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ON THE CONCEPT OF CONTIGUITY[1]

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The interrelationships among conditions for convergence in law of sequences of likelihood ratios and the concept of contiguity are explored. Related results of Le Cam (1960), Hájek and Šidák (1967) and Roussas (1972) are extended, modified and clarified. In particular, it is shown that if likelihood ratios converge in law under the numerator hypothesis, then they converge under the denominator hypothesis and the hypotheses are contiguous (numerator to denominator).

1. Results and discussion. Consider two sequences of absolutely continuous probability measures $\{P_n\}$ and $\{Q_n\}$ on measure spaces $(\Omega_n, \mathscr{B}_n, \mu_n)$ and denote $p_n = dP_n/d\mu_n$ and $q_n = dQ_n/d\mu_n$. Define the *likelihood ratio* as $L_n = q_n/p_n$ if $p_n < 0$ and $= 1$ or n if $q_n = p_n = 0$ or $q_n > p_n = 0$, respectively; the value n is more convenient than $+\infty$. The range space of L_n is then in $R^+ = [0, \infty)$. Let P_n' and Q_n', respectively, denote the induced probability measures for L_n on the Borel sets of R^+.

Following Hájek and Šidák (1967), we say that $\{Q_n\}$ *is contiguous to* $\{P_n\}$ if, for every sequence B_n $(\in \mathscr{B}_n)$ for which $P_n(B_n) \to 0$ is follows that $Q_n(B_n) \to 0$; this is a kind of asymptotic absolute continuity. This definition is not symmetric in P and Q; Le Cam (1960) introduced the concept, and he and Roussas (1972) adopted a symmetric definition—what might be termed *mutual contiguity*. Various implications regarding convergence in law of $\{L_n\}$, and contiguity, are presented in Chapter VI of Hájek and Šidák (especially *Le Cam's first and third lemmas*) and in Chapter 1 of Roussas. We explore these further.

Roussas implicitly avoids dealing with the possibility of 0 limiting values for L_n by always using $\log L_n$; in effect, he treats both extremes (0 and ∞) of the range of limiting forms of L_n similarly. Additional insight as well as generality may be obtained by the approach adopted here: we identify the possible difficulties at the origin.

We are motivated by statistical applications such as Hájek and Šidák (note Hall and Loynes (1975)). In such applications, we wish to consider settings in which $\{L_n\}$ converges in law under both $\{P_n\}$ and $\{Q_n\}$. It is apparent in Hájek and Šidák that contiguity is a sufficient condition to assure that convergence under $\{Q_n\}$ follows from that under $\{P_n\}$. (Indeed, the further power of the method is that the limit laws are simply and explicitly related.) Our primary new result is the converse: that convergence in law under $\{Q_n\}$ implies both

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contiguity (of $\{Q_n\}$ to $\{P_n\}$) *and* convergence in law under $\{P_n\}$ (Theorem 1). In other words, the only way L_n can converge under the "alternatives" Q_n, is for it to converge under the "hypothesis" P_n *and* for contiguity to hold; contiguity is *necessary*. Other results appear below.

Our various results are closely related to some in Roussas and in Hájek and Šidák. But the paper is self-contained. As our interest is in applications, and for reasons of clarity, our results are stated for convergence: in every case we may relax this to relative compactness, provided explicit limits that appear are taken to be limits of convergent subsequences.

We now label nine assertions to be investigated below:

A_1: $\{L_n\}$ *converges in law under* $\{P_n\}$—i.e., there is a probability measure P on the Borel sets of R^+ for which $P_n' \to P$ on P-continuity sets; let L denote a rv with distribution P.

A_2: $\{L_n\}$ *converges in law under* $\{Q_n\}$; denote the limiting probability measure by Q.

A_3: A_1 *and* A_2 *and* $dQ = L\,dP$.

A_4: $\{Q_n\}$ *is contiguous to* $\{P_n\}$.

A_5: $\{P_n\}$ *is contiguous to* $\{Q_n\}$.

A_6: A_1 *and* $E(L) = 1$.

A_7: A_1 *and* $P(\{0\}) = 0$.

