The Annals of Probability
1977, Vol. 5, No. 5, 724–739

THE LAW OF THE ITERATED LOGARITHM ON ARBITRARY SEQUENCES FOR STATIONARY GAUSSIAN PROCESSES AND BROWNIAN MOTION

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Let $X(t)$ be a stationary Gaussian process with continuous sample paths, mean zero, and a covariance function satisfying (a) $r(t) \sim 1 - C|t|^\alpha$ as $t \to 0$, $0 < \alpha \leqq 2$ and $C > 0$; and (b) $r(t) \log t = o(1)$ as $t \to \infty$. Let $\{t_n\}$ be any sequence of times with $t_n \uparrow \infty$. Then, for any nondecreasing function f, one obtains $P\{X(t_n) > f(t_n) \text{ i.o.}\} = 0$ or 1 according to a certain integral test. This result both combines and generalizes the law of iterated logarithm results for discrete and continuous time processes. In particular, it is shown that any sequence t_n satisfying $\limsup_{n\to\infty} (t_n - t_{n-1})(\log n)^{1/\alpha} < \infty$ captures continuous time in the sense that the upper and lower class functions for the law of the iterated logarithm of $X(t_n)$ are exactly the same as those for the continuous time $X(t)$.

Analogous results are obtained for Brownian motion.

1. Introduction. Let $\{B(t), t \geqq 0\}$ denote standard Brownian motion. Khintchine's law of iterated logarithm for Brownian motion states

$$\limsup_{t\to\infty} B(t)(2t \log \log t)^{-\frac{1}{2}} = 1 \quad \text{a.s.} \tag{1.1}$$

Lévy [3, page 88, 226 ff] discusses this result in terms of upper class and lower class functions and states that the best results are due to Petrowsky and Kolmorogov. The theorem communicated to Lévy by Kolmorogov is equivalent to the following.

THEOREM A. *For any monotone increasing positive function* $f(t)$,

$$P\{B(t) > t^{\frac{1}{2}}f(t) \text{ i.o. } \textit{for some sequence } t_n \to \infty\} = 0 \quad \textit{or} \quad 1 \tag{1.2}$$

according as the integral

$$K(f) = \int_a^\infty \frac{f(t)}{t} \exp(-f^2(t)/2)\, dt < \infty \quad \textit{or} \quad = \infty\,. \tag{1.3}$$

Now f belongs to the upper class of $B(t)$ if there exists a $t_0(\omega)$ for each sample path ω such that $B(t) \leqq t^{\frac{1}{2}}f(t)$ for all $t \geqq t_0$. According to Theorem A, this occurs if and only if the test integral converges. Since a proof was not given by Lévy, Sirao and Nisida [12] give a proof of this theorem using the direct method of Feller [1 and 2].

Received January 17, 1975; revised June 7, 1976.

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AMS 1970 *subject classifications.* Primary 60F10, 60F20, 60G10, 60G15, 60G17, 60J65.

Key words and phrases. Stationary process, Gaussian process, law of iterated logarithm, Brownian motion, sure sequences.

Feller's result for sums S_n of independent identically distributed random variables [1 and 2] stated for the Gaussian case where $S_n = B(n)$ becomes $P[B(n) > n^{\frac{1}{2}}f(n)$ i.o.$] = 0$ or 1 as

$$\sum_a^\infty \frac{f(n)}{n} \exp(-f^2(n)/2) < \infty \quad \text{or} \quad = \infty .$$

Since this series is finite if and only if $K(f) < \infty$, the upper and lower class functions for the discrete time $B(n)$ are identical to those for the continuous time $B(t)$.

Similar results have been obtained for stationary Gaussian processes. A recent result is given in Pathak and Qualls [6].

THEOREM B. *Let $\{X(t), t \geqq 0\}$ be a stationary Gaussian process with continuous sample paths and $EX(t) \equiv 0$. Suppose that the covariance function $r(t)$ satisfies*

(a) $r(t) = 1 - C|t|^\alpha + o(|t|^\alpha)$ *as $t \to 0$, for some α with $0 < \alpha \leqq 2$ and some $C > 0$; and*

(b) $r(t) \log t = O(1)$ *as $t \to \infty$.*

Then, for any nondecreasing positive function $f(t)$,

$$(1.4) \qquad P\{X(t) > f(t) \text{ i.o. for some sequence } t_n \to \infty\} = 0 \quad or \quad 1$$

according as the integral

$$(1.5) \qquad J(f) = \int_a^\infty (f(t))^{2/\alpha}\psi(f(t))\, dt < \infty \quad or \quad = \infty ,$$

where $\psi(x) = (2\pi)^{-\frac{1}{2}}x^{-1}\exp(-x^2/2)$.

Condition (b) is the conclusion of a sequence of improvements beginning with Watanabe [13]; see [9]. Condition (a) has been generalized in Qualls and Watanabe [10] where C is replaced by a slowly varying function $H(t)$. However, for the purpose of comparison with the result of this paper we keep the simpler condition (a) as stated. There are analogous results for discrete parameter stationary Gaussian processes. Improving results of Pickands [7] and Qualls, Simmons, and Watanabe [11], we cite the theorem from [6].

THEOREM C. *If $X(t)$ of Theorem* B *satisfies condition* (b), *then*

$$(1.6) \qquad P\{X(n) > f(n) \text{ i.o.}\} = 0 \quad or \quad 1 \qquad according\ as\ the\ integral$$

$$(1.7) \qquad I(f) = \int_a^\infty \psi(f(t))\, dt < \infty \quad or \quad = \infty .$$

Of course, the local condition (a) is not needed.

There were two observations that motivated the present paper. The proof of Theorem B mainly consists of approximating the continuous parameter $X(t)$ by $X(t)$ restricted to a particular sequence of time $\{t_n\}$ chosen to be more and more closely spaced as $t \to \infty$. The "law of iterated logarithm" was then proved for $\{X(t_n)\}$. (Note that we continue to say "iterated" even though Theorem B implies

$$\limsup_{t\to\infty} X(t)(2 \log t)^{-\frac{1}{2}} = 1 \quad \text{a.s.})$$

So, the upper and lower class functions for this discrete time $X(t_n)$ are identical to those of the continuous time $X(t)$. On the other hand, Theorem C is on the evenly spaced sequence $\{n\}$ and has a different integral test so that it is possible for a given f to belong to the upper class for $X(n)$ but belong to the lower class for the continuous time $X(t)$. The question becomes what is the result for arbitrary sequences of times and which sequences recapture the continuous time result of Theorem B? The main result of this paper, Theorem D of Section 2, answers this question; and Theorems B and C will be corollaries.

The second observation follows. The Markovian $B(t)$ and the family of stationary Gaussian $X(t)$ perhaps come closest to each other in the Uhlenbeck–Ornstein process $X(t)$ whose covariance function is $r(t) = \exp(-|t|)$. In fact we may write $B(t) = t^{\frac{1}{2}}X(\frac{1}{2}\log t)$. Noting that $P\{B(t) > t^{\frac{1}{2}}f(t) \text{ i.o. } t \uparrow \infty\} = P\{X(s) > f(\exp(2s)) \text{ i.o. } s \uparrow \infty\}$, we obtain Theorem A as a consequence of the case $\alpha = 1$ in Theorem B. Now can the discrete version of Theorem A be obtained from Theorem C? The answer is of course not, since $S_n/n^{\frac{1}{2}} = X(\frac{1}{2}\log n)$ and the sequence of times $t_n = \frac{1}{2}\log n$ are not evenly spaced. One needs Theorem D. The law of iterated logarithm on arbitrary sequences for Brownian motion is a consequence of Theorem D and is given in Section 3. Most proofs are in Section 4.

