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A CHARACTERIZATION OF VITALI CONDITIONS IN TERMS OF MAXIMAL INEQUALITIES[1]

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Vitali conditions V, V', V_p, $1 < p < \infty$, on σ-algebras indexed by a directed set, are shown to hold if and only if the maximal inequality

(1) $$P(\text{essential lim sup } X_t > \alpha) < K \text{ lim sup}_{T^*} E(X_\tau)/\alpha$$

holds for all adapted positive processes (X_t), and all positive numbers α. Here K is a constant which may be taken equal to 1, and T^* is the appropriate directed set of stopping times: for V, T^* is the set of simple stopping times; for V', T^* is the set of simple ordered stopping times; for V_p, T^* is the set of multivalued stopping times with overlap going to zero in L_p. The inequality (1) is true whatever be the σ-algebras, provided that essential lim sup is replaced by stochastic lim sup.

The Vitali conditions $V\ (= V_\infty)$, V', and V_p, $1 \leqslant p < \infty$, on σ-algebras indexed by a directed set J, are shown to hold if and only if maximal inequalities of the form

(1) $$P(e \text{ lim sup } X_t \geqslant \lambda) \leqslant \frac{C}{\lambda}$$

hold for all adapted positive processes (X_t) and all positive numbers λ. Here e lim sup is essential lim sup, and C is a constant of the form $K \text{ lim sup}_{T^*} E(X_\tau)$, where K is a constant which may be chosen equal to 1, and essential lim sup is taken over the appropriate directed set of stopping times: for V, $T^* = T$, the set of simple stopping times; for V', $T^* = T'$, the set of simple ordered (i.e., the range is ordered) stopping times; for V_p, $T^* = M_p$, the set of "multivalued stopping times with overlap going to 0 in L^p". Precise definitions are given below.

The asymptotic character of the maximal inequality (1) should not surprise, because Vitali conditions are asymptotic, but also the more usual (but not more useful in convergence proofs) form of (1), with sup replacing lim sup, is briefly discussed, and shown to correspond to nonasymptotic variations of Vitali conditions. The inequality (1) seems new even for martingales and submartingales, for which the right-hand side simplifies, but a particular case—martingales with countable index set—was recently proved by Gabriel [9].

Vitali conditions are introduced to insure *essential* convergence of classes of random variables-martingales, submartingales, appropriate classes of amarts, etc. (see [12], [10], [1], [15], [16]). *Stochastic* convergence holds for the same classes

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whatever be the structure of the σ-algebras, and therefore a stochastic maximal inequality may be expected to hold without any Vitali conditions. Such an inequality, featuring the stochastic lim sup, is indeed proved in Section 1. In Section 2 we continue to look at the stochastic lim sup, in particular observing that for "stochastically closed" classes of random variables, stochastic convergence implies a stochastic maximal inequality. This is only a slight modification of Burkholder's elegant theory [2] connecting a.s. convergence to a.s. maximal inequalities, but also some applications are given. In Section 3 maximal inequalities with essential lim sup are shown to be equivalent with V; so is the relation: stochastic $\lim_T \sup X_\tau$ = essential $\lim_J \sup X_t$. Analogous results for V' and V_p, $1 \leqslant p < \infty$, are given in Section 4.

1. Stochastic maximal inequalities. Let J be a directed set filtering to the right (i.e., a set of indices partially ordered by $\leqslant$, such that for each pair t_1, t_2 of elements of J, there exists an element t_3 of J such that $t_1 \leqslant t_3$ and $t_2 \leqslant t_3$). Let $(\Omega, \mathcal{F}, P)$ be a probability space. Sets and random variables are considered equal if they are equal almost surely. Let $\mathbf{X} = (X_t)$ be a family of random variables taking values in $\overline{\mathbb{R}}$. The *stochastic upper limit* of $\mathbf{X}$, $\tilde{X} = s \lim\sup X_t$, is the essential infimum of the set of random variables Y such that $\lim P(\{Y < X_t\}) = 0$. *The stochastic lower limit* of $\mathbf{X}$ is $s \lim\inf X_t = - s \lim\sup(-X_t)$. If $\mathbf{A} = (A_t)$ is a directed family of measurable sets, the *stochastic upper limit* of $\mathbf{A}$, $\tilde{A} = s \lim\sup A_t$, is defined by: $1_{s \lim\sup A_t} = s \lim\sup 1_{A_t}$.

In the following discussion the word "set" is often used for "measurable set". It is easy to see that $\tilde{A}$ is the smallest set C such that

$$\lim P(A_t \setminus C) = 0, \tag{2}$$

i.e., (2) holds iff $\tilde{A} \subset C$. Another characterization of $\tilde{A}$ is as follows.

