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ON THE INDEPENDENCE OF VELOCITIES IN A SYSTEM OF NONINTERACTING PARTICLES

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Consider a system of noninteracting particles moving at constant speeds in a Euclidean space. It is shown that, if at two or more epochs the velocities are independent of the positions and are independent and identically distributed, then the positions must be given by a Cox process of a very special structure.

1. Main result. Consider a particle system in R^d ($d \in N$) which is locally finite at each time, and let the positions at time 0 be given by a point process ξ. Further suppose that each particle moves with constant speed, and that the velocities at time 0 are independent of the positions and distributed according to some common nondegenerate probability measure μ on R^d. At first sight, one might expect the velocities to remain independent and identically distributed independently of the positions, but it turns out that the independence will normally be destroyed. In fact, we shall prove that independence with a common velocity distribution at two or more epochs implies that ξ is a Cox process (cf. [2]) of a very special structure. This strengthens a result by Thedéen ((1967), Corollary 5.1) (cf. Theorem 5.3 in [3]), who draws an equivalent conclusion under the additional hypothesis of time stationarity of the process of positions and velocities.

For a precise statement of our result, let S be the support of μ, and write H for the closed additive group generated by S-S. Note that H has more than one point since μ is nondegenerate. The positions and velocities of the particles at a fixed epoch are most conveniently described by a point process in the *phase space* $R^{2d} = R^d \times R^d = \{(p, q)\}$, where p and q denote the velocity and position of a particle. It is sometimes helpful to consider the evolution of the system as a line process in the *space-time diagram* $R^{d+1} = R^d \times R = \{(q, t)\}$. Note that a time shift corresponds to a p-preserving linear shear in the phase space and to a translation in the space-time diagram.

Theorem. *Let $t \neq 0$ be arbitrary but fixed. Then the velocities of the particles at time t are independent of the positions and mutually independent with a common distribution*, iff ξ *is a Cox process directed by some random measure η which is* a.s. *tH-invariant, apart from a factor of the form e^{rq} with $r \in R^d$.*

Note that, if the hypothesis of the theorem is fulfilled for two rationally

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independent values of t, then $e^{-rq}\eta(dq)$ must be a.s. $\mathscr{L}(H)$-invariant, where $\mathscr{L}(H)$ denotes the linear space spanned by H. In particular, $e^{-rq}\eta(dq)$ is a.s. proportional to Lebesgue measure whenever $\mathscr{L}(H) = R^d$, which occurs when μ is truly d-dimensional. In this case it is easily seen by looking at the shears in the phase space that time stationarity of the process of positions and velocities is only possible when $r = 0$. Thus Thedéen's result follows from ours:

COROLLARY. *Suppose that μ is truly d-dimensional. Then the system of positions and velocities is time stationary* iff *ξ is a mixed Poisson process.*

Various aspects of the asymptotic behavior of noninteracting particle systems (or the corresponding line processes) have been examined in the literature (see, e.g., [1, 7, 8, 9]), usually under the assumption of initial independence between positions and velocities, a rather dubious condition in view of our theorem. The case of general dependence will be discussed in [5].

The proof of our theorem will be given in Section 3 below, after some auxiliary results of some independent interest have been presented in Section 2. For the sake of brevity, we refer to [2] for general terminology and notation.

2. Random thinnings and displacements. Our first result applies to point processes on an arbitrary locally compact second countable Hausdorff space. Here and below, a slight abuse of language will be convenient. We shall say that ξ is a Cox process directed by η or an a-thinning of η, if *its distribution* is that of a Cox process or an a-thinning respectively. Thus such a phrase says nothing about the joint distribution of ξ and η.

LEMMA 1. *Let ξ_n be an a_n-thinning of η_n for $n \in N$, and suppose that two of the following statements hold*:

(i) $\xi_n \to_d$ *some* $\xi \neq_d 0$; (ii) $\eta_n \to_d$ *some* $\eta \neq_d 0$; (iii) $a_n \to$ *some* $a > 0$.

Then the third statement is also true, and ξ is an a-thinning of η.