2. Statement of theorem and discussion. Before the main theorem of this paper can be stated, it may be necessary to thin out the sequence of increasing times t_n. We let $t_n = \varphi(n)$, where φ is a change of time function such that $\varphi(t) \uparrow \infty$ as $t \uparrow \infty$. Beginning at an arbitrary j_0 where $\varphi(j_0) = b$, choose the new $\tilde{\varphi}(j_0) = \varphi(j_0)$. Now inductively choose $\tilde{\varphi}(j)$ for $j > j_0$ to be the first $\varphi(n_j)$ greater than $\tilde{\varphi}(j-1) + m(\log(j-1))^{-1/\alpha}$ where $m > 0$ is arbitrary for now but will be specified if the occasion arises. Without any loss in generality both φ and $\tilde{\varphi}$ can be taken here and in the following as strictly increasing continuous functions on $[j_0, \infty)$. In any case the new $\tilde{\Delta}(j) \equiv \tilde{\varphi}(j+1) - \tilde{\varphi}(j) > m(\log j)^{-1/\alpha}$ for $j > j_0$.

THEOREM D. *Let $\{X(t), t \geqq 0\}$ be a stationary Gaussian process with mean zero, continuous sample paths, and covariance function r satisfying*

(a) $r(t) = 1 - C \cdot |t|^\alpha + o(|t|^\alpha)$ *as* $t \to 0$, *for some* α *with* $0 < \alpha \leqq 2$ *and some* $C > 0$; *and*

(b′) $r(t)\log t = o(1)$ *as* $t \to \infty$.

Let $\varphi(t)$ be an arbitrary positive, increasing, continuous function on some interval $[a, \infty)$ such that $t_n = \varphi(n) \uparrow \infty$; and $f(t)$ be an arbitrary positive nondecreasing function on $[a, \infty)$ with $b = \varphi(a)$. Let $\Delta(n) = \varphi(n+1) - \varphi(n)$. Then

$$P[X(t_n) > f(t_n) \text{ i.o.}] = 0 \quad \text{or} \quad 1 \tag{2.1}$$

according as the integral

$$I(f \circ \tilde{\varphi}) = \int_a^\infty \psi(f \circ \tilde{\varphi}(t))\, dt < \infty \quad \text{or} \quad = \infty\,, \tag{2.2}$$

where the thinned function $\tilde{\varphi}$ *is defined above, and*

$$\psi(x) = (2\pi)^{-\frac{1}{2}}x^{-1}\exp(-x^2/2) .$$

Two particular subcases hold:

CASE 1. If $\liminf_{n\to\infty} \Delta(n)(\log n)^{1/\alpha} > 0$, then

$$I(f \circ \tilde{\varphi}) < \infty \quad \text{if and only if} \quad I(f \circ \varphi) = \int_b^\infty \psi(f(t))\, d\varphi^{-1}(t) < \infty . \tag{2.3}$$

CASE 2. If $\limsup_{n\to\infty} \Delta(n)(\log n)^{1/\alpha} < \infty$, then

$$I(f \circ \tilde{\varphi}) < \infty \quad \text{if and only if} \quad I(f \circ \hat{\varphi}) < \infty , \tag{2.4}$$

where $\hat{\varphi}(t) = t(\log t)^{-1/\alpha}$.

Since $\tilde{\varphi}$ is somewhat inaccessible, Cases 1 and 2 are appealing. When Cases 1 and 2 overlap, the integral tests are equivalent; this could be obtained from Proposition (c) below. The convergence of $I(f \circ \tilde{\varphi})$ is independent of the choice of (m, j_0) in the definition of $\tilde{\varphi}$. If $\liminf \Delta(n)(\log n)^{1/\alpha} > 0$ then no thinning is necessary and $\tilde{\varphi}$ can be chosen equal to φ for all sufficiently large t.

The following propositions answer several questions about Theorem D, and they are used later.

PROPOSITIONS.

(a) $I(f \circ g) = \int_a^\infty \psi(f \circ g(t))\, dt = \int_{g(a)}^\infty \psi(f(t))\, dg^{-1}(t)$, *for* $g = \varphi$, $\tilde{\varphi}$ *or* $\hat{\varphi}$.

(b) $I(f \circ \varphi) < \infty$ *if and only if* $\sum^\infty \psi(f \circ \varphi(k)) < \infty$.

(c) *The integral* $I(f \circ \hat{\varphi}) < \infty$, *where* $\hat{\varphi}(t) = t(\log t)^{-1/\alpha}$ *if and only if*

$$\int_a^\infty (2 \log t)^{1/\alpha}\psi(f(t))\, dt < \infty ; \qquad \textit{and}$$

(d) $I(f \circ \hat{\varphi}) < \infty$ *if and only if*

$$J(f) = \int_a^\infty (f(t))^{2/\alpha}\psi(f(t))\, dt < \infty .$$

PROOFS. (a) and (b) are easy since ψ is monotone.

(c) Now $D_t\hat{\varphi}^{-1}(t) = 1/\hat{\varphi}'(\hat{\varphi}^{-1}(t))$ and

$$\hat{\varphi}'(s) = (\log s)^{-1/\alpha}(1 - (\alpha \log s)^{-1}) = \hat{\varphi}(s)(1 - (\alpha \log s)^{-1})/s .$$

Consequently $D_t\hat{\varphi}^{-1}(t) = 1/\{t \cdot (1 - (\alpha \log \hat{\varphi}^{-1}(t))^{-1})/\hat{\varphi}^{-1}(t)\} \sim \hat{\varphi}^{-1}(t)/t$ as $t \to \infty$. Now in addition, for t sufficiently large,

$$t(\log t)^{1/\alpha} \leqq \hat{\varphi}^{-1}(t) \leqq \left(1 + 2/\alpha \frac{\log\log t}{\log t}\right)^{1/\alpha} \cdot t(\log t)^{1/\alpha} . \tag{2.5}$$

This may be checked since $\hat{\varphi}$ is monotone. So now

$$D_t\hat{\varphi}^{-1}(t) \sim (\log t)^{1/\alpha} \qquad \text{as} \quad t \to \infty . \tag{2.6}$$

The proof of (d) will be delayed. □

COROLLARY D1. *Let* $X(t)$ *satisfy the conditions of Theorem* D *except that* (a) *is replaced by* (a′) $r(0) = 1$. *Suppose the sequence* $t_n \uparrow \infty$ *satisfies* $\liminf_{k\to\infty} \Delta(k)(\log k)^{\frac{1}{2}} > 0$. *Then for any nondecreasing sequence of positive* a_n,

$$P\{X(t_n) > a_n \ \text{i.o.}\} = 0 \quad \textit{or} \quad 1 \tag{2.7}$$

according as the sum

(2.8) $$\sum_{n=1}^{\infty} \psi(a_n) < \infty \quad or \quad = \infty .$$

In particular, this result applies for any subsequence $\{n_k\}$ of the positive integers, since $\Delta(k) \geqq 1$.

PROOF. Note that if $X(t)$ satisfies condition (a) with $\alpha = 2$, then Corollary D1 is Theorem D, Case 1. However, every covariance function does satisfy

$$\lim_{t\to 0} \frac{1 - r(t)}{t^2} = \int_{-\infty}^{\infty} \lambda^2 F(d\lambda) > 0 ,$$

where F is the spectral distribution function and its second moment is possibly infinite. This implies $r(t) \leqq 1 - Ct^2$ for some $C > 0$ and all sufficiently small t. One can now verify that the proof of Theorem D, Case 1 (in the particular case $\alpha = 2$) applies here; we only require upper estimates of $r(t)$ near $t = 0$ in that proof. Now use Proposition (b) to obtain the test (2.8). ▯

REMARK 1. If $X(t)$ does satisfy condition (a) with $\alpha < 2$, then Theorem D describes the larger class of sequences $\{t_n\}$ that are admitted into Case 1.