LEMMA 1.1. $\tilde{A} = s \lim\sup A_t$ *is the largest set A such that for every nonempty subset B of A,*

$$\lim\sup P(A_t \cap B) > 0. \tag{3}$$

PROOF. Assume that $B \subset \tilde{A}$, and (3) fails. Since $P[A_t \setminus (\tilde{A} \setminus B)] \leqslant P(A_t \setminus \tilde{A}) + P(A_t \cap B)$, applying also (2) with $C = \tilde{A}$ we obtain that $\lim P[A_t \setminus (\tilde{A} \setminus B)] = 0$, hence $B = \varnothing$, because $\tilde{A}$ is the smallest set C satisfying (2). Conversely, let A be such that (3) holds for every nonempty subset B of A. Then $P[A_t \cap (A \setminus \tilde{A})] \leqslant P(A_t \setminus \tilde{A}) \to 0$ implies $A \setminus \tilde{A} = \varnothing$. $\square$

LEMMA 1.2. *Let $\tilde{A} = s \lim\sup A_t$, and let (s_n) be an arbitrary sequence of indices. Then there exists an increasing sequence of indices (t_n) such that $s_n \leqslant t_n$ and $\tilde{A} \subset \bigcup_{\mathbb{N}} A_{t_n}$.*

PROOF. Define on subsets of $\tilde{A}$ a function γ by $\gamma(B) = \lim\sup P(A_t \cap B)$. Set $B_1 = \tilde{A}$, and choose $t_1 \geqslant s_1$ such that $P(A_{t_1} \cap B_1) \geqslant \frac{1}{2}\gamma(B_1)$. Set $B_2 = \tilde{A} \setminus A_{t_1}$ and choose $t_2 \geqslant t_1$, $t_2 \geqslant s_2$, such that $P(A_{t_2} \cap B_2) \geqslant \frac{1}{2}\gamma(B_2)$. We define (t_n) and (B_n) by induction as follows: given $t_1, \cdots, t_n$ and $B_1, \cdots, B_n$, set $B_{n+1} = \tilde{A} \setminus \cup_{1 \leqslant i \leqslant n} A_{t_i}$ and choose $t_{n+1} \geqslant t_n$, $t_{n+1} \geqslant s_{n+1}$ and such that $P(A_{t_{n+1}} \cap B_{n+1}) \geqslant \frac{1}{2}\gamma(B_{n+1})$. Since $\cap B_n = \tilde{A} \setminus \cup A_{t_n}$, it suffices to show that $\cap B_n = \varnothing$. The sets $B_n \cap A_{t_n}$, $n \in \mathbb{N}$, are pairwise disjoint, hence

$$\limsup \gamma(B_n) \leqslant 2 \limsup P\big(B_n \cap A_{t_n}\big) = 0.$$

It follows that $\gamma(\cap B_n) = 0$, hence, by Lemma 1.1, $\cap B_n = \varnothing$. □

A *stochastic basis* $(\mathcal{F}_t)$ is an increasing family of sub σ-algebras of $\mathcal{F}$. A *process* $\mathbf{X} = (X_t)$ is a family of rv X_t such that each X_t is $\mathcal{F}_t$-measurable. The process is called *integrable* (*positive*) if for every t, X_t is integrable (positive). A family of sets $\mathbf{A} = (A_t)$ is *adapted* if for every t, $A_t \in \mathcal{F}_t$. A *stopping time* is a function $\tau : \Omega \to J$, such that for every $t \in J$, $\{\tau = t\} \in \mathcal{F}_t$. τ is called *simple* if it takes finitely many values; let T denote the set of simple stopping times; T is filtering to the right for the order $\leqslant$. An *ordered stopping time* is a simple stopping time τ such that the elements $t_1, t_2, \cdots, t_n$ in the range of τ are (linearly) ordered. Denote by T' the set of ordered stopping times. For any stopping time τ

$$X_\tau = \Sigma_t 1_{\{\tau = t\}} X_t,\ A_\tau = \cup_t\big[\{\tau = t\} \cap A_t\big],$$
$$\mathcal{F}_\tau = \{A \in \mathcal{F} | \forall t \in J, A \cap \{\tau = t\} \in \mathcal{F}_t\}.$$

We at first show that for every adapted family of sets $\mathbf{A}$, $s \limsup A_t$ can be approximated by A_τ with $\tau \in T'$, τ arbitrarily large. This may be interpreted to mean that the strongest "stochastic Vitali condition"—the stochastic analogue of the condition V'—holds for every stochastic basis.

PROPOSITION 1.3. *Let* $\mathbf{A} = (A_t)$ *be an adapted family of sets. For every* $\varepsilon > 0$ *and* $t_0 \in J$, *there exists a* $\tau \in T'$ *such that* $\tau \geqslant t_0$ *and* $P(s \limsup A_t \triangle A_\tau) < \varepsilon$.

PROOF. Given $\varepsilon > 0$, choose a sequence of indices (s_n) such that $s_n \geqslant t_0$ and $t \geqslant s_n$ implies $P(A_t \setminus \tilde{A}) \leqslant \varepsilon 2^{-(n+1)}$. Let (t_n) be a sequence obtained by application of Lemma 1.2, and choose k such that $P(\tilde{A} \setminus \cup_{1 \leqslant i \leqslant k} A_{t_i}) < \varepsilon/2$. Set $\tau = t_n$ on $A_{t_n} \setminus \cup_{i<n} A_{t_i}$ for $n \leqslant k$, and $\tau = t_{k+1}$ on $(\cup_{1 \leqslant n \leqslant k} A_{t_n})^c$. Then $P(\tilde{A} \triangle A_\tau) < \varepsilon$. □

We are now ready to prove the main result of the present section, a stochastic maximal inequality. Recall that T' is the smallest class of stopping times we consider—the class of simple ordered stopping times.