PROOF. If ξ is an a-thinning of η, then the corresponding Laplace transforms L_ξ and L_η are related by

$$L_\xi(f) = L_\eta(-\log[1 - a(1 - e^{-f})]), \tag{1}$$

(cf. [2], page 9). Thus the implication (ii) + (iii) $\Rightarrow$ (i) with the asserted connection between ξ, η, and a follows immediately from Theorem 4.2 in [2], even without the assumptions $\eta \neq_d 0$ and $a > 0$. (This part of the lemma is actually a special case of Theorem 8.1 in [2].) To prove that (i) + (ii) $\Rightarrow$ (iii) suppose that $a_n \to a$ along some subsequence, and conclude as above that (1) holds. But this determines a uniquely, provided $\eta \neq_d 0$. Finally suppose that (i) and (iii) hold. If $\eta_n \to_d \eta$ along some subsequence, then it is seen as above that ξ is an a-thinning of η, and since $a > 0$, this determines $P\eta^{-1}$ on account of Corollary 3.2 in [2]. It remains to prove that $\{\eta_n\}$ is relatively compact in distribution. But if it were not, then neither is $\{\xi_n\}$, since $\liminf a_n > 0$. $\square$

We shall write $\xi \sim \eta$ whenever ξ is an a-thinning of η or conversely, for some $a > 0$. Note that $\sim$ is an equivalence relation. By Lemma 1, the equivalence is persistent under convergence in distribution towards nonzero limits.

We shall further need the following extension to the lattice case of Dobrushin's theorem, (cf. [8] as well as Theorem 6.4.3 in [6]). Let ξ, μ, S and H be such as in Section 1, and let $\mathscr{I}_H$ denote the σ-field of those events which are invariant under H-translations.

LEMMA 2. *Let ξ be H-stationary, and suppose that $\eta = E[\xi \mid \mathscr{I}_H]$ is* a.s. *locally finite. Further suppose that $S \subset H$, and let $\xi_1, \xi_2, \cdots$ be obtained from ξ by successive independent displacements of the particles according to the distribution μ. Then ξ_n tends in distribution to a Cox process directed by η.*

Since this result may be established by exactly the same arguments as those employed in Section 1 of [4], we omit the proof.

We next state a simple continuity property for random displacements, (which extends with the same proof to general cluster fields).

LEMMA 3. *For $n \in N \cup \{\infty\}$, let ξ_n be obtained from ξ by independent displacements of the particles according to some distribution μ_n. Further suppose that the μ_n have uniformly bounded supports, and that $\mu_n \to_w \mu_\infty$. Then $\xi_n \to_d \xi_\infty$.*

PROOF. By conditioning, we may assume that ξ is nonrandom, and by a well-known theorem of Skorohod, we may further assume that the displacements converge a.s. Since the boundedness assumption implies that only finitely many atoms of ξ can contribute to $\{\xi_n B\}$ for every bounded set B, it follows easily that $\xi_n \to_v \xi_\infty$ a.s., which implies the asserted convergence in distribution. □

For the last result, let the subscripts a and μ denote the operations of a-thinning and independent displacements according to μ, respectively.

LEMMA 4. $(\xi_a)_\mu =_d (\xi_\mu)_a$.

PROOF. By Proposition 5.2.4 in [6], it is enough to consider the case when $\xi = \delta_0$. But then the statement follows trivially from the independence between thinning and displacement. □

3. Proof of the main result. Assume without loss that $t = 1$ and $\xi \neq_d 0$, and further that $S \subset H$. (If $S \not\subset H$, we may apply the statement for $S \subset H$ to the process defined by using $\delta_{-s} * \mu$ instead of μ for some fixed $s \in S$.) Let ν be the distribution of velocities at time 1 (which need not coincide with μ), and use the subscripts a and μ as in Lemma 4.

Because of the independence assumption, the position processes of particles at times 0 and 1 and with velocities in B are distributed like $\xi_{\mu B}$ and $(\xi_\mu)_{\nu B}$ respectively. Hence by Lemma 4, writing $B\mu$ for the restriction of μ to B and assuming that $\mu B > 0$,

$$(\xi_\mu)_{\nu B} =_d (\xi_{\mu B})_{B\mu/\mu B} =_d (\xi_{B\mu/\mu B})_{\mu B}\,,$$

which proves that $\xi_{B\mu/\mu B} \sim \xi_\mu$ whenever $\mu B > 0$. For any $s \in S$, let $B \downarrow \{s\}$ boundedly, and conclude from Lemma 3 and the remark following Lemma 1 that $\delta_s * \xi = \xi_{\delta_s} \sim \xi_\mu$. Since $s \in S$ was arbitrary, this proves that