A_8: $\{L_n\}$ *is uniformly integrable* (dP_n).

A_9: $Q_n\ (P_n = 0) \to 0$.

Whenever A_1 holds, we can define a measure (not necessarily a probability) Q^* by $dQ^* = L\,dP$—i.e., $Q^*(dx) = xP(dx)$. Under modest assumptions, Q^* will be shown to be the Q of A_2. Several properties of Q^* are immediate: (i) $Q^* \ll P$, (ii) $Q^*(\{0\}) = 0$, and (iii) $P \ll Q^*$ iff $P(\{0\}) = 0$. Thus, A_7 implies P and Q^* are *mutually absolutely continuous*; this is to be expected intuitively when A_1 and A_5 hold. Indeed, according to Proposition 2 below, under A_1 mutual contiguity implies A_7 and hence *mutual absolute continuity* of P and Q.

PROPOSITION 1. $A_4 \Leftrightarrow (A_8$ and $A_9)$.

The proof is deferred to Section 3.

THEOREM 1. $(A_1$ *and* $A_4) \Leftrightarrow A_2 \Leftrightarrow A_3 \Leftrightarrow A_6$.

Thus, contiguity and P_n-convergence (of L_n) are together equivalent to Q_n-convergence. That $A_6 \Rightarrow (A_1$ and $A_4)$ is labelled *Le Cam's first lemma* in Hájek and Šidák; that $(A_1$ and $A_4) \Rightarrow A_2$ is a version of *Le Cam's third lemma*. Some partially overlapping results also appear in Roussas (especially Proposition 3.1). Theorem 1 is proved in Section 3 below.

That A_2 is more powerful than A_1 is explained as follows: under each, escape of probability mass to infinity is ruled out by the convergence assumptions; under A_2, no concentration of probability at the origin is possible since q_n is in the numerator of L_n, while under A_1 such a possibility remains; A_4 is what is needed to prevent it.

An interesting and important special case is that in which $\log L_n$ is asymptotically normal. One half of the following result is well known (cf. Roussas, Corollaries 7.1 and 7.2).

COROLLARY. (i) *Suppose* $\{\log L_n\}$ *converges in law under* $\{P_n\}$ *to a normal distribution* $N(\mu, \sigma^2)$. *Then* $\mu \leqq -\frac{1}{2}\sigma^2 \leqq 0$. *Moreover,* $\{\log L_n\}$ *converges in law (to* Z, *say) under* $\{Q_n\}$ iff $\mu = -\frac{1}{2}\sigma^2$ *and, if so, then* Z *is* $N(\frac{1}{2}\sigma^2, \sigma^2)$ *and* A_1—A_9 *hold.*

(ii) *Suppose* $\{\log L_n\}$ *converges in law under* $\{Q_n\}$ *to a normal distribution* $N(\mu, \sigma^2)$. *Then* $\mu \geqq \frac{1}{2}\sigma^2 \geqq 0$, *and* $\{L_n\}$ *converges in law under* $\{P_n\}$ *to a mixture of* e^Z *(with probability* $p = e^{-\mu+\frac{1}{2}\sigma^2}$*) and* 0 *(with probability* $1 - p$*) where* Z *is* $N(\mu - \sigma^2, \sigma^2)$.

We close this section with several subsidiary results, some of which are used in the proofs and which help to show the exact relationship between results stated in terms of L_n and those in terms of $\log L_n$.

PROPOSITION 2. (A_1 *and* A_5) $\Leftrightarrow A_7$.

PROOF. (i) A_1 and $A_5 \Rightarrow A_7$: We have $Q_n(L_n = 0) \leqq Q_n(q_n = 0) = 0$, so that by A_5 $P_n(L_n = 0) \to 0$ which with A_1 implies A_7.

(ii) $A_7 \Rightarrow A_1$ and A_5: According to A_7 we have A_1 and $P(L = 0) = 0$. It follows easily (cf. the proof of Theorem 2 later) that M_n converges under P_n to L^{-1}, where M_n is the likelihood ratio of P_n to Q_n. But this is A_2 with $\{P_n\}$ and $\{Q_n\}$ interchanged, so that the interchanged version of A_4 (which is A_5) follows from Theorem 1.

