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ON THE RANGE OF RECURRENT MARKOV CHAINS

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Let R_n be the number of distinct elements among $X_0, X_1, \cdots, X_n$, where $\{X_n\}$ is an irreducible recurrent Markov chain. It is shown that, under an appropriate condition, $n^{-1}R_n \to 0$ a.s. (P_a) where a is any state and P_a is conditional probability measure given $X_0 = a$. We prove that any recurrent random walk satisfies our condition, so that the result contains the well-known random walk case. We also give an example of an irreducible recurrent chain for which the result fails to hold.

1. One of the first results for the range of a random walk was that $n^{-1}R_n \to P$ (escape from 0) a.s.; this was proved for simple random walk in [4] and then extended to arbitrary walks by Kesten, Spitzer and Whitman ([10], page 38) using Birkhoff's ergodic theorem. There has been much recent work on the range of random walks—for references see [10], page 35—and of course most of the results depend strongly on the fact that one is dealing with sums of independent random variables. However, the use of the ergodic theorem to get a general statement about $n^{-1}R_n$ valid for all random walks suggests that there may be related theorems for more general stationary processes. In this paper we confirm this conjecture for irreducible recurrent Markov chains.

Markov chain terminology follows [3] or [5]. Let $\{X_n\}$ be an irreducible recurrent Markov chain on a countable state space S which may be considered to be a subset of the integers. It is well known [3] that such a chain possesses an essentially unique σ-finite stationary measure π_0 with $\sum_i \pi_0(i) \leqq \infty$, that is,

$$\sum_{i \in S} \pi_0(i) P_{ij} = \pi_0(j)$$

is true for all states i and j, where P_{ij} is the one step transition probability for the process. $\pi_0(i) > 0$ for each $i \in S$ by irreducibility. The process may be represented on bilateral coordinate space Ω of points $\omega = (\cdots, x_{-1}, x_0, x_1, \cdots)$ with entries from S; π_0 and P_{ij} induce on Ω and the product σ-field a measure π which turns out to be invariant relative to the shift T that takes ω into the point obtained by shifting each coordinate of ω one step to the left. See e.g., [1], [9] for fuller discussions of sequence spaces and shifts.

Define, in the same way as for random walk ([10], page 35), random variables R_n, $n \geqq 0$ equal to the number of distinct elements among $X_0, X_1, \cdots, X_n$; R_n is called the range of the Markov chain in time n (we do not, as is customary in the random walk case, necessarily require $X_0 = 0$). Crucial to the discussion

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is the following stopping time:

$$N = \{\text{first index } n > 0 \text{ such that } X_n = 0\}$$

where "0" represents any fixed state of the process. N is finite a.e. (π) by the assumptions on the chain. Now define

$$W = R_N \quad \text{if} \quad X_0 = 0$$
$$= 0 \quad \text{if} \quad X_0 \neq 0 . \tag{1.1}$$

P_a will denote conditional probability measure, given $X_0 = a$. The main result is:

THEOREM 1. *Let $\{X_n\}$ be an irreducible recurrent Markov chain, and let π and W be as defined above. Then a sufficient condition for*

$$\lim_{n\to\infty} \frac{R_n}{n} = 0 \quad \text{a.s.} \quad (P_a) \tag{1.2}$$

for all states $a \in S$, is that

$$E_\pi W < \infty . \tag{1.3}$$

If π is a finite measure, then automatically (1.3) is satisfied, so that (1.2) is true for all Markov chains possessing a stationary probability measure.

REMARKS. Since $W = 0$ on the set $X_0 \neq 0$, (1.3) is the same as $E_{P_0} W < \infty$. Moreover, it is clear that π may replace P_a in (1.2). Theorem 1 contains the result for recurrent random walks by Theorem 2 and its corollary. The example of Section 3 shows that (1.2) may fail if (1.3) does not hold. Although we have not investigated the situation, we conjecture that an example may be constructed showing that (1.2) and (1.3) are not equivalent. Henceforth "E" without subscript will denote "E_π."

PROOF OF THEOREM 1.

CASE 1. $\pi(\Omega) = 1$. Notice that $W \leqq N \cdot 1_{(X_0=0)}$, and so $E_\pi W \leqq E_{P_0} N =$ mean recurrence time to $0 < \infty$ [5], page 356. This shows that (1.3) is automatically satisfied for finite π.