THEOREM 1.4. *Let* $\mathbf{X}$ *be a positive stochastic process. For every* $\lambda > 0$,

$$P\big[\{s \limsup X_t \geqslant \lambda\}\big] \leqslant \frac{1}{\lambda} \limsup_{\tau \in T'} EX_\tau.$$

PROOF. For every $\alpha > 0$, $\{s \limsup X_t > \lambda\} \subset s \limsup \{X_t > \lambda - \alpha\}$. Indeed, let $A_t = \{X_t > \lambda - \alpha\}$. Since $\tilde{A}$ is the smallest set C such that $\lim P(A_t \setminus C) = 0$, A^c is the largest set B such that $\lim P(B \cap \{X_t > \lambda - \alpha\}) = 0$. Hence on $\tilde{A}^c$ we have $\tilde{X} \leqslant \lambda - \alpha < \lambda$, so that $\tilde{A}^c \subset \{\tilde{X} < \lambda\}$. Given $\varepsilon > 0$ and $s \in J$, choose

$\tau \in T'$ such that $\tau \geq s$ and $P(\tilde{A} \setminus A_\tau) \leq \varepsilon$ (Proposition 1.3). Then $A_\tau = \{X_\tau > \lambda - \alpha\}$, hence,

$$P(\tilde{A}) \leq P(A_\tau) + \varepsilon \leq \frac{1}{\lambda - \alpha} EX_\tau + \varepsilon.$$

We deduce that for every $\varepsilon > 0$, $\alpha > 0$ and $s \in J$,

$$P(\tilde{X} \geq \lambda) \leq \frac{1}{\lambda - \alpha} \sup_{\tau \geq s} EX_\tau + \varepsilon.$$

The maximal inequality follows on letting $s \to \infty$, $\alpha \to 0$, $\varepsilon \to 0$. ▯

For submartingales and supermartingales, the net $(EX_\tau)_{\tau \in T'}$ is monotone, and $\lim_{T'} EX_\tau = \lim EX_t$ exists as an extended real number. Hence the corollary:

COROLLARY. *Let (X_t) be a positive submartingale or positive supermartingale. Then for every $\lambda > 0$*

$$P(s \limsup X_t \geq \lambda) \leq \frac{1}{\lambda} \lim E(X_t).$$

2. Maximal inequalities and convergence. Let $\mathbf{X}$ be a family of random variables. The essential upper limit of $\mathbf{X}$, $X^* = e \limsup X_t$, is defined by $X^* = e \inf_s (e \sup_{t \geq s} X_t)$. The *essential lower limit* of $\mathbf{X}$, $X_* = e \liminf X_t$, is $-e \limsup(-X_t)$. The directed family $\mathbf{X}$ is said to *converge essentially* if $X^* = X_*$; this common value is then called the *essential limit* of $\mathbf{X}$, $e \lim X_t$. If $\mathbf{A} = (A_t)$ is a directed family of measurable sets, the *essential upper limit* of $\mathbf{A}$, $A^* = e \limsup A_t$, is defined by $1_{A^*} = e \limsup 1_{A_t}$.

In [2] Burkholder proved that maximal inequalities for $\sup X_n$ could be deduced from almost sure convergence of the sequences (X_n) belonging to stochastically convex classes. The following generalizes Theorem 1 in [2]; the proof, similar to Burkholder's, is omitted. We consider processes (X_t) indexed by a fixed directed set J. The null process $\mathbf{X} = \mathbf{0}$ is defined by $X_t = 0$ for every t. Given two processes $\mathbf{X}$ and $\mathbf{Y}$, we write $\mathbf{X} \leq \mathbf{Y}$ iff $X_t \leq Y_t$ for every t, and $\mathbf{X} \sim \mathbf{Y}$ iff $\mathbf{X}$ and $\mathbf{Y}$ have the same joint distribution.

THEOREM 2.1. *Let F be a map from the set of positive processes into the set of measurable functions taking on values in $\mathbb{R}^+ \cup \{+\infty\}$. Assume that:*

(i) *for every positive process $\mathbf{X}$, $F(\mathbf{X})$ is measurable with respect to the σ-algebra generated by the family of* rv's X_t, $t \in J$;

(ii) $F(\mathbf{X}) \leq F(\mathbf{Y})$ *if* $\mathbf{X} \leq \mathbf{Y}$;

(iii) *for any* $\alpha \geq 0$ *and* $\mathbf{X} \geq \mathbf{0}$, $F(\alpha \mathbf{X}) = \alpha F(\mathbf{X})$.

