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# LINEAR BOUNDS ON THE EMPIRICAL DISTRIBUTION FUNCTION

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Let $\Gamma_n$ denote the empirical df of a sample from the uniform (0, 1) df $I$. Let $\xi_{nk}$ denote the $k$th smallest observation. Let $\lambda_n > 1$. Let $A_n$ denote the event that $\Gamma_n$ intersects the line $\lambda_n I$ on [0, 1] and let $B_n$ denote the event that $\Gamma_n$ intersects the line $I/\lambda_n$ on $[\xi_{n1}, 1]$. Conditions on $\lambda_n$ are given that determine whether $P(A_n$ i.o.) and $P(B_n$ i.o.) equal 0 or 1. Results for $A_n$ (for $B_n$) are related to upper class sequences for $1/(n\xi_{n1})$ (for $n\xi_{n2}$).

Upper class sequences for $n\xi_{nk}$, with $k > 1$, are characterized.

In the case of nonidentically distributed random variables, we present the result analogous to $P(A_n$ i.o.) $= 0$.

**1. Introduction and statement of the theorems.** Let $\xi_1, \cdots, \xi_n$ be independent uniform (0, 1) random variables having empirical df $\Gamma_n$ and whose ordered values are $0 \leqq \xi_{n1} \leqq \cdots \leqq \xi_{nn} \leqq 1$. The true df is the identity function on [0, 1], which we denote by $I$.

We let $\|f\|_a^b \equiv \sup_{a \leqq t \leqq b} |f(t)|$, and we simply write $\|f\|$ in case $a = 0$ and $b = 1$.

Note that $\Gamma_n$ lies entirely below the line $\lambda I$ if and only if $\|\Gamma_n/I\| \geqq \lambda$ a.s. for each $n$. We can not make $\Gamma_n$ lie entirely above any line through the origin with positive slope since $\Gamma_n(t) = 0$ for $0 \leqq t < \xi_{n1}$; however $\Gamma_n$ lies entirely above the line $I/\lambda$ on the interval $[\xi_{n1}, 1]$ if and only if $\|I/\Gamma_n\|_{\xi_{n1}}^1 \leqq \lambda$. Our main concern in this paper is bounding $\Gamma_n$ between straight lines through the origin. More precisely, we will characterize upper and lower class sequences for the random variables $\|\Gamma_n/I\|$ and $\|I/\Gamma_n\|_{\xi_{n1}}^1$.

"In probability upper and lower linear bounds" are well known (see Robbins (1954), Chang (1955) and Renyi (1973)); and see Shorack (1972) for applications. It is known that "a.s. linear bounds" do not exist (see Wellner (1977a)); also see Wellner (1977a) and (1977b) for applications of "a.s. nearly linear bounds."

Discussion of our theorems will be facilitated by contrasting them with the behavior of $\xi_{n1}$ and $\xi_{n2}$ that is set forth in Theorem 1.

THEOREM 1. *Let $k \geqq 1$ be a fixed integer.*

(i) (*Kiefer*). *If $c_n \searrow$ then*

$$P(n\xi_{nk} \leqq c_n \text{ i.o.}) = 0 \quad \textit{according as} \quad \sum_{n=1}^\infty \frac{c_n^{\,k}}{n} < \infty$$
$$= 1 \qquad\qquad\qquad\qquad = \infty .$$

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(ii) (*Robbins and Siegmund when* $k = 1$). *Let* $c_n/n \searrow$ *and suppose either* $c_n \nearrow$ *or* $\liminf_{n\to\infty} c_n/\log_2 n \geqq 1$. *Then*

$$P(n\xi_{nk} > c_n \text{ i.o.}) = 0 \quad \text{according as} \quad \sum_{n=1}^\infty \frac{c_n{}^k}{n} \exp(-c_n) < \infty$$
$$= 1 \qquad\qquad\qquad\qquad = \infty\,.$$

THEOREM 2. *Let* $n\lambda_n \nearrow$. *Then*

$$P(\|\Gamma_n/I\| \geqq \lambda_n \text{ i.o.}) = 0 \quad \text{according as} \quad \sum_{n=1}^\infty \frac{1}{n\lambda_n} < \infty$$
$$= 1 \qquad\qquad\qquad\qquad = \infty\,.$$

Note that $\|\Gamma_n/I\| = \max\{i/(n\xi_{ni}) \colon 1 \leqq i \leqq n\}$ is $\geqq \lambda_n$ if $n\xi_{n1}$ is $\leqq c_n \equiv 1/\lambda_n$. Comparing the series criteria of Theorem 1(i) with Theorem 2, it is seen that small values of $\xi_{n1}$ control large values of $\|\Gamma_n/I\|$. Note however that $\|\Gamma_n/I\|$ and $(n\xi_{n1})^{-1}$ have different limiting distributions.