PROPOSISION 3. $\{L_n\}$ *is relatively compact (tight) under* $\{P_n\}$.

This is so since $P_n(L_n > A) \leqq \int_{Ap_n<q_n} p_n \, d\mu_n \leqq A^{-1} \int_{Ap_n<q_n} q_n \, d\mu_n \leqq A^{-1}$.

PROPOSITION 4. $\{Q\}$ *is contiguous to* $\{P_n\}$ *if and only if* $\{\log L_n\}$ *(or equivalently* $\{L_n\}$*) is relatively compact under* $\{Q_n\}$.

This follows easily from the earlier results by dealing with subsequences. Plainly P_n and Q_n can be interchanged in the version involving $\log L_n$ (but not that involving L_n). These results show that the relationship between S_1 and S_2 of Proposition 3.1 of Roussas can be split into two natural parts.

2. Joint convergence with a statistic. The real power of the contiguity theory in applications (see Hájek and Šidák) it that convergence properties of other statistics can be determined under Q_n by proving their joint convergence together with $\log L_n$ under P_n (*Le Cam's third lemma* in Hájek and Šidák and Theorem 7.1 in Roussas). In analogy with part of Theorem 1 above, we now present a stronger version of this result in Theorem 2.

Let S_n denote a $\mathscr{B}_n$ measurable mapping from Ω_n to a metric space M. Let μ denote the product space $M \times R^+$ with its Borel sets—the range space of (S_n, L_n). Let P_n'' and Q_n'' be the induced probability measures for (S_n, L_n) on μ. Let B_1,

B_2 and B_3 correspond to A_1, A_2 and A_3 with (S_n, L_n) replacing L_n, and $dQ = L\,dP$ now means $Q(du, dv) = vP(du, dv)$.

THEOREM 2. (B_1 *and* A_4) $\Rightarrow$ B_3 *and* (B_2 *and* A_5) $\Rightarrow$ (B_3 *and* A_4).

COROLLARY. *If* $\{(S_n, \log L_n)\}$ *converges in law under* $\{P_n\}$ *and* A_4 *is assumed or if* $\{(S_n, \log L_n)\}$ *converges in law under* $\{Q_n\}$ *and* A_5 *is assumed, then it converges in law under both, and* B_1—B_3 *and* A_1—A_9 *hold.*

The proof of the theorem appears in Section 2 below; the corollary also requires the use of Proposition 2. (If $L_n = 0$ we may either take $\log L_n = -\infty$ and adopt the obvious conventions or arbitrarily redefine $\log L_n$, such as for example $\log L_n = -n$.)

Applications of the corollary appear in Hájek and Šidák and in Hall and Loynes (1975).

Counterexamples showing that no further implications exist (for example that B_2 does not imply B_1) are easily constructed by letting P_n and Q_n be (for every n) uniform distributions on two intervals, one a subset of the other (with $S_n = L_n^{-1}$ in the particular case mentioned).

3. Proofs of Proposition 1 and Theorems 1 and 2. We first state a necessary and sufficient condition for *uniform integrability* (u.i.) for use in several proofs.

LEMMA. *Suppose* X_n *is defined on* $(\Omega_n, \mathscr{B}_n, P_n)$ *for each* n. *Then* $\{X_n\}$ *is* u.i. iff (a) *and* (b) *hold*:

(a) $\sup_n E|X_n| < \infty$;

(b) *if* $B_n \in \mathscr{B}_n$ *and* $P_n(B_n) \to 0$, *then* $\int_{B_n} |X_n|\, dP_n \to 0$.

This is similar in content to problem 5, page 34, of Billingsley: the forward part is straightforward, while the converse is easily proved by contradiction. We now apply this to prove Proposition 1. Observe that

$$Q_n(B_n) = \int_{B_n} dQ_n = \int_{B_n \cap (p_n = 0)} dQ_n + \int_{B_n} L_n\, dP_n\,,$$

so that $\int_{B_n} L_n\, dP_n \leqq Q_n(B_n) \leqq Q_n(p_n = 0) + \int_{B_n} L_n\, dP_n$. As we also have $\int L_n\, dP_n = \int_{p_n > 0} dQ_n \leqq 1$ the proof is an easy consequence of the lemma.