Proposition (d) suggests Case 2 of Theorem D can be used to prove Theorem B, the continuous parameter law of the iterated logarithm. The proof of this will also be delayed to Section 4.

COROLLARY D2. *Theorem* B *is a consequence of Theorem* D.

REMARK 2. The functions $\varphi(t) = t(\log t)^{-\beta/2}$ where $0 \leqq \beta \leqq 2/\alpha$ includes the discrete parameter case at one extreme ($\beta = 0$) but captures the continuous parameter case at the other ($\beta = 2/\alpha$). The intermediate cases $0 < \beta < 2/\alpha$ fill the gap between Theorem C and Theorem B.

REMARK 3. The function $\varphi(t) = \frac{1}{2} \log t$ is clearly Case 2, which answers the question about S_n raised in the introduction.

EXAMPLES. CASE 1. For $\varphi(t) = t(\log t)^{-\beta/2}$, $\beta \leqq 2/\alpha$, and $t_k = \varphi(k)$, we have

(2.9) $$\limsup_{t_k\to\infty} (a(t_k))^{\frac{1}{2}}\{X(t_k) - (a(t_k))^{\frac{1}{2}}\}/\log_p t_k = 1 \quad \text{a.s.},$$

where

$$a(t_k) = (2 \log t_k + (1 + \beta) \log_2 t_k + \cdots + 2 \log_{p-1} t_k) ,$$

and $\log_p t$ denotes the pth iterated logarithm.

CASE 2. For all changes of time $\varphi(t)$ satisfy Case 2 of Theorem D, we have the Khintchine type result (2.9) where in the definition of $a(t_k)$ we replace β by $2/\alpha$.

OUTLINE OF PROOF OF 2.9. The changes of time $\varphi(t) = t(\log t)^{-\beta/2}$ for $\beta \leqq 2/\alpha$ satisfy Case 1 of Theorem D. For functions

$$f(t) = 2^{\frac{1}{2}}(\log t + (\tfrac{1}{2} + \beta/2) \log_2 t + \log_3 t + \cdots + (1 + \delta) \log_p t)^{\frac{1}{2}} ,$$

$I(f \circ \varphi) < \infty$ for $\delta > 0$ and $= \infty$ for $\delta \leqq 0$. This follows from the fact that

$t(\log t)^{\beta/2} \leqq \varphi^{-1}(t) \leqq (1+\varepsilon)t(\log t)^{\beta/2}$ for all t sufficiently large. This family of upper and lower class functions for $X(t_k)$ imply the Khintchine type result (2.9) as follows. For almost every sample path ω, consider the subsequence of $\{t_k\}$ (also called t_k's) on which

$$(a(t_k) + 2\log_p t_k)^{\frac{1}{2}} \leqq X(t_k) \leqq (a(t_k) + 2(1+\delta)\log_p t_k)^{\frac{1}{2}}, \qquad \delta > 0 .$$

The rest is algebra with the main step being

$$(a+b)^{\frac{1}{2}} - a^{\frac{1}{2}} = b/[(a+b)^{\frac{1}{2}} + a^{\frac{1}{2}}] \sim b/2a^{\frac{1}{2}} \qquad \text{when} \quad b/a \to 0 .$$

Next, for $\beta = 2/\alpha$, $\varphi(t) = t(\log t)^{-\beta/2}$ also satisfies Case 2 of Theorem D. Since in Case 2 the integral test is independent of φ and so the upper and lower class functions for $X(t_k)$ are independent of φ, the Khintchine result is (2.9) with $\beta = 2/\alpha$ for all φ satisfying Case 2. □

REMARK 4. Letting $Z_n = \max_{1\leqq k\leqq n} X(t_k)$, one can see that Khintchine type results like (2.9) apply equally to Z_n. We note here that under the conditions of Theorem C, Mittal [4] also obtains the lim inf behavior of $Z_n = \max_{1\leqq k\leqq n} X(k)$. Mittal and Ylvisaker [5] also study the a.s. asymptotic behavior of Z_n when condition (b) of Theorem C is violated.

3. Theorem for Brownian motion on arbitrary sequences. Write $B(t)/t^{\frac{1}{2}} = X(\frac{1}{2}\log t)$, where $X(s)$ is the Uhlenbech–Ornstein process with covariance $r(t) = \exp(-|t|)$. For an arbitrary sequence t_k, we wish to apply Theorem D with time change function $\mathscr{E}(k) = \frac{1}{2}\log t_k$ and $\Delta(k) = \frac{1}{2}\log(t_{k+1}/t_k)$.

REMARK 5. From Theorem D one can see that Cases 1 and 2 for Brownian motion can be decided by the asymptotic behavior of $\log(t_{k+1}/t_k)\log k$ as $k \to \infty$. However, by using the inequality $x/2 \leqq \log(1+x) \leqq x$, for $0 \leqq x \leqq 1$, with $x = \Delta t_k/t_k$ where $\Delta t_k = t_{k+1} - t_k$, we see that Case 1 and Case 2 can be decided by the asymptotic behavior of $\Delta t_k/t_k \log k$ instead. The possible exception where $\Delta t_k/t_k > 1$ on an infinite subsequence can be ignored since both limits equal ∞ on such a sequence.

In order to apply Theorem D, we may require the sequence t_k to be thinned. Writing $t_k = \varphi(k)$, begin with $\tilde{\varphi}(j_0) = \varphi(j_0)$. Thereafter define $\tilde{\varphi}(j+1)$ to be the first $\varphi(n_{j+1})$ greater than $\tilde{\varphi}(j)\exp(m/\log j)$ for some $m > 0$. We have the following theorem.

THEOREM E. *Let $\{B(t), t \geqq 0\}$ be standard Brownian motion. Let $\varphi(t)$ be an arbitrary positive, increasing, continuous function on some interval $[a, \infty)$ such that $t_n = \varphi(n) \uparrow \infty$, and $f(t)$ be an arbitrary positive nondecreasing function on $[b, \infty)$ with $b = \varphi(a)$. Let $\Delta(n) = \varphi(n+1) - \varphi(n)$. Then*

$$P\{B(t_k) > t_k^{\frac{1}{2}} f(t_k) \ \text{i.o.}\} = 0 \quad or \quad 1 \tag{3.1}$$

according as the integral

$$I(f\circ\tilde{\varphi}) = \textstyle\int_a^\infty \psi(f\circ\tilde{\varphi}(t))\,dt < \infty \quad or \quad = \infty , \tag{3.2}$$

where the thinned function $\tilde{\varphi}(t)$ was defined above.

Two particular subcases hold:

CASE 1. If $\liminf_{n\to\infty} (\Delta(n)/t_n) \log n > 0$, then

$$I(f\circ\tilde{\varphi}) < \infty \quad \text{if and only if} \quad I(f\circ\varphi) < \infty . \tag{3.3}$$

CASE 2. If $\limsup_{n\to\infty} (\Delta(n)/t_n) \log n < \infty$, then

$$I(f\circ\tilde{\varphi}) < \infty \quad \text{if and only if} \quad K(f) = \int_a^\infty \frac{f^2(t)}{t}\,\psi(f(t))\,dt < \infty . \tag{3.4}$$

PROOF OF THEOREM E. We have already indicated in the introduction how Theorem D applies; it only remains to derive the integral tests for Cases 1 and 2. The time change is $\mathscr{E}(t) = \frac{1}{2}\log\varphi(t)$ and the other function to be used in Theorem D is $f\circ\varphi\circ\mathscr{E}^{-1}$. For the Case 1 test, the composition function is $f\circ\varphi\circ\mathscr{E}^{-1}\circ\mathscr{E} = f\circ\varphi$ and the test is unchanged by the time change $\mathscr{E}(t)$. For Case 2, the integral $J(f) = \int^\infty f^2(t)\psi(f(t))\,dt$ in Theorem D modified by Proposition (d) becomes $J(f\circ\varphi\circ\mathscr{E}^{-1})$ for Theorem E. Now $\mathscr{E}^{-1}(t) = \varphi^{-1}\circ e^{2t}$ and

$$J(f\circ e^{2t}) = \int^\infty f^2(e^{2t})\psi(f(e^{2t}))\,dt = \int^\infty f^2(t)\psi(f(t))\,dt/2t = \tfrac{1}{2}K(f) . \qquad \square$$

The following propositions give equivalent integral tests.