Let $\mathcal{C}$ be a class of positive processes such that for every sequence $(\mathbf{X}_k)$ of elements of $\mathcal{C}$, there exists a sequence $(\mathbf{Y}_k)$ of processes satisfying the following conditions:

(a) *the processes $\mathbf{Y}_k$ are stochastically independent;*

(b) *for every $k = 1, 2, \cdots$, $\mathbf{X}_k \sim \mathbf{Y}_k$;*

(c) *for any sequence of positive numbers (α_k) such that $\sum_{k=1}^{\infty} \alpha_k = 1$, $P[\{\sup \alpha_k F(\mathbf{Y}_k) < \infty\}] > 0$.*

Then there exists a constant K such that for every $\lambda > 0$ *and every* $\mathbf{X} \in \mathcal{C}$, $P(\{F(\mathbf{X}) \geqslant \lambda\}) \leqslant K/\lambda$.

It is easy to see that the maps F defined by $F(\mathbf{X}) = e \sup X_t$, $F(\mathbf{X}) = e \limsup X_t$, and $F(\mathbf{X}) = s \limsup X_t$ satisfy the conditions (i), (ii) and (iii) of the theorem. Notice that the conditions (a), (b) and (c) on the class $\mathcal{C}$ hold if $\mathcal{C}$ is a *stochastically convex class* satisfying the finiteness condition $P(\{F(\mathbf{X}) < \infty\}) > 0$ for $\mathbf{X} \in \mathcal{C}$. A class $\mathcal{C}$ is called stochastically convex if it satisfies the conditions (a), (b) of Proposition 2.1, and the following condition (d): for any sequence (a_k) of positive numbers with $\Sigma_{k=1}^{\infty} a_k = 1$, there exists $\mathbf{Z} \in \mathcal{C}$ such that $\mathbf{Z} \sim \Sigma_{k=1}^{\infty} a_k \mathbf{Y}_k$, (see [2]).

As an application of Theorem 2.1, we consider the case of *weak martingales*, i.e., sequences (X_n) such that $E(X_p|X_n) = X_n$ for all $n, p \in N$, $n < p$ (cf. Nelson [19]). The class of positive weak martingales such that $E(X_n) = 1$ is stochastically convex; Burkholder's proof of stochastic convexity of *martingales* also proves this. Since L^1 bounded weak martingales converge stochastically (this nontrivial result was proved by F. Knight; for Burkholder's somewhat different proof see [19]), it follows that there exists a constant K such that for every positive weak martingale $\mathbf{X}$ and every constant $\lambda > 0$,

$$P(s \limsup X_n \geqslant \lambda) \leqslant \frac{K}{\lambda} E(X_1).$$

We now derive a stochastic maximal inequality for Cesàro averages of iterates of contractions of L^1.

PROPOSITION 2.2. *Let* $(\Omega, \mathcal{F}, P)$ *be the unit interval with Borel sets and Lebesgue measure. There exists a constant K such that for every linear contraction T of* L^1, *for every function* $f \in L^1$, *and for every constant* $\lambda > 0$,

$$P\left[\left\{s \limsup\left|\frac{1}{n}\Sigma_{i=0}^{n-1} T^i f\right| \geqslant \lambda\right\}\right] \leqslant \frac{K}{\lambda} E|f|.$$

PROOF. Krengel [11] showed that for every linear contraction T of L^1 and any function $f \in L^1$, the sequence $(1/n)\Sigma_{k=0}^{n-1} T^k f$ converges in probability. Considering, instead of the operator T, its modulus (cf. [4]), we can assume, without loss of generality, that T is positive. For a positive sequence $\mathbf{X} = (X_n)$, set $F(\mathbf{X}) = s \limsup X_n$. Let $\mathcal{C}$ be the class of sequences $(T^n f)$, where $0 \leqslant f$, $Ef \leqslant 1$, and T is a positive linear contraction of L^1 (which may change from sequence to sequence). If the class $\mathcal{C}$ is stochastically convex, so is the class $\bar{\mathcal{C}}$ of Cesàro averages of $T^n f$ under the same assumptions on f and T (cf. [2], Theorem 3). The proposition is then deduced from Theorem 2.1 applied to F and $\bar{\mathcal{C}}$. The proof of stochastic convexity of $\mathcal{C}$ is similar to the proof of Theorem 5, [2], and is therefore omitted. ▯

It would be of interest to give a direct proof of Proposition 2.2, independent of Krengel's theorem which at present can be only derived using deep pointwise results, e.g., the Chacon-Ornstein theorem. Such a proof may also give the best value of the constant K.

3. Vitali condition V and maximal inequalities. A stochastic basis $(\mathfrak{F}_t)$ satisfies the *Vitali condition* $V(=V_\infty)$ iff for every adapted family of sets (A_t) and for every $\varepsilon > 0$, there exists a simple stopping time $\tau \in T$ such that $P(e \limsup A_t \setminus A_\tau) < \varepsilon$. This condition, introduced by Krickeberg [12] (in an obviously equivalent form not involving stopping times; see also Neveu [20], page 99), may be characterized in terms of convergence of various classes of random variables (cf. [1], [15], [18]). Here we characterize V by maximal inequalities involving the essential lim sup; also other equivalent conditions are given. Recall that A^* is the set $e \limsup A_t$.