Theorem 2 yields the known result $\limsup_{n\to\infty} \log\|\Gamma_n/I\|/\log_2 n = 1$ a.s. In fact, $\log \lambda_n = \sum_{i=2}^{p-1} \log_i n + \tau \log_p n$, with $p \geqq 2$, is upper class or lower class for $\log\|\Gamma_n/I\|$ according as $\tau > 1$ or $\tau \leqq 1$.

THEOREM 3. *Let* $\lambda_n/n \searrow$ *and suppose either* $\lambda_n \nearrow$ *or* $\liminf_{n\to\infty} \lambda_n/\log_2 n \geqq 1$. *Then*

$$P(\|I/\Gamma_n\|_{\xi_{n1}}^1 \geqq \lambda_n \text{ i.o.}) = 0 \quad \text{according as} \quad \sum_{n=1}^\infty \frac{\lambda_n{}^2}{n} \exp(-\lambda_n) < \infty$$
$$= 1 \qquad\qquad\qquad\qquad = \infty\,.$$

Note that $\|I/\Gamma_n\|_{\xi_{n1}}^1 = \max\{n\xi_{n,i+1}/i \colon 1 \leqq i \leqq n\}$ is $\geqq \lambda_n$ if $n\xi_{n2}$ is $\geqq c_n \equiv \lambda_n$. (Here, and in the following, $\xi_{n,n+1} \equiv 1$ for all $n$.) Comparing the series criteria of Theorem 1(ii) with Theorem 3, it is seen that large values of $\xi_{n2}$ control large values of $\|I/\Gamma_n\|_{\xi_{n1}}^1$. Note however (see Renyi (1973)) that $\|I/\Gamma_n\|_{\xi_{n1}}^1$ and $n\xi_{n2}$ have different limiting distributions.

Theorem 3 yields the known result $\limsup_{n\to\infty} \|I/\Gamma_n\|_{\xi_{n1}}^1/\log_2 n = 1$ a.s. In fact, $\lambda_n = \log_2 n + 3\log_3 n + \sum_{i=4}^{p-1} \log_i n + \tau \log_p n$, with $p \geqq 4$, is upper class or lower class for $\|I/\Gamma_n\|_{\xi_{n1}}^1$ according as $\tau > 1$ or $\tau \leqq 1$.

## 2. Proofs.

Robbins (1954) showed that for any $n \geqq 1$

$$P(\|\Gamma_n/I\| \geqq \lambda) = 1/\lambda \qquad \text{for all} \quad \lambda > 1\,. \tag{1}$$

PROOF OF THEOREM 2. Suppose $\sum (n\lambda_n)^{-1} < \infty$. Let $n_k \equiv \operatorname{int}(\alpha^k)$ where $\alpha > 1$ is fixed, and where $\operatorname{int}(\cdot)$ denotes that greatest integer function. Note that

$$\text{(a)} \qquad \infty > \sum_{k=2}^\infty \sum_{j=n_{k-1}+1}^{n_k} (n\lambda_n)^{-1} \geqq \sum_{k=2}^\infty (n_k - n_{k-1})(n_k\lambda_{n_k})^{-1}$$
$$\geqq \text{constant} \cdot \sum_{k=2}^\infty (\lambda_{n_k})^{-1}\,.$$

Let $A_k \equiv [\max\{\|\Gamma_n/I\| \colon n_k < n \leqq n_{k+1}\} \geqq \lambda_n]$; and note that monotoneity of $n\Gamma_n$

and $n\lambda_n$ implies

$$\begin{aligned} P(A_k) &\leqq P(n_{k+1}\|\Gamma_{n_{k+1}}/I\| \geqq n_k \lambda_{n_k}) \\ &= n_{k+1}/(n_k \lambda_{n_k}) \qquad \text{by} \quad (1) \\ &\sim \alpha/\lambda_{n_k} \end{aligned}$$

so that (a) yields $\sum_1^\infty P(A_k) < \infty$. Thus $P(A_k \text{ i.o.}) = 0$ by Borel–Cantelli; and hence $P(\|\Gamma_n/I\| \geqq \lambda_n \text{ i.o.}) = 0$.