PROPOSITIONS.

(e) $K(f) < \infty$ *if and only if* $\int^\infty (\log\log t)\psi(f(t))\,dt/t < \infty$.
(f) $K(f) < \infty$ *if and only if* $I(f\circ\hat{\varphi}) < \infty$, *where* $\hat{\varphi}(t) = \exp\{2t(\log t)^{-1}\}$.

PROOF. Rather than prove these directly, we derive them from Propositions (c) and (d) by the change of time argument. $\square$

REMARK 6. Notice that the sequence of times t_k must be quite sparse, say $\varphi(t) = \exp\{2t\}$, before it fails to capture the continuous case result for Brownian motion. Since $\varphi(t) = \exp\{\log t\} = t$ is Case 2, this again explains why the continuous parameter $B(t)$ and the discrete parameter $B(n) = S_n$ have the same integral test. Since the integral test for Case 1 of Theorem E is unchanged from that of Theorem D, the upper and lower class functions for $B(t_k)/t_k^{\frac{1}{2}}$ are derived from those for $X(t_k)$ with $\alpha = 1$ by composition with $\mathscr{E}(t) = \frac{1}{2}\log t$.

EXAMPLES. CASE 1. In particular for $\varphi(t) = \exp\{2t(\log t)^{-\beta/2}\}$, $\beta \leqq 2$, and $t_k = \varphi(k)$, we have

$$\limsup_{t_k\to\infty} (a(t_k))^{\frac{1}{2}}\{t_k^{-\frac{1}{2}}B(t_k) - (a(t_k))^{\frac{1}{2}}\}/\log_p t_k = 1 \quad \text{a.s.,} \tag{3.5}$$

where $a(t_k) = (2\log_2 t_k + (1+\beta)\log_3 t_k + \cdots + 2\log_{p-1} t_k)$.

CASE 2. For all changes of time $\varphi(t)$ satisfying Theorem E, Case 2, we have the Khintchine result (3.5) where in the definition of $a(t_k)$ we replace β by 2.

4. Delayed proofs. We shall first prove Theorem D, Case 1. For this purpose we shall need the first five lemmas that follow.

LEMMA 1. *Let* $\{I_n : n \geqq 1\}$ *be a sequence of random indicators such that*

$\sum_1^\infty EI_k = \infty$. *If for some* A, $0 \leqq A < \infty$, $\operatorname{Var}(\sum_1^n I_k) \leqq A(\sum_1^n EI_k)^2$ *for all sufficiently large* n, *then*

$$P[\sum_1^\infty I_k = \infty] \geqq \frac{1}{1+A}. \tag{4.1}$$

PROOF. The proof is based on a lemma of Paley and Zygmund and is given in [6].

LEMMA 2. *Suppose* X_1, X_2 *are distributed according to the standard bivariate normal probability density* $n(x, y; \rho)$ *with covariance* ρ. *Then*

$$P[X_1 \geqq x, X_2 \geqq x] \leqq 2\psi(x)\{1 - \Phi(x((1-\rho)/(1+\rho))^{\frac{1}{2}})\} \tag{4.2}$$

where $\psi(x) = (2\pi)^{-\frac{1}{2}}x^{-1}\exp(-x^2/2)$ *and* Φ *is the standard normal cumulative distribution function.*

PROOF. This is Lemma 2.3 in Pickands [8].

LEMMA 3. *Let* I_1 *and* I_2 *be the indicators of the events* $[X_1 \geqq x]$ *and* $[X_2 \geqq y]$, *respectively, where the random variables* X_1, X_2 *are described in Lemma* 2 *above. Then*

$$|\operatorname{Cov}(I_1, I_2)| \leqq |\rho| n(x, y; |\rho|). \tag{4.3}$$

PROOF. This is a special case of Lemma 1.5 in Qualls and Watanabe [9].

LEMMA 4. *Suppose that* $X(t)$ *satisfies Theorem* D, *and for Case* 1 *that the conclusion* (2.1) (*where* $I(f \circ \varphi) = \infty$) *holds under the added restriction: for all large* t,

$$u(t)^{\frac{1}{2}} \leqq f \circ \varphi(t) \leqq v(t)^{\frac{1}{2}} \tag{4.4}$$

where $u(t) = 2\log t - \log\log t$ *and* $v(t) = 2\log t + 2A\log\log t$, $A > \frac{1}{2}$. *Then the conclusion of Theorem* D *Case* 1 *when* $I(f \circ \varphi) = \infty$ *holds without this restriction.*

PROOF. The proof is similar to that of Lemma 1.4 in [9].

LEMMA 5. *Let* A_n *and* B_n *be two sequences with* $B_n \uparrow \infty$ *as* $n \uparrow \infty$. *Define* $a_k = A_k - A_{k-1}$, $A_0 = 0$ *and* $b_k = B_k - B_{k-1}$, $B_0 = 0$. *If* $\limsup_{k\to\infty} a_k/b_k \leqq C$ *then* $\limsup_{n\to\infty} A_n/B_n \leqq C$.

PROOF. Easy.

The last lemma is needed in the proof of Theorem D and Theorem B.

LEMMA 6. *For* $a > 0$, $\gamma > 0$, *and* $2^{-\alpha/4} < b < 1$,

$$\limsup_{x\to\infty} \frac{P[X(0) \leqq x - \gamma/x, \max_{0\leqq t\leqq ax^{-2/\alpha}} X(t) > x]}{\psi(x)} \leqq M(C, a, \gamma), \tag{4.5}$$

where $M(C, a, \gamma)$ *is a finite positive constant.*

PROOF. This is Lemma 2.4 in [10].

PROOF OF THEOREM D CASE 1. In Case 1, suppose $I(f \circ \varphi) < \infty$. Let I_k be the indicator of $B_k = [X(\varphi(k)) > f(\varphi(k))]$. Since $1 - \Phi(x) \sim \psi(x)$ as $x \to \infty$, we have $\sum_1^\infty EI_k = \sum_1^\infty P(B_k) < \infty$ if and only if $\sum_1^\infty \psi(f \circ \varphi(k)) < \infty$. According to Propositions (a) and (b), $I(f \circ \varphi) < \infty$ implies $\sum_1^\infty EI_k < \infty$; and the Borel–Cantelli lemma says $E(\sum_1^\infty I_k) < \infty$ implies $\sum_1^\infty I_k < \infty$ a.s. So $P[\sum_1^\infty I_k = \infty] = P[X(\varphi(k)) > f \circ \varphi(k) \text{ i.o.}] = 0$.

Now we suppose $I(f \circ \varphi) = \infty$ and by the above argument $\sum_1^\infty EI_k = \infty$. The proof consists of using Lemma 1 as a converse to the Borel–Cantelli lemma by showing $\lim_{n\to\infty} \operatorname{Var}(\sum^n I_k)/(\sum^n EI_k)^2 = 0$. We first deal with several little-oh terms in the computation of $\operatorname{Var}(\sum I_k)$.

$$\operatorname{Var}\left(\sum_{\mu=j_0}^{n} I_\mu\right) = \sum_{\mu=j_0}^{n} \operatorname{Var} I_\mu + 2 \sum_{\mu=j_0}^{n-1} \sum_{k=1}^{n-\mu} \operatorname{Cov}(I_\mu, I_{\mu+k}) . \tag{4.6}$$

Since $\operatorname{Var} I_\mu \leqq EI_\mu^{\,2} = EI_\mu$ and $\sum_1^n EI_\mu \to \infty$, we have

$$\sum_1^n \operatorname{Var} I_\mu = o\left(\sum_1^n EI_\mu\right)^2 \qquad \text{as} \quad n \to \infty . \tag{4.7}$$

Without loss of generality, we invoke Lemma 4 for the remainder of the proof of Case 1. We also note $\Delta(k) > m(\log k)^{-1/\alpha}$ for some $m > 0$. We now need the following claim.