THEOREM 3.1. *Let $(\mathfrak{F}_t)$ be a stochastic basis. The following properties are equivalent.*

(1) *$(\mathfrak{F}_t)$ satisfies the Vitali condition V.*

(2) *For any adapted family of sets $\mathbf{A}$ and any $\varepsilon > 0$, there exists $\tau \in T$ such that $P(A^* \triangle A_\tau) < \varepsilon$.*

(3) *For any adapted family of sets $\mathbf{A}$ and any $\varepsilon > 0$, there exists $\tau \in T$ such that $P(A^*) - P(A_\tau) < \varepsilon$.*

(4) *There exists a constant α, $0 < \alpha < 1$, such that for each adapted family of sets $\mathbf{A}$, there exists $\tau \in T$ such that $P(A^* \cap A_\tau) \geqslant \alpha P(A^*)$.*

(5) *For each adapted family of sets $\mathbf{A}$,*

$$s \limsup_{\tau \in T} A_\tau = e \limsup A_t (= e \limsup_{\tau \in T} A_\tau).$$

(6) *For each stochastic process $\mathbf{X}$,*

$$s \limsup_{\tau \in T} X_\tau = e \limsup X_t (= e \limsup_{\tau \in T} X_\tau).$$

(7) *For each positive stochastic process $\mathbf{X}$,*

$$\forall \lambda > 0,\ P[\{e \limsup X_t \geqslant \lambda\}] \leqslant \frac{1}{\lambda} \limsup_{\tau \in T} EX_\tau.$$

(8) *There exists a constant $K > 0$ such that for every adapted family of sets $\mathbf{A} = (A_t)$,*

$$P(e \limsup A_t) \leqslant K \limsup_{\tau \in T} P(A_\tau).$$

PROOF. Obviously (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (3). It is easy to see that given any $s \in J$, we may require the stopping times given by the conditions (1), (2) or (3) to be larger than s. (Set $B_t = A_t$ if $t \geqslant s$, and $B_t = \varnothing$ otherwise.)

(3) $\Rightarrow$ (2). Given $\varepsilon > 0$, choose $s \in J$ such that $P[e \sup_{t \geqslant s} A_t \setminus A^*] \leqslant \varepsilon$. Then

$$\begin{aligned} P[A^* \triangle A_\tau] &= P(A^* \setminus A_\tau) + P(A_\tau \setminus A^*) \\ &\leqslant P[e \sup_{t \geqslant s} A_t \setminus A_\tau] + P[e \sup_{t \geqslant s} A_t \setminus A^*] \\ &\leqslant P[e \sup_{t \geqslant s} A_t] - P[A_\tau] + \varepsilon \\ &\leqslant P(A^*) - P(A_\tau) + 2\varepsilon \leqslant 3\varepsilon. \end{aligned}$$

(1) $\Rightarrow$ (5). Since $e \limsup A_t = e \limsup_{\tau \in T} A_\tau \supset s \limsup_{\tau \in T} A_\tau$, we only have to show that $e \limsup A_t \subset s \limsup_{\tau \in T} A_\tau$. Assume that $A^* \neq \varnothing$, and let B be a nonempty subset of A^*. Given any ε, $0 < \varepsilon < P(B)$, and any $s \in J$, there

exists $\tau \in T$ such that $P(B \setminus A_\tau) < \varepsilon$. Hence $P[B \cap A_\tau] \geqslant P(B) - \varepsilon$, so that $\limsup P[A_\tau \cap B] \geqslant P(B) - \varepsilon > 0$. Lemma 1.1 now implies $e \limsup A_t \subset s \limsup_{\tau \in T} A_\tau$.

(5) $\Rightarrow$ (6). Similarly, since $e \limsup X_t = e \limsup_{\tau \in T} X_\tau$, we only have to prove that $e \limsup X_t \leqslant s \limsup X_\tau$. Using (5), we easily obtain that for every λ

$$\{e \limsup X_t > \lambda\} \subset e \limsup\{X_t > \lambda\} \subset s \limsup_{\tau \in T}\{X_\tau > \lambda\}$$
$$\subset \{s \limsup_{\tau \in T} X_\tau \geqslant \lambda\}.$$

This implies that $e \limsup X_t \leqslant s \limsup_{\tau \in T} X_\tau$.

(6) $\Rightarrow$ (7). Apply Theorem 1.4 to the process $(X_\tau)_{\tau \in T}$, adapted to the stochastic basis $(\mathfrak{F}_\tau)$. Since $T'(\mathfrak{F}_\tau) \subset T(\mathfrak{F}_\tau)$, one has for each $\lambda > 0$

$$P[\{s \limsup X_\tau \geqslant \lambda\}] \leqslant \frac{1}{\lambda} \limsup_{\theta \in T(\mathfrak{F}_\tau)} EX_\theta.$$

Given any simple stopping time θ of the stochastic basis $(\mathfrak{F}_\tau)$, there exists a stopping time σ of the stochastic basis $(\mathfrak{F}_t)$ such that $(X_\tau)_\theta = (X_t)_\sigma$: set for each s, $\{\sigma = s\} = \cup_{\tau \in T}[\{\theta = \tau\} \cap \{\tau = s\}]$. Hence, for any positive process (X_t) and any $\lambda > 0$,

$$P[\{e \limsup X_t \geqslant \lambda\}] = P[\{s \limsup X_\tau \geqslant \lambda\}] \leqslant \frac{1}{\lambda} \limsup_{\tau \in T} EX_\tau.$$

(7) $\Rightarrow$ (8). Apply (7) with $(X_t) = (1_{A_t})$ and $\lambda = 1$.