Suppose $\sum (n\lambda_n)^{-1} = \infty$. Now

$$\begin{aligned} [\|\Gamma_n/I\| \geqq \lambda_n] &= [\sup\{\textstyle\sum_{i=1}^n I_{[0,t]}(\xi_i)/t\colon 0 < t \leqq 1\} \geqq n\lambda_n] \\ &\supset [\sup\{I_{[0,t]}(\xi_n)/t\colon 0 < t \leqq 1\} \geqq n\lambda_n] \\ &= [\xi_n \leqq (n\lambda_n)^{-1}]\,. \end{aligned}$$

Now the events $[\xi_n \leqq (n\lambda_n)^{-1}]$ are independent, and the sum of their probabilities equals $\sum_1^\infty (n\lambda_n)^{-1} = \infty$. Thus Borel–Cantelli yields $P(\xi_n \leqq (n\lambda_n)^{-1}$ i.o.$) = 1$; and hence $P(\|\Gamma_n/I\| \geqq \lambda_n \text{ i.o.}) = 1$. □

Before proving Theorem 3, we need the following probability bound. For all $n \geqq 1$ we have

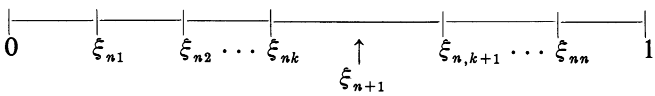
$$P(\|I/\Gamma_n\|_{\xi_{n1}}^1 \geqq \lambda) \leqq 16\lambda e^{-\lambda} \qquad \text{for all} \quad \lambda > 1\,. \tag{2}$$

The probability on the left-hand side of (2) is given in formula (17) on page 34 of Chang (1964); and for $\lambda \geqq 2$ Chang's next to the last formula on page 17 yields the bound $2^{\frac{1}{2}}(e\lambda e^{-\lambda})^k$, which when summed yields the right-hand side of (2). Note that (2) is trivial for $1 < \lambda \leqq 2$.

Proof of Theorem 3. Suppose $\sum_1^\infty (\lambda_n^2/n)\exp(-\lambda_n) < \infty$. Let $n_j \equiv \operatorname{int}(\exp(\alpha j/\log j))$ for $j \geqq 2$ with $\alpha > 0$ fixed. Let $A_n \equiv [M_n \geqq \lambda_n]$ where $M_n \equiv \|I/\Gamma_n\|_{\xi_{n1}}^1 = \max_{1 \leqq i \leqq n}(n\xi_{n,i+1}/i)$; and let $B_j \equiv [M_{n_j} \geqq (n_j/n_{j+1})\lambda_{n_{j+1}}]$. Note that

$$M_n/n = \max_{1 \leqq i \leqq n}(\xi_{n,i+1}/i) \qquad \text{is a} \quad \searrow \quad \text{function of} \quad n\,. \tag{3}$$

To see this, suppose $\xi_{n+1}$ falls between $\xi_{nk}$ and $\xi_{n,k+1}$; then




$$\frac{\xi_{n+1,i+1}}{i} = \frac{\xi_{n,i+1}}{i} \quad \text{for} \quad 1 \leqq i \leqq k-1\,, \qquad \frac{\xi_{n+1,k+1}}{k} = \frac{\xi_{n+1}}{k} \leqq \frac{\xi_{n,k+1}}{k}$$

and

$$\frac{\xi_{n+1,i+1}}{i} = \frac{\xi_{ni}}{i} \leqq \frac{\xi_{ni}}{i-1} \qquad \text{for} \quad k+1 \leqq i \leqq n$$

so that (3) is established. From (3) and $\lambda_n/n \searrow$ we get

$$\bigcup\{A_n\colon n_j \leqq n < n_{j+1}\} \subset \bigcup\{[M_n/n \geqq \lambda_{n_{j+1}}/n_{j+1}]\colon n_j \leqq n < n_{j+1}\} = B_j\,.$$