CLAIM 1. *Suppose $\Delta(j) = \varphi(j+1) - \varphi(j) \geqq m(\log j)^{-1/\alpha}$. Then for all μ, k such that $\varphi(k+\mu) - \varphi(\mu) \leqq B$, a constant, we have* $\lim_{\mu\to\infty} k/\mu = 0$.

PROOF OF CLAIM. Notice

$$B \geqq \varphi(\mu + k) - \varphi(\mu) = \sum_{j=\mu}^{\mu+k-1} \Delta(j) \geqq k \cdot m(\log(\mu + k))^{-1/\alpha} ,$$

so that $k/\mu \leqq (B/m\mu)(\log \mu(1 + k/\mu))^{1/\alpha}$. Considering this graphically, we see that $0 \leqq k/\mu \leqq x_\mu$ for fixed $\mu > 1$, where $x_\mu = (B/m\mu)(\log \mu(1 + x_\mu))^{1/\alpha}$, and that it suffices to show $x_\mu \to 0$ as $\mu \to \infty$.

Since $(x + y)^\alpha \leqq C_\alpha(x^\alpha + y^\alpha)$ and $\log(1 + y) < y$ for $x, y, \alpha > 0$,

$$\log(1 + x_\mu) = \frac{1}{\alpha} \log(1 + x_\mu)^\alpha \leqq \frac{1}{\alpha}[\log C_\alpha + x_\mu{}^\alpha] .$$

Consequently

$$x_\mu{}^\alpha = \left(\frac{B}{m\mu}\right)^\alpha [\log \mu + \log(1 + x_\mu)] \leqq \left(\frac{B}{m\mu}\right)^\alpha \left[\log \mu + \frac{1}{\alpha}\log C_\alpha + \frac{1}{\alpha} x_\mu{}^\alpha\right],$$

which in turn implies $x_\mu{}^\alpha \to 0$ as $\mu \to \infty$. □

Consider that portion $\sum^{(i)}$ of $\sum_{\mu=j_0}^{n-1} \sum_{k=1}^{n-\mu} \operatorname{Cov}(I_\mu, I_{\mu+k})$ where $\varphi(\mu + k) - \varphi(\mu) < \delta$ and $\delta > 0$ is arbitrary but chosen small in the following. By Lemma 2

$$\operatorname{Cov}(I_\mu, I_{\mu+k}) \leqq EI_\mu I_{\mu+k} \leqq 2\psi(x)\{1 - \Phi(x((1-\rho)/(1+\rho))^{\frac12})\} , \tag{4.8}$$

where $x = f \circ \varphi(\mu)$; and $1 - \rho(\varphi(\mu + k) - \varphi(\mu)) \geqq A_1(\delta) \cdot C \cdot |\varphi(\mu + k) - \varphi(\mu)|^\alpha$ by condition (a) of Theorem D, where $A_1(\delta) > 0$, for δ sufficiently small. Now for Case 1, $\varphi(\mu + k) - \varphi(\mu) \geqq mk(\log(\mu + k))^{-1/\alpha}$ and

$$x((1-\rho)/(1+\rho))^{\frac12} \geqq (A_1(\delta) \cdot C)^{\frac12}(mk)^{\alpha/2} f \circ \varphi(\mu)(2\log(\mu + k))^{-\frac12} . \tag{4.9}$$

By Lemma 4 and Claim 1,

$$\frac{f^2 \circ \varphi(\mu)}{2\log(\mu + k)} \geqq \frac{2\log \mu - \log\log \mu}{2\log \mu(1 + k/\mu)} \to 1 \qquad \text{as} \quad \mu \to \infty . \tag{4.10}$$

Since $f \circ \varphi(\mu) \to \infty$ and $\psi(x)(1 - x^{-2}) \leqq 1 - \Phi(x)$, we have $\psi(f \circ \varphi(\mu)) \leqq (1 + \varepsilon)EI_\mu$, and

$$
\begin{aligned}
&\textstyle\sum^{(i)} \operatorname{Cov}(I_\mu, I_{\mu+k}) \\
&\leqq \textstyle\sum_{\mu=j_0}^{n-1} \sum_{k=1}^{n-\mu} 2(1 + \varepsilon)EI_\mu\{1 - \Phi((A_1'C)^{\frac{1}{2}}(mk)^{\alpha/2})\} \\
&\leqq G_\alpha(m) \cdot \textstyle\sum_{\mu=j_0}^{n} EI_\mu = o(\sum_{j_0}^{n} EI_\mu)^2, \qquad \text{as} \quad n \to \infty,
\end{aligned} \tag{4.11}
$$

where $G_\alpha(m) \equiv 2(1 + \varepsilon) \sum_1^\infty\{1 - \Phi((A_1'C)^{\frac{1}{2}}(mk)^{\alpha/2})\} < \infty$, and $0 < A_1' < A_1(\delta)$.

Now consider that portion $\sum^{(ii)}$ of $\sum\sum \operatorname{Cov}(I_\mu, I_{\mu+k})$ where $\delta \leqq \varphi(\mu + k) - \varphi(\mu) < k^*\delta$, where the positive integer k^* is arbitrary now but will be chosen large later.

Again by Lemma 2

$$
\operatorname{Cov}(I_\mu, I_{\mu+k}) \leqq 2\psi(f \circ \varphi(\mu))\{1 - \Phi(x(1 - \rho)^{\frac{1}{2}}(1 + \rho)^{-\frac{1}{2}})\}, \tag{4.12}
$$

where $x = f \circ \varphi(\mu)$; and $1 - \rho >$ some $\kappa > 0$, since $\varphi(\mu + k) - \varphi(\mu) > \delta$. We also know $k \leqq (\delta k^*/m)(\log(\mu + k))^{1/\alpha}$, which is equivalent to $1 \leqq k \leqq b_\mu$ where $b_\mu = (\delta k^*/m)(\log(\mu + b_\mu))^{1/\alpha}$. Consequently,

$$
\begin{aligned}
&\textstyle\sum^{(ii)} \operatorname{Cov}(I_\mu, I_{\mu+k}) \\
&\leqq \textstyle\sum_{\mu=j_0}^{n-1} \sum_{k=1}^{b} 2(1 + \varepsilon)EI_\mu \cdot \psi(f \circ \varphi(\mu) \cdot (\kappa/2)^{\frac{1}{2}}) \\
&\leqq \textstyle\sum_{j_0}^{n} 2(1 + \varepsilon)EI_\mu \cdot \dfrac{\delta k^*}{m} (\log(\mu + b_\mu))^{1/\alpha}\psi(f \circ \varphi(\mu) \cdot (\kappa/2)^{\frac{1}{2}}) \\
&= o(\textstyle\sum_{j_0}^{n} EI_\mu)^2 \qquad \text{as} \quad n \to \infty, \quad \text{for every} \quad k^* \geqq 1,
\end{aligned} \tag{4.13}
$$

since $(\log(\mu + b_\mu))^{1/\alpha} \exp\{-\frac{1}{2} \cdot f^2 \circ \varphi(\mu) \cdot \kappa/2\} \leqq (\log(\mu + b_\mu))^{1/\alpha}(\log \mu)^{\kappa/4}\mu^{-\kappa/2} \to 0$ as $\mu \to \infty$ by Lemma 4 and Claim 1.