(8) $\Rightarrow$ (4). We may and do assume that K is $\geqslant 1$, and that given the family $\mathbf{A}$, $A^* \neq \varnothing$. Choose ε with $0 < \varepsilon < P(A^*)/3K$, and let s be such that $P(A^*) \geqslant P(e \sup_{t \geqslant s} A_t) - \varepsilon$. Choose $\tau \in T$ such that $\tau \geqslant s$ and $|P(A_\tau) - \limsup_{\sigma \in T} P(A_\sigma)| < \varepsilon$. Then

$$\begin{aligned} P(A^* \cap A_\tau) &\geqslant P[e \sup_{t \geqslant s} A_t \cap A_\tau] - P[e \sup_{t \geqslant s} A_t \setminus A^*] \\ &\geqslant P(A_\tau) - \varepsilon \\ &\geqslant \limsup_{\sigma \in T} P(A_\sigma) - 2\varepsilon \\ &\geqslant (1/K)P(A^*) - 2\varepsilon \geqslant (1/3K)P(A^*). \end{aligned}$$

(4) $\Rightarrow$ (1). Let $\tau_1 \in T$ be such that $P(A^* \cap A_{\tau_1}) \geqslant \alpha P(A^*)$. Let s_2 be larger than τ_1, and set $A_t^1 = A_t \setminus A_{\tau_1}$ for $t \geqslant s_2$, and $A_t^1 = \varnothing$ otherwise. Since $A^* \setminus A_{\tau_1} = e \limsup A_t^1$, there exists $\tau_2 \in T$ such that $\tau_2 \geqslant s_2$ and $P[(A^* \setminus A_{\tau_1}) \cap A_{\tau_2}] \geqslant \alpha P(A^* \setminus A_{\tau_1})$. One defines by induction a sequence (τ_n) of stopping times satisfying the relations $\tau_n \geqslant s_n$, and $P[A^* \setminus \cup_{j \leqslant n} A_{\tau_j}] \leqslant (1 - \alpha)^n P(A^*)$ for all n. Given $\varepsilon > 0$, choose n such that $(1 - \alpha)^n P(A^*) < \varepsilon$. Choose $s \geqslant \tau_n$. For every $j \leqslant n$ and every $t \in R(\tau_j)$, set $\{\tau = t\} = \{\tau_j = t\} \cap A_t \cap (\cup_{k<j} A_{\tau_k})^c$; set $\tau = s$ on $(\cup_{k \leqslant n} A_{\tau_k})^c$. Then $\tau \in T$ and $P(A \setminus A_\tau) \leqslant \varepsilon$. □

REMARK. Theorem 3.1 gives an alternative proof of the necessity of the Vitali condition V for the essential convergence of L^∞-bounded *amarts* proved by Astbury [1]. An integrable stochastic process $\mathbf{X}$ is an *amart* iff the net $(EX_\tau)_{\tau \in T}$ converges. Let $(\mathfrak{F}_t)$ be a stochastic basis such that every L^∞-bounded amart converges essentially, and let **A** be an adapted family of sets. Set $X_t = e\sup_{\tau \geqslant t} P^{\mathcal{F}_t} A_\tau$; since for each $\sigma \in T$ there exists a sequence $\tau_n \in T$, $\tau_n \geqslant \sigma$, such that $X_\sigma = \lim \nearrow P^{\mathcal{F}_\sigma} A_{\tau_n}$, the net $(EX_\tau)_{\tau \in T}$ is increasing. Thus, (X_t) is an amart. (In fact, it is also a supermartingale with the optional sampling property.) Hence X_t converges essentially, so that we deduce from Theorem 1.4 that

$$P[\{e \limsup X_t \geqslant 1\}] \leqslant \limsup_{\tau \in T} EX_\tau.$$

Since $X_t \geqslant 1_{A_t}$ for every t, $P[A^*] \leqslant \lim_T EX_\tau$. For every $\sigma \in T$ such that $\sigma \geqslant s$, there exists a sequence τ_n of stopping time such that $\tau_n \geqslant s$ and $EX_\sigma = \lim \nearrow P(A_{\tau_n})$. Hence $\sup_{\sigma \geqslant s} EX_\sigma \leqslant \sup_{\tau \geqslant s} P(A_\tau)$, so that

$$P(A^*) \leqslant \lim_{\tau \in T} EX_\tau \leqslant \limsup_{\tau \in T} P(A_\tau),$$

which shows that the conditon (8) of Theorem 3.1 is satisfied.