Let $\varphi_0 = 1$ and $\varphi_k(X_0, X_1, \cdots, X_k) = 1$ if $X_k \neq X_\nu$ for all $\nu = 0, 1, \cdots, k-1$, $k \geqq 1$ (see [10], page 36). Then

$$R_n = \sum_{k=0}^n \varphi_k .$$

Using the bilateral stationarity

$$E\varphi_k = \pi(X_k \neq X_{k-1}, X_k \neq X_{k-2}, \cdots, X_k \neq X_0)$$
$$= \pi(X_0 \neq X_{-1}, X_0 \neq X_{-2}, \cdots, X_0 \neq X_{-k}) \to 0$$

as $k \to \infty$ by recurrence (the reversed process is also recurrent). Therefore

$$n^{-1}ER_n = n^{-1} \sum_{k=0}^n E\varphi_k \to 0 . \tag{1.4}$$

Now let M be a fixed large positive integer. Define (compare [10], page 38)

$Z_k(M) =$ the number of distinct elements among $X_{kM}, X_{kM+1}, \cdots, X_{(k+1)M-1}$

for $k \geqq 0$. For the shift T we have

$$T^M Z_k(M) = Z_{k+1}(M)\,, \qquad k \geqq 0\,.$$

Moreover

$$R_n \leqq \sum_{k=0}^{[n/M]+1} Z_k(M) \tag{1.5}$$

where $[x]$ is the largest integer $\leqq x$. Apply Birkhoff's ergodic theorem on T^M to obtain

$$\limsup_{n\to\infty} n^{-1}R_n \leqq \lim_{n\to\infty} M^{-1}(Mn^{-1}) \sum_{k=0}^{[n/M]+1} Z_k(M) = M^{-1}Z^* \quad \text{a.s.} \quad (\pi)$$

where Z^* is a function satisfying $EZ^* = EZ_0(M)$. This implies

$$E \limsup_{n\to\infty} n^{-1}R_n \leqq M^{-1}EZ^* = M^{-1}EZ_0(M) = M^{-1}ER_{M-1}\,,$$

and letting $M \to \infty$ (1.4) proves $E \limsup_{n\to\infty} n^{-1}R_n = 0$, that is, $n^{-1}R_n \to 0$ a.s. (π). $\pi_0(a) > 0$ for each $a \in S$, so that the proof is complete for finite π.

CASE 2. $\pi(\Omega) = \infty$. This case is most easily handled by proving a series of lemmas. Define the successive hitting times of 0, that is, let $N_1 = N$, and let N_k be given by induction as the smallest integer $n > N_{k-1}$ with $X_n = 0$.

LEMMA 1. $\lim_{k\to\infty} k/N_k = 0$ a.e. (π).

PROOF. Let $V_j = 1_{(X_j=0)}$. Apply the ergodic theorem, the metric transitivity of T (by irreducibility, see e.g., Section 4 of [7]) and the infiniteness of π to obtain

$$n^{-1} \sum_{j=1}^{n} V_j \to 0 \quad \text{a.e.} \quad (\pi)\,. \tag{1.6}$$

Choosing the subsequence $\{N_k\}$, (1.6) yields

$$N_k^{-1} \sum_{j=1}^{N_k} V_j = N_k^{-1} \cdot k \to 0 \quad \text{a.e.} \quad (\pi)$$

proving the lemma.

Define the transformation T^N on almost all points of Ω by:

$$T^N(\omega) = T^{N(\omega)}(\omega)\,.$$

It can be shown that T^N is an a.e. measurable invertible transformation on the set $\{X_0 = 0\}$ which preserves π restricted to this set (see e.g., [6]). Let W be the random variable (1.1). Define

$$(T^N)^j W = W_j\,, \qquad j \geqq 0\ (W = W_0)\,.$$

LEMMA 2. $\lim_{k\to\infty} R_{N_k}/N_k = 0$ a.s. (P_0).