It is somewhat easier to obtain lower bounds on $\sum^{(i)}$ and $\sum^{(ii)}$.

$$
\begin{aligned}
-(\textstyle\sum^{(i)} + \sum^{(ii)}) \operatorname{Cov}(I_\mu, I_{\mu+k}) &\leqq (\textstyle\sum^{(i)} + \sum^{(ii)})EI_\mu EI_{\mu+k} \\
&\leqq \textstyle\sum_{\mu=j_0}^{n-1} \sum_{k=1}^{b} EI_\mu EI_{\mu+k} \\
&\leqq \textstyle\sum_{j_0}^{n} EI_\mu(b_\mu \cdot EI_\mu) = o(\sum_{j_0}^{n} EI_\mu)^2
\end{aligned} \tag{4.14}
$$

as $n \to \infty$, since $b_\mu \cdot EI_\mu \to 0$ as $\mu \to \infty$ by Lemma 4 and Claim 1.

It will be necessary to consider the sum $\sum_1^n I_k$ in blocks. Consider the block of times $\{\varphi(k) : \delta i \leqq \varphi(k) < \delta i + \delta\}$ for a small $\delta > 0$ to be chosen below. Write $k_i =$ first integer $\geqq \varphi^{-1}(\delta i)$ and $m_i = k_{i+1} - k_i$, so that the block of times $= \{\varphi(k), k_i \leqq k < k_{i+1}\}$ and contains m_i points. Define $L_i = \sum_{\mu=0}^{m_i-1} I_{k_i+\mu}$ and $M_n = \sum_{i=i_0}^{n} m_i$; if $m_i = 0$, take L_i to be empty. Consequently, for $M_{n-1} \leqq N < M_n$,

$$
\begin{aligned}
\operatorname{Var}(\textstyle\sum_{j_0}^{N} I_j) &= \operatorname{Var}(\textstyle\sum_{i_0}^{n} L_i) \\
&= \textstyle\sum_{i_0}^{n} \operatorname{Var}(L_i) + 2 \sum_{i=i_0}^{n-1} \sum_{k=1}^{n-i} \operatorname{Cov}(L_i, L_{i+k})
\end{aligned} \tag{4.15}
$$

is to be compared to $(\sum^n EL_i)^2$. Note that L_n may be incomplete, but that this does not affect the following proof. Now the results for $\sum^{(i)}$ and $\sum^{(ii)}$ obtained earlier can be summarized by

$$
\textstyle\sum_{i=i_0}^{n-1} \sum_{k=0}^{k^*-1} \operatorname{Cov}(L_i, L_{i+k}) = o(\sum^n EL_i)^2 \qquad \text{as} \quad n \to \infty \tag{4.16}
$$

for each fixed k^*.

In the remaining sum

$$(4.17)\qquad \sum_{i=i_0}^{n-1} \sum_{k=k^*}^{n-i} \operatorname{Cov}(L_i, L_{i+k}) = \sum_{i=i_0}^{n-k^*} \sum_{k=k^*}^{n-i} \sum_{\mu=0}^{m_i-1} \sum_{\nu=0}^{m_{i+k}-1} \operatorname{Cov}(I_{k_i+\mu}, I_{k_{i+k}+\nu})$$

we note $\varphi(k_{i+k}+\nu) - \varphi(k_i+\mu) \geqq \varphi(k_{i+k}) - \varphi(k_{i+1}-1) \geqq (k-1)\delta \geqq (k^*-1)\delta$. Since $k \geqq k^*$ and k^* can be chosen arbitrarily large, Lemma 3 and condition (b) of Theorem D imply

$$|\operatorname{Cov}(I_{k_i+\mu}, I_{k_{i+k}+\nu})| \leqq |r| \cdot n(x, y; |r|)$$

where $x = f \circ \varphi(k_i+\mu)$, $y = f \circ \varphi(k_{i+k}+\nu)$, and $|r| = |r(\varphi(k_i+\mu) - \varphi(k_{i+k}+\nu))| \leqq M/2 \log k\delta < \delta$, for any $M > 0$. Now

$$(4.18)\qquad |r| \cdot n(x, y; |r|) \leqq \frac{M}{2 \log k\delta} \cdot \frac{1}{2\pi(1-\delta^2)^{\frac12}} \exp\left\{-\frac{x^2 - 2|r|xy + y^2}{2(1-r^2)}\right\}$$
$$\leqq \frac{1}{(1-\delta^2)^{\frac12}} \frac{Mxy}{2 \log k\delta} \exp\left\{\frac{Mxy}{2 \log k\delta}\right\} \phi(x)\phi(y) .$$

Since $k_{i+k} + \nu \leqq k_{n+1}$ for $k \leqq n - i$, and

$$\frac{Mxy}{2 \log k\delta} \leqq \frac{Mf^2 \circ \varphi(k_{n+1})}{2 \log k\delta} \leqq \frac{Mv(k_{n+1})}{2 \log k\delta} ,$$

define

$$(4.19)\qquad \alpha_k{}^n = \frac{Mv(k_{n+1})}{2 \log k\delta} \exp\left\{\frac{Mv(k_{n+1})}{2 \log k\delta}\right\} ,$$

where v is given in Lemma 4. Note also that $EI_{k_i+\mu} \geqq \phi(x)(1 - x^{-2}) \geqq (1-\varepsilon)\phi(x)$ for $x = f \circ \varphi(k_i + \mu)$ sufficiently large. Consequently (4.17—4.19) yields

$$(4.20)\qquad \operatorname{Var}\left(\sum_{i=i_0}^{n} L_i\right) \leqq \frac{2}{(1-\varepsilon)^2(1-\delta^2)^{\frac12}} \sum_{i=i_0}^{n-k^*} \sum_{k=k^*}^{n-i} \alpha_k{}^n EL_i EL_{i+k} + o\left(\sum^n EL_i\right)^2 .$$

We now use the L'Hopital type result, Lemma 5, for the ratio of $\operatorname{Var}(\sum^n L_i)$, or rather its upper bound in (4.20), to $(\sum^n EL_i)^2$.

CLAIM 2. Let $A_n = \sum_{i=i_0}^{n-k^*} \sum_{k=k^*}^{n-i} f(i, k)$ and $B_n = (\sum_{i=i_0}^{n} h(i))^2$. Then

$$\frac{a_n}{b_n} \equiv \frac{A_n - A_{n-1}}{B_n - B_{n-1}} = \frac{\sum_{i=i_0}^{n-k^*} f(i, n-i)}{2h(n)\{\sum_{i_0}^{n} h(i)\} - h^2(n)} .$$

PROOF. Easy.

By this claim and Lemma 5, it suffices to study the ratio

$$(4.21)\qquad \frac{2/((1-\varepsilon)^2(1-\delta^2)^{\frac12}) \cdot \sum_{i=i_0}^{n-k^*} \alpha_{n-i}^n EL_i EL_n}{2EL_n\{\sum_{i=i_0}^{n} EL_i\} - (EL_n)^2}$$
$$\leqq \frac{1}{(1-\varepsilon)^2(1-\delta^2)^{\frac12}} \sum_{i=i_0}^{n-k^*} \alpha_{n-i}^n EL_i / \sum_{i=i_0}^{n-1} EL_i .$$

This study reduces to estimating α_{n-i}^n defined in (4.19).