PROPOSITION 3.2. *If V holds and (X_t) is a martingale taking values in a Banach space with norm $|\ |$, then for each $\lambda \geqslant 0$,*

$$P(\{e \limsup |X_t| \geqslant \lambda\}) \leqslant \frac{1}{\lambda} \lim E|X_t|.$$

PROOF. If (X_t) is a martingale, so is $(X_t)_{\tau \in T}$, hence $(|X_\tau|)_{\tau \in T}$ is a submartingale; therefore $\limsup E|X_\tau| = \lim E|X_t|$. Now apply (7) to the positive process $(|X_t|)$. ▯

4. Vitali conditions V' and V_p, $1 \leqslant p < \infty$. In this section we prove that the Vitali conditions V'(resp. V_p) can be characterized in terms of maximal inequalities for X_τ, where τ is an ordered (resp. multivalued) stopping time. Also nonasymptotic regularity conditions similar to Vitali conditions are shown to be equivalent to maximal (nonasymptotic) inequalities.

A stochastic basis $(\mathcal{F}_t)$ satisfies the *ordered Vitali condition V'* if for every adapted family of sets (A_t) and for every $\varepsilon > 0$, there exists an ordered stopping time $\tau \in T'$ such that $P[e \limsup A_t \setminus A_\tau] < \varepsilon$. This was shown by Krickeberg to be sufficient for essential convergence of L^1-bounded submartingales (cf. [10]); it is necessary and sufficient for essential convergence of L^1-bounded ordered amarts ([15], [18]). Given σ, τ in T', we write $\sigma <|< \tau$ if there exists $s \in J$ such that $\sigma \leqslant s \leqslant \tau$. For the partial order $<|<$, T' is a directed set filtering to the right.

THEOREM 4.1. *Identical to Theorem* 3.1, *with V' replacing V, and T' replacing T.*

PROOF. Analogous to that of Theorem 3.1. In particular, (6) ⇒ (7) still holds, because if θ is an ordered stopping time for the stochastic basis $(\mathcal{F}_\tau)_{\tau \in T'}$, then σ defined by

$$\{\sigma = s\} = \cup_{\tau \in T'} \{\theta = \tau\} \cap \{\tau = s\}$$

is an ordered stopping time for $(\mathcal{F}_t)$. ▯

Denote by $\mathcal{J}$ the set of finite subsets of J. A *multivalued stopping time* is a map τ from Ω to $\mathcal{J}$ such that for every $t \in J$, $\{\tau = t\} \equiv \{\omega \in \Omega | t \in \tau(\omega)\} \in \mathcal{F}_t$. Denote by M the set of multivalued stopping times; if σ and τ are elements of M, we say that $\sigma \leqslant \tau$ if $\forall s, \forall t, \{\sigma = s\} \cap \{\tau = t\} \neq \varnothing$ implies that $s \leqslant t$. Let $\tau \in M$; the *excess function* of τ is $e_\tau = \Sigma_{t \in J} 1_{\{\tau = t\}} - 1$; the *overlap of order* p of τ, $1 \leqslant p \leqslant \infty$, is $0_p(\tau) = \|e_\tau\|_p$. If $\mathbf{X} = (X_t)$ is a stochastic process, let $X_\tau = \Sigma_{t \in J} 1_{\{\tau = t\}} X_t$. For an adapted family of sets $\mathbf{A} = (A_t)$, $A_\tau = \cup_t [\{\tau = t\} \cap A_t]$. Set

$$\limsup_{\tau \in M_p} EX_\tau = \inf_{s \in J, \alpha > 0} \sup\left\{ EX_\tau | \tau \in M, \tau \geqslant s, 0_p(\tau) < \alpha \right\}.$$

Similarly, $s \limsup_{\tau \in M_p} X_\tau$ is the essential infimum of the set of random variables Y such that

$$\lim_{\tau \in M, 0_p(\tau) \to 0} P\left[\{X_\tau > Y\} \right] = 0,$$

(i.e., $\forall \varepsilon > 0, \exists \alpha > 0, \exists s \in J$ such that for every $\tau \in M$ satisfying $\tau \geqslant s$ and $0_p(\tau) < \alpha$, we have $P[\{X_\tau < Y\}] < \varepsilon$). Furthermore, $e \limsup_{\tau \in M_p} X_\tau = e \inf_{\alpha > 0, s \in J}[e \sup X_\tau, \tau \in M, \tau \geqslant s, 0_p(\tau) < \alpha]$. A stochastic basis $(\mathcal{F}_t)$ satisfies the *Vitali condition* V_p, $1 \leqslant p < \infty$, if for any adapted family of sets $\mathbf{A} = (A_t)$ and any $\varepsilon > 0$, there exists $\tau \in M$ such that $0_p(\tau) < \varepsilon$ and $P[e \limsup A_t \setminus A_\tau] < \varepsilon$. These conditions are sufficient (Krickeberg; see e.g., [10], page 169), and necessary ([10], page 170 and [14]) for the essential convergence of L^q-bounded martingales, and necessary and sufficient for the essential convergence of L^1-bounded amarts for M_p [16], [17]. The following theorem characterizes V_p in terms of maximal inequalities.

THEOREM 4.2. *Let* $(\mathcal{F}_t)$ *be a stochastic basis, and let* $1 \leqslant p < \infty$. *The following properties are equivalent*:

(1) $(\mathcal{F}_t)$ *satisfies the Vitali condition* V_p.

