate $EZ^{(n)2}$.

Proof of Theorem 5. For each n, $X^{(n)}$ is transient and (see Feller, Chapter 18, (4.9) and (3.2))

$$Et_i^{(n)} = \exp \textstyle\sum_{i=1}^{\infty} P(X(i/n) \leqq 0)/i < \infty , \tag{12}$$

and

$$\log (1 - Ee^{-t_1^{(n)}})^{-1} = \textstyle\sum_{i=1}^{\infty} P(X(i/n) > 0)e^{-i/n}/i . \tag{13}$$

Let $\{n\}$ denote a subsequence of the integers such that $T_n^+(\tau) \to T^+(\tau)$ almost surely, given by the corollary. As in the proof of Theorem 4 we have $ET^+(\tau) = \lim_{n\to\infty} \sum_{i=1}^{\tau a_n} Et_{i,n}^+$, if a finite limit exists, where $a_n = (1 - Ee^{-t_{1,n}^+})^{-1}$ and τa_n denotes its own integral part. Substitution of (12) and (13) gives

$$\begin{aligned} \textstyle\sum_{i=1}^{\tau a_n} Et_{i,n}^+ &\simeq \tau Et_{i,n}^+(1 - Ee^{-t_{1,n}^+})^{-1} \\ &= \frac{\tau}{n} \exp \textstyle\sum_{i=1}^{\infty} (P(X(i/n) < 0 + P(X(i/n) \geqq 0)e^{-i/n})/i \\ &\simeq \frac{\tau}{n} \exp \textstyle\int_{1/n}^{\varepsilon} (P(X_t < 0) + P(X_t \geqq 0)e^{-t})/t \, dt \end{aligned}$$

for any $\varepsilon > 0$. The numerator of the integrand is continuous and $\to 1$ as $t \to 0$. Consequently $ET^+(\tau) = \lim_{n\to\infty} (\tau/n) \exp(\log n) = \tau$.

4. Discussion. Spitzer's equation (4) and the method of Lemma 1 give a family of relations between the moments of the ladder variables and those of the increments of a random walk. For instance in the symmetric case we could conclude in Lemma 1 that

$$8EZEZ^3 - 6(EZ^2)^2 = \gamma .$$

All of these relations involve products of moments in such a way that in the nonsymmetric case we cannot obtain from them inequalities similar to (3) for higher moments of Z and $\tilde{Z}$. A possible method of evaluating EZ^2, for instance, in the nonsymmetric case would be analysis of $(\partial^2/\partial\zeta^2)\chi(s, \zeta)$ at $\zeta = 0$ as $s \to 1-$, where $\chi(s, \zeta) = E(s^Z e^{i\zeta S_Z})$. Such an evaluation might then be used to obtain a result like Theorem 4 for the nonsymmetric case.

The hypothesis that F is continuous enabled us to avoid the notions of strict and weak ladder variables (Feller) but is probably not necessary. Its removal from Lemma 1 would improve Theorem 1. The setting of Theorem 4 involves continuous F in any case.

The finiteness of the fourth moment of the process was needed not only to have EZ^2 finite but also to obtain the bound $EZ^2 \leqq (\gamma/2)^{\frac{1}{2}}$, an essential fact in our treatment of the continuous-time case. A different approach to the problem, not using ladder variables, might yield (1) under a weaker moment condition.

The proof of Theorem 3 is similar in outline to the proof of existence of a continuous additive functional whose potential is a given bounded excessive function, provided by Blumenthal and Getoor (1968). The points of similarity are the use of A_n and A in place of L_n and L and the manner of defining and using martingales.

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