Divide the sum in the numerator of (4.21) into two parts, for $z = k_{n+1}$,

(a) $i \ni k_{i_0} \leqq k_i \leqq \theta_z \cdot z$ or
(b) $i \ni \theta_z \cdot z < k_i < k_{n-k^*} < k_{n+1} = z$ (possibly empty),

where $\theta_z = 1 - z^{-\gamma}(\log z)^{-1}$ and γ satisfies $0 < 2\delta < \gamma < 1$. Let p denote the integer such that $k_p \leqq \theta_z z < k_{p+1}$. For part (b), we have $M/2 \log \delta(n-i) < \delta$, for $n - i \geqq k^*$ and k^* sufficiently large, so that

$$\frac{\sum^{(b)} \alpha_{n-i}^n EL_i}{\sum^{n-1} EL_i} \leqq \delta v(z) \exp[\delta v(z)] \frac{\sum^{(b)} EL_i}{\sum^{(a)} EL_i} . \tag{4.22}$$

Now

$$\sum^{(b)} EL_i \leqq \sum^{(b)} m_i \psi(f \circ \varphi(k_i)) \leqq \psi(f \circ \varphi(k_{p+1})) \sum_{p+1}^{n-k^*} m_i \leqq \psi(f \circ \varphi(k_{p+1}))[z - \theta_z z] ,$$

while

$$\sum^{(a)} EL_i \geqq \sum_{i_0}^{p} (1-\varepsilon) m_i \psi(f \circ \varphi(k_{i+1})) \geqq (1-\varepsilon)\psi(f \circ \varphi(k_{p+1}))[\theta_z \cdot z - k_{i_0}] .$$

Consequently, using Lemma 4,

$$\begin{aligned} \frac{\sum^{(b)} \alpha_{n-i}^n EL_i}{\sum^{(a)} EL_i} &\leqq \frac{\delta v(z) \exp[\delta v(z)](z - \theta_z z)}{(1-\varepsilon)[\theta_z \cdot z - k_{i_0}]} \\ &\sim \frac{2\delta \log z}{1-\varepsilon} [z^{2\delta}(\log z)^{2A\delta}][z^{-\gamma}(\log z)^{-1}] \\ &\to 0 \quad \text{as} \quad z = k_{n+1} \to \infty . \end{aligned} \tag{4.23}$$

For part (a),

$$\begin{aligned} \delta n - \delta i &\geqq \delta(n+1) - \delta(p+1) \geqq \varphi(k_{n+1} - 1) - \varphi(k_{p+1}) \\ &= \sum_{j=k_{p+1}}^{k_{n+1}-2} \Delta(j) \geqq (z - 1 - \theta_z z) m (\log z)^{-1/\alpha} , \end{aligned} \tag{4.24}$$

where $z = k_{n+1}$. Since $\log((z - 1 - \theta_z z)m(\log z)^{-1/\alpha}) \sim (1-\gamma)\log z$ as $z \to \infty$, Lemma 4 yields $\limsup_{z\to\infty} \alpha_{n-i}^n \leqq (M/(1-\gamma)) \exp(M/(1-\gamma))$ for part (a), and

$$\limsup_{N\to\infty} \frac{\operatorname{Var}(\sum^N I_k)}{(\sum^N EI_k)^2} \leqq \frac{1}{(1-\varepsilon)^2(1-\delta^2)^{\frac12}} \frac{M}{1-\gamma} \exp \frac{M}{1-\gamma} . \tag{4.25}$$

In fact we may let $\delta \to 0$, $\gamma \to 0$, $\varepsilon \to 0$ in (4.25), so that, from Lemma 1 and under the restrictions of Lemma 4,

$$P[\textstyle\sum_1^\infty I_k = \infty] \geqq \dfrac{1}{1 + M \exp(M)} > 0 . \tag{4.26}$$

Now condition (b) of Theorem D allows one to take $M = 0$. □

PROOF OF THEOREM D FOR ARBITRARY SEQUENCES. Recall the thinned sequence $\{\tilde\varphi(k)\} \subset \{\varphi(k)\}$ satisfies Case 1. By Proposition (b), we have $I(f \circ \tilde\varphi) < \infty$ if and only if $\sum_1^\infty \psi(f \circ \tilde\varphi(k)) < \infty$. Further, if $\hat f = \min(f, v \circ \tilde\varphi^{-1})$ with $v(t) = (3 \log t)^{\frac12}$, then

$$\sum_1^\infty \psi(f \circ \tilde\varphi(k)) < \infty \quad \text{if and only if} \quad \sum_1^\infty \psi(\hat f \circ \tilde\varphi(k)) < \infty . \tag{4.27}$$

This follows easily from the facts that $\psi(f \circ \tilde{\varphi}(k)) \leqq \psi(\hat{f} \circ \tilde{\varphi}(k)) \leqq \psi(f \circ \tilde{\varphi}(k)) + \psi(v(k))$ provided f is unbounded and k is sufficiently large, and that $\sum_1^\infty \psi(v(k)) < \infty$. If f is bounded, then both sums are infinite.

Suppose $\sum_1^\infty \psi(\hat{f} \circ \tilde{\varphi}(k)) < \infty$. Define the intervals $I_k = [\tilde{\varphi}(k), \tilde{\varphi}(k+1))$ and $J_k = [\tilde{\varphi}(k), \tilde{\varphi}(k) + d(k))$, where $d(k) = m(\log k)^{-1/\alpha}$. Note that $J_k \subset I_k$, but that every $\varphi(i) \in I_k$ is also in J_k. Consequently, we have

$$(4.28) \qquad A_k' \equiv [\sup_{\varphi(i) \in I_k} X(\varphi(i)) > f \circ \tilde{\varphi}(k)] \subset [\sup_{t \in J_k} X(t) > f \circ \tilde{\varphi}(k)] \equiv A_k \subset [\sup_{t \in J_k} X(t) > \hat{f} \circ \tilde{\varphi}(k)] \equiv \hat{A}_k .$$

By stationarity of $X(t)$,

$$(4.29) \qquad P\hat{A}_k = P[\sup_{0 \leqq t \leqq d(k)} X(t) > \hat{f} \circ \tilde{\varphi}(k)] \leqq P[\sup_{0 \leqq t \leqq ax^{-2/\alpha}} X(t) > x] , \qquad \text{for} \quad x = \hat{f} \circ \tilde{\varphi}(k) .$$

This last statement follows from $x = \hat{f} \circ \tilde{\varphi}(k) \leqq v(k)$, so that $\log k \geqq 3^{-1}x^2$ and $d(k) \leqq ax^{-2/\alpha}$ where $a = m3^{1/\alpha}$.

Since

$$[\sup_{0 \leqq t \leqq ax^{-2/\alpha}} X(t) > x] \subset [X(0) > x - \gamma/x] \cup [X(0) \leqq x - \gamma/x, \sup_{0 < t \leqq ax^{-2/\alpha}} X(t) > x]$$

for any $\gamma > 0$, and since $\psi(x - \gamma/x) \sim e^\gamma \psi(x)$ as $x \to \infty$, we obtain from Lemma 6,

$$(4.30) \qquad \limsup_{x \to \infty} \frac{P[\sup_{0 \leqq t \leqq ax^{-2/\alpha}} X(t) > x]}{\psi(x)} \leqq e^\gamma + M(C, a, \gamma) < \infty .$$

Consequently

$$(4.31) \qquad \sum PA_k' \leqq \sum_1^\infty PA_k \leqq \sum_1^\infty P\hat{A}_k \leqq \text{const. times } \sum_1^\infty \psi(\hat{f} \circ \tilde{\varphi}(k)) < \infty .$$

The Borel–Cantelli lemma now shows

$$O = P[A_k' \text{ i.o.}] \geqq P[X(\varphi(i)) > f \circ \varphi(i) \text{ i.o.}] .$$

For the second half of the proof, suppose $I(f \circ \tilde{\varphi}) = \infty$. Since $[X(\tilde{\varphi}(k)) > f \circ \tilde{\varphi}(k) \text{ i.o.}] \subset [X(\varphi(k)) > f \circ \varphi(k) \text{ i.o.}]$, an application of Theorem D Case 1 to the event on the left yields probability one for both events. □

Proof of Theorem D Case 2. Here there exists $m^* > 0$ such that $\Delta(k) \leqq m^*(\log k)^{-1/\alpha}$ for all k. Suppose the sequence $\{\varphi(k)\}$ is thinned to $\{\tilde{\varphi}(k)\}$ as discussed preceding Theorem D with $m = m^*$. Then $m^*(\log j)^{-1/\alpha} \leqq \tilde{\Delta}(j) \leqq 2m^*(\log j)^{-1/\alpha}$.