PROOF. In analogy to (1.5) write

$$1_{(X_0=0)} R_{N_k} \leqq \sum_{j=0}^{k} W_j\,.$$

Apply the ergodic theorem to T^N with reference to π restricted to $\{X_0 = 0\}$, obtaining

$$\limsup_{k\to\infty} k^{-1} 1_{(X_0=0)} R_{Nk} \leqq k^{-1} \sum_{j=0}^{k} W_j \to f \quad \text{a.e.} \quad (\pi) \tag{1.7}$$

for f an a.e. finite valued function. Therefore,

$$1_{(X_0=0)} N_k^{-1} R_{N_k} = (1_{(X_0=0)} k^{-1} R_{N_k}) \cdot (N_k^{-1} k) \to 0 \quad \text{a.e.} \quad (\pi) \tag{1.8}$$

as $k \to \infty$ by (1.7) and Lemma 1. This translates to the assertion of Lemma 2.

LEMMA 3. *There is a fixed set F with $P_0(F) = 1$ such that if $k(n)$ is any subsequence of the integers*

$$\lim_{n\to\infty} \frac{R_{N_{k(n)}}(\omega)}{N_{k(n)-1}(\omega)} = 0$$

for all $\omega \in F$.

PROOF. There is a P_0 full set F_1 on which all hitting times N_k are defined; there is a P_0 full set F_2 on which the convergence of Lemma 1 takes place; there is a P_0 full set F_3 on which the convergence of (1.7) takes place. Let $F = F_1 \cap F_2 \cap F_3$. For $\omega \in F$ (1.7) may be altered slightly to give

$$((k(n) - 1)^{-1}) 1_{(X_0=0)} R_{N_{k(n)}} \leqq (k(n) - 1)^{-1} \sum_{j=0}^{k(n)} W_j \to f$$

and then a variation of (1.8) completes the proof.

LEMMA 4. $\lim_{n\to\infty} R_n/n = 0$ a.s. (P_0).

PROOF. We show convergence on the set F of Lemma 3. For fixed $\omega \in F$ let $k(n)$ be the (random) subsequence defined by

$$k(n) = \text{the unique index} \quad j \quad \text{with} \quad N_{j-1} \leqq n < N_j .$$

Clearly

$$\frac{R_n(\omega)}{n} \leqq \frac{R_{N_{k(n)}}(\omega)}{N_{k(n)-1}(\omega)} \to 0$$

by Lemma 3 for $\omega \in F$, proving the lemma.

LEMMA 5. (1.2) *holds.*

PROOF. If (1.2) fails on a set of positive P_a measure for some $a \in S$, since almost all paths starting from a eventually visit 0 in finite time, (1.2) would then have to fail on some set of positive P_0 measure. This is impossible by Lemma 4.

2. In this section it will be shown that condition (1.3) is satisfied for recurrent random walk. We actually obtain a much stronger statement involving exponential convergence of the tail of the distribution of W.

THEOREM 2. *Let X_n be a recurrent random walk, and let h be a fixed positive integer satisfying $P_0(N \geqq h) = \gamma < 1$. Then*

$$P_0(W \geqq nh) \leqq \gamma^n , \qquad n \geqq 1 . \tag{2.1}$$

PROOF. Recurrent random walk has the stationary measure $\pi_0(i) = 1$ for each i. The following arguments will involve computations with respect to the measure π obtained from this π_0 and the transition probabilities as described in Section 1.

First, we make an observation: let U and V be a.e. finite valued stopping times ([1], page 131) satisfying $U = V = 0$ on $X_0 \neq 0$, and $1 \leqq U < V$ on $X_0 = 0$. Let

Λ be any set measurable with respect to X_j, $U \leqq j \leqq V$. Then

$$\begin{aligned}(2.2) \qquad \pi(\Lambda, X_U = a, X_V = b) &= \pi(\Lambda, X_V = b \mid X_U = a) \\ &= \pi(\Lambda, X_U = a \mid X_V = b),\end{aligned}$$

for every fixed $a, b \in S$, $a \neq b$. To prove this, note

$$\begin{aligned}\pi(X_U = a, U > 0) &= \textstyle\sum_{j\geqq 1} \pi(U = j)\pi(X_U = a \mid U = j) \\ (2.3) \qquad &= \textstyle\sum_{j\geqq 1} \pi(U = j)\pi(X_j = a) = \sum_{j\geqq 1} \pi(U = j) \\ &= \pi(X_0 = 0) = 1\,.\end{aligned}$$

Moreover, since $a \neq b$

$$\pi(\Lambda, U = 0, X_U = a, X_V = b) = \pi(\Lambda, U = 0, X_0 = a, X_0 = b) = 0\,.$$

Therefore,

$$\begin{aligned}\pi(\Lambda, X_U = a, X_V = b) &= \pi(\Lambda, U > 0, X_U = a, X_V = b) \\ &= \pi(X_U = a, U > 0)\pi(\Lambda, X_V = b \mid X_U = a, U > 0) \\ &= \pi(\Lambda, X_V = b \mid X_U = a),\end{aligned}$$

this last equality obtained from (2.3) and an easy generalization of a Markov property (see [3], page 5, relation (4)) to the strong Markov case. This proves one equality in (2.2). The other is proved in the same way by applying the above arguments to the reversed random walk.