$$\delta_h * \xi \sim \xi\,, \qquad h \in H\,, \tag{2}$$

$$\xi_\mu \sim \xi\,. \tag{3}$$

Let us first assume that $\delta_h * \xi =_d \xi$ for all $h \in H$, i.e., that ξ is H-stationary. Reversing time if necessary, we may further conclude from (3) that $\xi_\mu =_d \xi_a$ for some $a \leqq 1$. By Lemma 4 we get for all $n \in N$

$$\xi_{\mu^n} = (\xi_\mu)_{\mu^{n-1}} =_d (\xi_a)_{\mu^{n-1}} =_d (\xi_{\mu^{n-1}})_a\,,$$

(μ^n denoting a convolution power of μ), and hence by induction

$$\xi_{\mu^n} =_d \xi_{a^n}\,, \qquad n \in N\,. \tag{4}$$

But by Lemma 2, the sequence on the left is relatively compact in distribution iff $\eta = E[\xi \mid \mathscr{I}_H]$ is locally finite, and in that case

$$\xi_{\mu^n} \to_d \hat{\eta}\,, \tag{5}$$

where $\hat{\eta}$ is a Cox process directed by η. Since $\xi_{a^n} \to_d 0$ for $a < 1$, (5) is consistent with (4) only if $a = 1$, and in that case (4) and (5) yield $\xi =_d \hat{\eta}$, as desired.

Next suppose that ξ is an a-thinning of $\delta_h * \xi$ for some $h \in H$ and $a < 1$. By iterating this result, it is seen that ξ is an a^n-thinning for each $n \in N$, and so it follows by Corollary 8.5 in [2] that ξ is a Cox process. Then so is the phase space process ξ^* of positions and velocities at time 0, and if ξ is directed by the random measure η, than ξ^* is directed by $\eta \times \mu$. (There are many ways to see this, one being to verify from the form of the Laplace transforms (cf. [2], page 8) that an f-thinning of ξ is a Cox process directed by $f\eta$, even with f interpreted as a function on R^d. It follows that $P\{\xi^* B = 0\} \equiv Ee^{-(\eta\times\mu)B}$ for arbitrary Borel sets $B \subset R^{2d}$, which is enough by Theorem 3.3 in [2].) As pointed out above, the state $(\xi_\mu)^*$ of the system at time 1 is obtained from ξ^* by a p-preserving linear shear φ, and since measurable mappings preserve the Cox structure, it follows that $(\xi_\mu)^* \equiv \xi^* \varphi^{-1}$ is a Cox process directed by $(\eta \times \mu)\varphi^{-1}$. Applying the same argument as above to $\xi^*\varphi^{-1}$, it is seen that, if η_μ denotes the directing random measure of the Cox process ξ_μ, then

$$(\eta \times \mu)\varphi^{-1} = \eta_\mu \times \nu\,. \tag{6}$$

We may finally conclude from (2) that, for some positive constants a_h, $h \in H$,

$$\delta_h * \eta =_d a_h \eta\,, \qquad h \in H\,. \tag{7}$$

Now $H = H_1 \times H_2$, where H_1 and H_2 are isomorphic to R^k and Z^l for some $k, l \in Z_+$ with $1 \leqq k + l \leqq d$. For convenience of notation, we assume without loss that the $k + l$ first coordinate vectors $h_1, \cdots, h_{k+l}$ of R^d span H_1 and H_2.

Let r be the vector in R^d with components

$$(8)\qquad r_j = -\log a_{h_j}, \quad j = 1, \cdots, k + l; \qquad r_{k+l+1} = \cdots = r_d = 0,$$

and define the random measure ζ on R^d by $\zeta = e^{-rq}\eta$, i.e.,

$$\zeta(dq) = e^{-rq}\eta(dq), \qquad q \in R^d.$$

By (7) we then obtain for any $h \in H$

$$\delta_h * \zeta = \delta_h * (e^{-rq}\eta) = e^{-r(q-h)}\delta_h * \eta =_d e^{-r(q-h)}a_h\eta = a_h e^{rh}\zeta,$$

so

$$(9)\qquad \delta_h * \zeta =_d a_h e^{rh}\zeta \equiv a_h'\zeta, \qquad h \in H.$$

Now $a_h' = 1$ for $h = h_1, \cdots, h_{k+l}$ by (8), and furthermore a_h' is continuous in h and satisfies $a'_{g+h} \equiv a_g'a_h'$, so (9) reduces to $\delta_h * \zeta =_d \zeta$, $h \in H$, which means that ζ is H-stationary.