It suffices to show under Case 2 that

$$(4.32) \qquad I(f \circ \tilde{\varphi}) < \infty \qquad \text{if and only if} \quad I(f \circ \hat{\varphi}) < \infty .$$

By Proposition (c), consider, instead of $I(f \circ \hat{\varphi})$, the following

$$(4.33) \qquad \sum^\infty \psi(f(n))(2 \log (n-1))^{1/\alpha} \leqq \int_a^\infty \psi(f(t))(2 \log t)^{1/\alpha}\, dt \leqq \sum^\infty \psi(f(n))(2 \log (n+1))^{1/\alpha} .$$

On the other hand by Proposition (a), we have

$$(4.34)\qquad \sum^{\infty} \psi(f(n))[\tilde\varphi^{-1}(n) - \tilde\varphi^{-1}(n-1)] \leqq I(f \circ \tilde\varphi) \leqq \sum^{\infty} \psi(f(n))[\tilde\varphi^{-1}(n+1) - \tilde\varphi^{-1}(n)] .$$

Whatever continuous increasing version of $\tilde\varphi$ is chosen, we define k_n = first integer $\geqq \tilde\varphi^{-1}(n)$ and obtain

$$n - (n-1) \geqq \tilde\varphi(k_n - 1) - \tilde\varphi(k_{n-1}) = \sum_{j=k_{n-1}}^{k_n-2} \tilde\Delta(j) \geqq (k_n - 1 - k_{n-1}) m^*(\log (k_n - 1))^{-1/\alpha}$$

which implies

$$(4.35)\qquad \begin{aligned} \tilde\varphi^{-1}(n) - \tilde\varphi^{-1}(n-1) &\leqq k_n - (k_{n-1} - 1) = (k_n - 1) - k_{n-1} + 2 \\ &\leqq \frac{1}{m^*}(\log (k_n - 1))^{1/\alpha} + 2 \\ &\leqq \frac{1}{m^*}(\log \tilde\varphi^{-1}(n))^{1/\alpha} + 2 . \end{aligned}$$

Using the upper bound on $\tilde\Delta(j)$, we also obtain

$$(4.36)\qquad \tilde\varphi^{-1}(n) - \tilde\varphi^{-1}(n-1) \geqq \frac{1}{2m^*}(\log (\tilde\varphi^{-1}(n-1) - 1))^{1/\alpha} - 2 .$$

It then suffices to show $(\log \tilde\varphi^{-1}(n)/\log n) \to 1$ as $n \to \infty$. Letting

$$\tau = \tilde\varphi^{-1}(n) , \qquad \lim_{n\to\infty} \frac{\log \tilde\varphi^{-1}(n)}{\log n} = \lim_{\tau\to\infty} \frac{\log \tau}{\log \tilde\varphi(\tau)} .$$

Since

$$m^* \sum_{j_0}^{\tau} (\log j)^{-1/\alpha} \leqq \tilde\varphi(\tau + 1) - \tilde\varphi(j_0) \leqq 2m^* \sum_{j_0}^{\tau} (\log j)^{-1/\alpha} ,$$

one obtains

$$\log \tilde\varphi(\tau) \sim \log \left(\sum^{\tau} (\log j)^{-1/\alpha}\right) \sim \log \tau(\log \tau)^{-1/\alpha} \sim \log \tau \qquad \text{as} \quad \tau \to \infty . \qquad \square$$

It should be remarked that the proof of Theorem D for arbitrary sequences is sufficient to prove Theorem B as well.

PROOF OF THEOREM B. For the continuous parameter t, we select a discrete sequence $t_k = \varphi(k)$ as follows. Begin at any $t_0 = \varphi(j_0)$. Choose $\varphi(k)$ so that $\Delta(j) \equiv 2m^*(\log j)^{-1/\alpha}$ and choose $\tilde\varphi(k) = \varphi(k)$. The sequence $\{\varphi(k)\}$ is both Case 1 and Case 2 so that the integral test of Theorem B is identical to all those in Theorem D, provided we can prove Proposition (d) below.

As in the discussion following (4.27), let $J_k = I_k = [\varphi(k), \varphi(k) + d(k)]$ where $d(k) = 2m^*(\log k)^{-1/\alpha}$ and $A_k = [\sup_{t \in J_k} X(t) > f \circ \varphi(k)]$. We then showed in (4.31) that $\sum^{\infty} PA_k < \infty$ if $I(f \circ \varphi) < \infty$ which in turn completes one half of Theorem B.

Suppose $J(f) = \infty$. We now involve Lemma 1.4 of [9] for Theorem B, which is analogous to Lemma 4 in this paper. Under the restrictions of this lemma, we can apply Theorem D, Case 1 to the event $[X(t_n) > f(t_n) \text{ i.o.}] \subset [X(t) > f(t)$ i.o. as $t \uparrow \infty]$ obtaining

$$(4.37)\qquad 0 < \frac{1}{1 + M \exp M} \leqq P[X(t_n) < f(t_n) \text{ i.o.}] \leqq P[X(t) > f(t) \text{ i.o. as } t \uparrow \infty],$$

where M is the implied constant of the condition $r(t) \log t = O(1)$ as $t \to \infty$.

Now the condition that $r(t) \to 0$ as $t \to \infty$, or even the ergodic condition $(1/T) \int_0^T r(t)\, dt \to 0$ as $T \to \infty$, implies a.s. invariant events have 0 or 1 probability. Since f is nondecreasing, we have

$$A \equiv [X(t) > f(t) \text{ i.o. as } t \uparrow \infty] \subset A_\tau \equiv [X(t + \tau) > f(t) \text{ i.o. as } t \uparrow \infty] = [X(t) > f(t - \tau) \text{ i.o. as } t \uparrow \infty],$$

for any shift $\tau \geqq 0$.

By stationarity,

$$\begin{aligned} P(A \triangle A_\tau) &= PA_\tau - PA \\ &= P[X(t - \tau) > f(t - \tau) \text{ i.o. as } t \uparrow \infty] - PA \\ &= PA - PA = 0. \end{aligned}$$

Consequently, (4.37) implies $P(A) = 1$. □

PROOF OF PROPOSITION (d). For the integral $J(f) = \int_a^\infty f(t)^{2/\alpha} \psi(f(t))\, dt$, we show $J(f) < \infty$ if and only if $J(\hat{f}) < \infty$, where $\hat{f}(t) = \min\{f(t), v(t)\}$, $v(t) = (2 \log t + 2A \log\log t)^{\frac{1}{2}}$, $A > 1/\alpha + \frac{1}{2}$. The proof of this follows that of (4.27). Now let $\hat{f}(t) = \max\{f(t), u(t)\}$, where $u(t) = (2 \log t)^{\frac{1}{2}}$. We have shown elsewhere [9, Lemma 1.4] that $J(f) < \infty$ if and only if this $J(\hat{f}) < \infty$. Finally, we may without loss in generality take $f(t) \sim (2 \log t)^{\frac{1}{2}}$ as $t \to \infty$, and obtain $J(f) < \infty$ if and only if $\int_a^\infty (2 \log t)^{1/\alpha} \psi(f(t))\, dt < \infty$. Now Proposition (c) completes the proof. Note that the without loss in generality statement probably requires that we repeat the first two arguments of this proof for the new integral $\int^\infty (2 \log t)^{1/\alpha} \psi(f(t))\, dt$. □

Acknowledgments. I wish to thank Professor H. G. Tucker for several discussions related to these matters, and the referee for a thorough list of substantive and editorial suggestions.

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