We also need, in later proofs, that either A_2 or A_6 implies A_8.

(i) $A_2 \Rightarrow A_8$. We have $\int_{L_n > \alpha} L_n\, dP_n = \int_{L_n > \alpha, p_n > 0} L_n p_n\, d\mu_n \leqq \int_{L_n > \alpha} q_n\, d\mu_n = Q_n(L_n > \alpha)$, which under A_2 converges to 0 as $\alpha \to \infty$, uniformly in n.

(ii) $A_6 \Rightarrow A_8$. $E_{P_n}(L_n) = \int_{p_n > 0} L_n p_n\, d\mu_n = \int_{p_n > 0} q_n\, d\mu_n \leqq 1$. But since $\liminf EL_n \geqq EL$ (Billingsley, Theorem 5.3, page 32) and $EL = 1$, we have $EL_n \to EL$; A_8 follows (Billingsley, Theorem 5.4).

We now prove Theorem 1 by showing in turn: $A_3 \Rightarrow A_6 \Rightarrow$ (A_1 and A_4) and $A_2 \Rightarrow$ (A_1 and A_4); that (A_1 and A_4) $\Rightarrow A_3$ is a classical result, whose proof may be found in, e.g., Roussas.

(iii) $A_3 \Rightarrow A_6$. Obvious, since Q is a probability measure.

(iv) $A_6 \Rightarrow A_1$ and A_4. We have $Q_n(p_n = 0) = 1 - Q_n(p_n > 0) = 1 - E_{P_n}(L_n) \to 0$ as seen in the proof of (ii) above; use of Proposition 1 is now enough.

(v) $A_2 \Rightarrow A_4$. We have $\{p_n = 0\} \subset \{L_n = n\} \cup \{q_n = 0\}$, but since the Q_n measure of the last set is 0, it follows from A_2 that $Q_n(p_n = 0) \to 0$. Now we may recall (i) and apply Proposition 1.

(vi) $A_2 \Rightarrow A_1$. According to Proposition 3, and Theorem 2.3 (page 16) of Billingsley, it is sufficient to show that every convergent subsequence of $\{L_n\}$ has the same limit. Suppose for some subsequence $P_n' \to P$, say: then by what we have already proved $dQ = L\, dP$. Hence, on $(0, \infty)$ $dP = L^{-1}\, dQ$ (i.e., $P(dx) = x^{-1}Q(dx)$), while $P(\{0\}) = 1 - P((0, \infty))$. Thus, the limit is indeed the same for every subsequence.

For Theorem 2, that (B_1 and A_4) $\Rightarrow B_3$ is again essentially a classical result. For the rest of Theorem 2, first note that $B_2 \Rightarrow A_2 \Rightarrow A_4$ by Theorem 1. The remainder of the proof is slightly tedious, because we have to take account of values 0 and ∞ for likelihoods, but elementary. Let $L_n^* = L_n$ if $L_n > 0$, and $= n^{-1}$ if $L_n = 0$; let M_n be the likelihood ratio of P_n to Q_n; and let M_n^* be defined analogously to L_n^*. Then $M_n^* = (L_n^*)^{-1}$, and moreover $Q_n(L_n \neq L_n^*) \leqq Q_n(q_n = 0) = 0$, $Q_n(M_n \neq M_n^*) \leqq Q_n(L_n = n) \to 0$. Then (S_n, L_n, L_n) converges under $\{Q_n\}$ and hence in turn (S_n, L_n, L_n^*), (S_n, L_n, M_n^*), and (S_n, L_n, M_n) each converge under $\{Q_n\}$. But now the result proved at the beginning of this paragraph, with Q_n and P_n everywhere interchanged, and S_n replaced by (S_n, L_n) shows that (S_n, L_n, M_n) converges under $\{P_n\}$, and a fortiori (S_n, L_n) converges.

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