(2) *For any adapted family of sets* $\mathbf{A}$ *and any* $\varepsilon > 0$, *there exists* $\tau \in M$ *such that* $0_p(\tau) < \varepsilon$ *and* $P(A^* \setminus A_\tau) < \varepsilon$.

(3) *For any adapted family of sets* $\mathbf{A}$ *and any* $\varepsilon > 0$, *there exists* $\tau \in M$ *such that* $0_p(\tau) < \varepsilon$ *and* $P(A^*) - P(A_\tau) < \varepsilon$.

(4) *There exists a constant* α, $0 < \alpha < 1$, *such that for each adapted family of sets* $\mathbf{A}$ *and every* $\varepsilon > 0$, *there exists* $\tau \in M$ *such that* $0_p(\tau) < \varepsilon$ *and* $P(A^* \cap A_\tau) \geqslant \alpha P(A^*)$.

(5) *For each adapted family of sets* $\mathbf{A}$,

$$s \limsup_{\tau \in M_p} A_\tau = e \limsup A_t \left(= e \limsup_{\tau \in M_p} A_\tau\right).$$

(6) *For each positive stochastic process* $\mathbf{X}$ *and for every* $\lambda > 0$,

$$P\left[\{e \limsup X_t \geqslant \lambda\} \right] \leqslant \frac{1}{\lambda} \limsup_{\tau \in M_p} X_\tau.$$

(7) *There exists a constant* $K > 0$ *such that for every process* (1_{A_t}),

$$\forall \lambda > 0, P\left[e \limsup A_t \geqslant \lambda \right] \leqslant \frac{K}{\lambda} \limsup_{\tau \in M_p} P(A_\tau).$$

PROOF. The proof is similar to that of Theorem 3.1. We only observe that also for multivalued stopping times with overlap converging to zero, one has that if $\tau \geqslant s$, then $A_\tau \subset e \sup_{\tau \geqslant s} A_t$, and $e \limsup_{\tau \in M_p} A_\tau = e \limsup A_t$, $e \limsup_{\tau \in M_p} X_\tau$

$= e \lim \sup X_t$. A point where the proof is now slightly different is in the implication (4) $\Rightarrow$ (1): In addition to other properties, the sequence (τ_n) has now to be chosen so that $0_p(\tau_n) < \varepsilon 2^{-n}$. □

We finally introduce nonasymptotic analogues R_α of Vitali conditions. The σ-algebras $(\mathcal{F}_t)$ are said to satisfy the *regularity conditions* $R_\alpha(R'_\alpha)$ where α is a fixed positive number, if for every $\varepsilon > 0$ and every adapted family of sets $\mathbf{A} = (A_t)$, there exists $\tau \in T$ (resp. $\tau \in T'$), such that $P(A_\tau) \geqslant \alpha P(e \sup A_t) - \varepsilon$.

PROPOSITION 4.3. *A stochastic basis* $(\mathcal{F}_t)$ *satisfies* R_α (resp. R'_α) *if and only if for every positive stochastic process* (X_t) *and every* $\lambda > 0$,

$$P(e \sup X_t \geqslant \lambda) \leqslant \frac{1}{\lambda\alpha} \sup_{\tau \in T^*} EX_\tau,$$

where $T^* = T$ (resp. $T^* = T'$).

PROOF. Assume $(\mathcal{F}_t)$ satisfies R_α (R'_α). Given $\lambda > 0$ and ε such that $0 < \varepsilon < \lambda$, set $A_t = \{X_t > \lambda - \varepsilon\}$. Then there exists $\tau \in T^*$ such that $P(A_\tau) \geqslant \alpha P(e \sup A_t) - \varepsilon$. Hence

$$P(e \sup X_t \geqslant \lambda) \leqslant \frac{1}{\alpha} P(A_\tau) + \frac{\varepsilon}{\alpha} \leqslant \frac{1}{\alpha(\lambda - \varepsilon)} \sup_{\tau \in T^*} E(X_\tau) + \frac{\varepsilon}{\lambda}.$$

The maximal inequality follows on letting $\varepsilon \to 0$. Conversely, the maximal inequality applied to the process $(X_t) = (1_{A_t})$ with $\lambda = 1$ gives: $\alpha P(e \sup A_t) \leqslant \sup_{\tau \in T^*} P(A_\tau)$. Given $\varepsilon > 0$, choose $\tau \in T^*$ such that $\sup_{\sigma \in T^*} P(A_\sigma) - P(A_\tau) < \varepsilon$; now $P(A_\tau) \geqslant \alpha P(e \sup A_t) - \varepsilon$. □

If $J = \mathbb{N}$, then $T = T'$ and R_α holds with $\alpha = 1$. The corresponding maximal inequality was in that case observed in [5].

Added in proof. One can show that all L^1-bounded real valued martingales converge essentially if and only if for every real martingale (X_t) and every $a > 0$, one has $a\, P(e \lim \sup |X_t| \geqslant a) \leqslant \lim E|X_t|$. See our paper "On convergence of L^1-bounded martingales indexed by directed sets", to appear in the first volume of the new Polish J. Probability Math. Statist., 1980.

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