The proof is by induction. A fixed positive h satisfying $P_0(N \geqq h) = \gamma < 1$ surely exists. As noted previously $W \leqq N \cdot 1_{(X_0=0)}$, so that

$$P_0(W \geqq h) \leqq P_0(N \geqq h) = \gamma$$

proving (2.1) for $n = 1$. Let us define

$$U_1 = \text{first index} \quad n > 0 \quad \text{with} \quad X_n \neq X_\nu \quad \text{for all} \quad n > \nu \geqq 0\,; \quad X_0 = 0\,,$$

and in general,

$$U_j = \text{first index} \quad n > U_{j-1} \quad \text{with} \quad X_n \neq X_\nu \quad \text{for all} \quad n > \nu \geqq 0\,; \quad X_0 = 0\,,$$

and let $U_j = 0$ on $X_0 \neq 0$ for all $j \geqq 1$.

By the strong Markov property

$$\begin{aligned}&\pi(W \geqq nh) \\ (2.4) \qquad &= \textstyle\sum_{\rho_i \neq 0} \sum_{\omega_i \neq 0} \pi(X_j \neq 0, 0 < j \leqq U_{(n-1)h}, X_{U_{(n-1)h}} = \omega_i \mid X_0 = 0)\,. \\ &\pi(X_j \neq 0, \rho_i; U_{(n-1)h} \leqq j < U_{nh}, X_{U_{nh}} = \rho_i \mid X_{U_{(n-1)h}} = \omega_i)\,. \\ &\pi(X_j \neq 0, U_{nh} \leqq j < N, X_N = 0 \mid X_{U_{nh}} = \rho_i) = \textstyle\sum\sum \text{(I)} \cdot \text{(II)} \cdot \text{(III)}\,.\end{aligned}$$

From (2.2), term III $= \pi(X_j \neq 0, U_{nh} \leqq j < N, X_{U_{nh}} = \rho_i \mid X_N = 0)$. Again using (2.2) and homogeneity of random walk we obtain

$$\begin{aligned}\text{(II)} &= \pi(X_j \neq 0, \rho_i; U_{(n-1)h} \leqq j < U_{nh}, X_{U_{(n-1)h}} = \omega_i, X_{U_{nh}} = \rho_i) \\ &\leqq \textstyle\sum_{k\geqq 1} \pi(U_{nh} = k)\pi(X_j \neq 0, \rho_i; U_{(n-1)h} \leqq j < U_{nh}, U_{nh} = k, X_k = \rho_i) \\ &\leqq \textstyle\sum_{k\geqq 1} \pi(U_{nh} = k)\pi(X_j \neq 0, U_{(n-1)h} \leqq j < U_{nh}, U_{nh} = k, X_k = 0)\end{aligned}$$

$$= \sum_{k\geq 1} \pi(U_{nh} = k)\pi(X_j \neq 0,\ U_{(n-1)h} \leqq j < U_{nh},\ U_{nh} = k \mid X_k = 0)$$
$$\leqq \sum_{k\geq 1} \pi(U_{nh} = k)\pi(\text{reversed random walk starting from } 0 \text{ avoids } 0 \text{ for at least } h \text{ consecutive steps})$$
$$= \sum_{k\geq 1} \pi(U_{nh} = k)\cdot\gamma = \gamma\ ,$$

where the last step uses the easily verified fact that the distribution of the first return time to 0 is the same for the reversed walk as it is for the forward one. Summing first over ρ_i in (2.4) we obtain

$$\pi(W \geqq nh) \leqq \sum_{\omega_i\neq 0} \pi(X_j \neq 0,\ 0 \leqq j \leqq U_{(n-1)h},\ X_{U_{(n-1)h}} = \omega_i \mid X_0 = 0)\cdot\gamma$$
$$= \gamma\pi(W \geqq (n-1)h) = \gamma\cdot\gamma^{n-1} = \gamma^n$$

by the induction assumption. Since $\pi(W \geqq nh) = P_0(W \geqq nh)$, the proof is complete.