Since $\varphi(p, q) \equiv (p, q + p)$ and hence $\varphi^{-1}(p, q) \equiv (p, q - p)$, is follows from (6) that

$$\begin{aligned}(\zeta \times \mu)\varphi^{-1} &= (e^{-rq}\eta \times \mu)\varphi^{-1} = e^{-r(q-p)}(\eta \times \mu)\varphi^{-1} = e^{-r(q-p)}(\eta_\mu \times \nu)\\ &= (e^{-rq}\eta_\mu) \times (e^{rp}\nu),\end{aligned}$$

so $(\zeta \times \mu)\varphi^{-1}$ is like $(\eta \times \mu)\varphi^{-1}$ a product measure. Here the factor $e^{rp}\nu$ may be unbounded, but we may then consider the measure $(\zeta \times C\mu)\varphi^{-1}$, which admits a similar factorization with a normalizable second factor, provided C is bounded with $\mu C > 0$. Letting $\mathfrak{Z}$ be a Cox process directed by $\zeta \times C\mu$, it is seen that $\mathfrak{Z}$ fulfills all the requirements imposed on ξ^*, and that in addition $\mathfrak{Z}$ is H-stationary. We may then conclude from the first part of the proof and from Corollary 3.2 in [2] that ζ is distributed like an a.s. H-invariant random measure. Since the set of H-invariant measure is measurable, this shows that ζ itself is a.s. H-invariant.

This completes the proof of the necessity. To prove the sufficiency suppose that ξ is a Cox process directed by $\eta = e^{rq}\zeta$, where ζ is a.s. H-invariant. Then a.s.

$$(\eta \times \mu)\varphi^{-1} = (e^{rq}\zeta \times \mu)\varphi^{-1} = e^{r(q-p)}\zeta(dq - p)\mu(dp) = e^{rq}\zeta(dq)e^{-rp}\mu(dp),$$

since $\zeta(dq - p) = \zeta(dq)$ for $p \in H$ and outside a fixed null-event. Moreover, the measure $e^{-rp}\mu$ can be normalized, say to ν, since ξ_μ is locally finite by assumption. Hence the velocities at time 1 are mutually independent and independent of the positions with ν as a common distribution. □

REFERENCES

[1] BREIMAN, L. (1963). The Poisson tendency in traffic distribution. *Ann. Math. Statist.* **34** 308-311.

[2] KALLENBERG, O. (1976a). *Random Measures*. Akademie-Verlag and Academic Press, Berlin-London.

[3] KALLENBERG, O. (1976b). On the structure of stationary flat processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **37** 157–174.
[4] KALLENBERG, O. (1977). Stability of critical cluster fields. *Math. Nachr.* **77** 7–43.
[5] KALLENBERG, O. (1978). On the asymptotic behavior of line processes and systems of non-interacting particles. *Z. Warscheinlichkeitstheorie und Verw. Gebiete* **43** 65–95.
[6] KERSTAN, J., MATTHES, K. and MECKE, J. (1974). *Unbegrenzt teilbare Punktprozesse.* Akademie, Berlin.
[7] PAPANGELOU, F. (1974). On an unfinished manuscript of R. Davidson's. In *Stochastic Geometry* (E. F. Harding and D. G. Kendall, eds.), 252–255. Wiley, London.
[8] STONE, C. (1968). On a theorem by Dobrushin. *Ann. Math. Statist.* **39** 1391–1401.
[9] SZATZSCHNEIDER, W. (1975). A more deterministic version of Harris–Spitzer's "random constant velocity" model for infinite systems of particles. In *Probability Winter School, Karpacz.* Lecture Notes in Math. No. 472. Springer, Berlin.
[10] THEDÉEN, T. (1967). Convergence and invariance questions for point systems in R_1 under random motion. *Ark. Mat.* **7** 211–239.

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