Thus to establish $P(A_n \text{ i.o.}) = 0$, it suffices to show $\sum_2^\infty P(B_j) < \infty$. Let $d_n \equiv \lambda_n \wedge 2\log_2 n$. Then

$$d_n/n \searrow, \quad d_n \to \infty \qquad \text{and} \qquad \textstyle\sum_1^\infty (d_n^2/n)\exp(-d_n) < \infty \tag{b}$$

since $d_n^2 \exp(-d_n) \leqq \lambda_n^2 \exp(-\lambda_n) + (2 \log_2 n)^2 \exp(-2 \log_2 n)$. Since $B_j \subset D_j \equiv [M_{n_j} \geqq (n_j/n_{j+1}) d_{n_{j+1}}]$, it suffices to show that $\sum_2^\infty P(D_j) < \infty$. Now

$$\sum_{j=2}^\infty P(D_j) \leqq \sum_{j=2}^\infty 16(n_j/n_{j+1}) d_{n_{j+1}} \exp(-d_{n_{j+1}}) \exp\left(\left(1 - \frac{n_j}{n_{j+1}}\right) d_{n_{j+1}}\right)$$

by (2)

$$\leqq (\text{Constant}_\alpha) \sum_{j=2}^\infty d_{n_{j+1}} \exp(-d_{n_{j+1}}) \quad \text{as in (2.45) of [6]}$$

$$< \infty \quad \text{in complete analogy with Lemma 8 of [6] and using (b).}$$

This completes the convergence half of the proof.

Suppose $\sum_1^\infty (\lambda_n^2/n) \exp(-\lambda_n) = \infty$. Note that $\|I/\Gamma_n\|_{\xi_{n1}}^1 \geqq n\xi_{n2}$, and Theorem 1(ii) shows that $P(n\xi_{n2} \geqq \lambda_n \text{ i.o.}) = 1$. □

REMARK. Now $\{n\xi_{n,i+1}/i \colon 1 \leqq i \leqq n\}$ is a reverse submartingale. This yields

$$P(\|I/\Gamma_n\|_{\xi_{n1}}^1 \geqq \lambda) \leqq \inf_{r>0} E(\exp(rn\xi_{n2}))/\exp(r\lambda) \leqq 14\lambda^2 \exp(-\lambda)$$

for all $\lambda > 1$. This will only yield $P(\|I/\Gamma_n\|_{\xi_{n1}}^1 \geqq \lambda_n \text{ i.o.}) = 0$ in Theorem 3 in case $\sum_1^\infty (\lambda_n^3/n) \exp(-\lambda_n) < \infty$.

PROOF OF THEOREM 1. (i) See Kiefer (1972). (ii) See Robbins and Siegmund (1971) for the case $k = 1$. See Frankel (1976) for a statement of this result when $k > 1$ and $c_n \nearrow \infty$; Frankel gives references to his 1972 thesis for a proof. It would appear that Frankel's technique is similar to that of Wichura (1973); using diffusion processes and speed measure, Wichura establishes some results very closely related to the present ones.

The authors' original version of this manuscript included a very long proof of Theorem 1(ii); it is available upon request. It uses only elementary techniques, and is a straightforward generalization of the proof of Robbins and Siegmund; the details are quite heavy. □

**3. The case of arbitrary** df's. Suppose $X_{n1}, \cdots, X_{nn}$ are independent with completely arbitrary df's $F_{n1}, \cdots, F_{nn}$ on $(-\infty, \infty)$. Let $\bar{F}_n = n^{-1} \sum_1^n F_{ni}$ denote the average df, and let $\mathbb{F}_n$ denote the empirical df of the observations.

THEOREM 4. *Let* $n\lambda_n \nearrow$. *Then* $\sum_{n=1}^\infty (n\lambda_n)^{-1} < \infty$ *implies* $P(\|\mathbb{F}_n/\bar{F}_n\| \geqq \lambda_n \text{ i.o.}) = 0$.

PROOF. By Theorem 1.1.1 and Corollary 1.3.1 of van Zuylen (1976) we have

$$P(\|\mathbb{F}_n/\bar{F}_n\| \geqq \lambda) \leqq 2\pi^2/3\lambda \qquad \text{for all } \lambda > 1\,. \tag{4}$$

We can now just recopy the proof of Theorem 2, except that an appeal to (4) replaces the appeal to (1). □

We did not generalize Theorem 3 to the present case. It is possible to obtain an exponential bound in place of the bound in van Zuylen's equation (1.1.4) by applying a binomial exponential bound to the probability $P(\sum_{i=1}^n z_i > n - j + 1)$ of his proof. However, the resulting bound is not as strong as (2); and so we omit the resulting weak generalization of Theorem 3 that we can prove in the present case.

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