COROLLARY 1. *Recurrent random walks satisfy* (1.3).

PROOF. From (2.1) it is easy to see that

$$\sum_{n=1}^{\infty} P_0(W \geqq n) \leqq h + \sum_{n=0}^{\infty} h\gamma^n < \infty\ .$$

This is sufficient to imply $E_\pi W = E_{P_0} W < \infty$, as follows, e.g., from [8], page 111.

3. We exhibit an example showing that (1.2) may fail for an irreducible recurrent Markov chain.

Define a subsequence $u(n)$, $n \geqq 1$, by induction as follows: $u(1) = 1$, and given $u(n)$, let $u(n+1) > u(n)$ be selected so that $u(n+1)/u(n) = r(n)$, an integer, and

$$2^n(1 - 2^{-n})^{r(n)} < n^{-2}\ .$$

We define a chain with behavior described roughly as follows: a particle starting at 0 moves deterministically one step to the right unless it is at $u(n)$; at $u(n)$ it may return to 0 or else continue moving to the right, each choice having positive probability. More precisely, the state space is the set of nonnegative integers with one step transition function p_{ij} given by:

$$p_{i,i+1} = 1\ , \qquad i \neq u(n),\ n \geqq 1$$
$$p_{u(n),0} = 2^{-n}\ ,$$
$$p_{u(n),u(n)+1} = 1 - 2^{-n}\ , \qquad n \geqq 1\ .$$

This chain is easily seen to be irreducible and recurrent. Let $\{X_n\}$ be the variables of the process. Define the events

$$A_k = \{X_0 = 0,\ X_{u(k)+1} = 0 \text{ and } X_j \neq 0 \text{ for } 1 \leqq j < u(k) + 1\}$$
$$A_{n+1}^* = \{X_0 = 0,\ \text{first return to } 0 \text{ occurs before the first hitting of } u(n+1)\}\ ,$$

observe

$$A_{n+1}^* = \bigcup_{k=1}^{n} A_k\ , \qquad \text{and so}$$
$$P(A_{n+1}^*) = P(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k) \leqq \sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}\ .$$

Since the process "starts afresh upon returning to 0," the resulting independence yields

$$\begin{aligned}(3.1)\quad & P\{X_0 = 0,\ \text{at least}\ r(n)\ \text{returns to}\ 0,\ \text{each such return before} \\ & \quad \text{first hitting}\ u(n+1)\} \\ & \leqq \textstyle\sum_{r=r(n)}^{\infty} (1-2^{-n})^r = (1-2^{-n})^{r(n)}[1 + (1-2^{-n}) + \cdots] \\ & = 2^n(1-2^{-n})^{r(n)} < n^{-2},\end{aligned}$$

and, calling W_n the event on the left side of (3.1),

$$\textstyle\sum_{n=1}^{\infty} P(W_n) < \sum_{n=1}^{\infty} n^{-2} < \infty,$$

proving that W_n occurs at most finitely often for almost all sample sequences. Now let

$$t_n(\omega) = \text{first hitting time of}\ \ u(n+1).$$

Then

$$\frac{R_{t_n}}{t_n} = \frac{u(n+1)}{t_n}$$

where

$$(3.2)\qquad t_n(\omega) = l_n(\omega) + u(n+1).$$

Now suppose that

$$(3.3)\qquad \frac{R_{t_n}}{t_n} < \tfrac{1}{2}.$$

Then $u(n+1) < l_n(\omega)$, from (3.2). Since, by (3.2), the excess $l_n(\omega)$ is the time spent on the way to $u(n+1)$ in states that were already visited, it follows that there must be at least

$$\frac{l_n(\omega)}{u(n)} > \frac{u(n+1)}{u(n)} = r(n)$$

returns to the origin before reaching $u(n+1)$. According to (3.1) this can happen only finitely often a.s., so that (3.3) can happen only finitely often a.s., and thus $\limsup n^{-1}R_n \geqq \frac{1}{2}$ a.s., showing the failure of (1.2). The preceding example can be modified to give an irreducible recurrent Markov chain for which $\limsup n^{-1}R_n = 1$ a.s.

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